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Multi-Toeplitz operators and free pluriharmonic functions ☆

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A R T I C L E I N F O

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A B S T R A C T

We initiate the study of weighted multi-Toeplitz operators associated with noncommutative regular domains $\mathbf{D}_q^m(\mathcal{H}) \subset B(\mathcal{H})^n$, $m, n \geq 1$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. These operators are acting on the full Fock space with n generators and have as symbols free pluriharmonic functions on the interior of the domain $\mathbf{D}_q^m(\mathcal{H})$. We prove that the set of all weighted multi-Toeplitz operators coincides with

$$\overline{\mathcal{A}(\mathbf{D}_q^m)^* + \mathcal{A}(\mathbf{D}_q^m)}^{WOT},$$

where the domain algebra $\mathcal{A}(\mathbf{D}_q^m)$ is the norm-closed unital non-selfadjoint algebra generated by the universal model $(W_1, \ldots, W_n)$ of the noncommutative domain $\mathbf{D}_q^m(\mathcal{H})$. These results are used to study the class of free pluriharmonic functions on $\mathbf{D}_q^m(\mathcal{H})^\circ$. Several classical results from complex analysis concerning harmonic functions have analogues in our noncommutative setting. In particular, we show that the bounded free pluriharmonic functions are precisely those which are noncommutative Berezin transforms of weighted multi-Toeplitz operators, and solve the Dirichlet extension problem in this setting. Using noncommutative Cauchy transforms, we provide a free analytic functional calculus for n-tuples of operators, which extends to free pluriharmonic functions. Our study of weighted multi-Toeplitz operators on Fock spaces is a blend of multi-variable operator theory, noncommutative function theory, operator spaces, and harmonic analysis.

0. Introduction

Let $H^2(\mathbb{D})$ be the Hardy space of all analytic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ with square-summable coefficients. An operator $T \in B(H^2(\mathbb{D}))$ is called Toeplitz if

$$Tf = P_+(\varphi f), \qquad f \in H^2(\mathbb{T}),$$

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for some $\varphi \in L^\infty(\mathbb{T})$, where P_+ is the orthogonal projection of the Lebesgue space $L^2(\mathbb{T})$ onto the Hardy space $H^2(\mathbb{T})$, which is identified with $H^2(\mathbb{D})$. Brown and Halmos [3] proved that a necessary and sufficient condition that an operator on the Hardy space $H^2(\mathbb{D})$ be a Toeplitz operator is that its matrix $[\lambda_{ij}]$ with respect to the standard basis $e_k(z) = z^k$, $k \in \{0, 1, \dots\}$, be a Toeplitz matrix, i.e.

$$\lambda_{i+1,j+1} = \lambda_{ij}, \qquad i, j \in \{0, 1, \dots\},$$

which is equivalent to $S^*TS = T$, where S is the unilateral shift on $H^2(\mathbb{D})$. In this case, $\lambda_{ij} = a_{i-j}$, where $\varphi = \sum_{k\in\mathbb{Z}} a_k\chi_k$ is the Fourier expansion of the symbol $\varphi \in L^\infty(\mathbb{T})$. The class of Toeplitz operators originates with O. Toeplitz [38] and has been studied extensively over the years, starting with Hartman and Wintner [10] and the seminal paper of Brown and Halmos [3]. The study of Toeplitz operators on the Hardy space $H^2(\mathbb{D})$ was extended to Hilbert spaces of holomorphic functions on the unit disc (see [11]) such as the Bergman space and weighted Bergman space, and also to higher dimensional setting involving holomorphic functions in several complex variables on various classes of domains in $\mathbb{C}^n$ (see Upmeier's book [39]).

The class of Toeplitz operators is one of the most important classes of non-selfadjoint operators having applications in index theory and noncommutative geometry, prediction theory, boundary values problems for analytic functions, probability, information theory and control theory, and several other fields. We refer the reader to [2], [7], [35], and [11] for a comprehensive account on Toeplitz operators.

A polynomial $q \in \mathbb{C}\langle Z_1, \dots, Z_n\rangle$ in n noncommutative indeterminates is called *positive regular* if all its coefficients are positive, $q(0) = 0$, and the coefficients of the linear terms $Z_1, \dots, Z_n$ are different from zero. If $q = \sum_\alpha a_\alpha Z_\alpha$ and $X = (X_1, \dots, X_n) \in B(\mathcal{H})^n$, we define the completely positive map

$$\Phi_{q,X} : B(\mathcal{H}) \to B(\mathcal{H}), \qquad \Phi_{q,X}(Y) := \sum_\alpha a_\alpha X_\alpha Y X_\alpha^*.$$

For each $m \geq 1$, we define the *noncommutative regular domain*

$$\mathbf{D}_q^m(\mathcal{H}) := \left\{X := (X_1, \dots, X_n) \in B(\mathcal{H})^n : \ (id - \Phi_{q,X})^k(I) \geq 0 \text{ for } 1 \leq k \leq m\right\}.$$

According to [30] and [28], each such a domain has a universal model $(W_1, \dots, W_n)$ consisting of weighted left creation operators acting on the full Fock space with n generators. We mention a few remarkable particular cases.

Single variable case: $n = 1$.

(i) If $m = 1$ and $q = Z$, the corresponding domain $\mathbf{D}_q^m(\mathcal{H})$ coincides with the closed unit ball $[B(\mathcal{H})]_1 := \{X \in B(\mathcal{H}) : \ \|X\| \leq 1\}$, the study of which has generated the Nagy–Foiaş theory of contractions (see [37]). In this case, the universal model is the unilateral shift S acting on the Hardy space $H^2(\mathbb{D})$. The Toeplitz operators on the Hardy space $H^2(\mathbb{D})$ have been studied extensively (see for example [7], [35]).

(ii) If $m \geq 2$ and $q = Z$, the corresponding domain coincides with the set of all m-hypercontractions studied by Agler in [1], and recently by Olofsson [19], [20]. The corresponding universal model is the unilateral shift acting on the weighted Bergman space $A_m(\mathbb{D})$, the Hilbert space of all analytic functions on the unit disc $\mathbb{D}$ with

$$\|f\|^2 := \frac{m-1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{m-2} dz < \infty.$$

In [16], Louhichi and Olofsson obtain a Brown–Halmos type characterization of Toeplitz operators with harmonic symbols on $A_m(\mathbb{D})$, which can be seen as a reproducing kernel Hilbert space with reproducing kernel given by $\kappa_m(z, w) := (1 - z\bar{w})^{-m}$, $z, w \in \mathbb{D}$. Their result was recently extended by Eschmeier and Langendörfer [9] to the analytic functional Hilbert space $H_m(\mathbb{B})$ on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ given by the reproducing kernel $\kappa_m(z, w) := (1 - \langle z, w\rangle)^{-m}$ for $z, w \in \mathbb{B}$, where $m \geq 1$.

Multivariable noncommutative case: $n \geq 2$.

(i) When $m = 1$ and $q = Z_1 + \cdots + Z_n$, the noncommutative domain $\mathbf{D}_q^m(\mathcal{H})$ coincides with the closed unit ball $[B(\mathcal{H})^n]_1 := \{(X_1, \ldots, X_n) : X_1X_1^* + \cdots + X_nX_n^* \leq I\}$, the study of which has generated a free analogue of Nagy–Foiaş theory. The corresponding universal model is the n-tuple of left creation operators $(S_1, \ldots, S_n)$ acting on the full Fock space with n generators. A study of unweighted multi-Toeplitz operators on the full Fock space with n generators was initiated in [23], [24] and has had an important impact in multivariable operator theory and the structure of free semigroups algebras (see [4], [5], [6], [27], [29], [14], [15], [17], [18]).

(ii) When $m \geq 1$, $n \geq 1$, and q is any positive regular polynomial the domain $\mathbf{D}_q^m(\mathcal{H})$ was studied in [30] (when $m = 1$), and in [28] (when $m \geq 2$). In this case, the corresponding universal model is an n-tuple of weighted left creation operators acting on the full Fock space. We remark that, in the particular case when $m \geq 2$ and $q = Z_1 + \cdots + Z_n$, the corresponding domain can be seen as a noncommutative m-hyperball, the elements of which can be viewed as multivariable noncommutative analogues of Agler's m-hypercontractions. As far as we know, Toeplitz operators have not been introduced or studied in this very general setting.

The goal of the present paper is to initiate the study of weighted multi-Toeplitz operators associated with noncommutative regular domains $\mathbf{D}_q^m(\mathcal{H}) \subset B(\mathcal{H})^n$, $m, n \geq 1$, when $q \in \mathbb{C}\langle Z_1, \ldots, Z_n\rangle$ is any positive regular polynomial in noncommutative indeterminates. This is accompanied by the study of their symbols which are free pluriharmonic functions on the interior of the domain $\mathbf{D}_q^m(\mathcal{H})$.

In Section 1, we present some background from [30] and [28] on the noncommutative domains $\mathbf{D}_q^m(\mathcal{H})$, their universal models, and the associated noncommutative Berezin transforms.

In Section 2, we introduce the weighted multi-Toeplitz operators which are acting on the full Fock space $F^2(H_n)$ with n generators and are associated with the noncommutative domain $\mathbf{D}_q^m(\mathcal{H}) \subset B(\mathcal{H})^n$. We show that they are uniquely determined by their free pluriharmonic symbols

$$\varphi(X_1, \ldots, X_n) = \sum_{k=1}^{\infty} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} b_\alpha X_\alpha^* + \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} a_\alpha X_\alpha, \qquad a_\alpha, b_\alpha \in \mathbb{C},$$

where $\mathbb{F}_n^+$ is the unital free semigroup with n generators and the convergence of the series is in the operator norm topology for any n-tuple $(X_1, \ldots, X_n)$ in the interior of $\mathbf{D}_q^m(\mathcal{H})$. We prove that the set of all weighted multi-Toeplitz operators coincides with

$$\overline{\mathcal{A}(\mathbf{D}_q^m)^* + \mathcal{A}(\mathbf{D}_q^m)}^{WOT},$$

where the domain algebra $\mathcal{A}(\mathbf{D}_q^m)$ is the norm-closed unital non-selfadjoint algebra generated by the universal model $(W_1, \ldots, W_n)$ of the noncommutative domain $\mathbf{D}_q^m(\mathcal{H})$. In the particular case when $n = 1$ and $q = Z$, we obtain a characterization of the Toeplitz operators with harmonic symbol on the Bergman space $A_m(\mathbb{D})$, which should be compared with the corresponding result from [16].

In Section 3, we provide basic results concerning the free pluriharmonic functions on the noncommutative domain $\mathbf{D}_q^m(\mathcal{H})^\circ$ and show that they are characterized by a mean value property. This result is used to obtain an analogue of Weierstrass theorem for free pluriharmonic functions and to show that the set $Har((\mathbf{D}_q^m)^\circ)$ of all pluriharmonic functions is a complete metric space with respect to an appropriate metric ρ. We also obtain, in this section, a Schur type result [36] characterizing the free pluriharmonic functions with positive real parts in terms of positive semi-definite weighted multi-Toeplitz kernels.

Section 4 concerns the space $Har^\infty((\mathbf{D}_q^m)^\circ)$ of all bounded free pluriharmonic functions on $\mathbf{D}_q^m(\mathcal{H})^\circ$. One of the main results states that $F \in Har^\infty((\mathbf{D}_q^m)^\circ)$ if and only if it is the noncommutative Berezin transform of a weighted multi-Toeplitz operator. Moreover, we prove that the map

$$\Phi : Har^\infty((\mathbf{D}_q^m)^\circ) \to \overline{\mathcal{A}(\mathbf{D}_q^m)^* + \mathcal{A}(\mathbf{D}_q^m)}^{WOT}$$

defined by $\Phi(F) := \text{SOT-}\lim_{r\to 1} F(rW)$ is a completely isometric isomorphism of operator spaces. A noncommutative version of the Dirichlet extension problem for harmonic functions (see [13]) is also provided. We prove that $F \in Har((\mathbf{D}_q^m)^\circ)$ has a continuous extension in the operator norm topology to $\mathbf{D}_q^m(\mathcal{H})$ if and only if there exists a multi-Toeplitz operator $\psi \in \overline{\mathcal{A}(\mathbf{D}_q^m)^* + \mathcal{A}(\mathbf{D}_q^m)}^{\|\cdot\|}$ such that F is the noncommutative Berezin transform of ψ.

In Section 5, using noncommutative Cauchy transforms associated with the domain $\mathbf{D}_q^m(\mathcal{H})$, we provide a free analytic functional calculus for n-tuples of operators $X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ with the spectral radius of the reconstruction operator $R_{\tilde{q},X}$ strictly less than 1. This extends to free pluriharmonic functions, proving that the map

$$\Psi_{q,X} : \big(Har((\mathbf{D}_q^m)^\circ), \rho\big) \to (B(\mathcal{H}), \|\cdot\|)$$

defined by $\Psi_{q,X}(G) := G(X)$ is continuous and its restriction $\Psi_{q,X}|_{Hol((\mathbf{D}_q^m)^\circ)}$ is a continuous unital algebra homomorphism. Several consequences of this result are also provided.

We should mention that our results are presented in the more general setting of weighted multi-Toeplitz matrices with operator-valued entries and free pluriharmonic functions with operator-valued coefficients, while the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ is generated by any positive regular free holomorphic functions f in a neighborhood of the origin.

In a forthcoming paper [34], we obtain a Brown–Halmos characterization of the weighted multi-Toeplitz operators associated with the noncommutative m-hyperball (the case when $q = Z_1 + \cdots + Z_n, m \geq 2$) which is a noncommutative version of Eschmeier and Langendörfer recent commutative result [9]. This result shows that the weighted multi-Toeplitz are characterized by an algebraic equation involving the universal model $(W_1, \ldots, W_n)$ of the noncommutative m-hyperball. It remains to be seen if this characterization extends to the more general domains $\mathbf{D}_q^m$, where q is any positive regular polynomial.

1. Noncommutative domains, universal models, and Berezin transforms

This section contains some definitions and the necessary background from [30] and [28] on the noncommutative regular domains $\mathbf{D}_f^m(\mathcal{H})$, their universal models, and the associated noncommutative Berezin transforms.

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$. If $Z_1, \ldots, Z_n$ are noncommutative indeterminates, we denote $Z_\alpha := Z_{i_1} \ldots Z_{i_k}$ if $\alpha = g_{i_1} \ldots g_{i_k} \in \mathbb{F}_n^+$, $i_1, \ldots i_k \in \{1, \ldots, n\}$, and $Z_{g_0} := 1$. Similarly, if $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$, we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_\mathcal{H}$. A formal power series $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$, $a_\alpha \in \mathbb{C}$, in noncommutative indeterminates $Z_1, \ldots, Z_n$, is called *free holomorphic function* on the noncommutative ball $[B(\mathcal{H})^n]_\rho$ for some $\rho > 0$, where

$$[B(\mathcal{H})^n]_\rho := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \|X_1X_1^* + \cdots + X_nX_n^*\|^{1/2} < \rho\},$$

if the series $\sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha$ is convergent in the operator norm topology for any $(X_1,\dots,X_n) \in [B(\mathcal{H})^n]_\rho$. According to [26], f is a free holomorphic function on $[B(\mathcal{H})^n]_\rho$ for any Hilbert space $\mathcal{H}$ if and only if

$$\limsup_{k\to\infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2 \right)^{1/2k} \leq \frac{1}{\rho}.$$

Throughout this paper, we assume that $a_\alpha \geq 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, and $a_{g_i} > 0$ if $i \in \{1,\dots,n\}$. A function f satisfying all these conditions on the coefficients is called *positive regular free holomorphic function on* $[B(\mathcal{H})^n]_\rho$. Let $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ be the completely positive linear map given by $\Phi_{f,X}(Y) := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha Y X_\alpha^*$ for $Y \in B(\mathcal{H})$, where the convergence is in the weak operator topology, and define the *noncommutative regular domain*

$$\mathbf{D}_f^m(\mathcal{H}) := \left\{ X := (X_1,\dots,X_n) \in B(\mathcal{H})^n : \ (id - \Phi_{f,X})^k(I) \geq 0 \text{ for } 1 \leq k \leq m \right\}.$$

We saw in [28], that $X \in \mathbf{D}_f^m(\mathcal{H})$ if and only if $\Phi_{f,X}(I) \leq I$ and $(id - \Phi_{f,X})^m(I) \geq 0$. The *abstract noncommutative domain* $\mathbf{D}_f^m$ is the disjoint union $\coprod_{\mathcal{H}\in\mathcal{A}} \mathbf{D}_f^m(\mathcal{H})$, where $\mathcal{A}$ is a set of Hilbert spaces including one which is separable and infinite dimensional. We associate with the abstract domain $\mathbf{D}_f^m$ a unique n-tuple $(W_1,\dots,W_n)$ of weighted shifts, as follows. Define $b_{g_0}^{(m)} := 1$ and

$$b_\alpha^{(m)} = \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1\cdots\gamma_j=\alpha \\ |\gamma_1|\geq 1,\dots,|\gamma_j|\geq 1}} a_{\gamma_1}\cdots a_{\gamma_j} \begin{pmatrix} j+m-1 \\ m-1 \end{pmatrix} \qquad \text{if } \ \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1, \tag{1.1}$$

(see Lemma 1.1 from [28]). Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \dots, e_n$, where $n \in \mathbb{N} := \{1,2,\dots\}$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k\geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$ and $H_n^{\otimes k}$ is the Hilbert tensor product of k copies of H_n. Let $D_i : F^2(H_n) \to F^2(H_n)$, $i \in \{1,\dots,n\}$, be the diagonal operators defined by setting

$$D_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}} e_\alpha, \qquad \alpha \in \mathbb{F}_n^+,$$

where $\{e_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is the orthonormal basis of the full Fock space $F^2(H_n)$. The *weighted left creation operators* $W_i : F^2(H_n) \to F^2(H_n)$ associated with $\mathbf{D}_f^m$ are defined by $W_i := S_i D_i$, where $S_1,\dots,S_n$ are the left creation operators on the full Fock space $F^2(H_n)$, i.e.

$$S_i\varphi := e_i \otimes \varphi, \qquad \varphi \in F^2(H_n), \ i \in \{1,\dots,n\}.$$

A simple calculation reveals that

$$W_\beta e_\gamma = \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\beta\gamma}^{(m)}}} e_{\beta\gamma} \quad \text{and} \quad W_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = \beta\gamma \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$. We recall from [28] that the weighted left creation operators $W_1, \dots, W_n$ have the following properties:

(i) $\sum_{|\beta|\geq 1} a_\beta W_\beta W_\beta^* \leq I$, where the convergence is in the strong operator topology;

(ii) $(id - \Phi_{f,W})^m (I) = P_{\mathbb{C}}$, where $P_{\mathbb{C}}$ is the orthogonal projection of $F^2(H_n)$ on $\mathbb{C}1 \subset F^2(H_n)$, and the map $\Phi_{f,W} : B(F^2(H_n)) \to B(F^2(H_n))$ is defined by

$$\Phi_{f,W}(Y) := \sum_{|\alpha|\geq 1} a_\alpha W_\alpha Y W_\alpha^*,$$

where the convergence is in the weak operator topology;

(iii) $W := (W_1, \dots, W_n)$ is a *pure* element of the domain $\mathbf{D}_f^m(F^2(H_n))$, i.e. $\lim_{p\to\infty} \Phi_{f,W}^p(I) = 0$ in the strong operator topology.

