



Asymptotic behavior for coupled systems of second order abstract evolution equations with one infinite memory [☆]



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ARTICLE INFO

Article history:

Received 8 April 2017

Available online 25 February 2019

Submitted by A. Lunardi

Keywords:

Abstract evolution equations

Coupled systems

Asymptotic behavior

Energy decay estimate

Damping effect

Infinite memory

ABSTRACT

We are concerned with the asymptotic behavior for a coupled system of two second order abstract evolution equations with one infinite memory and the damping effect given by only one equation, which describes the dynamics of many viscoelastic systems and covers the well-known Timoshenko system. We present an optimal energy decay estimate under very weak conditions. More specifically, we prove that the energy of the system decays to zero at least at the rate of t^{-1} only under the condition that the related kernel functions are non-increasing and integrable. Moreover, we apply our abstract results to three classes of coupled systems of second-order partial differential equations arising in viscoelasticity.

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1. Introduction

We consider the following Cauchy problem for a coupled system of two second order abstract evolution equations with one infinite memory:

$$u''(t) + A_1 u(t) - \int_0^{+\infty} g(s) A u(t-s) ds + B v(t) = 0, \quad (1.1)$$

$$v''(t) + A_2 v(t) + B u(t) = 0, \quad (1.2)$$

$$u(-t) = u_0(t), \quad \forall t \geq 0, \quad u'(0) = u_1, \quad (1.3)$$

$$v(0) = v_0, \quad v'(0) = v_1. \quad (1.4)$$

[☆] The work was supported partly by the NSF of China (11571229, 11771091, 11831011).

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Here A, A_1, A_2 are positive self-adjoint linear operators in a Hilbert space H , B a positive self-adjoint, bounded linear operator in H , and $g(t)$ a nonnegative, locally absolutely continuous, and non-increasing function on $[0, +\infty)$.

The purpose of this paper is to study the asymptotic behavior of the above system when time goes to infinity. It is well known that the infinite memory term $\int_0^{+\infty} g(s)Au(t-s)ds$ produces damping mechanism.

In [8], Rivera and Naso studied the following single equation

$$Du''(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds + Cu'(t) = 0;$$

here A, B, C, D are four positive definite self-adjoint operators, and $\int_0^{+\infty} g(s)Bu(t-s)ds$, $u'(t)$ are two damping terms. Under some restrictions on A, B, C, D and the condition

$$g'(t) \leq -kg(t), \quad k > 0, \quad (1.5)$$

some decay rates were established (see [8, Theorems 2 and 9] for details).

In [7,11,12], the authors considered the abstract system

$$\begin{aligned} u''(t) + Au(t) - \int_0^t g(t-s)Au(s)ds + \alpha u(t) + \beta Bv(t) &= 0, \\ v''(t) + Av(t) + \beta Bu(t) &= 0 \end{aligned}$$

with one finite memory, where A is a positive self-adjoint linear operator in H , and B a symmetric linear operator in H comparable to $\sqrt{A}$. The Timoshenko system in mechanics is a prototype of the above abstract system.

Moreover, Almeida and Santos ([1]) investigated the stability of the C_0 -semigroups associated with a coupled system of wave equations (only one equation containing memory), and they obtained the decay rate $\frac{1}{t}$ under some conditions. The model in [1] describes the evolution of a system consisting of two elastic membranes subject to an elastic force that attracts one membrane to the other, and the integral term in the model acts on one of the membranes as a stabilizer.

Recently, Guesmia ([5]) studied the well-posedness and asymptotic behavior for two coupled abstract linear evolution equations, which describes the behaviors of some viscoelastic solids, a generalized Kirchhoff viscoelastic beam with memory, and systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model. Under a boundedness condition on the past history data, Guesmia proved that the stability of the system holds for convolution kernels having much weaker decay rates than the exponential one.

Actually, over the last decade, people have established many results for the case of the nonnegative, integrable, decreasing kernels $g(t)$ with some additional assumptions on its derivative $g'(s)$, for example, assumptions as follows (see [1–5,9–12] and references therein):

(1) the kernel $g(t) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is nonnegative, with some other conditions, especially, it satisfies

$$g'(t) \leq -kg^p(t), \quad k > 0, \quad 1 \leq p < 2,$$

or

(2) there exists an increasing, strictly convex, and continuously differentiable function G on $[0, g(0)]$, with $G(0) = G'(0) = 0$, $\lim_{t \rightarrow +\infty} G'(t) = +\infty$ such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty, \quad (1.6)$$

or

- (3) there exists a nonnegative, strictly convex, and continuously differentiable function K on $[0, g(0)]$, with $K(0) = K'(0) = 0$, such that

$$g'(s) \leq -K(g(t)).$$

But for the more general decreasing/integrable kernels without restrictions on $g'(t)$, there is not so much work (cf., e.g., [6,7]). Especially, what are the asymptotic behaviors of the system (1.1)-(1.4) with these general kernel functions? It is still an unsolved problem. In this paper, by seeking some new auxiliary functions and using higher order energy methods, we obtain a solution for this problem and show, for general decreasing/integrable kernel functions, that the energy of the system decays at the rate of $\frac{1}{t}$, which is optimal.

The paper is organized as follows. Section 2 gives an existence result for solutions of the system (1.1)-(1.2). The optimal energy decay estimate for this system (the main theorem) is presented in Section 3. Then, Section 4 is devoted to the proof of the theorem. In the final section, we apply our abstract result to three classes of coupled systems of second order partial differential equations arising in viscoelasticity.

2. Existence of solutions

To start with, we make the following assumptions:

- (I₁) A, A_1, A_2 and B are positive self-adjoint linear operators in a Hilbert space H , with domains

$$\mathcal{D}(A_2) \subseteq \mathcal{D}(A_1) = \mathcal{D}(A) \subseteq \mathcal{D}(B) = H,$$

satisfying

$$\begin{aligned} a_i \langle A_i u, u \rangle &\geq \|u\|^2, & u &\in \mathcal{D}(A_i), \\ c_1 \left\| \sqrt{A_1} u \right\|^2 &\leq \left\| \sqrt{A} u \right\|^2 \leq \left\| \sqrt{A_1} u \right\|^2, & u &\in \mathcal{D}(A), \\ \left\| \sqrt{A_1} u \right\|^2 &\leq c_2 \left\| \sqrt{A_2} u \right\|^2, & u &\in \mathcal{D}(A_2), \\ d_1 \|u\|^2 &\leq \left\| \sqrt{B} u \right\|^2 \leq d_2 \|u\|^2, & u &\in H, \end{aligned} \quad (2.1)$$

where $a_i > 0, c_i > 0, d_i > 0$ ($i = 1, 2$) are constants.

Note that condition (2.1) implies $\mathcal{D}(\sqrt{A_1}) = \mathcal{D}(\sqrt{A})$.

- (I₂) $g(s): [0, \infty) \rightarrow [0, \infty)$ is a non-increasing and locally absolutely continuous function satisfying

$$g(0) > 0, \quad 1 - h(0) - a_1 a_2 d_2^2 > 0,$$

and $m(\Lambda) = 0$; here $\Lambda := \{s \geq 0; g(s) > 0, g'(s) = 0\}$, and $h(0) := \int_0^{+\infty} g(s) ds$.

A pair (u, v) of functions is called a (classical) solution of (1.1)-(1.4) on $[0, +\infty)$, if

$$\begin{aligned} u &\in C^2([0, +\infty), H) \cap C^1([0, +\infty), [\mathcal{D}(\sqrt{A_1})]) \cap C([0, +\infty), [\mathcal{D}(A_1)]), \\ v &\in C^2([0, +\infty), H) \cap C^1([0, +\infty), [\mathcal{D}(\sqrt{A_2})]) \cap C([0, +\infty), [\mathcal{D}(A_2)]), \end{aligned}$$

satisfying (1.1)-(1.4) for $t \in [0, +\infty)$.

