Doctopic: Miscellaneous ARTICLE IN PRESS YJMAA:23245

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Congruences related to an eighth order mock theta function of Gordon and McIntosh

Eduardo H.M. Brietzke [a], Robson da Silva [b,*], James A. Sellers [c]

[a] Departmento de Matemática, Universidade Federal do Rio Grande do Sul, C. P. 15080, Porto Alegre, CEP 91509-900, Brazil
[b] Universidade Federal de São Paulo, Av. Cesare M. G. Lattes, 1201, São José dos Campos, SP, 12247-014, Brazil
[c] Department of Mathematics, Penn State University, 104 McAllister Building, University Park, PA 16802, USA

A R T I C L E I N F O

*Article history:*
Received 28 June 2018
Available online xxxx
Submitted by B.C. Berndt

*Keywords:*
Partition
Congruences
Mock theta function

A B S T R A C T

In this paper, we study arithmetic properties of a partition function related to an eighth order mock theta function of Gordon and McIntosh. Via elementary generating function manipulations, a complete characterization of the parity of this function is presented, from which infinitely many Ramanujan-like congruences modulo 2 are obtained. In addition, many congruences modulo certain numbers of the form $2^\alpha 3^\beta 5$ are presented.



## 1. Introduction

The notion of a mock theta function was introduced by Ramanujan in his last letter to Hardy in 1920 (see [19, p. 127–131]). He listed 17 such functions having orders 3, 5, and 7. Since then, other mock theta functions have been found, including eight having order 8 by Gordon and McIntoch [10].

The mock theta functions have been the subject of intense study. Their combinatorial aspects have been investigated by many authors, including [1,6,21]. When studying their arithmetic properties, many authors have found congruence properties of partition functions connected with mock theta functions. For instance, recently in [3] the authors found a number of congruences for the partition functions $p_\omega(n)$ and $p_\nu(n)$, introduced in [2], associated with the third order mock theta functions $\omega(q)$ and $\nu(q)$ defined, respectively, by

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(q;q^2)_{n+1}^2} \quad \text{and} \quad \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

* Corresponding author.
*E-mail addresses:* brietzke@mat.ufrgs.br (E.H.M. Brietzke), silva.robson@unifesp.br (R. da Silva), sellersj@psu.edu (J.A. Sellers).

https://doi.org/10.1016/j.jmaa.2019.06.016

Very recently, Wang [22] presented some additional congruences for both $p_\omega(n)$ and $p_\nu(n)$.

This paper is devoted to exploring arithmetic properties of a partition function, namely $v_0(n)$, associated to the mock theta function $V_0(q)$ of order 8 (see [10]) defined by

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{(-q\,;q^2)_n q^{n^2}}{(q\,;q^2)_n} = -1 + 2\sum_{n=0}^{\infty} \frac{(-q^2\,;q^4)_n q^{2n^2}}{(q\,;q^2)_{2n+1}}, \tag{1}$$

where we use the following standard $q$-series notation:

$$\begin{aligned}
&(a;q)_0 = 1\\
&(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \forall n \geq 1,\\
&(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, |q| < 1,\\
&(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_k;q)_\infty.
\end{aligned}$$

Here we consider instead the function

$$\sum_{n=0}^{\infty} v_0(n)q^n = \sum_{n=0}^{\infty} \frac{(-q\,;q^2)_n q^{n^2}}{(q\,;q^2)_n} = \frac{V_0(q)+1}{2}. \tag{2}$$

Except for $v_0(0) = 1$, $v_0(n)$ is half the coefficient of $q^n$ in $V_0(q)$. This function has been combinatorially interpreted in terms of split $n$-color partitions in [1] and in terms of signed partitions in [20]. Using the objects called overpartitions, introduced in [9], we note that $v_0(n)$ is also the number of overpartitions of $n$ into odd parts without gaps between the non-overlined parts. Indeed, in

$$\frac{(-q\,;q^2)_n q^{n^2}}{(q\,;q^2)_n} = \frac{(1+q)(1+q^3)\cdots(1+q^{2n-1})}{(1-q)(1-q^3)\cdots(1-q^{2n-1})} q^{1+3+\cdots+2n-1}$$

the term $q^{1+3+\cdots+2n-1}$ generates one copy of each odd number from 1 to $2n-1$, the numerator generates the overlined parts, and the denominator gives us additional copies of the non-overlined odd numbers from 1 to $2n-1$. Arithmetic properties for the number of overpartitions of $n$, $\overline{p}(n)$, and the number of overpartitions into odd parts, $\overline{p}_o(n)$, have been studied by many authors, including [8,13–17]. In [8] and [14], a number of Ramanujan-like congruences modulo 32 and 64 for $\overline{p}_o(n)$ were proven.