The right creation operators are defined by $R_i\varphi := \varphi \otimes e_i$, $i \in \{1, \dots, n\}$. We can also define the *weighted right creation operators* $\Lambda_i : F^2(H_n) \to F^2(H_n)$ by setting $\Lambda_i := R_i G_i$, $i = 1, \dots, n$, where each diagonal operator G_i, $i = 1, \dots, n$, is given by

$$G_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{\alpha g_i}^{(m)}}} e_\alpha, \quad \alpha \in \mathbb{F}_n^+,$$

where the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$, are described by relation (1.1). In this case, we have

$$\Lambda_\beta e_\gamma = \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\gamma\tilde{\beta}}^{(m)}}} e_{\gamma\tilde{\beta}} \quad \text{and} \quad \Lambda_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = \gamma\tilde{\beta} \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$, where $\tilde{\beta}$ denotes the reverse of $\beta = g_{i_1} \cdots g_{i_k}$, i.e., $\tilde{\beta} = g_{i_k} \cdots g_{i_1}$. As in the case of weighted left creation operators, one can show that

$$\sum_{|\beta|\geq 1} a_{\tilde{\beta}} \Lambda_\beta \Lambda_\beta^* \leq I \quad \text{and} \quad \left(id - \Phi_{\tilde{f},\Lambda}\right)^m (I) = P_{\mathbb{C}}, \tag{1.4}$$

where $\tilde{f}(Z) := \sum_{|\alpha|\geq 1} a_{\tilde{\alpha}} Z_\alpha$, $\tilde{\alpha}$ denotes the reverse of α, and $\Phi_{\tilde{f},\Lambda}(Y) := \sum_{|\alpha|\geq 1} a_{\tilde{\alpha}} \Lambda_\alpha Y \Lambda_\alpha^*$ for any $Y \in B(F^2(H_n))$, with the convergence is in the weak operator topology.

Let $X := (X_1, \dots, X_n) \in \mathbf{D}_f^m(\mathcal{H})$ and let $K_{f,X}^{(m)} : \mathcal{H} \to F^2(H_n) \otimes \overline{\Delta_{m,X}(\mathcal{H})}$ be the *noncommutative Berezin kernel* defined by

$$K_{f,X}^{(m)} h := \sum_{\alpha\in\mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} e_\alpha \otimes \Delta_{m,X} X_\alpha^* h, \qquad h \in \mathcal{H},$$

where $\Delta_{m,X} := [(I - \Phi_{f,X})^m(I)]^{1/2}$ and the coefficients $b_\alpha^{(m)}$ are given by relation (1.1). We know that

$$K_{f,X}^{(m)} X_i^* = (W_i^* \otimes I) K_{f,X}^{(m)} \qquad i \in \{1, \dots, n\}.$$

Assume that X is a *pure* n-tuple, i.e. $\Phi_{f,X}^k(I) \to 0$ strongly, as $k \to \infty$. Then $K_{f,X}^{(m)}$ is an isometry and the n-tuple $W := (W_1, \dots, W_n)$ plays the role of the *universal model for the noncommutative domain* $\mathbf{D}_f^m$.

Let $\varphi(W) := \sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta$, $c_\beta \in \mathbb{C}$, be a formal sum with the property that $\sum_{\beta\in\mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} < \infty$. In [28], we proved that $\sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta(p) \in F^2(H_n)$ for any $p \in \mathcal{P}$, where $\mathcal{P} \subset F^2(H_n)$ is the set of all polynomial in e_α, $\alpha \in \mathbb{F}_n^+$. If

$$\sup_{p\in\mathcal{P},\|p\|\le 1} \left\| \sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta(p) \right\| < \infty,$$

then there is a unique bounded operator acting on $F^2(H_n)$, which we should also denote by $\varphi(W)$, such that

$$\varphi(W)p = \sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta(p) \quad \text{for any } \ p \in \mathcal{P}.$$

The set of all operators $\varphi(W) \in B(F^2(H_n))$ satisfying the above-mentioned properties is denoted by $F^\infty(\mathbf{D}_f^m)$. One can prove that $F^\infty(\mathbf{D}_f^m)$ is a Banach algebra, which we call Hardy algebra associated with the noncommutative domain $\mathbf{D}_f^m$. We introduce the domain algebra $\mathcal{A}(\mathbf{D}_f^m)$ to be the norm closure of all polynomials in the weighted left creation operators $W_1, \dots, W_n$ and the identity. Using the weighted right creation operators associated with $\mathbf{D}_f^m$, one can also define the corresponding domain algebra $\mathcal{R}(\mathbf{D}_f^m)$.

In a similar manner, using the weighted right creation operators $\Lambda := (\Lambda_1, \dots, \Lambda_n)$ associated with $\mathbf{D}_f^m$, one can define the corresponding Hardy algebra $R^\infty(\mathbf{D}_f^m)$. More precisely, if $g(\Lambda) = \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde{\beta}} \Lambda_\beta$ is a formal sum with the property that $\sum_{\beta\in\mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} < \infty$ and such that

$$\sup_{p\in\mathcal{P},\|p\|\le 1} \left\| \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde{\beta}} \Lambda_\beta(p) \right\| < \infty,$$

then there is a unique bounded operator on $F^2(H_n)$, which we also denote by $g(\Lambda)$, such that

$$g(\Lambda)p = \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde{\beta}} \Lambda_\beta(p) \quad \text{for any } \ p \in \mathcal{P}.$$

The set of all operators $g(\Lambda) \in B(F^2(H_n))$ satisfying the above-mentioned properties is denoted by $R^\infty(\mathbf{D}_f^m)$. We proved in [28] that $F^\infty(\mathbf{D}_f^m)' = R^\infty(\mathbf{D}_f^m)$ and $F^\infty(\mathbf{D}_f^m)'' = F^\infty(\mathbf{D}_f^m)$, where $'$ stands for the commutant.

Let us recall some definitions concerning completely bounded maps on operator spaces. We identify $M_k(B(\mathcal{H}))$, the set of $k \times k$ matrices with entries in $B(\mathcal{H})$, with $B(\mathcal{H}^{(k)})$, where $\mathcal{H}^{(k)}$ is the direct sum of k copies of $\mathcal{H}$. If $\mathcal{X}$ is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_k(\mathcal{X})$ as a subspace of $M_k(B(\mathcal{H}))$ with the induced norm. Let $\mathcal{X}, \mathcal{Y}$ be operator spaces and $u : \mathcal{X} \to \mathcal{Y}$ be a linear map. Define the map $u_k : M_k(\mathcal{X}) \to M_k(\mathcal{Y})$ by

$$u_k([x_{ij}]_k) := [u(x_{ij})]_k.$$

We say that u is completely bounded if $\|u\|_{cb} := \sup_{k\ge 1} \|u_k\| < \infty$. When $\|u\|_{cb} \le 1$ (resp. u_k is an isometry for any $k \ge 1$) then u is completely contractive (resp. isometric). We call u completely positive if u_k is positive for all $k \ge 1$. For more information on completely bounded (resp. positive) maps, we refer to [21] and [22].

Definition 1.1. An n-tuple $X \in \mathbf{D}_f^m(\mathcal{H})$ is called pure if $\lim_{p\to\infty} \Phi_{f,T}^p(I) = 0$ in the strong operator topology.

The *noncommutative Berezin transform at* $X \in \mathbf{D}_f^m(\mathcal{H})$, where X is a pure element, is the map $\mathbf{B}_X^{(m)} : B(F^2(H_n)) \to B(\mathcal{H})$ defined by

$$\mathbf{B}_X^{(m)}[g] := K_{f,X}^{(m)*}(g \otimes I_{\mathcal{H}})K_{f,X}^{(m)}, \quad g \in B(F^2(H_n)),$$

where the $K_{f,X}^{(m)} : \mathcal{H} \to F^2(H_n)\otimes\mathcal{H}$ is noncommutative Berezin kernel. Let $\mathcal{P}(W)$ be the set of all polynomials $p(W)$ in the operators W_i, $i \in \{1,\dots,k\}$, and the identity. If g is in the operator space

$$\mathcal{S} := \overline{\operatorname{span}}\{p(W)q(W)^* : \ p(W), q(W) \in \mathcal{P}(W)\},$$

where the closure is in the operator norm, we define the Berezin transform at $X \in \mathbf{D}_f^m(\mathcal{H})$, by

$$\mathbf{B}_X^{(m)}[g] := \lim_{r\to1} K_{f,rX}^{(m)}{}^*(g \otimes I_{\mathcal{H}})K_{f,rX}^{(m)}, \qquad g \in \mathcal{S},$$

where the limit is in the operator norm topology. In this case, the Berezin transform at X is a unital completely positive linear map such that

$$\mathbf{B}_X^{(m)}(W_\alpha W_\beta^*) = X_\alpha X_\beta^*, \qquad \alpha, \beta \in \mathbb{F}_n.$$

If, in addition, X is a pure n-tuple in $\mathbf{D}_f^m(\mathcal{H})$, then $\lim_{r\to1} \mathbf{B}_{rX}^{(m)}[g] = \mathbf{B}_X^{(m)}[g]$, $g \in \mathcal{S}$. More on noncommutative Berezin transforms and their applications can be found in [25], [28], [30], [31], and [32].

2. Weighted multi-Toeplitz operators on Fock spaces

In this section, we introduce the weighted multi-Toeplitz operators associated with the noncommutative domain $\mathbf{D}_f^m(\mathcal{H}) \subset B(\mathcal{H})^n$. We show that they are uniquely determined by their free pluriharmonic symbols and provide a characterization in terms of the domain algebra $\mathcal{A}(\mathbf{D}_f^m)$.

In what follows, we need some notation. If $\omega, \gamma \in \mathbb{F}_n^+$, we say that $\omega \geq_r \gamma$ if there is $\sigma \in \mathbb{F}_n^+$ such that $\omega = \sigma\gamma$. In this case we set $\omega\backslash_r\gamma := \sigma$. If $\sigma \neq g_0$ we write $\omega >_r \gamma$. We say that ω and γ are *comparable* if either $\omega \geq_r \gamma$ or $\gamma >_r \omega$. If $\omega, \gamma \in \mathbb{F}_n^+$ are comparable, we consider the weights

$$\lambda_{\omega,\gamma} := \begin{cases} \sqrt{\frac{b_\omega^{(m)}}{b_\gamma^{(m)}}}, & \text{if } \ \omega \geq_r \gamma, \\ \sqrt{\frac{b_\gamma^{(m)}}{b_\omega^{(m)}}}, & \text{if } \ \gamma >_r \omega, \end{cases}$$

where the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$, are given by relation (1.1). Let $\mathcal{E}$ be a separable Hilbert space and let $[C_{\omega,\gamma}]_{\mathbb{F}_n^+\times\mathbb{F}_n^+}$ be the operator matrix representation of $T \in B(\mathcal{E}\otimes F^2(H_n))$, i.e.

$$\langle C_{\omega,\gamma}x, y\rangle := \langle T(x \otimes e_\gamma), y \otimes e_\omega\rangle$$

for any $\omega, \gamma \in \mathbb{F}_n^+$ and $x, y \in \mathcal{E}$.

Definition 2.1. We say that T is a weighted right multi-Toeplitz operator if for each $i \in \{1,\dots,n\}$ and $\omega, \gamma, \alpha, \beta \in \mathbb{F}_n^+$,

$$\lambda_{\omega g_i,\gamma g_i} C_{\omega g_i,\gamma g_i} = \lambda_{\omega,\gamma} C_{\omega,\gamma}, \qquad \text{if } \ \omega, \gamma \ \text{ are comparable},$$

and $C_{\alpha,\beta} = 0$ if α, β are not comparable.

We remark that when $n = m = 1$, $f = Z$, and $\mathcal{E} = \mathbb{C}$ we recover the classical Toeplitz operators on the Hardy space $H^2(\mathbb{D})$. Also if $n \geq 2$, $m = 1$, and $f = Z_1 + \cdots + Z_n$ we obtain the unweighted right multi-Toeplitz operators on the full Fock space $F^2(H_n)$ (see [23], [24] and [29]). In this case, we have $b_\alpha^{(m)} = 1$ for any $\alpha \in \mathbb{F}_n^+$ and the condition above becomes

$$C_{\omega g_i, \gamma g_i} = \begin{cases} C_{\omega,\gamma}, & \text{if } \ \omega \geq_r \gamma \text{ or } \ \gamma >_r \omega, \\ 0, & \text{otherwise}, \end{cases}$$

and $C_{\alpha,\beta} = 0$ if α, β are not comparable.

For an equivalent and more transparent definition of weighted right multi-Toeplitz operators on the full Fock space $F^2(H_n)$ see the remarks following the next theorem.

Theorem 2.2. *Any weighted right multi-Toeplitz operator $T \in B(\mathcal{E} \otimes F^2(H_n))$ has a formal Fourier representation*

$$\varphi(W) := \sum_{|\alpha| \geq 1} B_{(\alpha)} \otimes W_\alpha^* + A_{(0)} \otimes I + \sum_{|\alpha| \geq 1} A_{(\alpha)} \otimes W_\alpha,$$

where $\{A_{(\alpha)}\}_{\alpha \in \mathbb{F}_n^+}$ and $\{B_{(\alpha)}\}_{\alpha \in \mathbb{F}_n^+ \setminus \{g_0\}}$ are some operators on the Hilbert space $\mathcal{E}$, such that

$$Tq = \varphi(W)q, \qquad q = \sum_{|\alpha| \leq k} h_\alpha \otimes e_\alpha,$$

for any $h_\alpha \in \mathcal{E}$ and $k \in \mathbb{N}$. If T_1, T_2 are weighted right multi-Toeplitz operators having the same formal Fourier representation, then $T_1 = T_2$.

Proof. First, we note that, using Definition 2.1, one can prove that $T \in B(\mathcal{E} \otimes F^2(H_n))$ is a weighted right multi-Toeplitz operator if and only if the entries of its matrix representation $[C_{\omega,\gamma}]_{\mathbb{F}_n^+ \times \mathbb{F}_n^+}$ satisfy the following relations:

(i) $C_{\sigma\gamma,\gamma} = \sqrt{\frac{b_\sigma^{(m)} b_\gamma^{(m)}}{b_{\sigma\gamma}^{(m)}}} C_{\sigma, g_0}$ for any $\sigma, \gamma \in \mathbb{F}_n^+$;

(ii) $C_{\gamma,\sigma\gamma} = \sqrt{\frac{b_\sigma^{(m)} b_\gamma^{(m)}}{b_{\sigma\gamma}^{(m)}}} C_{g_0,\sigma}$ for any $\sigma, \gamma \in \mathbb{F}_n^+$;

(iii) $C_{\alpha,\beta} = 0$ if $(\alpha, \beta) \in \mathbb{F}_n^+ \times \mathbb{F}_n^+$ is not of the form $(\sigma\gamma, \gamma)$ or $(\gamma, \sigma\gamma)$ for $\sigma, \gamma \in \mathbb{F}_n^+$.