We define the energy of a solution (u, v) of the system (1.1)-(1.4) as

$$\begin{aligned} E(t) &= \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|v'(t)\|^2 + \frac{1}{2} \left\| \sqrt{A_1} u(t) \right\|^2 - \frac{h(0)}{2} \left\| \sqrt{A} u(t) \right\|^2 + \frac{1}{2} \left\| \sqrt{A_2} v(t) \right\|^2 \\ &\quad + \frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A} u(t-s) - \sqrt{A} u(t) \right\|^2 ds + \langle Bu(t), v(t) \rangle. \end{aligned}$$

From Assumptions (I₁) and (I₂), it is clear that there are positive constants $c_0, d_0 > 0$ such that

$$c_0 \tilde{E}(t) \leq E(t) \leq d_0 \tilde{E}(t), \quad (2.2)$$

where

$$\begin{aligned} \tilde{E}(t) &:= \|u'(t)\|^2 + \|v'(t)\|^2 + \left\| \sqrt{A_1} u(t) \right\|^2 + \left\| \sqrt{A_2} v(t) \right\|^2 \\ &\quad + \int_0^{+\infty} g(s) \left\| \sqrt{A} u(t-s) - \sqrt{A} u(t) \right\|^2 ds. \end{aligned}$$

Now, denote the space

$$L_g^2(\mathbb{R}^+, \mathcal{D}(\sqrt{A})) = \left\{ w : \mathbb{R}^+ \rightarrow \mathcal{D}(\sqrt{A}); \int_0^{+\infty} g(t) \left\| \sqrt{A} w(t) \right\|^2 dt < +\infty \right\},$$

and $\mathcal{U} = (u, v, u_t, v_t, \eta(t, \cdot))^T$, $\mathcal{U}_0 = (u_0(0), v(0), u_1, v_1, \eta_0)^T \in \mathcal{H}$, where

$$\begin{cases} \eta(t, s) := u(t) - u(t-s), & \forall t, s \in \mathbb{R}^+, \\ \eta_0(s) := \eta(0, s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}^+, \end{cases}$$

and $\mathcal{H} := \mathcal{D}(\sqrt{A_1}) \times \mathcal{D}(\sqrt{A_2}) \times H \times H \times L_g^2(\mathbb{R}^+, \mathcal{D}(\sqrt{A}))$.

Then, we can formulate the system (1.1)-(1.4) as following:

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t), & t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \quad (2.3)$$

where $\mathcal{A}$ is a linear operator given by

$$\mathcal{A} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ (-A_1 + h(0)A)w_1 - \int_0^{+\infty} g(s)Aw_5(s)ds - Bw_2 \\ -A_2w_2 - Bw_1 \\ -\partial_s w_5 + w_3 \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathcal{H}; \quad \omega_1 \in D(A), w_2 \in \mathcal{D}(A_2), \\ w_3 \in \mathcal{D}(\sqrt{A_1}), w_4 \in \mathcal{D}(\sqrt{A_2}), \omega_5(\cdot) \in D(A), \partial_s w_5 \in L_g^2(\mathbb{R}^+, \mathcal{D}(\sqrt{A})), \\ w_5(0) = 0, (A_1 - h(0)A)w_1 + \int_0^{+\infty} g(s)Aw_5(s)ds \in H \end{array} \right\}.$$

Then, we have the following global existence result.

Lemma 2.1. *Let the assumptions (I₁) – (I₂) hold and $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n)$. Then the system (2.3) (i.e. (1.1)-(1.4)) has a unique solution*

$$\mathcal{U} \in \bigcap_{k=0}^n C^k(\mathbb{R}^+, \mathcal{D}(\mathcal{A}^{n-k})),$$

and its energy satisfies

$$E'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds, \quad t \geq 0. \quad (2.4)$$

The proof of the above lemma is similar to the one given in [5], and so we omit it here.

3. Main theorem

Firstly, denote

$$\mathfrak{H}_n = \left\{ \mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n); \quad \int_0^{+\infty} h(s) \left\| \sqrt{A} \partial_s^i u_0(s) \right\|^2 ds < +\infty, \quad i = 0, 1, \dots, n \right\}, \quad (3.1)$$

where $h(t) := \int_t^{+\infty} g(s)ds$.

Theorem 3.1. *Suppose that the assumptions (I₁) – (I₂) hold, and let $\mathcal{U}_0 \in \mathfrak{H}_3$. Then, there exists a positive constant $C > 0$ such that the solution energy $E(t)$ of the system (1.1)-(1.4) satisfies*

$$E(t) \leq C(t+1)^{-1}, \quad t \geq 0, \quad (3.2)$$

where C depends on $E(0), E_2(0), E_3(0), E_4(0)$ and $\int_0^{+\infty} h(s) \left\| \sqrt{A} \partial_s^i u_0(s) \right\|^2 ds$ ($i = 0, 1, 2, 3$), and E_k ($k = 2, 3, 4$) denote the high-order energies (the details will be seen in Section 4). Moreover

$$tE(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Remark 3.2. If

$$A_1 = A_2 = A, \quad B = \alpha I$$

in system (1.1)-(1.4), and

$$u_0(0), v_0 \in \mathcal{D}(A^2), \quad u_1, v_1 \in \mathcal{D}(A^{\frac{3}{2}}), \quad \eta_0 \in L_g^2(\mathbb{R}^+, \mathcal{D}(A^2)),$$

we have $\mathcal{U}_0 \in \mathfrak{H}_3$.

Remark 3.3. In [5], the following abstract system is considered

$$u''(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds + \tilde{B}v(t) = 0, \quad (3.3)$$

$$v''(t) + \tilde{A}v(t) + \tilde{B}u(t) = 0, \quad (3.4)$$

where $A, \tilde{A}, B, \tilde{B}$ are positive self-adjoint linear operators, and $\tilde{B}$ is bounded. The energy decay rate $\frac{1}{t}$ of the system (3.3)-(3.4) was obtained under condition (1.5), and a weaker decay rate also derived under condition (1.6) (see [5, Theorem 3.1] for details, which is a generalization of [1, Theorem 3.1]). Moreover, the single equation with $\tilde{B} = 0$ in (3.3) was also considered and some decay estimates were also obtained under condition (1.6) in [4].

It is clear that Theorem 3.1 improves [5, Theorem 3.1] (without the condition 3.3 in [5]), in the sense that assuming much weaker conditions on the convolution kernel $g(s)$, the theorem here gives stronger conclusions (explicit decay rates do not appear in [5, Theorem 3.1]). It tells us that the energy decays at rate t^{-1} for any kernel $g(s)$ satisfying Assumption (I₂), which is nearly the basic conditions ensuring the well-posedness of the problem, and also a very weak condition without restrictions on $g'(s)$ like in [5] and the related references therein.

Remark 3.4. From [7, Remark 3.3 (ii)] and [3, Theorem 4.1], it is easy to see that the decay rate t^{-1} given in Theorem 3.1 is optimal in the polynomial sense.

Remark 3.5. For system (1.1)-(1.2), when the infinite integral $\int_0^{+\infty}$ is replaced by the finite one $\int_0^t$, Theorem 3.1 still holds; one just needs to do a minor modification in the proof, that is, replace the estimates of $\int_0^{+\infty} g(s) \|x(t-s) - x(t)\|^2 ds$ with those of

$$\int_0^t g(t-s) \|x(s) - x(t)\|^2 ds.$$

4. Proof of Theorem 3.1

In this section, we let the assumptions in Theorem 3.1 be satisfied.