In this paper, we present a complete characterization of the parity of $v_0(n)$, from which infinitely many Ramanujan-like congruences mod 2 are obtained. Namely, for $n \geq 0$, $v_0(pn + r) \equiv 0 \pmod 2$, where $r$ is a quadratic nonresidue modulo the prime $p > 2$. We also prove higher moduli congruences such as

$$v_0(4(pn + r)) \equiv 0 \pmod 4,$$

if $p > 2$ is a prime and $r$ is a quadratic nonresidue modulo $p$, and many congruences modulo certain numbers of the form $2^\alpha 3^\beta 5$, including

$$\begin{aligned}
&v_0(12n+9) \equiv 0 \pmod{3^2}, && v_0(16n+12) \equiv 0 \pmod{2^4},\\
&v_0(80n+52) \equiv 0 \pmod{5}, && v_0(40n+37) \equiv 0 \pmod{2^2 5},\\
&v_0(36n+21) \equiv 0 \pmod{2^2 3^2}, && v_0(24n+21) \equiv 0 \pmod{2^2 3^2},\\
&v_0(48n+40) \equiv 0 \pmod{2^4 3}, && v_0(32n+28) \equiv 0 \pmod{2^6},\\
&v_0(60n+57) \equiv 0 \pmod{2^2 3^2 5}, && v_0(96n+28) \equiv 0 \pmod{2^6 3}.
\end{aligned}$$

Our proof techniques are elementary, utilizing dissections of generating functions as well as theta series identities.

Please cite this article in press as: E.H.M. Brietzke et al., Congruences related to an eighth order mock theta function of Gordon and McIntosh, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.06.016

This paper is organized as follows. In Section 2, we recall some basic properties of Ramanujan's theta functions and we also present some useful lemmas including the 2- and 3-dissections of $v_0(2n)$. Section 3 is devoted to presenting the characterization of the parity of $v_0(n)$ from which we derive infinitely many Ramanujan-like congruences for $v_0(n)$ modulo 2. Congruences modulo certain powers of 2 and related identities are discussed in Section 4, while congruences modulo powers of 3 are the subject of Section 5. In Section 6, we prove a number of arithmetic properties of $v_0(n)$ moduli certain numbers of the form $2^\alpha 3^\beta 5$.

## 2. Preliminaries

We recall Ramanujan's theta functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \text{ for } |ab|<1,$$

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{4}$$

These functions satisfy many interesting properties (see Entries 18, 19, and 22 in [4]), including:

$$f(1,q) = 2\psi(q), \tag{5}$$

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \quad \text{(Jacobi's Triple Product)} \tag{6}$$

$$f(-q) := f(-q,-q^2) = (q;q)_\infty, \quad \text{(Euler Pentagonal Theorem)} \tag{7}$$

and

$$\phi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \tag{8}$$

$$\psi(-q) = \frac{(q;q)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty}. \tag{9}$$

By Entry 25 (i), (ii), (v), and (vi) in [4, p. 40], we have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{10}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{11}$$

We also recall the following version of the Quintuple Product Identity (see [5, Eq. (1.3.54), p. 19])

$$f(-\lambda^2 x^3, -\lambda x^6) + x f(-\lambda, -\lambda^2 x^9) = \frac{f(-x^2,-\lambda x)\, f(-\lambda x^3, -\lambda^2 x^6)}{f(-x,-\lambda x^2)}. \tag{12}$$

An additional relation involving these theta functions is presented in the next lemma.

**Lemma 2.1.** *We have*

$$\frac{\psi(q)^3}{\psi(q^3)} = \frac{\phi(-q^3)^3}{\phi(-q)} + q\frac{\psi(q^3)^3}{\psi(q)}. \tag{13}$$

**Proof.** We recall from [12, Chapter 22] the following identities:

$$\frac{\phi(-q^3)^3}{\phi(-q)} = \frac{1}{3}\left(a(q) + 2a(q^2)\right), \quad ([12, \text{Eq. } 22.11.8]) \tag{14}$$

$$\frac{\psi(q)^3}{\psi(q^3)} = \frac{1}{2}\left(a(q) + a(q^2)\right), \quad ([12, \text{Eq. } 22.11.9]) \tag{15}$$

$$q\frac{\psi(q^3)^3}{\psi(q)} = \frac{1}{6}\left(a(q) - a(q^2)\right), \quad ([12, \text{Eq. } 22.11.10]) \tag{16}$$

where

$$a(q) = \frac{\psi(q)^3}{\psi(q^3)} + 3q\frac{\psi(q^3)^3}{\psi(q)}. \quad ([12, \text{Eq. } 22.11.6])$$

Identity (13) follows directly from (14), (15), and (16). □

Throughout the remainder of this paper, we define

$$f_k := (q^k; q^k)_\infty$$

in order to shorten the notation.