Consequently, $T \in B(\mathcal{E} \otimes F^2(H_n))$ is a weighted right multi-Toeplitz if and only if

$$\langle T(x \otimes e_\gamma), y \otimes e_\omega \rangle = \begin{cases} \frac{\sqrt{b_{\omega \backslash_r \gamma}^{(m)}} \sqrt{b_\gamma^{(m)}}}{\sqrt{b_\omega^{(m)}}} \left\langle T(x \otimes 1), y \otimes e_{\omega \backslash_r \gamma} \right\rangle, & \text{if } \ \omega \geq_r \gamma, \\ \frac{\sqrt{b_{\gamma \backslash_r \omega}^{(m)}} \sqrt{b_\omega^{(m)}}}{\sqrt{b_\gamma^{(m)}}} \left\langle T(x \otimes e_{\gamma \backslash_r \omega}), y \otimes 1 \right\rangle, & \text{if } \ \gamma >_r \omega, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

We define the formal Fourier representation of T by setting

$$\varphi(W) := \sum_{|\alpha| \geq 1} B_{(\alpha)} \otimes W_\alpha^* + A_{(0)} \otimes I + \sum_{|\alpha| \geq 1} A_{(\alpha)} \otimes W_\alpha,$$

where the coefficients are given by

$$\begin{aligned}\left\langle A_{(\alpha)}x, y\right\rangle &:= \sqrt{b_\alpha^{(m)}}\left\langle T(x\otimes 1), y\otimes e_\alpha\right\rangle, \quad \alpha\in\mathbb{F}_n^+,\\ \left\langle B_{(\alpha)}x, y\right\rangle &:= \sqrt{b_\alpha^{(m)}}\left\langle T(x\otimes e_\alpha), y\otimes 1\right\rangle, \quad \alpha\in\mathbb{F}_n^+\backslash\{g_0\},\end{aligned} \tag{2.2}$$

for any $x, y\in\mathcal{E}$. We also set $A_{(0)} := A_{(g_0)}$. Hence, we deduce that

$$T(x\otimes 1) = \sum_{\alpha\in\mathbb{F}_n^+}\frac{1}{\sqrt{b_\alpha^{(m)}}}A_{(\alpha)}x\otimes e_\alpha$$

and

$$T^*(x\otimes 1) = \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|\ge 1}\frac{1}{\sqrt{b_\alpha^{(m)}}}B_{(\alpha)}^*x\otimes e_\alpha$$

for any $x\in\mathcal{E}$. As a consequence, we can see that $\sum_{|\alpha|\ge 1}\frac{1}{b_\alpha^{(m)}}A_{(\alpha)}^*A_{(\alpha)}$ and $\sum_{|\alpha|\ge 1}\frac{1}{b_\alpha^{(m)}}B_{(\alpha)}B_{(\alpha)}^*$ are WOT convergent series. We note that

$$\varphi(W)(x\otimes e_\beta) := \sum_{|\alpha|\ge 1}(B_{(\alpha)}\otimes W_\alpha^*)(x\otimes e_\beta) + \sum_{\alpha\in\mathbb{F}_n^+}(A_{(\alpha)}\otimes W_\alpha)(x\otimes e_\beta),$$

is well-defined as a vector in $\mathcal{E}\otimes F^2(H_n)$. Indeed, the first sum consists of finitely many terms, while the second one is equal to $\sum_{\alpha\in\mathbb{F}_n^+}A_{(\alpha)}x\otimes\sqrt{\frac{b_\beta^{(m)}}{b_{\alpha\beta}^{(m)}}}e_{\alpha\beta}$. Using the definition of the coefficients $b_\alpha^{(m)}$, one can easily see that $b_\alpha^{(m)}b_\beta^{(m)}\le\binom{|\beta|+m-1}{m-1}b_{\alpha\beta}^{(m)}$. This implies

$$\sum_{\alpha\in\mathbb{F}_n^+}\|A_{(\alpha)}x\|^2\frac{b_\beta^{(m)}}{b_{\alpha\beta}^{(m)}}\le\binom{|\beta|+m-1}{m-1}\sum_{\alpha\in\mathbb{F}_n^+}\|A_{(\alpha)}x\|^2\frac{1}{b_\alpha^{(m)}}<\infty.$$

Since T is a weighted right multi-Toeplitz operator, we can use relations (2.1) and (2.2), to obtain

$$\langle T(x\otimes e_\gamma), y\otimes e_\omega\rangle = \begin{cases}\frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\omega^{(m)}}}\left\langle A_{\omega\backslash_r\gamma}x, y\right\rangle, & \text{if } \omega\ge_r\gamma,\\ \frac{\sqrt{b_\omega^{(m)}}}{\sqrt{b_\gamma^{(m)}}}\left\langle B_{\gamma\backslash_r\omega}x, y\right\rangle, & \text{if } \gamma>_r\omega,\\ 0, & \text{otherwise.}\end{cases} \tag{2.3}$$

Now, note that

$$\langle\varphi(W)(x\otimes e_\gamma), y\otimes e_\omega\rangle = \sum_{|\alpha|\ge 1}\left\langle B_{(\alpha)}x, y\right\rangle\langle W_\alpha^*e_\gamma, e_\omega\rangle + \sum_{\alpha\in\mathbb{F}_n^+}\left\langle A_{(\alpha)}x, y\right\rangle\langle W_\alpha e_\gamma, e_\omega\rangle.$$

Due to the definition of the weighted left creation operators $W_1,\dots,W_n$, we have

$$\langle W_\alpha e_\gamma, e_\omega\rangle = \begin{cases}\frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\alpha\gamma}^{(m)}}}, & \text{if } \omega=\alpha\gamma,\\ 0, & \text{otherwise,}\end{cases}$$

for any $\alpha\in\mathbb{F}_n^+$, and

$$\langle W_\alpha^* e_\gamma, e_\omega\rangle = \begin{cases} \frac{\sqrt{b_\omega^{(m)}}}{\sqrt{b_{\alpha\omega}^{(m)}}}, & \text{if } \gamma = \alpha\omega, \\ 0, & \text{otherwise} \end{cases}$$

for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geq 1$. Using these relations, we deduce that

$$\langle \varphi(W)(x \otimes e_\gamma), y \otimes e_\omega\rangle = \begin{cases} \left\langle \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\alpha\gamma}^{(m)}}} A_{(\alpha)} x, y \right\rangle, & \text{if } \omega = \alpha\gamma, \alpha \in \mathbb{F}_n^+, \\ \left\langle \frac{\sqrt{b_\omega^{(m)}}}{\sqrt{b_{\alpha\omega}^{(m)}}} B_{(\alpha)} x, y \right\rangle, & \text{if } \gamma = \alpha\omega, \alpha \in \mathbb{F}_n^+ \text{ with } |\alpha| \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Comparing these relations with (2.3), we conclude that

$$\langle T(x \otimes e_\gamma), y \otimes e_\omega\rangle = \langle \varphi(W)(x \otimes e_\gamma), y \otimes e_\omega\rangle$$

for any $x, y \in \mathcal{E}$ and $\gamma, \omega \in \mathbb{F}_n^+$. Consequently, we obtain $T(x \otimes e_\gamma) = \varphi(W)(x \otimes e_\gamma)$. The last part of the theorem is now straightforward. The proof is complete. □

Let $F^2_{f,m}$ be the Hilbert space of formal power series in noncommutative indeterminates $Z_1, \ldots, Z_n$ with complete orthogonal basis $\{Z_\alpha : \ \alpha \in \mathbb{F}_n^+\}$ and such that $\|Z_\alpha\|_{f,m} := \frac{1}{\sqrt{b_\alpha^{(m)}}}$. It is clear that

$$F^2_{f,m} = \left\{ \varphi := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha : \ a_\alpha \in \mathbb{C} \text{ and } \ \|\varphi\|^2_{f,m} := \sum_{\alpha \in \mathbb{F}_n^+} \frac{1}{b_\alpha^{(m)}} |a_\alpha|^2 < \infty \right\}.$$

The left multiplication operators $L_1, \ldots, L_n$ are defined by $L_i \xi := Z_i \xi$ for all $\xi \in F^2_{f,m}$. Note that the operator $U_{f,m} : F^2(H_n) \to F^2_{f,m}$ defined by $U_{f,m}(e_\alpha) := \sqrt{b_\alpha^{(m)}} Z_\alpha$, $\alpha \in \mathbb{F}_n^+$, is unitary and $U_{f,m} W_i = L_i U_{f,m}$ for any $i \in \{1, \ldots, n\}$. A straightforward calculation reveals that $T \in B(\mathcal{E} \otimes F^2(H_n))$ is a weighted right multi-Toeplitz operator if and only if $A := (I \otimes U_{f,m}) T (I \otimes U^*_{f,m})$ satisfies the condition

$$\langle A(x \otimes Z_\gamma), y \otimes Z_\omega\rangle = \begin{cases} \frac{1}{b_\omega^{(m)}} \left\langle A_{(\omega \backslash_r \gamma)} x, y \right\rangle, & \text{if } \ \omega \geq_r \gamma, \\ \frac{1}{b_\gamma^{(m)}} \left\langle B_{(\gamma \backslash_r \omega)} x, y \right\rangle, & \text{if } \ \gamma >_r \omega, \\ 0, & \text{otherwise,} \end{cases}$$

for some operators $\{A_{(\alpha)}\}_{\alpha \in \mathbb{F}_n^+}$ and $\{B_{(\alpha)}\}_{\alpha \in \mathbb{F}_n^+ \backslash \{g_0\}}$ in $B(\mathcal{E})$. Note that the Hilbert space $F^2_{f,m}$ can be seen as a weighted Fock space. In the particular case when $n = 1$ and $q = Z$, it coincides with the weighted Bergman space $A_m(\mathbb{D})$, while A is a Toeplitz operator with operator-valued bounded harmonic symbol (see [16] for the scalar case when $\mathcal{E} = \mathbb{C}$). All the results of the present paper can be written in the setting of multi-Toeplitz operators on weighted Fock spaces. However, dictated by technical aspects, we preferred this time to put the weights on the left creation operators instead on the full Fock space.

We denote by $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)$ the spatial tensor product $B(\mathcal{E}) \otimes_{min} \mathcal{A}(\mathbf{D}_f^m)$, where $\mathcal{A}(\mathbf{D}_f^m)$ is the noncommutative domain algebra. Let $\mathcal{P} \subset F^2(H_n)$ be the set of all polynomials in e_α, $\alpha \in \mathbb{F}_n^+$.

The main result of this section is the following characterization of the weighted right multi-Toeplitz operators in terms of their Fourier representations, which can be viewed as their symbols.

Theorem 2.3. *Let $\{A_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+}$ and $\{B_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+\setminus\{g_0\}}$ be two sequences of operators on a Hilbert space $\mathcal{E}$. Then*

$$\varphi(W) := \sum_{|\alpha|\geq 1} B_{(\alpha)} \otimes W_\alpha^* + A_{(0)} \otimes I + \sum_{|\alpha|\geq 1} A_{(\alpha)} \otimes W_\alpha$$

is the formal Fourier representation of a weighted right multi-Toeplitz operator $T \in B(\mathcal{E} \otimes F^2(H_n))$ if and only if

(i) $\sum_{|\alpha|\geq 1} \frac{1}{b_\alpha^{(m)}} A_{(\alpha)}^* A_{(\alpha)}$ *and* $\sum_{|\alpha|\geq 1} \frac{1}{b_\alpha^{(m)}} B_{(\alpha)} B_{(\alpha)}^*$ *are WOT convergent series, and*
(ii) $\sup_{0\leq r<1} \|\varphi(rW)\| < \infty$.

Moreover, in this case,

(a) *for each $r \in [0,1)$, the operator*

$$\varphi(rW) := \sum_{k=1}^\infty \sum_{|\alpha|=k} B_{(\alpha)} \otimes r^{|\alpha|} W_\alpha^* + A_{(0)} \otimes I + \sum_{k=1}^\infty \sum_{|\alpha|=k} A_{(\alpha)} \otimes r^{|\alpha|} W_\alpha$$

is in the operator space $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)$, where the series are convergent in the operator norm topology;
(b) $T = \text{SOT-}\lim_{r\to 1} \varphi(rW)$, *and*
(c) $\|T\| = \sup_{0\leq r<1} \|\varphi(rW)\| = \lim_{r\to 1} \|\varphi(rW)\| = \sup_{q\in\mathcal{E}\otimes\mathcal{P}, \|q\|\leq 1} \|\varphi(W)q\|$.

Proof. Assume that $T \in B(\mathcal{E} \otimes F^2(H_n))$ is a weighted right multi-Toeplitz operator and that $\varphi(W)$ is its formal Fourier representation. Note that part (i) was proved in the proof of Theorem 2.2.

Using the definition of the weighted left creation operators and the fact that

$$b_\alpha^{(m)} b_\beta^{(m)} \leq \binom{|\beta|+m-1}{m-1} b_{\alpha\beta}^{(m)},$$

we deduce that

$$\|W_\alpha\| \leq \frac{1}{\sqrt{b_\alpha^{(m)}}} \binom{|\alpha|+m-1}{m-1}^{1/2}.$$

Since the operators $W_\alpha, \alpha \in \mathbb{F}_n^+, |\alpha| = k$ have orthogonal ranges, we deduce that

$$\left\|\sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} b_\alpha^{(m)} W_\alpha W_\alpha^*\right\| \leq \binom{k+m-1}{m-1}.$$

Consequently

$$\left\|\sum_{|\alpha|=k} A_{(\alpha)} \otimes r^{|\alpha|} W_\alpha\right\| = r^k \left\|\sum_{|\alpha|=k} \frac{1}{b_\alpha^{(m)}} A_{(\alpha)}^* A_{(\alpha)}\right\|^{1/2} \binom{k+m-1}{m-1}^{1/2}$$

which implies the convergence of the series $\sum_{k=0}^{\infty}\left\|\sum_{|\alpha|=k} A_{(\alpha)}\otimes r^{|\alpha|}W_\alpha\right\|$. A similar result holds for the operators $B_{(\alpha)}$. Using part (i), we can easily see that $\varphi(rW)$ is in $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)$, where the series in the definition of $\varphi(rW)$ are convergent in the operator norm topology. This shows that part (a) holds. Now, we prove that, for any $r\in[0,1)$,

$$(I_{\mathcal{E}}\otimes K_{f,rW}^{(m)})^*(T\otimes I_{F^2(H_n)})(I_{\mathcal{E}}\otimes K_{f,rW}^{(m)})=\varphi(rW), \tag{2.5}$$

where $K_{f,rW}^{(m)}: F^2(H_n)\to F^2(H_n)\otimes\mathcal{D}_{rW}$ is the noncommutative Berezin kernel defined by

$$K_{f,rW}^{(m)}\xi=\sum_{\beta\in\mathbb{F}_n^+} e_\beta\otimes\Delta_{m,rW}^{1/2}W_\beta^*\xi,\qquad \xi\in F^2(H_n),$$

and $\mathcal{D}_{rW}:=\overline{\Delta_{m,rW}^{1/2}(F^2(H_n))}$. Let $\gamma,\omega\in\mathbb{F}_n^+$, set $s:=\max\{|\gamma|,|\omega|\}$ and define

$$\varphi_s(W):=\sum_{1\le|\alpha|\le s} B_{(\alpha)}\otimes W_\alpha^*+\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|\le s} A_{(\alpha)}\otimes W_\alpha.$$

Note that $W_\alpha^*(e_\gamma)=0$ if $|\alpha|>s$ and, similarly, $W_\beta^*(e_\omega)=0$ if $|\beta|>s$. Consequently, using Theorem 2.2, careful computation reveals that

$$\begin{aligned}
&\left\langle(I_{\mathcal{E}}\otimes K_{f,rW}^{(m)})^*(T\otimes I_{F^2(H_n)})(I_{\mathcal{E}}\otimes K_{f,rW}^{(m)})(x\otimes e_\gamma),y\otimes e_\omega\right\rangle\\
&\quad=\left\langle(T\otimes I_{F^2(H_n)})(x\otimes K_{f,rW}^{(m)}e_\gamma),y\otimes K_{f,rW}^{(m)}e_\omega\right\rangle\\
&\quad=\left\langle(T\otimes I_{F^2(H_n)})\left(\sum_{\alpha\in\mathbb{F}_n^+} x\otimes\sqrt{b_\alpha^{(m)}}e_\alpha\otimes\Delta_{m,rW}^{1/2}W_\alpha^*(e_\gamma)\right),\sum_{\beta\in\mathbb{F}_n^+} y\otimes\sqrt{b_\beta^{(m)}}e_\beta\otimes\Delta_{m,rW}^{1/2}W_\beta^*(e_\omega)\right\rangle\\
&\quad=\left\langle\sum_{\alpha\in\mathbb{F}_n^+}\sqrt{b_\alpha^{(m)}b_\beta^{(m)}}T(x\otimes e_\alpha)\otimes\Delta_{m,rW}^{1/2}W_\alpha^*(e_\gamma),\sum_{\beta\in\mathbb{F}_n^+} y\otimes e_\beta\otimes\Delta_{m,rW}^{1/2}W_\beta^*(e_\omega)\right\rangle\\
&\quad=\sum_{\alpha\in\mathbb{F}_n^+}\sum_{\beta\in\mathbb{F}_n^+}\sqrt{b_\alpha^{(m)}b_\beta^{(m)}}\left\langle T(x\otimes e_\alpha),y\otimes e_\beta\right\rangle\left\langle\Delta_{m,rW}^{1/2}W_\alpha^*(e_\gamma),\Delta_{m,rW}^{1/2}W_\beta^*(e_\omega)\right\rangle\\
&\quad=\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|\le s}\sum_{\beta\in\mathbb{F}_n^+,|\beta|\le s}\sqrt{b_\alpha^{(m)}b_\beta^{(m)}}\left\langle T(x\otimes e_\alpha),y\otimes e_\beta\right\rangle\left\langle\Delta_{m,rW}^{1/2}W_\alpha^*(e_\gamma),\Delta_{m,rW}^{1/2}W_\beta^*(e_\omega)\right\rangle\\
&\quad=\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|\le s}\sum_{\beta\in\mathbb{F}_n^+,|\beta|\le s}\sqrt{b_\alpha^{(m)}b_\beta^{(m)}}\left\langle\varphi_q(W)(x\otimes e_\alpha),y\otimes e_\beta\right\rangle\left\langle\Delta_{m,rW}^{1/2}W_\alpha^*(e_\gamma),\Delta_{m,rW}^{1/2}W_\beta^*(e_\omega)\right\rangle\\
&\quad=\sum_{\alpha\in\mathbb{F}_n^+}\sum_{\beta\in\mathbb{F}_n^+}\sqrt{b_\alpha^{(m)}b_\beta^{(m)}}\left\langle\varphi_q(W)(x\otimes e_\alpha),y\otimes e_\beta\right\rangle\left\langle\Delta_{m,rW}^{1/2}W_\alpha^*(e_\gamma),\Delta_{m,rW}^{1/2}W_\beta^*(e_\omega)\right\rangle\\
&\quad=\left\langle(\varphi_s(W)\otimes I_{F^2(H_n)})(x\otimes K_{f,rW}^{(m)}e_\gamma),y\otimes K_{f,rW}^{(m)}e_\omega\right\rangle\\
&\quad=\left\langle(I_{\mathcal{E}}\otimes K_{f,rW}^{(m)})^*(\varphi_s(W)\otimes I_{F^2(H_n)})(I_{\mathcal{E}}\otimes K_{f,rW}^{(m)})(x\otimes e_\gamma),y\otimes e_\omega\right\rangle\\
&\quad=\langle\varphi_s(rW)(x\otimes e_\gamma),y\otimes e_\omega\rangle\\
&\quad=\langle\varphi(rW)(x\otimes e_\gamma),y\otimes e_\omega\rangle
\end{aligned}$$

for any $x,y\in\mathcal{E}$. This shows that relation (2.5) holds. Since $K_{f,rW}^{(m)}$ is an isometry for any $r\in[0,1)$, we deduce that

$$\|\varphi(rW)\| \leq \|T\|, \qquad r \in [0,1), \tag{2.6}$$

which completes the proof of part (ii). Now, we show that $T = \text{SOT-}\lim_{r\to 1} \varphi(rW)$. Indeed, first note that, due to part (i) and the proof of Theorem 2.2, we deduce that

$$\left\| \sum_{\alpha \in \mathbb{F}_n^+} (A_{(\alpha)} \otimes W_\alpha)(x \otimes e_\beta) \right\|^2 \leq \binom{|\beta| + m - 1}{m-1} \sum_{\alpha \in \mathbb{F}_n^+} \|A_{(\alpha)} x\|^2 \frac{1}{b_\alpha^{(m)}} < \infty$$

and also a similar result involving the other series in the definition of $\varphi(W)$. Consequently,

$$\|\varphi(rW)p - \varphi(W)p\| \to 0, \qquad \text{as } r \to 1, \tag{2.7}$$

for any $p := \sum_{|\alpha| \leq k} h_{(\alpha)} \otimes e_\alpha$, where $h_{(\alpha)} \in \mathcal{E}$ and $k \in \mathbb{N}$. Let $x \in \mathcal{E} \otimes F^2(H_n)$ and choose p as above such that $\|x - p\| \leq \frac{\epsilon}{2\|T\|}$. Using relation (2.6) and Theorem 2.2, we obtain

$$\begin{aligned} \|\varphi(rW)x - Tx\| &\leq \|\varphi(rW)(x-p)\| + \|\varphi(rW)p - \varphi(W)p\| + \|\varphi(W)p - Tx\| \\ &\leq 2\|T\|\|x-p\| + \|\varphi(rW)p - \varphi(W)p\| \\ &\leq \epsilon + \|\varphi(rW)p - \varphi(W)p\|. \end{aligned}$$

Now, relation (2.7) implies $\limsup_{r\to 1} \|\varphi(rW)x - Tx\| \leq \epsilon$ for any $\epsilon > 0$. Hence $\lim_{r\to 1} \|\varphi(rW)x - Tx\| = 0$ for any $x \in \mathcal{E} \otimes F^2(H_n)$, which proves part (b). To prove part (c), let $\epsilon > 0$ and choose $p \in \mathcal{E} \otimes F^2(H_n)$ be a polynomial such that $\|p\| = 1$ and $\|Tp\| > \|T\| - \epsilon$. Theorem 2.2 and relation (2.7) imply that there is $t \in (0,1)$ such that $\|\varphi(tW)p\| > \|T\| - \epsilon$. This shows that $\sup_{r\in[0,1)} \|\varphi(rW)\| \geq \|T\|$. Since the reverse inequality holds due to relation (2.6), we deduce that

$$\sup_{r\in[0,1)} \|\varphi(rW)\| = \|T\|. \tag{2.8}$$

Now, let $t_1, t_2 \in [0,1)$ such that $t_1 < t_2$. Since $\varphi(t_2 W)$ is in $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)$ we can use the noncommutative Berezin transform to deduce that

$$(I_{\mathcal{E}} \otimes K_{f,rW}^{(m)})^* \left(\varphi(t_2 W) \otimes I_{F^2(H_n)}\right) (I_{\mathcal{E}} \otimes K_{f,rW}^{(m)}) = \varphi(t_2 rW)$$

for any $r \in [0,1)$. Taking $r := \frac{t_1}{t_2}$ and employing the fact that $K_{f,rW}^{(m)}$ is an isometry, we obtain

$$\|\varphi(t_1 W)\| \leq \|\varphi(t_2 W)\|,$$

which together with relation (2.8) show that $\|T\| = \lim_{r\to 1} \|\varphi(rW)\|$. On the other hand, the fact that $\|T\| = \sup_{q \in \mathcal{E}\otimes\mathcal{P}, \|q\|\leq 1} \|\varphi(W)q\|$ is a consequence of Theorem 2.2.