Differentiating the system (1.1)-(1.2) with respect to t yields that

$$u'''(t) + A_1u'(t) - \int_0^{+\infty} g(s)Au'(t-s)ds + Bv'(t) = 0, \quad (4.1)$$

$$v'''(t) + A_2v'(t) + Bu'(t) = 0,$$

and

$$u''''(t) + A_1u''(t) - \int_0^{+\infty} g(s)Au''(t-s)ds + Bv''(t) = 0, \quad (4.2)$$

$$v''''(t) + A_2v''(t) + Bu''(t) = 0.$$

Define high-order energy functions by $E_2(t) = E(u'(t), v'(t), t)$, $E_3(t) = E(u''(t), v''(t), t)$, and $E_4(t) = E(u'''(t), v'''(t), t)$. Similar to Lemma 2.1, the following equalities hold:

$$E'_2(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \left\| \sqrt{A}u'(t-s) - \sqrt{A}u'(t) \right\|^2 ds, \quad (4.3)$$

$$E'_3(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \left\| \sqrt{A}u''(t-s) - \sqrt{A}u''(t) \right\|^2 ds, \quad (4.4)$$

and

$$E'_4(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \left\| \sqrt{A}u'''(t-s) - \sqrt{A}u'''(t) \right\|^2 ds. \quad (4.5)$$

Set

$$H(t) = \int_0^{+\infty} h(s) \left\| \sqrt{A}u(t-s) \right\|^2 ds,$$

and

$$H_1(t) = M(\delta) \left(\delta \int_0^{+\infty} h(s) \left\| \sqrt{A}u(t-s) \right\|^2 ds + E(t) \right);$$

here

$$M(\delta) := \int_0^{+\infty} \frac{g(s)}{K_\delta(s)} ds, \quad K_\delta(s) := \frac{-g'(s)}{g(s)} + \delta,$$

and $\delta \in (0, 1)$ is a constant which will be specified later.

Lemma 4.1. *For $t \geq 0$, we have,*

$$H'(t) \leq - \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) \right\|^2 ds + h(0) \left\| \sqrt{A_1}u(t) \right\|^2, \quad (4.6)$$

or

$$H'(t) \leq -\frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds + 2h(0) \left\| \sqrt{A_1}u(t) \right\|^2, \quad (4.7)$$

and

$$\begin{aligned} H'_1(t) &\leq -\frac{1}{2} M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds \\ &\quad + 2\delta M(\delta) h(0) \left\| \sqrt{A_1}u(t) \right\|^2. \end{aligned} \quad (4.8)$$

Proof. Noticing $h'(t) = -g(t)$ and Assumption (I₁), we get, via integration by parts,

$$\begin{aligned} H'(t) &= \int_0^{+\infty} h(s) \partial_t \left(\left\| \sqrt{A}u(t-s) \right\|^2 \right) ds \\ &= - \int_0^{+\infty} h(s) \partial_s \left(\left\| \sqrt{A}u(t-s) \right\|^2 \right) ds \\ &= - \int_0^{+\infty} g(s) \left(\left\| \sqrt{A}u(t-s) \right\|^2 \right) ds + h(0) \left\| \sqrt{A}u(t) \right\|^2 \\ &\leq - \int_0^{+\infty} g(s) \left(\left\| \sqrt{A}u(t-s) \right\|^2 \right) ds + h(0) \left\| \sqrt{A_1}u(t) \right\|^2. \end{aligned}$$

Furthermore, since $-(p-q)^2 \leq -\frac{1}{2}p^2 + q^2$, we have

$$\begin{aligned} H'(t) &\leq - \int_0^{+\infty} g(s) \left(\left\| \sqrt{A}u(t-s) \right\|^2 \right) ds + h(0) \left\| \sqrt{A_1}u(t) \right\|^2 \\ &\leq - \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) + \sqrt{A}u(t) \right\|^2 ds + h(0) \left\| \sqrt{A_1}u(t) \right\|^2 \\ &\leq -\frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds + \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t) \right\|^2 ds \\ &\quad + h(0) \left\| \sqrt{A_1}u(t) \right\|^2 \\ &\leq -\frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds + 2h(0) \left\| \sqrt{A_1}u(t) \right\|^2. \end{aligned}$$

Moreover, by (2.4) we get

$$\begin{aligned} H'_1(t) &\leq -\frac{1}{2}M(\delta) \int_0^{+\infty} \delta g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds + 2\delta M(\delta)h(0) \left\| \sqrt{A_1}u(t) \right\|^2 \\ &\quad + \frac{1}{2}M(\delta) \int_0^{+\infty} g'(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds \\ &\leq -\frac{1}{2}M(\delta) \int_0^{+\infty} (-g'(s) + \delta g(s)) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds \\ &\quad + 2\delta M(\delta)h(0) \left\| \sqrt{A_1}u(t) \right\|^2 \end{aligned}$$

$$\leq -\frac{1}{2}M(\delta) \int_0^{+\infty} K_\delta(s)g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds + 2\delta M(\delta)h(0) \left\| \sqrt{A_1}u(t) \right\|^2.$$

This completes the proof of Lemma 4.1. $\square$

Furthermore, define

$$\begin{aligned}\tilde{H}(t) &= \int_0^{+\infty} h(s) \left\| \sqrt{A}u'(t-s) \right\|^2 ds, \\ \hat{H}(t) &= \int_0^{+\infty} h(s) \left\| \sqrt{A}u''(t-s) \right\|^2 ds, \\ H_2(t) &= M(\delta) \left(\delta \int_0^{+\infty} h(s) \left\| \sqrt{A}u'(t-s) \right\|^2 ds + E_2(t) \right), \\ H_3(t) &= M(\delta) \left(\delta \int_0^{+\infty} h(s) \left\| \sqrt{A}u''(t-s) \right\|^2 ds + E_3(t) \right),\end{aligned}$$

and

$$H_4(t) = M(\delta) \left(\delta \int_0^{+\infty} h(s) \|u'''(t-s)\|^2 ds + \frac{a_1}{c_1} E_4(t) \right).$$

Lemma 4.2. For $t \geq 0$, we have

$$\tilde{H}'(t) \leq - \int_0^{+\infty} g(s) \left\| \sqrt{A}u'(t-s) \right\|^2 ds + h(0) \left\| \sqrt{A_1}u'(t) \right\|^2, \quad (4.9)$$

$$\hat{H}'(t) \leq - \int_0^{+\infty} g(s) \left\| \sqrt{A}u''(t-s) \right\|^2 ds + h(0) \left\| \sqrt{A_1}u''(t) \right\|^2, \quad (4.10)$$

$$\begin{aligned}H_2'(t) &\leq -\frac{1}{2}M(\delta) \int_0^{+\infty} K_\delta(s)g(s) \left\| \sqrt{A}u'(t-s) - \sqrt{A}u'(t) \right\|^2 ds \\ &\quad + 2\delta M(\delta)h(0) \left\| \sqrt{A_1}u'(t) \right\|^2,\end{aligned} \quad (4.11)$$

$$\begin{aligned}H_3'(t) &\leq -\frac{1}{2}M(\delta) \int_0^{+\infty} K_\delta(s)g(s) \left\| \sqrt{A}u''(t-s) - \sqrt{A}u''(t) \right\|^2 ds \\ &\quad + 2\delta M(\delta)h(0) \left\| \sqrt{A_1}u''(t) \right\|^2,\end{aligned} \quad (4.12)$$

and

$$\begin{aligned}H_4'(t) &\leq -\frac{1}{2}M(\delta) \int_0^{+\infty} K_\delta(s)g(s) \|u'''(t-s) - u'''(t)\|^2 ds \\ &\quad + 2\delta M(\delta)h(0) \|u'''(t)\|^2.\end{aligned} \quad (4.13)$$

Next, we set

$$F_1(t) = \langle u'(t), u(t) \rangle - \langle v'(t), A_2^{-1}Bu(t) \rangle + \frac{1}{2}H(t).$$