It remains to prove the converse of the theorem. Assume that $\{A_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+}$ and $\{B_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+\setminus\{g_0\}}$ are two sequences of operators on a Hilbert space $\mathcal{E}$ satisfying conditions (i) and (ii), where $\varphi(W)$ is the formal series

$$\sum_{|\alpha|\geq 1} B_{(\alpha)} \otimes W_\alpha^* + \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes W_\alpha.$$

As is the first part of the proof, we can show that $\varphi(rW)$ is in the operator space $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)$, and $\varphi(W)p$ makes sense for any polynomial in $\mathcal{E} \otimes F^2(H_n)$. Note that item (ii) implies

$$\sup_{q\in \mathcal{E}\otimes \mathcal{P}, \|q\|\leq 1} \|\varphi(W)q\|<\infty. \tag{2.9}$$

Indeed, if $M>0$ and there is a polynomial $p_0\in \mathcal{E}\otimes \mathcal{P}$ such that $\|\varphi(W)p_0\|>M$. Using the fact that $\|\varphi(rW)p_0-\varphi(W)p_0\|\to 0$ as $r\to 1$, we find $t\in (0,1)$ such that $\|\varphi(tW)p_0\|>M$, which implies that $\|\varphi(W)\|>M$. Since M is arbitrary, we get a contradiction. Therefore, relation (2.9) holds. Consequently, there is a unique operator $T\in B(\mathcal{E}\otimes F^2(H_n))$ such that $Tp=\varphi(W)p$ for any polynomial $p\in \mathcal{E}\otimes F^2(H_n)$. Now one can easily see that relation (2.4) holds and

$$\begin{aligned}
\langle T(x\otimes e_\gamma), y\otimes e_\omega\rangle &= \langle \varphi(W)(x\otimes e_\alpha), y\otimes e_\omega\rangle \\
&= \begin{cases} \left\langle \dfrac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\alpha\gamma}^{(m)}}} A_{(\alpha)}x, y\right\rangle, & \text{if } \omega=\alpha\gamma,\\ \left\langle \dfrac{\sqrt{b_\omega^{(m)}}}{\sqrt{b_{\alpha\gamma}^{(m)}}} B_{(\alpha)}x, y\right\rangle, & \text{if } \gamma=\alpha\omega,\\ 0, & \text{otherwise,}\end{cases}\\
&= \begin{cases} \dfrac{\sqrt{b_{\omega\backslash_r\gamma}^{(m)}}\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\omega^{(m)}}} \left\langle T(x\otimes 1), y\otimes e_{\omega\backslash_r \gamma}\right\rangle, & \text{if } \ \omega\geq_r\gamma,\\ \dfrac{\sqrt{b_{\gamma\backslash_r\omega}^{(m)}}\sqrt{b_\omega^{(m)}}}{\sqrt{b_\gamma^{(m)}}} \left\langle T(x\otimes e_{\gamma\backslash_r\omega}), y\otimes 1\right\rangle, & \text{if } \ \gamma>_r\omega,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

This shows that T is a weighted right multi-Toeplitz operator and completes the proof. $\square$

Corollary 2.4. *The set of all weighted right multi-Toeplitz operators on $\mathcal{E}\otimes F^2(H_n)$ coincides with*

$$\overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{WOT}=\overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{SOT},$$

where $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m):=B(\mathcal{E})\otimes_{min}\mathcal{A}(\mathbf{D}_f^m)$ and $\mathcal{A}(\mathbf{D}_f^m)$ is the noncommutative domain algebra.

Proof. Let $\mathcal{M}_T$ be the set of all weighted right multi-Toeplitz operators and note that the inclusion $\mathcal{M}_T\subseteq \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{SOT}$ holds due to Theorem 2.3. Since

$$\overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{SOT}\subseteq\overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{WOT},$$

it remains to show that

$$\overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^*+\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{WOT}\subseteq \mathcal{M}_T.$$

To this end, note that, for any operator $A\in B(\mathcal{E})$ and $\alpha\in \mathbb{F}_n^+$, the operators $A\otimes W_\alpha^*$ and $A\otimes W_\alpha$ are multi-Toeplitz. On the other hand, if $\{T_i\}$ is a net of weighted right multi-Toeplitz operators such that $T_i\to T$ in the weak operator topology, passing to the limit in relation (2.1), written for T_i, shows that T is a weighted right multi-Toeplitz operator as well. This completes the proof. $\square$

Next, we show that for certain noncommutative domains $\mathbf{D}_f^m$, the corresponding set of weighted right multi-Toeplitz operators does not contain any nonzero compact operator.

Theorem 2.5. *Let $\mathbf{D}_f^m$ be a noncommutative domain where the coefficients $b_\alpha^{(m)}$ associated to f satisfy the condition*

$$\sup_{\alpha\in\mathbb{F}_n^+} \frac{b_{g_i\alpha}^{(m)}}{b_\alpha^{(m)}} < \infty, \qquad i \in \{1,\dots,n\}.$$

Then there is no nonzero compact weighted right multi-Toeplitz operator on the full Fock space $F^2(H_n)$.

Proof. First, note that

$$\sup_{\alpha\in\mathbb{F}_n^+} \frac{b_{\sigma\alpha}^{(m)}}{b_\alpha^{(m)}} < \infty \quad \text{for each} \quad \sigma\in\mathbb{F}_n^+. \tag{2.10}$$

Indeed, if $\sigma = g_{i_1}\cdots g_{i_q}$, $i_1,\dots,i_q \in \{1,\dots,n\}$, then

$$\frac{b_{\sigma\gamma}}{b_\gamma} = \frac{b_{g_{i_1}\cdots g_{i_q}\gamma}}{b_{g_{i_2}\cdots g_{i_q}\gamma}} \frac{b_{g_{i_2}\cdots g_{i_q}\gamma}}{b_{g_{i_3}\cdots g_{i_q}\gamma}} \cdots \frac{b_{g_{i_q}\gamma}}{b_\gamma}.$$

Using now the condition in the theorem, the assertion follows. Assume that T is a compact weighted right multi-Toeplitz operator on $F^2(H_n)$. Then, we have

$$\begin{aligned} \langle Te_\gamma, e_{\sigma\gamma}\rangle &= \sqrt{\frac{b_\sigma^{(m)} b_\gamma^{(m)}}{b_{\sigma\gamma}^{(m)}}}\, \langle Te_{g_0}, e_\sigma\rangle \\ \langle Te_{\sigma\gamma}, e_\gamma\rangle &= \sqrt{\frac{b_\sigma^{(m)} b_\gamma^{(m)}}{b_{\sigma\gamma}^{(m)}}}\, \langle Te_\sigma, e_{g_0}\rangle \end{aligned} \tag{2.11}$$

for any $\sigma\gamma \in \mathbb{F}_n^+$, and $\langle Te_\alpha, e_\beta\rangle = 0$ if $(\alpha,\beta)\in\mathbb{F}_n^+\times\mathbb{F}_n^+$ is not of the form $(\sigma\gamma,\gamma)$ or $(\gamma,\sigma\gamma)$ for $\sigma,\gamma\in\mathbb{F}_n^+$. Since a compact operator maps weakly convergent sequences to norm convergent sequences, we deduce that $\langle Te_\gamma, e_{\sigma\gamma}\rangle \to 0$ and $\langle Te_{\sigma\gamma}, e_\gamma\rangle \to 0$ as $|\gamma|\to\infty$. Consequently, using relations (2.10) and (2.11), we conclude that $\langle Te_\gamma, e_{\sigma\gamma}\rangle = \langle Te_{\sigma\gamma}, e_\gamma\rangle = 0$ for any $\sigma\in\mathbb{F}_n^+$. Now, using again relation (2.11), we deduce that $T = 0$, which completes the proof. □

We shall present a concrete class of noncommutative domains for which the theorem above holds. Consider the case when $f = Z_1 + \cdots + Z_n$ and $m\in\mathbb{N}$. The corresponding domain $\mathbf{D}_f^m$ is the *noncommutative m-hyperball*, which is defined by

$$\left\{X := (X_1,\dots,X_n) \in B(\mathcal{H})^n : \ (id - \Phi_X)^k(I) \geq 0 \text{ for } 1\leq k\leq m\right\},$$

where $\Phi_X : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by $\Phi_X(Y) := \sum_{i=1}^n X_i Y X_i^*$ for $Y\in B(\mathcal{H})$. In this case, we have $b_{g_0}^{(m)} = 1$ and $b_\alpha^{(m)} = \binom{|\alpha|+m-1}{m-1}$ if $\alpha\in\mathbb{F}_n^+, |\alpha|\geq 1$. Consequently,

$$\frac{b_{\alpha g_i}^{(m)}}{b_\alpha^{(m)}} = \frac{\binom{|\alpha|+m}{m-1}}{\binom{|\alpha|+m-1}{m-1}} \to 1, \qquad \text{as } \ |\alpha|\to\infty.$$

This shows that Theorem 2.5 holds for the weighted right multi-Toeplitz operators associated with the noncommutative m-hyperball. It would be interesting to know if there are examples of domains $\mathbf{D}_f^m$ which admit nonzero compact multi-Toeplitz operators.

3. Free pluriharmonic functions on the noncommutative domain $\mathbf{D}^m_{f,rad}$

In this section, we provide basic results concerning the free pluriharmonic functions on the noncommutative domain $\mathbf{D}^m_{f,rad}(\mathcal{H})$ and show that they are characterized by a mean value property. This result is used to obtain an analogue of Weierstrass theorem for free pluriharmonic functions and to show that the set of all pluriharmonic functions is a complete metric space with respect to an appropriate metric. A Schur type result in this setting is also presented.

Since the domain $\mathbf{D}^m_f$ is radial (see Theorem 1.4 from [32]), i.e. $rX \in \mathbf{D}^m_f(\mathcal{H})$ for any $X \in \mathbf{D}^m_f(\mathcal{H})$ and any $r \in [0,1)$, we can introduce the radial part of the domain $\mathbf{D}^m_f(\mathcal{H})$, i.e.

$$\mathbf{D}^m_{f,\text{rad}}(\mathcal{H}) := \bigcup_{0\le t<1} t\mathbf{D}^m_f(\mathcal{H}) \subseteq \mathbf{D}^m_f(\mathcal{H}).$$

Note that, in general, we have

$$\text{Int}\mathbf{D}^m_f(\mathcal{H}) \subseteq \mathbf{D}^m_{f,\text{rad}}(\mathcal{H}) \subseteq \mathbf{D}^m_f(\mathcal{H}) \subseteq \mathbf{D}^m_{f,\text{rad}}(\mathcal{H})^-.$$

In the particular case when q is a positive regular noncommutative polynomial, we have

$$\text{Int}\mathbf{D}^m_q(\mathcal{H}) = \mathbf{D}^m_{q,\text{rad}}(\mathcal{H}) \quad \text{and} \quad \mathbf{D}^m_{q,\text{rad}}(\mathcal{H})^- = \mathbf{D}^m_q(\mathcal{H}) = \mathbf{D}^m_q(\mathcal{H})^-.$$

Let $Z_1,\ldots,Z_n$ be noncommutative indeterminates, set $Z_\alpha := Z_{i_1}\cdots Z_{i_k}$ if $\alpha = g_{i_1}\cdots g_{i_k} \in \mathbb{F}_n^+$, and $Z_{g_0} := 1$.

Definition 3.1. A formal power series $F := \sum_{\alpha\in\mathbb{F}_n} A_{(\alpha)} \otimes Z_\alpha$ with $A_{(\alpha)} \in B(\mathcal{E})$ is called free holomorphic function on the abstract domain $\mathbf{D}^m_{f,\text{rad}} := \coprod_{\mathcal{H}} \mathbf{D}^m_{f,\text{rad}}(\mathcal{H})$, if the series

$$F(X) := \sum_{k=0}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} A_{(\alpha)} \otimes X_\alpha$$

is convergent in the operator norm topology for any $X \in \mathbf{D}^m_{f,\text{rad}}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$.

We remark that it is enough to assume in Definition 3.1 that $\mathcal{H}$ is an arbitrary infinite dimensional separable Hilbert space. Unless otherwise specified, we assume throughout this paper that $\mathcal{H}$ has this property.

We denote by $Hol_{\mathcal{E}}(\mathbf{D}^m_f)$ the set of all free holomorphic functions on the abstract domain $\mathbf{D}^m_{f,\text{rad}}$ with operator coefficients in $B(\mathcal{E})$. Let $F := \sum_{\alpha\in\mathbb{F}_n} A_{(\alpha)} \otimes Z_\alpha$ with $A_{(\alpha)} \in B(\mathcal{E})$ be a formal power series and define $\gamma \in [0,\infty]$ by setting

$$\frac{1}{\gamma} := \limsup_{k\in\mathbb{Z}_+} \left\| \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} A_{(\alpha)} \otimes W_\alpha \right\|^{\frac{1}{k}}.$$

According to [33], the series

$$\sum_{k=1}^\infty \left\| \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha \right\|, \qquad X \in \gamma\mathbf{D}^m_{f,\text{rad}}(\mathcal{H}),$$

is convergent. Moreover, if $\gamma > 0$ and $r \in [0, \gamma)$, then the convergence is uniform on $r\mathbf{D}_f^m(\mathcal{H})$. In addition, if $\gamma \in [0, \infty)$ and $s > \gamma$, then there is a Hilbert space $\mathcal{H}$ and $Y \in s\mathbf{D}_f^m(\mathcal{H})$ such that the series

$$\sum_{k=1}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes Y_\alpha$$

is divergent in the operator norm topology. As a consequence of these results, we deduce the following.

Proposition 3.2. *$F = \sum_{\alpha \in \mathbb{F}_n} A_{(\alpha)} \otimes Z_\alpha$ is free holomorphic on $\mathbf{D}_{f,\mathrm{rad}}^m$ if and only if*

$$\limsup_{k \in \mathbb{Z}_+} \left\| \sum_{\beta \in \mathbb{F}_n^+, |\beta|=k} \omega_\beta A_{(\beta)}^* A_{(\beta)} \right\|^{1/2k} \leq 1,$$

where $\omega_\beta := \sup_{\gamma \in \mathbb{F}_n^+} \frac{b_\gamma^{(m)}}{b_{\beta\gamma}^{(m)}}$ and $b_\gamma^{(m)}$ is given by relation (1.1).

Proof. Due to the results preceding the proposition, we have that F is free holomorphic on $\mathbf{D}_{f,\mathrm{rad}}^m$ if and only if $\limsup_{k \in \mathbb{Z}_+} \left\| \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} A_{(\alpha)} \otimes W_\alpha \right\|^{\frac{1}{k}} \leq 1$. On the other hand, due to relation (1.2), we have

$$W_\beta W_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\alpha & \text{if } \alpha = \beta\gamma, \\ 0 & \text{otherwise}, \end{cases}$$

which implies $\|W_\beta W_\beta^*\| = \omega_\beta$. Since the operators W_β, with $\beta \in \mathbb{F}_n^+$ and $|\beta| = k$, have orthogonal ranges, we deduce that

$$\left\| \sum_{\beta \in \mathbb{F}_n^+, |\beta|=k} A_{(\beta)} \otimes W_\beta \right\| = \left\| \sum_{\beta \in \mathbb{F}_n^+, |\beta|=k} A_{(\beta)}^* A_{(\beta)} \otimes W_\beta W_\beta^* \right\|^{1/2} = \left\| \sum_{\beta \in \mathbb{F}_n^+, |\beta|=k} \omega_\beta A_{(\beta)}^* A_{(\beta)} \right\|^{1/2}.$$

The proof is complete. □

Definition 3.3. A map $G : \mathbf{D}_{f,rad}^m(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is called self-adjoint free pluriharmonic function on $\mathbf{D}_{f,rad}^m(\mathcal{H})$ with coefficients in $B(\mathcal{E})$ if there is a free holomorphic function F on $\mathbf{D}_{f,rad}^m(\mathcal{H})$ such that $G = \Re F$. Any linear combination of self-adjoint free pluriharmonic functions is called free pluriharmonic function.

We remark that if $G = \Re F$ as in the latter definition, then G determines F up to an operator $A_{(g_0)} \in B(\mathcal{E})$ with $\Re A_{(g_0)} = 0$. Indeed, assume that $\Re F = 0$. Then $F(rW) = -F(rW)^*$, $r \in [0, 1)$. If F has the representation $F = \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$, the relation above implies

$$F(rW)(x \otimes 1) = -F(rW)^*(x \otimes 1) = -A_{(g_0)}^* x, \qquad x \in \mathcal{E}.$$

Hence, we deduce that $A_{(\alpha)} = 0$ if $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geq 1$ and $\Re A_{(g_0)} = 0$. Therefore, $F = A_{(g_0)} \otimes I$. On the other hand, it is easy to see that any free pluriharmonic function H has a representation of the form $H = H_1 + iH_2$, where H_1 and H_2 are self-adjoint free pluriharmonic functions. Note also that any free holomorphic function F is a free pluriharmonic function, due to the decomposition $F = \frac{F+F^*}{2} + i\frac{F-F^*}{2i}$.

Using Proposition 3.2, one can easily prove the following characterization of free pluriharmonic functions on $\mathbf{D}^m_{f,rad}$.

Proposition 3.4. *A map $G : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is a free pluriharmonic function on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ with coefficients in $B(\mathcal{E})$ if and only if there exist two sequences $\{A_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+} \subset B(\mathcal{E})$ and $\{B_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+\setminus\{g_0\}} \subset B(\mathcal{E})$ such that*

$$\limsup_{k\in\mathbb{Z}_+}\left\|\sum_{\beta\in\mathbb{F}_n^+,|\beta|=k}\omega_\beta A^*_{(\beta)}A_{(\beta)}\right\|^{1/2k} \le 1 \quad \textit{and} \quad \limsup_{k\in\mathbb{Z}_+}\left\|\sum_{\beta\in\mathbb{F}_n^+,|\beta|=k}\omega_\beta B_{(\beta)}B^*_{(\beta)}\right\|^{1/2k} \le 1$$

and

$$G(X) = \sum_{k=1}^\infty\sum_{|\alpha|=k} B_{(\alpha)}\otimes X^*_\alpha + A_{(0)}\otimes I + \sum_{k=1}^\infty\sum_{|\alpha|=k} A_{(\alpha)}\otimes X_\alpha,$$

where the series are convergent in the operator norm topology for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. Moreover, the representation of G is unique.

The *extended noncommutative Berezin transform at $X \in \mathbf{D}^m_f(\mathcal{H})$*, where X is a pure element, is the map $\widetilde{\mathbf{B}}^{(m)}_X : B(\mathcal{E}\otimes F^2(H_n)) \to B(\mathcal{E}\otimes\mathcal{H})$ defined by

$$\widetilde{\mathbf{B}}^{(m)}_X[g] := \left(I_\mathcal{E}\otimes K^{(m)*}_{f,X}\right)(g\otimes I_\mathcal{H})\left(I_\mathcal{E}\otimes K^{(m)}_{f,X}\right), \quad g\in B(\mathcal{E}\otimes F^2(H_n)),$$

where the $K^{(m)}_{f,X} : \mathcal{H} \to F^2(H_n)\otimes\mathcal{H}$ is noncommutative Berezin kernel.

We denote by $\mathcal{P}_\mathcal{E}(W)$ the set of all operators of the form $\sum_{|\alpha|\le k} A_{(\alpha)}\otimes W_\alpha$, where $k\in\mathbb{N}$ and $A_{(\alpha)}\in B(\mathcal{E})$. The following result extends Theorem 2.4 from [32]. We include it for completeness.

Theorem 3.5. *If $X \in \mathbf{D}^m_f(\mathcal{H})$ and $\mathcal{S}_\mathcal{E} := \overline{\mathcal{P}_\mathcal{E}(W)^* + \mathcal{P}_\mathcal{E}(W)}^{\|\cdot\|}$, then there is a unital completely contractive linear map $\Phi_{f,X} : \mathcal{S}_\mathcal{E} \to B(\mathcal{E})\otimes_{min} B(\mathcal{H})$ such that*

$$\Phi_{f,X}(\varphi) = \lim_{r\to 1}\widetilde{\mathbf{B}}^{(m)}_{rX}[\varphi], \qquad \varphi\in\mathcal{S}_\mathcal{E},$$

where the limit is in the operator norm topology. If, in addition, X is a pure n-tuple in $\mathbf{D}^m_f(\mathcal{H})$, then

$$\lim_{r\to 1}\widetilde{\mathbf{B}}^{(m)}_{rX}[\varphi] = \widetilde{\mathbf{B}}^{(m)}_X[\varphi], \qquad \varphi\in\mathcal{S}_\mathcal{E}.$$

Proof. Let $\varphi\in\mathcal{S}_\mathcal{E}$ and let $\{q_k(W,W^*)\}_{k=1}^\infty \subset \mathcal{P}_\mathcal{E}(W)^*+\mathcal{P}_\mathcal{E}(W)$ be such that $q_k(W,W^*)\to\varphi$ in the operator norm, as $k\to\infty$. For any $X\in\mathbf{D}^m_f(\mathcal{H})$, the noncommutative von Neumann inequality (see [28]) implies $\|q_k(X,X^*)-q_j(X,X^*)\| \le \|q_k(W,W^*)-q_j(W,W^*)\|$ for any $k,j\in\mathbb{N}$. Consequently, since $\{q_k(W,W^*)\}_{k=1}^\infty$ is a Cauchy sequence, so is the sequence $\{q_k(X,X^*)\}_{k=1}^\infty$. Therefore,

$$\Phi_{f,X}(\varphi) := \lim_{k\to\infty} q_k(X,X^*), \qquad X\in\mathbf{D}^m_f(\mathcal{H}), \tag{3.1}$$

exists in the operator norm and $\|\Phi_{f,X}(\varphi)\| \le \|\varphi\|$. Now, we show that

$$\Phi_{f,X}(\varphi) = \lim_{r\to 1}\widetilde{\mathbf{B}}^{(m)}_{rX}[\varphi], \qquad \varphi\in\mathcal{S}_\mathcal{E},$$

where the limit is in the operator norm topology. Since

$$\widetilde{\mathbf{B}}_{rX}^{(m)}[q_k(W,W^*)] = (I_{\mathcal{E}} \otimes K_{f,rX}^{(m)})^*(q_k(W,W^*) \otimes I_{\mathcal{H}})(I_{\mathcal{E}} \otimes K_{f,rX}^{(m)}) = q_k(rX, rX^*),$$

relation (3.1) implies

$$\Phi_{f,rX}(\varphi) = \widetilde{\mathbf{B}}_{rX}^{(m)}[\varphi], \qquad r \in [0,1). \tag{3.2}$$

Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\|q_N(W,W^*) - \varphi\| < \frac{\epsilon}{3}$. Note that

$$\Phi_{f,rX}(\varphi) - q_N(rX, rX^*) = \lim_{k\to\infty} q_k(rX, rX^*) - q_N(rX, rX^*)$$

and

$$\|\Phi_{f,rX}(\varphi) - q_N(rX, rX^*)\| \le \|\varphi - q_N(W,W^*)\| < \frac{\epsilon}{3}, \qquad r \in [0,1]. \tag{3.3}$$

Note also that there is a $\delta \in (0,1)$ such that

$$\|q_N(rX, rX^*) - q_N(X, X^*)\| < \frac{\epsilon}{3}, \qquad r \in (\delta, 1). \tag{3.4}$$

Using the relations (3.1), (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
\|\Phi_{f,X}(\varphi) - \widetilde{\mathbf{B}}_{rX}^{(m)}[\varphi]\| &= \|\Phi_{f,X}(\varphi) - \Phi_{f,rX}(\varphi)\| \\
&\le \|\Phi_{f,X}(\varphi) - q_N(X,X^*)\| + \|q_N(X,X^*) - q_N(rX,rX^*)\| \\
&\quad + \|q_N(rX,rX^*) - \Phi_{f,rX}(\varphi)\| < \epsilon
\end{aligned}$$

for any $r \in (\delta, 1)$. Therefore,

$$\Phi_{f,X}(\varphi) = \lim_{r\to 1} \widetilde{\mathbf{B}}_{rX}^{(m)}[\varphi], \qquad \varphi \in \mathcal{S}_{\mathcal{E}},$$

where the limit is in the operator norm topology. Similarly, one can prove that, for any $k \times k$ matrix $[\varphi_{ij}]_{k\times k}$ with entries in $\mathcal{S}_{\mathcal{E}}$,

$$[\Phi_{f,X}(\varphi_{ij})]_{k\times k} = \lim_{r\to 1} \left[\widetilde{\mathbf{B}}_{rX}^{(m)}[\varphi_{ij}]\right]_{k\times k}$$

in the operator norm. Using the properties of the noncommutative Berezin kernel, we deduce that

$$\left\|[\Phi_{f,X}(\varphi_{ij})]_{k\times k}\right\| \le \|[\varphi_{ij}]_{k\times k}\|\,.$$

This proves that $\Phi_{f,X}$ is a unital completely contractive linear map. Now, assume that $X \in \mathbf{D}_f^m(\mathcal{H})$ is a pure n-tuple of operators. Then we have

$$\widetilde{\mathbf{B}}_{X}^{(m)}[q_k(W,W^*)] = (I_{\mathcal{E}} \otimes K_{f,X}^{(m)})^*(q_k(W,W^*) \otimes I_{\mathcal{H}})(I_{\mathcal{E}} \otimes K_{f,X}^{(m)}) = q_k(X, X^*).$$

Since $q_k(W,W^*) \to \varphi$ in the operator norm, as $k \to \infty$, we can use the relation above and (3.1) to deduce that

$$\widetilde{\mathbf{B}}_{X}^{(m)}[\varphi] = \lim_{k\to\infty} q_k(X,X^*) = \Phi_{f,X}(\varphi) = \lim_{r\to 1} \widetilde{\mathbf{B}}_{rX}^{(m)}[\varphi].$$

The proof is complete. $\square$

Next, we show that the free pluriharmonic functions are characterized by a mean value property.

Theorem 3.6. *If $G : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is a free pluriharmonic function, then it has the mean value property, i.e.*

$$G(X) = \widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}[G(rW)], \qquad X \in r\mathbf{D}^m_f(\mathcal{H}), r \in (0,1).$$

Conversely, if a function $\varphi : [0,1) \to \overline{\boldsymbol{\mathcal{A}}_n(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_n(\mathbf{D}^m_f)}^{\|\cdot\|}$ satisfies the relation

$$\varphi(r) = \widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\varphi(t)], \qquad \text{for any} \quad 0 \le r < t < 1,$$

then the map $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ defined by

$$F(X) := \widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}[\varphi(r)], \qquad X \in r\mathbf{D}^m_f(\mathcal{H}), r \in (0,1),$$

is a free pluriharmonic function. Moreover, $F(rW) = \varphi(r)$ for any $r \in [0,1)$. In particular, $F \ge 0$ if and only if $\varphi \ge 0$.

Proof. Assume that $G : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is a free pluriharmonic function with representation

$$G(X) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes X^*_\alpha + A_{(0)} \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha,$$

where the series are convergent in the operator norm topology for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. Since the universal model $W = (W_1, \ldots, W_n)$ is in $\mathbf{D}^m_f(F^2(H_n))$, for any $r \in [0,1)$, we have

$$G(rW) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes r^{|\alpha|} W^*_\alpha + A_{(0)} \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes r^{|\alpha|} W_\alpha,$$

where the convergence is in the operator norm topology. If $X \in r\mathbf{D}^m_f(\mathcal{H})$, $r \in (0,1)$, then $\frac{1}{r}X \in \mathbf{D}^m_f(\mathcal{H})$ and

$$\begin{aligned}
\widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}[G(rW)] &= \lim_{\delta \to 1} \widetilde{\mathbf{B}}^{(m)}_{\frac{\delta}{r}X}[G(rW)] \\
&= \lim_{\delta \to 1} G(\delta X) = G(X),
\end{aligned}$$

where the limits are in the operator norm topology. The latter equality is due to the continuity of G on $r\mathbf{D}^m_f(\mathcal{H})$. To prove the converse, assume that the function $\varphi : [0,1) \to \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$ satisfies the relation

$$\varphi(r) = \widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\varphi(t)], \qquad \text{for any} \quad 0 \le r < t < 1.$$

Due to Theorem 2.3 and Corollary 2.4, $\varphi(r)$ is a weighed right multi-Toeplitz operator and has a unique formal Fourier representation

$$\sum_{|\alpha| \ge 1} r^{|\alpha|} B_{(\alpha)}(r) \otimes W^*_\alpha + \sum_{|\alpha| \ge 0} r^{|\alpha|} A_{(\alpha)}(r) \otimes W_\alpha$$

for some operators $\{A_{(\alpha)}(r)\}_{\alpha \in \mathbb{F}^+_n}$ and $\{B_{(\alpha)}(r)\}_{\alpha \in \mathbb{F}^+_n \setminus \{g_0\}}$. Moreover, setting

$$\varphi_\delta(r) := \sum_{|\alpha|\geq 1} r^{|\alpha|} B_{(\alpha)}(r) \otimes \delta^{|\alpha|} W_\alpha^* + \sum_{|\alpha|\geq 0} r^{|\alpha|} A_{(\alpha)}(r) \otimes \delta^{|\alpha|} W_\alpha, \qquad \delta \in [0,1),$$

where the convergence of the series is in the operator norm topology, we have

$$\varphi(r) = \text{SOT-}\lim_{\delta\to 1} \varphi_\delta(r) \quad \text{and} \quad \sup_{\delta\in[0,1)} \|\varphi_\delta(r)\| = \|\varphi(r)\|. \tag{3.5}$$

Due to similar reasons, $\varphi(t)$ is a weighted right multi-Toeplitz operator and has a unique formal Fourier representation

$$\sum_{|\alpha|\geq 1} t^{|\alpha|} B_{(\alpha)}(t) \otimes W_\alpha^* + \sum_{|\alpha|\geq 0} t^{|\alpha|} A_{(\alpha)}(t) \otimes W_\alpha$$

for some operators $\{A_{(\alpha)}(t)\}_{\alpha\in\mathbb{F}_n^+}$ and $\{B_{(\alpha)}(t)\}_{\alpha\in\mathbb{F}_n^+\setminus\{g_0\}}$. Moreover, setting

$$\psi_\delta(t) := \sum_{|\alpha|\geq 1} t^{|\alpha|} B_{(\alpha)}(t) \otimes \delta^{|\alpha|} W_\alpha^* + \sum_{|\alpha|\geq 0} t^{|\alpha|} A_{(\alpha)}(t) \otimes \delta^{|\alpha|} W_\alpha, \qquad \delta \in [0,1),$$

where the convergence of the series is in the operator norm topology, we have

$$\varphi(t) = \text{SOT-}\lim_{\delta\to 1} \psi_\delta(t) \quad \text{and} \quad \sup_{\delta\in[0,1)} \|\psi_\delta(t)\| = \|\varphi(t)\|.$$

Now, since the map $Y \to Y \otimes I$ is SOT-continuous on bounded sets, so is the noncommutative Berezin transform $\widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}$. Using the results above, we deduce that

$$\begin{aligned}\widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\varphi(t)] &= \text{SOT-}\lim_{\delta\to 1} \widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\psi_\delta(t)] \\ &= \text{SOT-}\lim_{\delta\to 1} \left(\sum_{|\alpha|\geq 1} r^{|\alpha|} B_{(\alpha)}(t) \otimes \delta^{|\alpha|} W_\alpha^* + \sum_{|\alpha|\geq 0} r^{|\alpha|} A_{(\alpha)}(t) \otimes \delta^{|\alpha|} W_\alpha \right).\end{aligned}$$

Consequently, using the fact that $\varphi(r) = \widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\varphi(t)]$ and relation (3.5), we can easily see that $B_{(\alpha)}(t) = B_{(\alpha)}(r)$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geq 1$ and $A_{(\alpha)}(t) = A_{(\alpha)}(r)$ for any $\alpha \in \mathbb{F}_n^+$. Therefore,

$$\varphi(r) = \sum_{|\alpha|\geq 1} r^{|\alpha|} B_{(\alpha)} \otimes W_\alpha^* + \sum_{|\alpha|\geq 0} r^{|\alpha|} A_{(\alpha)} \otimes W_\alpha, \qquad r \in [0,1),$$

for some operators $\{A_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+}$ and $\{B_{(\alpha)}\}_{\alpha\in\mathbb{F}_n^+\setminus\{g_0\}}$, where the convergence is in the operator norm topology. Now, for any $X \in r\mathbf{D}_f^m(\mathcal{H})$, $r \in (0,1)$, we define

$$F(X) := \widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}[\varphi(r)].$$

Hence, we deduce that

$$F(X) = \sum_{|\alpha|\geq 1} B_{(\alpha)} \otimes X_\alpha^* + \sum_{|\alpha|\geq 0} A_{(\alpha)} \otimes X_\alpha, \qquad X \in \mathbf{D}^m_{f,rad}(\mathcal{H}),$$

and $F(rW) = \varphi(r)$ for any $r \in [0,1)$. The proof is complete. $\square$

Let $Hol^+_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ be the set of all free holomorphic functions $F \in Hol_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ such that $\Re F \geq 0$. If $F := \sum_{\alpha\in\mathbb{F}_n} A_{(\alpha)} \otimes Z_\alpha$, we associated the kernel $\Gamma_{rF} : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \to B(\mathcal{E})$ defined by

$$\Gamma_{rF}(\omega,\gamma) := \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\alpha\gamma}^{(m)}}} r^{|\alpha|} A_{(\alpha)}, & \text{if } \omega = \alpha\gamma,\ |\alpha| \geq 1, \\ A_{(g_0)} + A^*_{(g_0)}, & \text{if } \omega = \gamma, \\ \frac{\sqrt{b_\omega^{(m)}}}{\sqrt{b_{\alpha\omega}^{(m)}}} r^{|\alpha|} A^*_{(\alpha)}, & \text{if } \gamma = \alpha\omega,\ |\alpha| \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We will use the notation $\mathcal{S}^+_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ for the set of all free holomorphic functions $F \in Hol_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ such that the weighted multi-Toeplitz kernels $\Gamma_{rF} : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \to B(\mathcal{E})$, $r \in [0,1)$, are positive semidefinite.