Lemma 4.3. *For any $\xi_1 > 0$, there is a constant $C(\xi_1) > 0$ such that*

$$\begin{aligned} F_1'(t) \leq & -\left(1 - h(0) - a_1a_2d_2^2\right) \left\| \sqrt{A_1}u(t) \right\|^2 + \xi_1 \|v'(t)\|^2 \\ & + C(\xi_1) \|u'(t)\|^2, \quad t \geq 0. \end{aligned} \quad (4.14)$$

Proof. Thanks to the assumption (I_1) ,

$$\left\| A_2^{-\frac{1}{2}}Bu(t) \right\|^2 \leq \left\| A_2^{-\frac{1}{2}} \right\|^2 \|B\|^2 \|u(t)\|^2 \leq a_1a_2d_2^2 \left\| \sqrt{A_1}u(t) \right\|^2.$$

Then, from (1.1), we have

$$\begin{aligned} & (\langle u'(t), u(t) \rangle - \langle v'(t), A_2^{-1}Bu(t) \rangle)' \\ &= -\left\| \sqrt{A_1}u(t) \right\|^2 + \left\langle \int_0^{+\infty} g(s)\sqrt{A}u(t-s)ds, \sqrt{A}u(t) \right\rangle + \|u'(t)\|^2 \\ & \quad + \left\| A_2^{-\frac{1}{2}}Bu(t) \right\|^2 - \langle v'(t), A_2^{-1}Bu'(t) \rangle \\ & \leq -\left(1 - \frac{h(0)}{2} - a_1a_2d_2^2\right) \left\| \sqrt{A_1}u(t) \right\|^2 + \frac{1}{2h(0)} \left\| \int_0^{+\infty} g(s)\sqrt{A}u(t-s)ds \right\|^2 \\ & \quad + \|u'(t)\|^2 + \xi_1 \|v'(t)\|^2 + \frac{1}{4\xi_1} \left\| A_2^{-1}Bu'(t) \right\|^2 \\ & \leq -\left(1 - \frac{h(0)}{2} - a_1a_2d_2^2\right) \left\| \sqrt{A_1}u(t) \right\|^2 + \frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) \right\|^2 ds \\ & \quad + \xi_1 \|v'(t)\|^2 + \left(1 + \frac{a_2^2d_2^2}{4\xi_1}\right) \|u'(t)\|^2. \end{aligned}$$

This and (4.6) finish the proof. $\square$

In addition, define

$$F_2(t) = \langle u''(t), u'(t) \rangle + \frac{1}{2}\tilde{H}(t),$$

and

$$F_3(t) = \langle u'''(t), u''(t) \rangle + \langle Bv'(t), u''(t) \rangle + \frac{1}{2}\hat{H}(t).$$

Lemma 4.4. *For any $t \geq 0$,*

$$\begin{aligned} F_2'(t) \leq & -(1 - h(0)) \left\| \sqrt{A_1}u'(t) \right\|^2 + \frac{d_1}{5h(0)} \|v'(t)\|^2 \\ & + \frac{5h(0)d_2^2}{4d_1} \|u'(t)\|^2 + \|u''(t)\|^2, \end{aligned} \quad (4.15)$$

and for any $\xi_2 > 0$, there exists a constant $C(\xi_2) > 0$ such that

$$\begin{aligned} F'_3(t) \leq & -(1 - h(0)) \left\| \sqrt{A_1} u''(t) \right\|^2 + \xi_2 \|v'(t)\|^2 \\ & + C(\xi_2) \|u'''(t)\|^2, \quad t \geq 0. \end{aligned} \quad (4.16)$$

Proof. From (4.1) and (4.2), we have

$$\begin{aligned} & \langle u''(t), u'(t) \rangle' \\ &= - \left\| \sqrt{A_1} u'(t) \right\|^2 + \left\langle \int_0^{+\infty} g(s) \sqrt{A} u'(t-s) ds, \sqrt{A} u'(t) \right\rangle + \|u''(t)\|^2 \\ & \quad - \langle Bv'(t), u'(t) \rangle \\ &\leq - \left(1 - \frac{h(0)}{2} \right) \left\| \sqrt{A_1} u'(t) \right\|^2 + \frac{1}{2h(0)} \left\| \int_0^{+\infty} g(s) \sqrt{A} u'(t-s) ds \right\|^2 \\ & \quad + \|u''(t)\|^2 + \frac{d_1}{5h(0)d_2} \left\| \sqrt{B} v'(t) \right\|^2 + \frac{5h(0)d_2}{4d_1} \left\| \sqrt{B} u'(t) \right\|^2 \\ &\leq - \left(1 - \frac{h(0)}{2} \right) \left\| \sqrt{A_1} u'(t) \right\|^2 + \frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A} u'(t-s) \right\|^2 ds \\ & \quad + \|u''(t)\|^2 + \frac{d_1}{5h(0)} \|v'(t)\|^2 + \frac{5h(0)d_2^2}{4d_1} \|u'(t)\|^2, \end{aligned}$$

and

$$\begin{aligned} & (\langle u'''(t), u''(t) \rangle + \langle Bv'(t), u''(t) \rangle)' \\ &= - \left\| \sqrt{A_1} u''(t) \right\|^2 + \left\langle \int_0^{+\infty} g(s) \sqrt{A} u''(t-s) ds, \sqrt{A} u''(t) \right\rangle + \|u'''(t)\|^2 \\ & \quad + \langle Bv'(t), u'''(t) \rangle \\ &\leq - \left(1 - \frac{h(0)}{2} \right) \left\| \sqrt{A_1} u''(t) \right\|^2 + \frac{1}{2h(0)} \left\| \int_0^{+\infty} g(s) \sqrt{A} u''(t-s) ds \right\|^2 \\ & \quad + \|u'''(t)\|^2 + \frac{\xi_2}{d_2} \left\| \sqrt{B} v'(t) \right\|^2 + \frac{d_2}{4\xi_2} \left\| \sqrt{B} u'''(t) \right\|^2 \\ &\leq - \left(1 - \frac{h(0)}{2} \right) \left\| \sqrt{A_1} u''(t) \right\|^2 + \frac{1}{2} \int_0^{+\infty} g(s) \left\| \sqrt{A} u''(t-s) \right\|^2 ds \\ & \quad + \xi_2 \|v'(t)\|^2 + \left(1 + \frac{d_2^2}{4\xi_2} \right) \|u'''(t)\|^2. \end{aligned}$$

This and (4.9)-(4.10) together lead to the desired estimate. $\square$

Next, we define a functional $J_1(t)$ by

$$J_1(t) = \left\langle u'(t), \int_0^{+\infty} g(s)(u(t-s) - u(t))ds \right\rangle.$$

Lemma 4.5. For any $\xi_3, \xi_4 > 0$, there exists $C(\xi_3, \xi_4) > 0$ such that for $t \geq 0$,

$$\begin{aligned} J_1'(t) &\leq -\frac{h(0)}{2} \|u'(t)\|^2 + \xi_3 \left\| \sqrt{A_1}u(t) \right\|^2 + \xi_4 \left\| \sqrt{A_2}v(t) \right\|^2 \\ &\quad + C(\xi_3, \xi_4)M(\delta) \int_0^{+\infty} K_\delta(s)g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds \\ &\quad - \frac{a_1g(0)}{2c_1h(0)} \int_0^{+\infty} g'(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds. \end{aligned} \quad (4.17)$$

Proof. From the definitions of $M(\delta)$, $K_\delta(s)$, it is easy to check that

$$\begin{aligned} &\left\| \int_0^{+\infty} g(s) \left(\sqrt{A}u(t-s) - \sqrt{A}u(t) \right) ds \right\| \\ &\leq M(\delta) \int_0^{+\infty} K_\delta(s)g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^{+\infty} g(s)\partial_t u(t-s)ds &= - \int_0^{+\infty} g(s)\partial_s u(t-s)ds \\ &= - \int_0^{+\infty} g(s)\partial_s (u(t-s) - u(t))ds \\ &= \int_0^{+\infty} g'(s) (u(t-s) - u(t))ds. \end{aligned}$$

Therefore, from (1.1), by a direct calculation we have

$$\begin{aligned} &J_1'(t) \\ &= \left\langle u''(t), \int_0^{+\infty} g(s)(u(t-s) - u(t))ds \right\rangle \\ &\quad + \left\langle u'(t), \int_0^{+\infty} g(s)\partial_t u(t-s)ds \right\rangle - \left\langle u'(t), \int_0^{+\infty} g(s)u'(t)ds \right\rangle \\ &= -h(0) \|u'(t)\|^2 - \left\langle A_1u(t) - h(0)Au(t), \int_0^{+\infty} g(s)(u(t-s) - u(t))ds \right\rangle \end{aligned}$$

$$\begin{aligned}
& + \left\langle \int_0^{+\infty} g(s) (Au(t-s) - Au(t)) ds, \int_0^{+\infty} g(s) (u(t-s) - u(t)) ds \right\rangle \\
& - \left\langle Bv(t), \int_0^{+\infty} g(s) (u(t-s) - u(t)) ds \right\rangle \\
& + \left\langle u'(t), \int_0^{+\infty} g'(s) (u(t-s) - u(t)) ds \right\rangle \\
& \leq -h(0) \|u'(t)\|^2 + \frac{\xi_3}{2} \|\sqrt{A_1}u(t)\|^2 + \frac{\xi_3}{2} \|\sqrt{A}u(t)\|^2 \\
& + \frac{1}{2\xi_3} M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \|\sqrt{A_1}u(t-s) - \sqrt{A_1}u(t)\|^2 ds \\
& + \left(1 + \frac{h^2(0)}{2\xi_3}\right) M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \|\sqrt{A}u(t-s) - \sqrt{A}u(t)\|^2 ds \\
& + \frac{\xi_4}{a_2 d_2} \|\sqrt{B}v(t)\|^2 + \frac{a_2 d_2}{4\xi_4} M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \|\sqrt{B}u(t-s) - \sqrt{B}u(t)\|^2 ds \\
& + \frac{h(0)}{2} \|u'(t)\|^2 - \frac{g(0)}{2h(0)} \int_0^{+\infty} g'(s) \|u(t-s) - u(t)\|^2 ds \\
& \leq -\frac{h(0)}{2} \|u'(t)\|^2 + \xi_3 \|\sqrt{A_1}u(t)\|^2 + \xi_4 \|\sqrt{A_2}v(t)\|^2 \\
& + \left(1 + \frac{1}{2c_1\xi_3} + \frac{h^2(0)}{2\xi_3} + \frac{a_1 a_2 d_2^2}{4c_1\xi_4}\right) M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \|\sqrt{A}u(t-s) - \sqrt{A}u(t)\|^2 ds \\
& - \frac{a_1 g(0)}{2c_1 h(0)} \int_0^{+\infty} g'(s) \|\sqrt{A}u(t-s) - \sqrt{A}u(t)\|^2 ds.
\end{aligned}$$

This leads to the conclusion. $\square$

Now, we define $J_2(t)$

$$J_2(t) = \left\langle u''(t), \int_0^{+\infty} g(s) (u'(t-s) - u'(t)) ds \right\rangle.$$

Similar to Lemma 4.5, we have the following lemma.

Lemma 4.6. *We have, for $t \geq 0$,*

$$\begin{aligned}
J_2'(t) & \leq -\frac{h(0)}{2} \|u''(t)\|^2 + \frac{h(0)(1-h(0))}{16} \|\sqrt{A_1}u'(t)\|^2 + \frac{d_1}{20} \|v'(t)\|^2 \\
& + C_1 \int_0^{+\infty} K_\delta(s) g(s) \|\sqrt{A}u'(t-s) - \sqrt{A}u'(t)\|^2 ds
\end{aligned}$$

$$-\frac{a_1 g(0)}{2c_1 h(0)} \int_0^{+\infty} g'(s) \left\| \sqrt{A} u'(t-s) - \sqrt{A} u'(t) \right\|^2 ds, \quad (4.18)$$

where

$$C_1 := 1 + \frac{8}{h(0)(1-h(0))} \left(\frac{1}{c_1} + h^2(0) \right) + \frac{5a_1 a_2 d_2^2}{c_1 d_1}.$$

Next, define

$$\begin{aligned} J_3(t) = & \left\langle u'''(t), \int_0^{+\infty} g(s)(u''(t-s) - u''(t))ds \right\rangle \\ & + \left\langle Bv'(t), \int_0^{+\infty} g(s)(u''(t-s) - u''(t))ds \right\rangle. \end{aligned}$$

Lemma 4.7. For any $\xi_5, \xi_6 > 0$, there are $C(\xi_5) > 0, C(\xi_6) > 0$ such that for $t \geq 0$,

$$\begin{aligned} J_3'(t) \leq & -\frac{3h(0)}{4} \|u'''(t)\|^2 + \xi_5 \left\| \sqrt{A_1} u''(t) \right\|^2 + \xi_6 \|v'(t)\|^2 \\ & + C(\xi_5) M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \left\| \sqrt{A} u''(t-s) - \sqrt{A} u''(t) \right\|^2 ds \\ & + C(\xi_6) M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \|u'''(t-s) - u'''(t)\|^2 ds \\ & - \frac{a_1 g(0)}{h(0)c_1} \int_0^{+\infty} g'(s) \left\| \sqrt{A} u''(t-s) - \sqrt{A} u''(t) \right\|^2 ds. \end{aligned} \quad (4.19)$$

Proof. From (4.2), a direct calculation yields

$$\begin{aligned} J_3'(t) = & -h(0) \|u'''(t)\|^2 - \left\langle A_1 u''(t), \int_0^{+\infty} g(s)(u''(t-s) - u''(t))ds \right\rangle \\ & + h(0) \left\langle A u''(t), \int_0^{+\infty} g(s)(u''(t-s) - u''(t))ds \right\rangle \\ & + \left\langle \int_0^t g(s)(A u''(t-s) - A u''(t))ds, \int_0^{+\infty} g(s)(u''(t-s) - u''(t))ds \right\rangle \\ & + \left\langle Bv'(t), \int_0^{+\infty} g(s)(u'''(t-s) - u'''(t))ds \right\rangle \\ & + \left\langle u'''(t), \int_0^{+\infty} g'(s)(u''(t-s) - u''(t))ds \right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq -h(0) \|u'''(t)\|^2 + \frac{\xi_5}{2} \left\| \sqrt{A_1} u''(t) \right\|^2 + \frac{\xi_5}{2} \left\| \sqrt{A} u''(t) \right\|^2 \\
&\quad + \frac{1}{2\xi_5} M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \left\| \sqrt{A_1} u''(t-s) - \sqrt{A_1} u''(t) \right\|^2 ds \\
&\quad + \left(1 + \frac{h^2(0)}{2\xi_5} \right) M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \left\| \sqrt{A} u''(t-s) - \sqrt{A} u''(t) \right\|^2 ds \\
&\quad + \frac{\xi_6}{d_2} \left\| \sqrt{B} v'(t) \right\|^2 + \frac{d_2}{4\xi_6} M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \left\| \sqrt{B} u'''(t-s) - \sqrt{B} u'''(t) \right\|^2 ds \\
&\quad + \frac{h(0)}{4} \|u'''(t)\|^2 - \frac{g(0)}{h(0)} \int_0^{+\infty} g'(s) \|u''(t-s) - u''(t)\|^2 ds \\
&\leq -\frac{3h(0)}{4} \|u'''(t)\|^2 + \xi_5 \left\| \sqrt{A_1} u''(t) \right\|^2 + \xi_6 \|v'(t)\|^2 \\
&\quad + \left(1 + \frac{1}{2c_1\xi_5} + \frac{h^2(0)}{2\xi_5} \right) M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \left\| \sqrt{A} u''(t-s) - \sqrt{A} u''(t) \right\|^2 ds \\
&\quad + \frac{d_2^2}{4\xi_6} M(\delta) \int_0^{+\infty} K_\delta(s) g(s) \|u'''(t-s) - u'''(t)\|^2 ds \\
&\quad - \frac{a_1 g(0)}{h(0)c_1} \int_0^{+\infty} g'(s) \left\| \sqrt{A} u''(t-s) - \sqrt{A} u''(t) \right\|^2 ds.
\end{aligned}$$