Now, we prove a Schur type result for free pluriharmonic functions with positive real parts.

Theorem 3.7. $Hol^+_{\mathcal{E}}(\mathbf{D}^m_{f,rad}) = \mathcal{S}^+_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$.

Proof. Assume that $F \in Hol^+_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ has the representation $F = \sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)} \otimes Z_\alpha$ and let $h_\beta \in \mathcal{E}$, for $\beta \in \mathbb{F}_n^+$ with $|\beta| \leq q \in \mathbb{N}$. Note that, using relation (1.2), we have

$$\begin{aligned} &\left\langle \sum_{s=1}^\infty \sum_{|\alpha|=s} A_{(\alpha)} \otimes r^{|\alpha|} W_\alpha \left(\sum_{|\beta|\leq q} h_\beta \otimes e_\beta \right), \sum_{|\gamma|\leq q} h_\gamma \otimes e_\gamma \right\rangle \\ &= \sum_{s=1}^\infty \sum_{|\alpha|=s} \left\langle \sum_{|\beta|\leq q} A_{(\alpha)} h_\beta \otimes r^{|\alpha|} W_\alpha e_\beta, \sum_{|\gamma|\leq q} h_\gamma \otimes e_\gamma \right\rangle \\ &= \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|\geq 1} \sum_{|\beta|,|\gamma|\leq q} r^{|\alpha|} \left\langle A_{(\alpha)} h_\beta, h_\gamma \right\rangle \left\langle \frac{\sqrt{b_\beta^{(m)}}}{\sqrt{b_{\alpha\beta}^{(m)}}} e_{\alpha\beta}, e_\gamma \right\rangle \\ &= \sum_{|\beta|,|\gamma|\leq q, \gamma >_r \beta} r^{|\gamma\backslash_r \beta|} \left\langle A_{(\gamma\backslash_r\beta)} h_\beta, h_\gamma \right\rangle \frac{\sqrt{b_\beta^{(m)}}}{\sqrt{b_\gamma^{(m)}}} \\ &= \sum_{|\beta|,|\gamma|\leq q, \gamma >_r \beta} \left\langle \Gamma_{rF}(\gamma,\beta) h_\beta, h_\gamma \right\rangle. \end{aligned}$$

In a similar manner, one can prove that

$$\left\langle \sum_{s=1}^\infty \sum_{|\alpha|=s} A^*_{(\alpha)} \otimes r^{|\alpha|} W^*_\alpha \left(\sum_{|\beta|\leq q} h_\beta \otimes e_\beta \right), \sum_{|\gamma|\leq q} h_\gamma \otimes e_\gamma \right\rangle = \sum_{|\beta|,|\gamma|\leq q, \gamma >_r \beta} \left\langle \Gamma_{rF}(\gamma,\beta) h_\beta, h_\gamma \right\rangle.$$

Note also that, for any $\beta \in \mathbb{F}_n$, $\Gamma_{rF}(\beta,\beta) = \Gamma_{rF}(g_0,g_0) = A_{g_0} + A^*_{g_0}$. Taking into account the relations above, we deduce that

$$\left\langle (F(rW)^* + F(rW)) \left(\sum_{|\beta|\leq q} h_\beta \otimes e_\beta \right), \sum_{|\gamma|\leq q} h_\gamma \otimes e_\gamma \right\rangle = \sum_{|\beta|,|\gamma|\leq q} \left\langle \Gamma_{rF}(\gamma,\beta) h_\beta, h_\gamma \right\rangle.$$

Consequently, $F(rW)^* + F(rW) \geq 0$ for any $r \in [0,1)$ if and only if Γ_{rF} is a positive semidefinite kernel for any $r \in [0,1)$.

On the other hand, if $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$, then there is $r \in (0,1)$ such that $X \in r\mathbf{D}^m_f(\mathcal{H})$. Due to Theorem 3.6, we have

$$F(X)^* + F(X) = \widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}\left[F(rW)^* + F(rW)\right].$$

Since the noncommutative Berezin transform is a positive map, we deduce that $F(X)^* + F(X) \geq 0$ for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ whenever $F(rW)^* + F(rW) \geq 0$ for any $r \in [0,1)$. The converse is obviously true. Putting all these things together we complete the proof. $\square$

The next result is an analogue of Weierstrass theorem for free pluriharmonic functions on the noncommutative domain $\mathbf{D}^m_{f,rad}(\mathcal{H})$.

Theorem 3.8. *Let $F_k : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, $k \in \mathbb{N}$, be a sequence of free pluriharmonic functions such that, for any $r \in [0,1)$, the sequence $\{F_k(rW)\}_{k=1}^\infty$ is convergent in the operator norm topology. Then there is a free pluriharmonic function $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ such that $F_k(rW)$ converges to $F(rW)$, as $k \to \infty$, for any $r \in [0,1)$. In particular, F_k converges to F uniformly on any domain $r\mathbf{D}^m_f(\mathcal{H})$, $r \in [0,1)$.*

Proof. Assume that F_k has the representation

$$F_k(X) = \sum_{|\alpha|\geq 1} B_{(\alpha)}(k) \otimes X^*_\alpha + \sum_{|\alpha|\geq 0} A_{(\alpha)}(k) \otimes X_\alpha, \qquad X \in \mathbf{D}^m_{f,rad}(\mathcal{H}),$$

for some operators $\{A_{(\alpha)}(k)\}_{\alpha\in\mathbb{F}_n^+}$ and $\{B_{(\alpha)}(k)\}_{\alpha\in\mathbb{F}_n^+\setminus\{g_0\}}$, where the convergence is in the operator norm topology. According to Theorem 2.3, for any $r \in [0,1)$, $F_k(rW)$ is in the operator space $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)$. Define the function $\varphi : [0,1) \to \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$ by setting

$$\varphi(r) := \lim_{k\to\infty} F_k(rW), \qquad r \in [0,1). \tag{3.6}$$

Let $0 \leq r < t < 1$ and note that

$$\widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\varphi(t)] = \lim_{k\to\infty} \widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[F_k(tW)] = \lim_{k\to\infty} F_k(rW) = \varphi(r),$$

where the limits are in the operator norm topology. According to Theorem 3.6 the map $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ defined by

$$F(X) := \widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}[\varphi(r)], \qquad X \in r\mathbf{D}^m_f(\mathcal{H}), r \in (0,1),$$

is a free pluriharmonic function and $F(rW) = \varphi(r)$ for any $r \in [0,1)$. Using relation (3.6), we obtain $F(rW) = \lim_{k\to\infty} F_k(rW)$, $r \in [0,1)$. Since

$$\sup_{X\in r\mathbf{D}^m_f(\mathcal{H})} \|F(X) - F_k(X)\| = \|F(rW) - F_k(rW)\|,$$

we deduce that F_k converges to F uniformly on any domain $r\mathbf{D}^m_f(\mathcal{H})$, $r \in [0,1)$. The proof is complete. $\square$

Corollary 3.9. *Let $F_k : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, $k \in \mathbb{N}$, be a sequence of free pluriharmonic functions such that $\{F_k(0)\}$ is a convergent sequence in the operator norm topology and*

$$F_1 \leq F_2 \leq \cdots.$$

Then F_k converges to a free pluriharmonic function on $\mathbf{D}^m_{f,rad}(\mathcal{H})$.

Proof. We may assume that $F_1 \geq 0$, otherwise we take $G_k := F_k - F_1$, $k \in \mathbb{N}$. Due to Harnack type inequality for positive free pluriharmonic functions on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ (see Theorem 6.1 from [33]), if $k \geq q \in \mathbb{N}$, then we have

$$\|F_k(X) - F_q(X)\| \leq \|F_k(0) - F_q(0)\| \frac{1-r}{1+r}$$

for any $X \in r\mathbf{D}^m_f(\mathcal{H})$. Since $\{F_k(0)\}$ is a Cauchy sequence in the operator norm, we deduce that $\{F_k\}$ is a uniformly Cauchy sequence on $r\mathbf{D}^m_f(\mathcal{H})$. Hence $\{F_k(rW)\}$ is a Cauchy sequence and, therefore, convergent in the operator norm topology. Applying Theorem 3.8, we find a free pluriharmonic function $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ such that $F_k(rW)$ converges to $F(rW)$, as $k \to \infty$, for any $r \in [0, 1)$. In particular, F_k converges to F uniformly on any domain $r\mathbf{D}^m_f(\mathcal{H})$, $r \in [0, 1)$. The proof is complete. $\square$

Let $Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ denote the set of all free pluriharmonic functions $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$. If $F, G \in Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ and $0 < r < 1$, we define

$$d_r(F, G) := \|F(rW) - G(rW)\|.$$

If $\mathcal{H}$ is an infinite dimensional Hilbert space, the noncommutative von Neumann inequality for the n-tuples in the domain $\mathbf{D}^m_f(\mathcal{H})$ implies

$$d_r(F, G) = \sup_{X \in r\mathbf{D}^m_f(\mathcal{H})} \|F(X) - G(X)\|.$$

Let $\{r_m\}_{m=1}^\infty$ be an increasing sequence of positive numbers convergent to 1. For any $F, G \in Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$, we define

$$\rho(F, G) := \sum_{k=1}^\infty \left(\frac{1}{2}\right)^k \frac{d_{r_k}(F, G)}{1 + d_{r_k}(F, G)}.$$

Using standard arguments, one can show that ρ is a metric on $Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$.

Theorem 3.10. $\left(Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad}), \rho\right)$ *is a complete metric space.*

Proof. It is easy to see that if $\epsilon > 0$, then there exists $\delta > 0$ and $N \in \mathbb{N}$ such that, for any $F, G \in Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$, $d_{r_N}(F, G) < \delta \implies \rho(F, G) < \epsilon$. Conversely, if $\delta > 0$ and $N \in \mathbb{N}$ are fixed, then there is $\epsilon > 0$ such that, for any $F, G \in Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$, $\rho(F, G) < \epsilon \implies d_{r_N}(F, G) < \delta$.

Let $\{G_k\}_{k=1}^\infty \subset Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ be a Cauchy sequence in the metric ρ. A consequence of the remark above is that $\{G_k(r_N W)\}_{k=1}^\infty$ is a Cauchy sequence in $B(\mathcal{E} \otimes F^2(H_n))$, for any $N \in \mathbb{N}$. Consequently, for each $N \in \mathbb{N}$, the sequence $\{G_k(r_N W)\}_{k=1}^\infty$ is convergent in the operator norm. Using Theorem 3.8, we find a free pluriharmonic function $G \in Har_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ such that $G_k(rW)$ converges to $G(rW)$ for any $r \in [0, 1)$. By the observation made at the beginning of this proof, we conclude that $\rho(G_k, G) \to 0$, as $k \to \infty$, which completes the proof. $\square$

4. Bounded free pluriharmonic functions and Dirichlet extension problem

In this section, we characterize the bounded pluriharmonic functions on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ as noncommutative Berezin transforms of weighted right multi-Toeplitz operators and present a noncommutative version of Dirichlet extension problem.

A free pluriharmonic function $G : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is called bounded if

$$\|G\| := \sup \|G(X)\| < \infty,$$

where the supremum is taken over all n-tuples $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. Due to the noncommutative von Neumann inequality for elements in $\mathbf{D}^m_{f,rad}(\mathcal{H})$, it is enough to assume, throughout this section, that the Hilbert space $\mathcal{H}$ is separable and infinite dimensional. Denote by $Har^\infty_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ the set of all bounded free pluriharmonic functions on $\mathbf{D}^m_{f,rad}$ with coefficients in $B(\mathcal{E})$, where $\mathcal{E}$ is a separable Hilbert space. For each $k = 1, 2, \ldots$, we define the norms $\|\cdot\|_k : M_k\left(Har^\infty_{\mathcal{E}}(\mathbf{D}^m_{f,rad})\right) \to [0, \infty)$ by setting

$$\|[F_{ij}]_k\|_k := \sup \|[F_{ij}(X)]_k\|,$$

where the supremum is taken over all n-tuples $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_k$, $k = 1, 2, \ldots$, determine an operator space structure on $Har^\infty_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$, in the sense of Ruan (see e.g. [8]).

Theorem 4.1. *If $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, then the following statements are equivalent:*

(i) *F is a bounded free pluriharmonic function on $\mathbf{D}^m_{f,rad}(\mathcal{H})$;*
(ii) *there exists $\psi \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{SOT}$ such that $F(X) = \widetilde{\mathbf{B}}^{(m)}_X[\psi]$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$,*

where $\widetilde{\mathbf{B}}^{(m)}_X$ is the noncommutative Berezin transform at X. In this case, $\psi = \text{SOT-}\lim_{r\to 1} F(rW)$. Moreover, the map

$$\Phi : Har^\infty_{\mathcal{E}}(\mathbf{D}^m_{f,rad}) \to \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{SOT} \quad \textit{defined by} \quad \Phi(F) := \psi$$

is a completely isometric isomorphism of operator spaces, where $\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f) := B(\mathcal{E}) \otimes_{min} \mathcal{A}(\mathbf{D}^m_f)$ and $\mathcal{A}(\mathbf{D}^m_f)$ is the noncommutative domain algebra.

Proof. Let $F \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and note that, due to Proposition 3.4, it has a representation

$$F(X) = \sum_{k=1}^\infty \sum_{|\alpha|=k} B_{(\alpha)} \otimes X^*_\alpha + A_{(0)} \otimes I + \sum_{k=1}^\infty \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha,$$

where the series are convergent in the operator norm topology for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$. Consequently, we have $F(rW) \in \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)$ for any $r \in [0, 1)$, and $\sup_{r\in[0,1)} \|F(rW)\| < \infty$. Applying Theorem 2.3, we find a unique weighted right multi-Toeplitz operator $T \in B(\mathcal{E} \otimes F^2(H_n))$ such that

$$T = \text{SOT-}\lim_{r\to 1} F(rW) \quad \text{and} \quad \|T\| = \sup_{r\in[0,1)} \|F(rW)\|. \tag{4.1}$$

Therefore, $T \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{SOT}$. Now, we prove that $F(X) = \widetilde{\mathbf{B}}^{(m)}_X[T]$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$. Indeed, since $F(rW) \in \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)$, we have

$$F(rX) = (I_{\mathcal{E}} \otimes K^{(m)}_{f,X})^*(F(rW) \otimes I_{\mathcal{H}})(I_{\mathcal{E}} \otimes K^{(m)}_{f,X})$$

for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and $r \in [0, 1)$. Since the map $Y \mapsto Y \otimes I$ is SOT-continuous on bounded sets, we use relation (4.1) to deduce that $\text{SOT-}\lim_{r\to 1} F(rX) = \widetilde{\mathbf{B}}^{(m)}_X[T]$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$. On the other hand, since F

is continuous on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ with respect to the operator norm topology, we conclude that $F(X) = \widetilde{\mathbf{B}}^{(m)}_X[T]$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$, which shows that item (ii) holds. Conversely, assume that (ii) holds. Then $F(X) = \widetilde{\mathbf{B}}^{(m)}_X[\psi]$ for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$ and some $\psi \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{SOT}$. Due to Corollary 2.4, ψ is a weighted right multi-Toeplitz operator on $\mathcal{E} \otimes F^2(H_n)$. Applying Theorem 2.3, we find be two sequences $\{A_{(\alpha)}\}_{\alpha \in \mathbb{F}_n^+}$ and $\{B_{(\alpha)}\}_{\alpha \in \mathbb{F}_n^+ \setminus \{g_0\}}$ of operators on a Hilbert space $\mathcal{E}$ such that, $\psi = \text{SOT-}\lim_{r\to\infty} G(rW)$, where

$$G(rW) := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes r^{|\alpha|} W_\alpha^* + A_{(0)} \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes r^{|\alpha|} W_\alpha,$$

with the convergence is in the operator norm topology. Moreover, we have $\sup_{r\in[0,1)} \|G(rW)\| = \|\psi\|$. Define the bounded free pluriharmonic function $G : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ by setting

$$G(X) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes X_\alpha^* + A_{(0)} \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha, \qquad X \in \mathbf{D}^m_{f,rad}(\mathcal{H}),$$

where the series are convergent in the operator norm topology. Note that

$$\begin{aligned}
\widetilde{\mathbf{B}}^{(m)}_X[\psi] &= \text{SOT-}\lim_{r\to 1} \widetilde{\mathbf{B}}^{(m)}_X[G(rW)] \\
&= \text{SOT-}\lim_{r\to 1} \left(\sum_{k=1}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} B_{(\alpha)} \otimes X_\alpha^* + A_{(0)} \otimes I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes r^{|\alpha|} X_\alpha \right) \\
&= \text{SOT-}\lim_{r\to 1} G(rX) = G(X),
\end{aligned}$$

where the last equality is due to the continuity of G. Therefore, $G(X) = F(X)$ for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$.

Now, let $[F_{ij}]_{k\times k}$ be a $k \times k$ matrix with entries in $Har^\infty_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$. As in the case when $k = 1$, we can use the noncommutative von Neumann inequality for the domain $\mathbf{D}^m_f$, to show that

$$\|[F_{ij}]_{k\times k}\| = \sup_{r\in[0,1)} \|[F_{ij}(rW)]_{k\times k}\|$$

and that $T_{ij} := \text{SOT-}\lim_{r\to 1} F_{ij}(rW)$ are weighted right multi-Toeplitz operators. Since

$$(I_{\mathcal{E}} \otimes K^{(m)}_{f,rW})^* (T_{ij} \otimes I_{\mathcal{H}})(I_{\mathcal{E}} \otimes K^{(m)}_{f,rW}) = F_{ij}(rW), \qquad r \in [0,1),$$

we deduce that $\|[F_{ij}(rW)]_{k\times k}\| \le \|[T_{ij}]_{k\times k}\|$, $r \in [0,1)$, which, due to the convergence above, implies $\|[F_{ij}(rW)]_{k\times k}\| = \|[T_{ij}]_{k\times k}\|$. This completes the proof. □

A consequence of Theorem 4.1 and Corollary 2.4 is the following noncommutative version of Herglotz theorem (see [12], [13]).