This enables us to get the estimate. $\square$

Furthermore, we define

$$\begin{aligned}
\tilde{J}_1(t) &= J_1(t) + 2C(\xi_3, \xi_4)H_1(t) + \frac{a_1 g(0)}{c_1 h(0)} E(t), \\
\tilde{J}_2(t) &= J_2(t) + 2C_1 H_1(t) + \frac{a_1 g(0)}{c_1 h(0)} E_2(t),
\end{aligned}$$

and

$$\tilde{J}_3(t) = J_3(t) + 2C(\xi_5)H_3(t) + 2C(\xi_6)H_4(t) + \frac{2a_1 g(0)}{c_1 h(0)} E_3(t).$$

From (2.4), (4.3)-(4.5), (4.8), (4.11)-(4.13), and (4.17)-(4.19), we have,

$$\begin{aligned}
\tilde{J}'_1(t) &\leq -\frac{h(0)}{2} \|u'(t)\|^2 + (\xi_3 + 4\delta M(\delta)h(0)C(\xi_3, \xi_4)) \left\| \sqrt{A_1} u(t) \right\|^2 \\
&\quad + \xi_4 \left\| \sqrt{A_2} v(t) \right\|^2,
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\tilde{J}'_2(t) &\leq -\frac{h(0)}{2} \|u''(t)\|^2 + \left(4\delta M(\delta)h(0)C_1 + \frac{h(0)(1-h(0))}{16} \right) \left\| \sqrt{A_1} u'(t) \right\|^2 \\
&\quad + \frac{d_1}{20} \|v'(t)\|^2,
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \tilde{J}_3(t) \leq & - \left(\frac{3h(0)}{4} - 4\delta M(\delta)h(0)C(\xi_6) \right) \|u'''(t)\|^2 \\ & + (\xi_5 + 4\delta M(\delta)h(0)C(\xi_5)) \left\| \sqrt{A_1}u''(t) \right\|^2 + \xi_6 \|v'(t)\|^2. \end{aligned} \quad (4.22)$$

Next, let us consider the functional

$$R_1(t) = \langle v'(t), v(t) \rangle.$$

Lemma 4.8. *For any $t \geq 0$, we have*

$$R_1'(t) \leq -\frac{7}{8} \left\| \sqrt{A_2}v(t) \right\|^2 + 2a_1a_2d_2^2 \left\| \sqrt{A_1}u(t) \right\|^2 + \|v'(t)\|^2. \quad (4.23)$$

Proof. From (1.2),

$$R_1'(t) = - \left\| \sqrt{A_2}v(t) \right\|^2 - \langle Bu(t), v(t) \rangle + \|v'(t)\|^2.$$

Thus, using Cauchy-Schwarz's inequality and Assumption (I₁) we finish the proof. $\square$

Finally, we introduce the auxiliary functional

$$\begin{aligned} R_2(t) = & \langle (A_1 - h(0)A)u'(t), v(t) \rangle \\ & - \left\langle \int_0^{+\infty} g(s) (Au'(t-s) - Au'(t)) ds, v(t) \right\rangle + \frac{16c_2}{d_1} H_3(t). \end{aligned}$$

Lemma 4.9. *For any $t \geq 0$,*

$$\begin{aligned} R_2'(t) \leq & -\frac{3d_1}{4} \|v'(t)\|^2 + \frac{1}{d_1} \|u'''(t)\|^2 + \frac{3d_1}{32} \left\| \sqrt{A_2}v(t) \right\|^2 \\ & + \left(C_2 + \frac{32c_2}{d_1} \delta M(\delta)h(0) \right) \left\| \sqrt{A_1}u''(t) \right\|^2, \end{aligned} \quad (4.24)$$

where $C_2 := \frac{8c_2(1+h^2(0))}{d_1}$.

Proof. From (4.1) and (4.12), we infer

$$\begin{aligned} R_2'(t) = & - \left\| \sqrt{B}v'(t) \right\|^2 - \langle u'''(t), v'(t) \rangle + \langle (A_1 - h(0)A)u''(t), v(t) \rangle \\ & - \left\langle \int_0^{+\infty} g(s) (Au''(t-s) - Au''(t)) ds, v(t) \right\rangle + \frac{16c_2}{d_1} H_3'(t) \\ \leq & -d_1 \|v'(t)\|^2 + \frac{d_1}{4} \|v'(t)\|^2 + \frac{1}{d_1} \|u'''(t)\|^2 \\ & + \frac{d_1}{32c_2} \left\| \sqrt{A_1}v(t) \right\|^2 + \frac{8c_2}{d_1} \left\| \sqrt{A_1}u''(t) \right\|^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{d_1}{32c_2} \left\| \sqrt{A}v(t) \right\|^2 + \frac{8c_2h^2(0)}{d_1} \left\| \sqrt{A}u''(t) \right\|^2 \\
& + \frac{d_1}{32c_2} \left\| \sqrt{A}v(t) \right\|^2 + \frac{32c_2}{d_1} \delta M(\delta)h(0) \left\| \sqrt{A_1}u''(t) \right\|^2 \\
& \leq -\frac{3d_1}{4} \|v'(t)\|^2 + \frac{1}{d_1} \|u'''(t)\|^2 + \frac{3d_1}{32} \left\| \sqrt{A_2}v(t) \right\|^2 \\
& + \left(\frac{8c_2(1+h^2(0))}{d_1} + \frac{32c_2}{d_1} \delta M(\delta)h(0) \right) \left\| \sqrt{A_1}u''(t) \right\|^2. \quad \square
\end{aligned}$$

In order to determine the values of the coefficients, we give the following lemma (see [7, Page 1525, Line 10]).

Lemma 4.10. *If $g(s)$ satisfies the assumption (I_2) , then*

$$\delta M(\delta) \rightarrow 0, \text{ as } \delta \rightarrow 0. \quad (4.25)$$

Proof. Notice that

$$\delta M(\delta) = \int_0^{+\infty} \frac{\delta g^2(s)}{\delta g(s) - g'(s)} ds,$$

and $g(s)$ is a control function. Then, by Lebesgue's dominated convergence theorem, we get (4.25). $\square$

Proof of Theorem 3.1. First, we construct an auxiliary function

$$\begin{aligned}
L(t) = & N_1 F_1(t) + \frac{h(0)}{4} F_2(t) + N_2 F_3(t) + N_3 \tilde{J}_1(t) + \tilde{J}_2(t) + N_4 \tilde{J}_3(t) \\
& + \frac{d_1}{4} R_1(t) + R_2(t) + \frac{1}{2h(0)} H(t),
\end{aligned}$$

where N_1, N_2, N_3 and N_4 are some positive constants to be determined. Then, from (4.7), (4.14)-(4.16), and (4.20)-(4.24), we deduce

$$\begin{aligned}
& L'(t) \\
& \leq - \left(\frac{h(0)}{2} N_3 - \frac{5h^2(0)d_2^2}{16d_1} - N_1 C(\xi_1) \right) \|u'(t)\|^2 \\
& - \left(\frac{8d_1}{20} - N_1 \xi_1 - N_2 \xi_2 - N_4 \xi_6 \right) \|v'(t)\|^2 \\
& - \left(N_1 (1 - h(0) - a_1 a_2 d_2^2) - N_3 (\xi_3 + 4\delta M(\delta)h(0)C(\xi_3, \xi_4)) - 1 - \frac{a_1 a_2 d_1 d_2^2}{2} \right) \\
& \times \left\| \sqrt{A_1}u(t) \right\|^2 \\
& - \left(\frac{d_1}{8} - N_3 \xi_4 \right) \left\| \sqrt{A_2}v(t) \right\|^2 \\
& - \frac{1}{4h(0)} \int_0^{+\infty} g(s) \left\| \sqrt{A}u(t-s) - \sqrt{A}u(t) \right\|^2 ds
\end{aligned}$$

$$\begin{aligned}
& -\frac{h(0)}{4} \|u''(t)\|^2 \\
& -\left(N_4 \left(\frac{3h(0)}{4} - 4\delta M(\delta)h(0)C(\xi_6)\right) - N_2C(\xi_2) - \frac{1}{d_1}\right) \|u'''(t)\|^2 \\
& -\left(\frac{3h(0)(1-h(0))}{16} - 4\delta M(\delta)h(0)C_1\right) \left\|\sqrt{A_1}u'(t)\right\|^2 \\
& -\left(N_2(1-h(0)) - N_4(\xi_5 + 4\delta M(\delta)h(0)C(\xi_5)) - \left(C_2 + \frac{32c_2}{d_1}\delta M(\delta)h(0)\right)\right) \\
& \times \left\|\sqrt{A_1}u''(t)\right\|^2.
\end{aligned}$$

Take

$$\begin{aligned}
N_1 &= \frac{a_1a_2d_1d_2^2 + 4}{1-h(0) - a_1a_2d_2^2}, & N_2 &= \frac{4+C_2}{1-h(0)}, & \xi_1 &= \frac{d_1}{20N_1}, & \xi_2 &= \frac{d_1}{20N_2}, \\
N_3 &= \frac{h(0)d_2^2}{d_1} + \frac{2N_1C(\xi_1) + 2}{h(0)}, & N_4 &= \frac{2N_2C(\xi_2) + \frac{2}{d_1} + 2}{h(0)}, & \xi_3 &= \frac{1}{N_3}, \\
& \xi_4 = \frac{d_1}{16N_3}, & \xi_5 &= \frac{1}{N_4}, & \xi_6 &= \frac{d_1}{20N_4}.
\end{aligned}$$

According to Lemma 4.10, there exists $\delta > 0$ such that

$$\delta M(\delta) \leq \min \left\{ \frac{1}{4h(0)N_3C(\xi_3, \xi_4)}, \frac{1}{16C(\xi_6)}, \frac{1-h(0)}{64C_1}, \frac{1}{4h(0)N_4C(\xi_5)}, \frac{d_1}{32c_2h(0)} \right\}.$$

Hence,

$$\begin{aligned}
& \frac{h(0)}{2}N_3 - \frac{5h^2(0)d_2^2}{16d_1} - N_1C(\xi_1) = 1 + \frac{3h^2(0)d_2^2}{16d_1} \geq 1, \\
& \frac{8d_1}{20} - N_1\xi_1 - N_2\xi_2 - N_4\xi_6 = \frac{d_1}{4}, \\
& N_1(1-h(0) - a_1a_2d_2^2) - N_3(\xi_3 + 4\delta M(\delta)h(0)C(\xi_3, \xi_4)) - 1 - \frac{a_1a_2d_1d_2^2}{2} \geq 1 + \frac{a_1a_2d_1d_2^2}{2} \geq 1, \\
& \frac{d_1}{8} - N_3\xi_4 = \frac{d_1}{16}, \\
& N_4 \left(\frac{3h(0)}{4} - 4\delta M(\delta)h(0)C(\xi_6)\right) - N_2C(\xi_2) - \frac{1}{d_1} \geq 1, \\
& \frac{3h(0)(1-h(0))}{16} - 4\delta M(\delta)h(0)C_1 \geq \frac{h(0)(1-h(0))}{8}, \\
& N_2(1-h(0)) - N_4(\xi_5 + 4\delta M(\delta)h(0)C(\xi_5)) - \left(C_2 + \frac{32c_2}{d_1}\delta M(\delta)h(0)\right) \geq 1.
\end{aligned}$$

Therefore, we have

$$L'(t) \leq -\epsilon \tilde{E}(t),$$

where $0 < \epsilon := \min \left\{ 1, \frac{d_1}{16}, \frac{1}{4h(0)} \right\}$. By (2.2), there is a constant $\epsilon_0 > 0$ such that

$$L'(t) \leq -\epsilon_0 E(t).$$

Hence, for a positive constant $C_3 > 0$,

$$\int_0^t E(s) ds \leq C_3 (L(0) - L(t)). \quad (4.26)$$

In addition, it is not difficult to see that

$$L(0) \leq \sum_{i=0}^3 k_0 \int_0^{+\infty} h(s) \left\| \sqrt{A} \partial_s^i u_0(s) \right\|^2 ds + k_1 (E(0) + E_2(0) + E_3(0) + E_4(0)), \quad (4.27)$$

$$k_2 (E(t) + E_2(t) + E_3(t) + E_4(t)) \leq L(t), \quad (4.28)$$

for constants $k_0, k_1 > 0$, $k_2 \in \mathbb{R}$.

Thus, from (4.26), (4.27), (4.28) and (3.1) it follows that

$$\int_0^{+\infty} E(t) dt \leq C_4,$$

for a constant $C_4 > 0$ which depends on $E(0), E_2(0), E_3(0), E_4(0)$ and $\int_0^{+\infty} h(s) \left\| \sqrt{A} \partial_s^i u_0(s) \right\|^2 ds$, $i = 0, 1, 2, 3$. As $E'(t) \leq 0$, we have

$$((t+1)E(t))' \leq E(t);$$

integrating from 0 to t yields

$$(t+1)E(t) \leq C_4 + E(0), \quad t \geq 0.$$

Hence, the estimate (3.2) holds. Furthermore, since the integral $\int_0^{+\infty} E(t) dt$ is convergent, it follows that

$$tE(t) \leq 2 \int_{\frac{t}{2}}^t E(s) ds \rightarrow 0, \quad t \rightarrow +\infty,$$

via the Cauchy convergence principle. This completes the proof of Theorem 3.1. $\square$

Remark 4.1. From Lemma 4.10 and the proof of Theorem 3.1, the condition $m(\Lambda) = 0$ in Assumption (I₂) can be relaxed. Indeed, $m(\Lambda)$ may be allowed to have small values (which could be obtained by some more fine estimates for the coefficients).

5. Applications

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial\Omega$, and $\mu(x), \mu_i(x)$ ($i = 1, 2, 3, 4, 5$) be variable coefficients depending only on the space variable with

$$\mu(x) \geq \mu_0, \quad \mu_1(x) \geq \mu_2(x), \quad \frac{\mu_i(x)}{\mu(x)} = \mu_{i0}, \quad i = 1, 2, 3, 4, 5; \quad (5.1)$$

here $\mu_0, \mu_{10}, \mu_{20}, \mu_{40}$ are four positive constants, and μ_{30}, μ_{50} are two constants with $\mu_{30}\mu_{50} > 0$.

The coupled systems considered in this section are:

Coupled system I).