Corollary 4.2. *Any non-negative bounded free pluriharmonic function on $\mathbf{D}^m_{f,rad}$ is the Berezin transform of a positive weighted right multi-Toeplitz operator on $\mathcal{E} \otimes F^2(H_n)$.*

Corollary 4.3. *If $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is a bounded free pluriharmonic function and $Y \in \mathbf{D}^m_f(\mathcal{H})$ is a pure n-tuple of operators, then $\lim_{r\to 1} F(rY)$ exists in the strong operator topology.*

Proof. Assume that F has the representation

$$F(X) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes X_\alpha^* + \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha, \qquad X \in \mathbf{D}_{f,rad}^m(\mathcal{H}),$$

where the series are convergent in the operator norm topology. Due to Theorem 4.1, we find a unique weighted right multi-Toeplitz operator $T \in B(\mathcal{E} \otimes F^2(H_n))$ such that

$$T = \text{SOT-}\lim_{r\to 1} F(rW) \quad \text{and} \quad \|T\| = \sup_{r\in[0,1)} \|F(rW)\|. \tag{4.2}$$

Let $Y \in \mathbf{D}_f^m(\mathcal{H})$ be a pure n-tuple of operators and let $r \in [0,1)$. Then we have

$$\begin{aligned} F(rY) &= \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes r^{|\alpha|} Y_\alpha^* + \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes r^{|\alpha|} Y_\alpha \\ &= \mathbf{B}_Y^{(m)}[F(rW)] \\ &= (I_\mathcal{E} \otimes K_{f,Y}^{(m)})^*(F(rW) \otimes I_\mathcal{H})(I_\mathcal{E} \otimes K_{f,Y}^{(m)}), \end{aligned}$$

where the convergence of the series is in the operator norm topology. Consequently, since the map $A \mapsto A \otimes I$ is SOT-continuous on bounded sets, relation (4.2) implies that SOT-$\lim_{r\to 1} F(rY)$ exists and it is equal to $(I_\mathcal{E} \otimes K_{f,Y}^{(m)})^*(T \otimes I_\mathcal{H})(I_\mathcal{E} \otimes K_{f,Y}^{(m)})$. The proof is complete. $\square$

Corollary 4.4. *Given a function $F : \mathbf{D}_{f,rad}^m(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, the following statements are equivalent:*

(i) *F is a bounded free plurihamonic function.*

(ii) *There is a bounded function $\varphi : [0,1) \to \overline{\boldsymbol{\mathcal{A}}_\mathcal{E}(\mathbf{D}_f^m)^* + \boldsymbol{\mathcal{A}}_\mathcal{E}(\mathbf{D}_f^m)}^{\|\cdot\|}$ which satisfies the relation*

$$\varphi(r) = \widetilde{\mathbf{B}}_{\frac{r}{t}W}^{(m)}[\varphi(t)], \qquad \textit{for any } \ 0 \le r < t < 1,$$

and $F(X) := \widetilde{\mathbf{B}}_{\frac{1}{r}X}^{(m)}[\varphi(r)]$ for any $X \in r\mathbf{D}_f^m(\mathcal{H})$ and $r \in (0,1)$.

Moreover, F and φ uniquely determine each other and $F(rW) = \varphi(r)$ for any $r \in [0,1)$.

Proof. Assume that F is a bounded free pluriharmonic function and has representation

$$F(X) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes X_\alpha^* + \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha, \qquad X \in \mathbf{D}_{f,rad}^m(\mathcal{H}),$$

where the series are convergent in the operator norm topology. Then $\sup_{r\in[0,1)} \|F(rW)\| < \infty$ and

$$F(rW) = \widetilde{\mathbf{B}}_{\frac{r}{t}W}^{(m)}[F(tW)], \qquad 0 \le r < t < 1.$$

Define $\varphi : [0,1) \to \overline{\boldsymbol{\mathcal{A}}_\mathcal{E}(\mathbf{D}_f^m)^* + \boldsymbol{\mathcal{A}}_\mathcal{E}(\mathbf{D}_f^m)}^{\|\cdot\|}$ by setting $\varphi(r) := F(rW)$. Note that, if $X \in r\mathbf{D}_f^m(\mathcal{H})$, then

$$\begin{aligned} F(X) &= \lim_{\delta\to 1} \left(\sum_{k=1}^{\infty} \sum_{|\alpha|=k} B_{(\alpha)} \otimes \delta^{|\alpha|} X_\alpha^* + \sum_{k=0}^{\infty} \sum_{|\alpha|=k} A_{(\alpha)} \otimes \delta^{|\alpha|} X_\alpha \right) \\ &= \lim_{\delta\to 1} \widetilde{\mathbf{B}}_{\frac{\delta}{r}X}^{(m)}[\varphi(r)] = \widetilde{\mathbf{B}}_{\frac{1}{r}X}^{(m)}[\varphi(r)]. \end{aligned}$$

Conversely, assume that item (ii) holds. Applying Theorem 3.6 to φ, we deduce that F is a free pluriharmonic function and $F(rW) = \varphi(r)$ for any $r \in [0,1)$. Since φ is bounded, we also have $\|F\| \leq \sup_{r\in[0,1)} \|F(rW)\| < \infty$. This completes the proof. $\square$

We denote by $Har^c_{\mathcal{E}}(\mathbf{D}^m_{f,rad})$ the set of all free pluriharmonic functions on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ with operator-valued coefficients in $B(\mathcal{E})$, which have continuous extensions (in the operator norm topology) to the domain $\mathbf{D}^m_f(\mathcal{H})$. Here is our noncommutative version of the Dirichlet extension problem for harmonic functions [13].

Theorem 4.5. *If $F : \mathbf{D}^m_{f,rad}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, then the following statements are equivalent:*

(i) *F is a free pluriharmonic function on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ which has a continuous extension (in the operator norm topology) to the domain $\mathbf{D}^m_f(\mathcal{H})$;*
(ii) *F is a free pluriharmonic function on $\mathbf{D}^m_{f,rad}(\mathcal{H})$ such that $F(rW)$ converges in the operator norm topology, as $r \to 1$;*
(iii) *there exists $\psi \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$ such that $F(X) = \widetilde{\mathbf{B}}^{(m)}_X[\psi]$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$;*

In this case, $\psi = \lim\limits_{r\to1} F(rW)$, where the convergence is in the operator norm. Moreover, the map $\Phi : Har^c_{\mathcal{E}}(\mathbf{D}^m_{f,rad}) \to \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$ defined by $\Phi(F) := \psi$ is a completely isometric isomorphism of operator spaces.

Proof. The implication (i) $\Longrightarrow$ (ii) is clear. Assume that (ii) holds and note that the function $\varphi : [0,1] \to \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$ given by $\varphi(r) := F(rW)$ if $r \in [0,1)$ and $\varphi(1) := \lim_{r\to1} F(rW)$ is continuous and bounded. Setting $\psi := \varphi(1)$ and using Theorem 4.1, we deduce that $F(X) = \widetilde{\mathbf{B}}^{(m)}_X(\psi)$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$. Therefore, the implication (ii) $\Longrightarrow$ (iii) holds true. Now, we prove the implication (iii) $\Longrightarrow$ (i). Assume that item (iii) holds. Thus there exists $\psi \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$ such that $F(X) = \widetilde{\mathbf{B}}^{(m)}_X(\psi)$ for $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$. According to Theorem 4.1, F is a bounded free pluriharmonic function on $\mathbf{D}^m_{f,rad}(\mathcal{H})$, $\|F\| = \|\psi\|$, and $\psi = \text{SOT-}\lim_{r\to1} F(rW)$. In what follows, we show that $\psi = \lim_{r\to1} F(rW)$ in the operator norm topology. Indeed, let $\sum_{k=1}^\infty \sum_{|\alpha|=k} B_{(\alpha)} \otimes W^*_\alpha + \sum_{k=0}^\infty \sum_{|\alpha|=k} A_{(\alpha)} \otimes W_\alpha$ be the Fourier representation of ψ and note that

$$F(X) = \sum_{k=1}^\infty \sum_{|\alpha|=k} B_{(\alpha)} \otimes X^*_\alpha + \sum_{k=0}^\infty \sum_{|\alpha|=k} A_{(\alpha)} \otimes X_\alpha, \qquad X \in \mathbf{D}^m_{f,rad}(\mathcal{H}),$$

where the series are convergent in the operator norm topology. Then for any $r \in [0,1)$, $F(rW) \in \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)$ and $F(rW) = \widetilde{\mathbf{B}}^{(m)}_{rW}(\psi)$. Since $\psi \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$, Theorem 3.5 implies that $\psi = \lim_{r\to1} \widetilde{\mathbf{B}}^{(m)}_{rW}(\psi)$ in the operator norm topology. Consequently, $\lim_{r\to1} F(rW) = \psi$ in the operator norm topology, which proves our assertion.

Let $Y \in \mathbf{D}^m_f(\mathcal{H})$ and define $\widetilde{F}(Y) := \widetilde{\mathbf{B}}^{(m)}_{rY}(\psi)$ We remark that, due to Theorem 3.5, the latter limit exists in the operator norm topology. It remains to prove that $\widetilde{F}|_{\mathbf{D}^m_{f,rad}(\mathcal{H})} = F$ an $\widetilde{F}$ is continuous on $\mathbf{D}^m_f(\mathcal{H})$. Indeed, if $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$, then X is a pure n-tuple and Theorem 3.5 implies that $\lim_{r\to1} \widetilde{\mathbf{B}}^{(m)}_{rX}(\psi) = \mathbf{B}^{(m)}_X(\psi)$. Consequently, $\widetilde{F}(X) = F(X)$ for any $X \in \mathbf{D}^m_{f,rad}(\mathcal{H})$. Now, we prove the continuity of $\widetilde{F}$ on $\mathbf{D}^m_f(\mathcal{H})$. Since $\psi = \lim_{r\to1} F(rW)$ in the operator norm topology, for any $\epsilon > 0$ there exists $r_0 \in (0,1)$ such that $\|\psi - F(r_0W)\| < \frac{\epsilon}{3}$. Since $\psi - F(r_0W) \in \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}^m_f)}^{\|\cdot\|}$, we can use again Theorem 3.5 and deduce that

$$\lim_{r\to1} \widetilde{\mathbf{B}}^{(m)}_{rY}[\psi - F(r_0W)] = \widetilde{F}(Y) - F(r_0Y)$$

and

$$\|\widetilde{F}(Y) - F(r_0 Y)\| \leq \|\psi - F(r_0 W)\| < \frac{\epsilon}{3}, \qquad Y \in \mathbf{D}_f^m(\mathcal{H}).$$

On the other hand, since F is continuous on $\mathbf{D}_{f,rad}^m(\mathcal{H})$, there is $\delta > 0$ such that $\|F(r_0 Y) - F(r_0 Z)\| < \frac{\epsilon}{3}$ for any $Z \in \mathbf{D}_f^m(\mathcal{H})$ such that $\|Y - Z\| < \delta$. Using the estimations above, we note that

$$\|\widetilde{F}(Y) - \widetilde{F}(Z)\| \leq \|\widetilde{F}(Y) - F(r_0 Y)\| + \|F(r_0 Y) - F(r_0 Z)\| + \|F(r_0 Z) - \widetilde{F}(Z)\| < \epsilon$$

for any $Y, Z \in \mathbf{D}_f^m(\mathcal{H})$ such that $\|Y - Z\| < \delta$. The last part of the theorem follows from Theorem 4.1. The proof is complete. □

Corollary 4.6. *Given a function $F : \mathbf{D}_{f,rad}^m(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, the following statements are equivalent:*

(i) *F is a free plurihamonic function which has continuous extension to $\mathbf{D}_f^m(\mathcal{H})$.*

(ii) *There is a continuous function $\varphi : [0, 1] \to \overline{\boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)^* + \boldsymbol{\mathcal{A}}_{\mathcal{E}}(\mathbf{D}_f^m)}^{\|\cdot\|}$ in the operator norm topology which satisfies the relation*

$$\varphi(r) = \widetilde{\mathbf{B}}^{(m)}_{\frac{r}{t}W}[\varphi(t)], \qquad \text{for any} \;\; 0 \leq r < t < 1,$$

and $F(X) := \widetilde{\mathbf{B}}^{(m)}_{\frac{1}{r}X}[\varphi(r)]$ for any $X \in r\mathbf{D}_f^m(\mathcal{H})$ and $r \in (0, 1)$.

Moreover, F and φ uniquely determine each other and $F(rW) = \varphi(r)$ for any $r \in [0, 1)$.

Proof. The proof is similar to that of Corollary 4.4, but uses Theorem 4.5. We leave it to the reader. □

5. Cauchy transforms and functional calculus for noncommuting operators

In this section, we use noncommutative Cauchy transforms associated with the domain $\mathbf{D}_f^m(\mathcal{H})$, to provide a free analytic functional calculus for n-tuples of operators $X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ with the spectral radius of the reconstruction operator strictly less than 1. This extends to free pluriharmonic functions and has several consequences.

Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$, $\alpha \in \mathbb{C}$, be a positive regular free holomorphic. For any n-tuple of operators $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ such that $\sum_{|\alpha| \geq 1} a_\alpha X_\alpha X_\alpha^*$ is SOT-convergent, we define the joint spectral radius of X with respect to the noncommutative domain $\mathbf{D}_f^m$ to be

$$r_f(X) := \lim_{k \to \infty} \|\Phi_{f,X}^k(I)\|^{1/2k},$$

where the positive linear map $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\Phi_{f,X}(Y) := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha Y X_\alpha^*, \quad Y \in B(\mathcal{K}),$$

and the convergence is in the weak operator topology. In the particular case when $f := Z_1 + \cdots + Z_n$, we obtain the usual definition of the joint operator radius for n-tuples of operators.

Since $\sum_{|\alpha| \geq 1} a_{\tilde{\alpha}} \Lambda_\alpha \Lambda_\alpha^*$ is SOT convergent, one can easily see that the series $\sum_{|\alpha| \geq 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes X_{\tilde{\alpha}}^*$ is SOT-convergent in $B(F^2(H_n) \otimes \mathcal{H})$. We call the operator

$$R_{\tilde{f},X} := \sum_{|\alpha|\geq 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes X_{\tilde{\alpha}}^*$$

the *reconstruction operator* associated with the n-tuple $X := (X_1, \dots, X_n)$ and the noncommutative domain $\mathbf{D}_f^m$. Note that

$$\left\| R_{\tilde{f},X}^k \right\| \leq \left\| \Phi_{\tilde{f},\Lambda}^k(I) \right\|^{1/2} \left\| \Phi_{f,X}^k(I) \right\|^{1/2}, \quad k \in \mathbb{N},$$

where $\tilde{f} := \sum_{|\alpha|\geq 1} a_{\tilde{\alpha}} Z_\alpha$ and $\Phi_{\tilde{f},\Lambda}(Y) := \sum_{|\alpha|\geq 1} a_{\tilde{\alpha}} \Lambda_\alpha Y \Lambda_\alpha^*$. Consequently, we deduce that

$$r\left(R_{\tilde{f},X}\right) \leq r_{\tilde{f}}(\Lambda) r_f(X),$$

where $r(A)$ denotes the usual spectral radius of an operator A. Since $\left\| \Phi_{\tilde{f},\Lambda}(I) \right\| \leq 1$ (see relation (1.4)), we deduce that $r_{\tilde{f}}(\Lambda) \leq 1$. This implies

$$r(R_{\tilde{f},X}) \leq r_f(X).$$

Assume now that $X := (X_1, \dots, X_n) \in B(\mathcal{H})^n$ is an n-tuple of operators with $r(R_{\tilde{f},X}) < 1$. Note that the latter condition holds if $r_f(X) < 1$. We introduce the *Cauchy kernel* associated with X to be the operator

$$C_{f,X}^{(m)} := \left(I - R_{\tilde{f},X}\right)^{-m},$$

which is well-defined and

$$C_{f,X}^{(m)} = \left(\sum_{k=0}^\infty R_{\tilde{f},X}^k\right)^m,$$

where the convergence is in the operator norm topology.

We remark that $C_{f,X}^{(m)} \in R^\infty(\mathbf{D}_f^m) \bar{\otimes} B(\mathcal{H})$, the WOT-closed operator algebra generated by the spatial tensor product. Moreover, its Fourier representation is

$$C_{f,X}^{(m)} = \sum_{\beta\in\mathbb{F}_n^+} \Lambda_\beta \otimes b_{\tilde{\beta}}^{(m)} X_{\tilde{\beta}}^*, \tag{5.1}$$

where the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$ are given by relation (1.1). In the particular case when f is a polynomial, the Cauchy kernel is in $\mathcal{R}(\mathbf{D}_f^m) \bar{\otimes}_{min} B(\mathcal{H})$.

Given an n-tuple of operators $X := (X_1, \dots, X_n) \in B(\mathcal{H})^n$ with $r(R_{\tilde{f},X}) < 1$, we define the *Cauchy transform* at X to be the mapping

$$\mathcal{C}_{f,X}^{(m)} : B(F^2(H_n)) \to B(\mathcal{H})$$

defined by

$$\left\langle \mathcal{C}_{f,X}^{(m)}(A)x, y \right\rangle := \left\langle (A \otimes I_\mathcal{H})(1 \otimes x), C_{f,X}^{(m)}(1 \otimes y) \right\rangle, \qquad x, y \in \mathcal{H}.$$

The operator $\mathcal{C}_{f,X}^{(m)}(A)$ is called the Cauchy transform of A at X.

In what follows, we provide a *free analytic functional calculus* for n-tuples of operators $X \in B(\mathcal{H})^n$ with $r(R_{\tilde{f},X}) < 1$.