For $(x, t) \in \Omega \times (0, +\infty)$,

$$\begin{aligned} u_{tt} - \operatorname{div}(\mu(x)\nabla u_{tt}) - \operatorname{div}(\mu_1(x)\nabla u) + \int_0^{+\infty} g(s)\operatorname{div}(\mu_2(x)\nabla u(t-s)) ds \\ - \operatorname{div}(\mu_3(x)\nabla v) = 0, \\ v_{tt} - \operatorname{div}(\mu(x)\nabla v_{tt}) - \operatorname{div}(\mu_4(x)\nabla v) - \operatorname{div}(\mu_5(x)\nabla u) = 0, \end{aligned}$$

with Dirichlet boundary conditions

$$u(x, t) = v(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty),$$

where and in the sequel, $(\cdot)_{tt}$ denotes the second derivative with respect to t .

Coupled system II).

For $(x, t) \in \Omega \times (0, +\infty)$,

$$\begin{aligned} u_{tt} - \operatorname{div}(\mu(x)\nabla u_{tt}) - \operatorname{div}(\mu_1(x)\nabla u) + \int_0^{+\infty} g(s)\operatorname{div}(\mu_2(x)\nabla u(t-s)) ds \\ - \operatorname{div}(\mu_3(x)\nabla v) = 0, \\ v_{tt} - \operatorname{div}(\mu(x)\nabla v_{tt}) + \Delta(\mu_4(x)\Delta v) - \operatorname{div}(\mu_5(x)\nabla u) = 0, \end{aligned}$$

with the homogeneous boundary conditions

$$u(x, t) = v(x, t) = \frac{\partial v(x, t)}{\partial \nu} = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty),$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative.

Coupled system III).

For $(x, t) \in \Omega \times (0, +\infty)$,

$$\begin{aligned} u_{tt} - \operatorname{div}(\mu(x)\nabla u_{tt}) + \Delta(\mu_1(x)\Delta u) - \int_0^{+\infty} g(s)\Delta(\mu_2(x)\Delta u(t-s)) ds \\ - \operatorname{div}(\mu_3(x)\nabla v) = 0, \\ v_{tt} - \operatorname{div}(\mu(x)\nabla v_{tt}) + \Delta(\mu_4(x)\Delta v) - \operatorname{div}(\mu_5(x)\nabla u) = 0, \end{aligned}$$

with the homogeneous boundary conditions

$$u(x, t) = v(x, t) = \frac{\partial u(x, t)}{\partial \nu} = \frac{\partial v(x, t)}{\partial \nu} = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty).$$

Clearly, if we set

$$\tilde{u} = (I - \operatorname{div}(\mu(x)\nabla))u, \quad \tilde{v} = \sqrt{\frac{\mu_{30}}{\mu_{50}}}(I - \operatorname{div}(\mu(x)\nabla))v,$$

$$A = \mu_{20} \Delta (\mu(x) \Delta) (I - \operatorname{div} (\mu(x) \nabla))^{-1}, \quad (5.2)$$

$$A_1 = \mu_{10} \Delta (\mu(x) \Delta) (I - \operatorname{div} (\mu(x) \nabla))^{-1}, \quad (5.3)$$

$$A_2 = \mu_{40} \Delta (\mu(x) \Delta) (I - \operatorname{div} (\mu(x) \nabla))^{-1}, \quad (5.4)$$

$$B = -\sqrt{\mu_{30}\mu_{50}} \operatorname{div} (\mu(x) \nabla) (I - \operatorname{div} (\mu(x) \nabla))^{-1}, \quad (5.5)$$

then the system (III) can be transformed into the abstract system

$$\begin{aligned} \tilde{u}''(t) + A_1 \tilde{u}(t) - \int_0^{+\infty} g(s) A \tilde{u}(t-s) ds + B \tilde{v}(t) &= 0, \\ \tilde{v}''(t) + A_2 \tilde{v}(t) + B \tilde{u}(t) &= 0. \\ \tilde{u}(-t) &= \tilde{u}_0(t), \quad \forall t \geq 0, \quad \tilde{u}'(0) = \tilde{u}_1, \\ \tilde{v}(0) &= \tilde{v}_0, \quad \tilde{v}'(0) = \tilde{v}_1. \end{aligned}$$

By (5.1), we know that

$$\begin{aligned} &-\operatorname{div} (\mu_1(x) \nabla), \quad -\operatorname{div} (\mu_2(x) \nabla), \quad -\operatorname{div} (\mu_3(x) \nabla), \quad -\operatorname{div} (\mu_4(x) \nabla), \\ &-\operatorname{div} (\mu_5(x) \nabla), \quad \Delta (\mu_1(x) \Delta), \quad \Delta (\mu_2(x) \Delta), \quad \Delta (\mu_4(x) \Delta), \quad (I - \operatorname{div} (\mu(x) \nabla))^{-1} \end{aligned}$$

are commutative. Moreover A, A_1, A_2, B in (5.2)–(5.5) are positive self-adjoint linear operators. So, it is not difficult to verify that **Coupled system III** satisfies Assumption (I₁). Therefore, Theorem 3.1 tells us that the solution energy of the above systems decays at the rate of t^{-1} , once the kernel $g(s)$ satisfies Assumption (I₂).

Coupled system I and **Coupled system II** are similar to **Coupled system III**; we omit the details here.

Furthermore, by Theorem 3.1, we can also obtain directly the optimal energy decay estimate for the following concrete coupled systems (I), (II), and (III).

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and $\lambda(x)$, $\lambda_1(x)$, $\lambda_2(x)$ be variable coefficients depending only on the space variable with

$$\lambda_1(x) \geq \lambda(x) \geq \lambda_{10}, \quad \lambda_2(x) \geq \lambda_{20};$$

where $\lambda_{10}, \lambda_{20}$ are positive constants, and β_1, β_2 are constants with $\beta_1 \beta_2 > 0$.

(I) For $(x, t) \in \Omega \times (0, +\infty)$,

$$\begin{aligned} u_{tt} - \operatorname{div} (\lambda_1(x) \nabla u) + \int_0^{+\infty} g(s) \operatorname{div} (\lambda(x) \nabla u(t-s)) ds + \beta_1 v &= 0, \\ v_{tt} - \operatorname{div} (\lambda_2(x) \nabla v) + \beta_2 u &= 0, \end{aligned}$$

with the homogeneous Dirichlet boundary conditions

$$u(x, t) = v(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty).$$

(II) For $(x, t) \in \Omega \times (0, +\infty)$,

$$\begin{aligned} u_{tt} - \operatorname{div} (\lambda_1(x) \nabla u) + \int_0^{+\infty} g(s) \operatorname{div} (\lambda(x) \nabla u(t-s)) ds + \beta_1 v &= 0, \\ v_{tt} + \Delta (\lambda_2(x) \Delta v) + \beta_2 u &= 0, \end{aligned}$$

with the homogeneous boundary conditions

$$u(x, t) = v(x, t) = \frac{\partial v(x, t)}{\partial \nu} = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty).$$

(III) For $(x, t) \in \Omega \times (0, +\infty)$,

$$u_{tt} + \Delta(\lambda_1(x)\Delta u) - \int_0^{+\infty} g(s)\Delta(\lambda(x)\Delta u(t-s))ds + \beta_1 v = 0,$$

$$v_{tt} + \Delta(\lambda_2(x)\Delta v) + \beta_2 u = 0,$$

with the homogeneous boundary conditions

$$u(x, t) = v(x, t) = \frac{\partial u(x, t)}{\partial \nu} = \frac{\partial v(x, t)}{\partial \nu} = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty).$$

It is not difficult to verify the systems (I)-(III) satisfy Assumption (I₁) if we take $\tilde{u} = u$, $\tilde{v} = \sqrt{\frac{\beta_1}{\beta_2}}$. Therefore, if the kernel $g(s)$ satisfies Assumption (I₂), we can we utilize Theorem 3.1 to conclude that the whole energy of each of systems (I)-(III) decays at the rate of t^{-1} .

Acknowledgments

The authors would like to thank the referee very much for valuable and helpful comments and suggestions.

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