Theorem 5.1. *Let $p \in \mathbb{C}\langle Z_1, \dots, Z_n\rangle$ be a positive regular noncommutative polynomial and let $X := (X_1, \dots, X_n) \in B(\mathcal{H})^n$ be an n-tuple of operators with $r(R_{\tilde{p},X}) < 1$. If*

$$G := \sum_{|\alpha|\geq 1} d_\alpha Z_\alpha^* + \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha$$

is a free pluriharmonic function on the noncommutative domain $\mathbf{D}_p^m(\mathcal{H})$, then

$$G(X) := \sum_{s=1}^\infty \sum_{|\alpha|=s} d_\alpha X_\alpha^* + \sum_{s=0}^\infty \sum_{|\alpha|=s} c_\alpha X_\alpha$$

is convergent in the operator norm of $B(\mathcal{H})$ and the map

$$\Psi_{p,X} : (Har(\mathbf{D}_{p,rad}^m), \rho) \to (B(\mathcal{H}), \|\cdot\|) \quad \text{defined by} \quad \Psi_{p,X}(G) := G(X)$$

is continuous. In particular, $\Psi_{p,X}|_{Hol(\mathbf{D}_{p,rad}^m)}$ is a continuous unital algebra homomorphism. Moreover, the free analytic functional calculus on $Hol(\mathbf{D}_{p,rad}^m)$ is uniquely determined by the map

$$Z_i \mapsto X_i, \qquad i \in \{1, \dots, n\}.$$

Proof. Note that, using relations (5.1), (1.2), (1.3), we obtain

$$\begin{aligned}
\left\langle C_{p,X}^{(m)}(W_\alpha)x, y\right\rangle &= \left\langle (W_\alpha \otimes I_\mathcal{H})(1\otimes x), C_{p,X}^{(m)}(1\otimes y)\right\rangle \\
&= \left\langle \frac{1}{\sqrt{b_\alpha^{(m)}}} e_\alpha \otimes x, \sum_{\beta\in\mathbb{F}_n^+} b_{\tilde{\beta}}^{(m)}(\Lambda_\beta \otimes X_{\tilde{\beta}}^*)(1\otimes y)\right\rangle \\
&= \left\langle \frac{1}{\sqrt{b_\alpha^{(m)}}} e_\alpha \otimes x, \sum_{\beta\in\mathbb{F}_n^+} \sqrt{b_{\tilde{\beta}}^{(m)}} e_{\tilde{\beta}} \otimes X_{\tilde{\beta}}^* y\right\rangle \\
&= \langle X_\alpha x, y\rangle
\end{aligned}$$

for any $x, y \in \mathcal{H}$. Hence we deduce that, for any polynomial $q \in \mathbb{C}\langle Z_1, \dots, Z_n\rangle$,

$$\langle q(X)x, y\rangle = \left\langle (q(W)\otimes I_\mathcal{H})(1\otimes x), C_{p,X}^{(m)}(1\otimes y)\right\rangle$$

and

$$\|q(X)\| \leq \|q(W)\|\|C_{p,X}^{(m)}\|. \tag{5.2}$$

Since $F := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha$ is a free holomorphic function on $\mathbf{D}_{p,rad}^m$, the series $F(rW) := \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} W_\alpha$, $r \in [0, 1)$, converges in the operator norm topology. Now, using relation (5.2), we deduce that $F(rX) := \sum_{s=0}^\infty \sum_{|\alpha|=s} c_\alpha r^{|\alpha|} X_\alpha$ converges in the operator norm topology of $B(\mathcal{H})$,

$$\|F(rX)\| \leq \|F(rW)\|\|C_{p,X}^{(m)}\|, \tag{5.3}$$

and

$$\langle F(rX)x,y\rangle=\left\langle (F(rW)\otimes I_{\mathcal{H}})(1\otimes x), C_{p,X}^{(m)}(1\otimes y)\right\rangle \tag{5.4}$$

for any $x,y\in\mathcal{H}$ and $r\in[0,1)$.

In what follows, we prove that if $r(R_{\tilde{p},X})<1$, then there is $t>1$ such that $r(R_{\tilde{p},tX})<1$. Indeed, since the spectrum of an operator is upper continuous, so is the spectral radius. Consequently, for any $\delta>0$, there is $\epsilon>0$ such that if $\|X-tX\|<\epsilon$, then $r(R_{\tilde{p},tX})<r(R_{\tilde{p},X})+\delta$. Hence, using the fact that $r(R_{\tilde{p},X})<1$, we deduce that there is $t>1$ such that $r(R_{\tilde{p},tX})<1$. Using relations (5.3) and (5.4) in the particular case when $r=\frac{1}{t}$ and when X is replaced by tX, we deduce that $F(X):=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}c_\alpha X_\alpha$ is convergent in the operator norm topology and

$$\langle F(X)x,y\rangle=\left\langle (F\left(\frac{1}{t}W\right)\otimes I_{\mathcal{H}})(1\otimes x), C_{p,tX}^{(m)}(1\otimes y)\right\rangle,\qquad x,y\in\mathcal{H}. \tag{5.5}$$

Hence, we obtain

$$\|F(X)\|\leq\left\|F\left(\frac{1}{t}W\right)\right\|\|C_{p,tX}^{(m)}\|.$$

Similar results hold true for the free holomorphic function $E:=\sum_{\alpha\in\mathbb{F}_n^+}\overline{d}_\alpha Z_\alpha$. Combining the results, we deduce that

$$G(X):=\sum_{s=1}^{\infty}\sum_{|\alpha|=s}d_\alpha X_\alpha^*+\sum_{s=1}^{\infty}\sum_{|\alpha|=s}c_\alpha X_\alpha$$

is convergent in the operator norm of $B(\mathcal{H})$ and

$$\|G(X)\|\leq\left(\left\|E\left(\frac{1}{t}W\right)\right\|+\left\|F\left(\frac{1}{t}W\right)\right\|\right)\|C_{p,tX}^{(m)}\|. \tag{5.6}$$

To prove the continuity of $\Psi_{p,X}$, let G_k and G be in $Har(\mathbf{D}_{p,rad}^m)$ such that $G_k\to G$, as $m\to\infty$, in the metric ρ of $Har(\mathbf{D}_{p,rad}^m)$. This is equivalent to the fact that, for each $r\in[0,1)$,

$$G_k(rW)\to G(rW),\qquad \text{as}\ \ k\to\infty,$$

where the convergence is in the operator norm of $B(F^2(H_n))$. Employing relation (5.6), when G is replaced by G_k-G, we deduce that

$$\|G_k(X)-G(X)\|\to 0,\qquad \text{as}\ \ k\to\infty,$$

which proves the continuity of $\Psi_{p,X}$.

Let $F_j:=\sum_{s=0}^{\infty}\sum_{|\alpha|=s}c_\alpha^{(j)}Z_\alpha$, $j\in\{1,2\}$, be free holomorphic functions on $\mathbf{D}_{f,rad}^m$. Recall that $\mathcal{A}(\mathbf{D}_f^m)$ is the noncommutative domain algebra and $F_1(rW)F_2(rW)=(F_1F_2)(rW)$ for any $r\in[0,1)$. Setting $p_{j,k}:=\sum_{s=0}^{k}\sum_{|\alpha|=s}c_\alpha^{(j)}Z_\alpha$, we have $p_{j,k}(X)\to F_j(X)$, as $k\to\infty$, in the operator norm for any $X\in\mathbf{D}_{p,rad}^m(\mathcal{H})$. Using relation (5.5), we obtain

$$\langle p_{1,k}(X)p_{2,k}(X)x,y\rangle=\left\langle (p_{1,k}\left(\frac{1}{t}W\right)(p_{2,k}\left(\frac{1}{t}W\right)\otimes I_{\mathcal{H}})(1\otimes x), C_{p,tX}^{(m)}(1\otimes y)\right\rangle,\qquad x,y\in\mathcal{H}.$$

Passing to the limit as $k \to \infty$ and using again relation (5.5), we obtain

$$\begin{aligned}\langle F_1(X)F_2(X)x, y\rangle &= \left\langle (F_1\left(\frac{1}{t}W\right)(F_2\left(\frac{1}{t}W\right)\otimes I_{\mathcal{H}})(1\otimes x), C_{p,tX}^{(m)}(1\otimes y)\right\rangle \\ &= \left\langle (F_1F_2)\left(\frac{1}{t}W\right)\otimes I_{\mathcal{H}})(1\otimes x), C_{p,tX}^{(m)}(1\otimes y)\right\rangle \\ &= \langle (F_1F_2)(X)x, y\rangle\end{aligned}$$

for any $x, y \in \mathcal{H}$. Consequently, $\Psi_{p,X}|_{Hol(\mathbf{D}_{p,rad}^m)}$ is a unital algebra homomorphism.

To prove the uniqueness of the free analytic functional calculus, assume that $\Phi : Hol(\mathbf{D}_{p,rad}^m) \to B(\mathcal{H})$ is a continuous unital algebra homomorphism such that $\Phi(Z_i) = T_i$, $i = 1, \ldots, n$. It is clear that

$$\Psi_{p,X}(q) = \Phi(q) \tag{5.7}$$

for any polynomial $q \in \mathbb{C}\langle Z_1, \ldots, Z_n\rangle$. Let $F = \sum_{s=0}^{\infty}\sum_{|\alpha|=s} c_\alpha Z_\alpha$ be an element in $Hol(\mathbf{D}_{p,rad}^m)$ and let $Q_k := \sum_{s=0}^{k}\sum_{|\alpha|=s} c_\alpha Z_\alpha$, $k \in \mathbb{N}$. Since

$$F(rW) = \sum_{s=0}^{\infty}\sum_{|\alpha|=s} r^s c_\alpha W_\alpha$$

and the series $\sum_{s=0}^{\infty} r^s \left\|\sum_{|\alpha|=s} c_\alpha W_\alpha\right\|$ converges, we deduce that $Q_k(rW) \to F(rW)$ in the operator norm, as $k \to \infty$, which shows that $Q_k \to F$ in the metric ρ of $Hol(\mathbf{D}_{p,rad}^m)$. Hence, using relation (5.7) and the continuity of Φ and $\Psi_{p,T}$, we deduce that $\Phi = \Psi_{p,T}$. This completes the proof. $\square$

Corollary 5.2. *Let $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ be an n-tuple of operators with $r(R_{\tilde{p},X}) < 1$ and let $F \in Hol(\mathbf{D}_{p,rad}^m)$. If $t > 1$ is such that $r(R_{\tilde{p},tX}) < 1$, then*

$$F(X) = \mathcal{C}_{p,tX}^{(m)}\left[F\left(\frac{1}{t}W\right)\right],$$

where $F(X)$ is defined by the free analytic functional calculus. If, in addition, F is bounded, then

$$F(X) = \mathcal{C}_{p,X}^{(m)}(\widetilde{F}), \qquad \textit{where}\ \ \widetilde{F} = \text{SOT-}\lim_{r\to 1} F(rW).$$

Proof. The first part of the corollary is due to Theorem 5.1 (see relation (5.5)). Now, we assume that F is bounded and has the representation $F := \sum\limits_{k=0}^{\infty}\sum\limits_{|\alpha|=k} c_\alpha Z_\alpha$. Then, we have

$$F(rW) := \lim_{k\to\infty}\sum_{s=0}^{k} r^s \sum_{|\alpha|=s} c_\alpha W_\alpha, \qquad 0 < r < 1,$$

in the operator norm of $B(F^2(H_n))$, and

$$\lim_{k\to\infty}\sum_{s=0}^{k} r^s \sum_{|\alpha|=s} c_\alpha X_\alpha = F(rX)$$

in the operator norm of $B(\mathcal{H})$. Now, due to the continuity of the noncommutative Cauchy transform in the operator norm, we deduce that

$$F(rX) = \mathcal{C}_{p,X}(F(rW)), \qquad r \in [0,1).$$

Since F is bounded, we know that $\widetilde{F} := \lim_{r\to 1} F(rW)$ exists in the strong operator topology. Since $\|F(rW)\| \le \|\widetilde{F}\|$, $r \in [0,1)$, we deduce that

$$\text{SOT-}\lim_{r\to 1}[F(rW) \otimes I_{\mathcal{H}}] = \widetilde{F} \otimes I_{\mathcal{H}}.$$

According to the proof of Theorem 5.1, we have

$$\|F(X)\| \le \left\| F\left(\frac{1}{t}W\right)\right\| \|C_{p,tX}^{(m)}\|.$$

Using this relation, we deduce that

$$\|F(X) - F(\delta X)\| \le \left\| F\left(\frac{1}{t}W\right) - F\left(\frac{\delta}{t}W\right)\right\| \|C_{p,tX}^{(m)}\|, \qquad \delta \in (0,1).$$

Since $\left\|F\left(\frac{1}{t}W\right) - F\left(\frac{\delta}{t}W\right)\right\| \to 0$, as $\delta \to 1$, we obtain $\lim_{\delta\to 1}\|F(X) - F(\delta X)\| = 0$. On the other hand, since

$$\langle F(rX)x, y\rangle = \left\langle (F(rW) \otimes I_{\mathcal{H}})(1 \otimes x), C_{p,X}^{(m)}(1 \otimes y)\right\rangle$$

for any $x, y \in \mathcal{H}$ and $r \in [0,1)$, we can pass to the limit as $r \to 1$ and obtain $F(X) = \mathcal{C}_{p,X}^{(m)}[\widetilde{F}]$. This completes the proof. □

Corollary 5.3. *Let $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ be an n-tuple of operators with $r(R_{\tilde{p},X}) < 1$.*

(i) *If $\{G_k\}_{k=1}^\infty$ and G are free pluriharmonic functions in $Har(\mathbf{D}_{p,rad}^m)$ such that $\|G_k - G\|_\infty \to 0$, as $k \to \infty$, then $G_k(X) \to G(X)$ in the operator norm of $B(\mathcal{H})$.*

(ii) *Let $\{G_k\}_{k=1}^\infty$ and G be bounded free holomorphic functions on $\mathbf{D}_{p,rad}^m$ and let $\{\widetilde{G}_k\}_{k=1}^\infty$ and $\widetilde{G}$ be the corresponding boundary operators in the noncommutative Hardy algebra $F^\infty(\mathbf{D}_p^m)$. If $\widetilde{G}_k \to \widetilde{G}$ in the w^*-topology (or strong operator topology) and $\|G_k\|_\infty \le M$ for any $k \in \mathbb{N}$, then $G_k(X) \to G(X)$ in the weak operator topology.*

Using Theorem 5.1 one can deduce the following.

Corollary 5.4. *For any n-tuple of operators $(X_1, \ldots, X_n) \in \mathbf{D}_{p,rad}^m(\mathcal{H})$, the free analytic functional calculus coincides with the $F^\infty(\mathbf{D}_p^m)$-functional calculus (see [28]).*

Using Corollary 5.2, we can obtain the following.

Corollary 5.5. *Let $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ be an n-tuple of operators with $r(R_{\tilde{p},X}) < 1$. Then, the map $\Psi_{p,X} : F^\infty(\mathbf{D}_p^m) \to B(\mathcal{H})$ defined by*

$$\Psi_{p,X}(\widetilde{G}) := \mathcal{C}_{p,X}^{(m)}[\widetilde{G}],$$

for any $\widetilde{G} \in F_n^\infty(\mathbf{D}_p^m)$, is a unital WOT continuous homomorphism such that $\Psi_{f,X}(W_\alpha) = X_\alpha$ for any $\alpha \in \mathbb{F}_n^+$. Moreover,

$$\|\Psi_{p,X}(\widetilde{G})\| \leq \left(\sum_{k=0}^\infty \left\|R_{\tilde{f},X}^k\right\|\right)^m \|\widetilde{G}\|.$$

Definition 5.6. Let H_1 and H_2 be two self-adjoint free pluriharmonic functions on $\mathbf{D}_{f,rad}^m$ with scalar coefficients. We say that H_2 is the pluriharmonic conjugate of H_1, if $H_1 + iH_2$ is a free holomorphic function on $\mathbf{D}_{f,rad}^m$.

Proposition 5.7. *The free pluriharmonic conjugate of a self-adjoint free pluriharmonic function on $\mathbf{D}_{f,rad}^m$ is unique up to an additive real constant.*

Proof. Assume that $G = \Re F$ with $F \in Hol(\mathbf{D}_{f,rad}^m)$, and let H be a self-adjoint free pluriharmonic function such that $G + iH = \Omega \in Hol(\mathbf{D}_{f,rad}^m)$. Then $H = \frac{2\Omega - F - F^*}{2i}$ and the equality $H = H^*$ implies $\Re(\Omega - F) = 0$. Consequently, due to the remarks following Definition 3.3, we have $\Omega - F = \lambda$ with λ is an imaginary complex number. Now, it is clear that $H = \frac{F - F^*}{2i} - i\lambda$. The proof is complete. □

Theorem 5.8. *Let $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ be an n-tuple of operators with $r(R_{\tilde{p},X}) < 1$ and let $F \in Hol(\mathbf{D}_{p,rad}^m)$ be such that $F(0)$ is real. If $G = \Re F$ and $t > 1$ is such that $r(R_{\tilde{p},tX}) < 1$, then*

$$\langle F(X)x, y\rangle = \left\langle (G\left(\frac{1}{t}W\right) \otimes I_{\mathcal{H}})(1 \otimes x), \left[2C_{p,tX}^{(m)} - I\right](1 \otimes y)\right\rangle, \qquad x, y \in \mathcal{H}.$$

If, in addition, F is bounded, then

$$\langle F(X)x, y\rangle = \left\langle (\widetilde{G} \otimes I_{\mathcal{H}})(1 \otimes x), \left[2C_{p,X}^{(m)} - I\right](1 \otimes y)\right\rangle, \qquad x, y \in \mathcal{H},$$

where $\widetilde{G} := \text{SOT-}\lim_{r\to 1} G(rW)$.

Proof. Using the proof of Theorem 5.1, we deduce that

$$\begin{aligned}
&\left\langle (F\left(\frac{1}{t}W\right) \otimes I_{\mathcal{H}})(1 \otimes x), \left[2C_{p,tX}^{(m)} - I\right](1 \otimes y)\right\rangle \\
&= 2\left\langle (F\left(\frac{1}{t}W\right) \otimes I_{\mathcal{H}})(1 \otimes x), C_{p,tX}^{(m)}(1 \otimes y)\right\rangle - \left\langle F\left(\frac{1}{t}W\right) \otimes I_{\mathcal{H}})(1 \otimes x), 1 \otimes y\right\rangle \\
&= 2\langle F(X)x, y\rangle - F(0)\langle x, y\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle (F\left(\frac{1}{t}W\right)^* \otimes I_{\mathcal{H}})(1 \otimes x), \left[2C_{p,tX}^{(m)} - I\right](1 \otimes y)\right\rangle \\
&= \left\langle (\overline{F(0)} \otimes I_{\mathcal{H}})(1 \otimes x), \left[2C_{p,tX}^{(m)} - I\right](1 \otimes y)\right\rangle \\
&= \overline{F(0)}\langle x, y\rangle.
\end{aligned}$$

Taking into account that $F(0) \in \mathbb{R}$ and adding the relations above, we obtain

$$2\langle F(X)x,y\rangle = \left\langle \left[\left(F\left(\frac{1}{t}W\right)^* + F\left(\frac{1}{t}W\right)\right)\otimes I_{\mathcal{H}}\right](1\otimes x), \left[2C_{p,tX}^{(m)} - I\right](1\otimes y)\right\rangle,$$

which proves the first part of the theorem. In a similar manner, but using Corollary 5.2, one can prove the second part of the theorem. □

We remark that the free pluriharmonic conjugate H of G can be expressed in terms of G, due to the fact that $H = \frac{F-F^*}{2i} - i\lambda$, where λ is an imaginary complex number.

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