**Lemma 2.2.** *The following identities hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{17}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{18}$$

$$\frac{1}{f_1^6} = \frac{f_4^{14} f_8}{f_2^{19} f_{16}^2} + 2q\frac{f_4^{16} f_{16}^2}{f_2^{19} f_8^5} + 4q\frac{f_4^2 f_8^9}{f_2^{15} f_{16}^2} + 8q^2\frac{f_4^4 f_8^3 f_{16}^2}{f_2^{15}}, \tag{19}$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_4^4 f_8^8}{f_2^{20}}, \tag{20}$$

$$\frac{1}{f_1^{12}} = \frac{f_4^{42}}{f_2^{42} f_8^{12}} + 12q\frac{f_4^{30}}{f_2^{38} f_8^4} + 48q^2\frac{f_4^{18} f_8^4}{f_2^{34}} + 64q^3\frac{f_4^6 f_8^{12}}{f_2^{30}}, \tag{21}$$

$$\begin{aligned}\frac{1}{f_1^{14}} &= \frac{f_4^{42}}{f_2^{47} f_8^7 f_{16}^2} + 2q\frac{f_4^{44} f_{16}^2}{f_2^{47} f_8^{13}} + 12q\frac{f_4^{30} f_8}{f_2^{43} f_{16}^2} + 48q^2\frac{f_4^{18} f_8^9}{f_2^{39} f_{16}^2} \\ &\quad + 24q^2\frac{f_4^{32} f_{16}^2}{f_2^{43} f_8^5} + 96q^3\frac{f_4^{20} f_8^3 f_{16}^2}{f_2^{39}} + 64q^3\frac{f_4^6 f_8^{17}}{f_2^{35} f_{16}^2} + 128q^4\frac{f_4^8 f_8^{11} f_{16}^2}{f_2^{35}}.\end{aligned} \tag{22}$$

**Proof.** By (3) and (4) we can rewrite (10) in the form

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8},$$

from which we obtain (17) after multiplying both sides by $\frac{f_4^2}{f_2^5}$.

By (3) and (4) we can rewrite (11) in the form

$$\frac{f_2^{10}}{f_1^4 f_4^4} = \frac{f_4^{10}}{f_2^4 f_8^4} + 4q\frac{f_8^4}{f_4^2},$$

from which we obtain (18). Squaring (18) we obtain (20).

Multiplying equations (17) and (18) we obtain (19). Identity (21) follows after multiplying (18) and (20) while (22) can be obtained by multiplying (17) and (21). □

We recall Eq. (5.15) from [11] (also Eq. (2.9) in [18]):

$$V_0(q) = (-q^2; q^4)_\infty^4 (q^8; q^8)_\infty + 2qB(q^2), \tag{23}$$

where $B(q)$ (see [11, Eq. (5.1)]) is given by

$$B(q) = \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n(n+1)}}{(q; q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^n}{(q; q^2)_{n+1}}.$$

Bringmann, Ono, and Rhoades [7] obtained the identity

$$B(q) + B(-q) = 2\frac{(q^4; q^4)_\infty^5}{(q^2; q^2)_\infty^4}. \tag{24}$$

It follows from (23) that the even part of $V_0(q)$ is given by

$$\begin{aligned} \frac{V_0(q) + V_0(-q)}{2} &= (-q^2; q^4)_\infty^4 (q^8; q^8)_\infty \\ &= \frac{(q^4; q^8)_\infty^4 (q^8; q^8)_\infty}{(q^2; q^4)_\infty^4} \\ &= \frac{(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4 (q^8; q^8)_\infty^3}, \end{aligned} \tag{25}$$

while its odd part is given by

$$\frac{V_0(q) - V_0(-q)}{2} = 2qB(q^2). \tag{26}$$

Using (8), we can deduce from (25) and (26) the following result.

**Proposition 2.3.** *We have*

$$\frac{1}{2} + \sum_{n=1}^{\infty} v_0(2n) q^n = \frac{1}{2} \frac{\phi(q)\phi(-q^2)}{\phi(-q)}, \tag{27}$$

$$\sum_{n=0}^{\infty} v_0(2n+1) q^n = B(q). \tag{28}$$

In the following three lemmas we establish the 3-dissections of $\psi(q)$, $\phi(q)$ and $1/\phi(-q)$.

**Lemma 2.4.** *We have*

$$\psi(q) = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}.$$

**Proof.** This identity follows from the first equality in Corollary (ii) [4, p. 49]. □

**Lemma 2.5.** *We have*

$$\phi(q) = \phi(q^9) + 2q\Omega(q^3), \tag{29}$$

*where*

$$\Omega(q) := f(q, q^5) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n} = \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6}. \tag{30}$$

**Proof.** By Corollary (i) [4, p. 49], we have (29) with $\Omega(q) = f(q, q^5)$. Using Jacobi's triple product identity (6), we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} q^{3n^2+2n} &= \sum_{n=-\infty}^{\infty} q^{\frac{5n(n+1)}{2}} q^{\frac{n(n-1)}{2}} \\
&= f(q, q^5) \\
&= (-q, -q^5, q^6; q^6)_\infty \\
&= \frac{(-q, -q^3, -q^5; q^6)_\infty (q^6; q^6)_\infty}{(-q^3; q^6)_\infty} \\
&= \frac{(-q; q^2)_\infty (q^6; q^6)_\infty (q^3; q^6)_\infty}{(q^6; q^{12})_\infty} \\
&= \frac{(q^2; q^4)_\infty (q^3; q^3)_\infty (q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (q; q^2)_\infty} \\
&= \frac{(q^2; q^2)_\infty^2 (q^3; q^3)_\infty (q^{12}; q^{12})_\infty}{(q; q)_\infty (q^4; q^4)_\infty (q^6; q^6)_\infty},
\end{aligned}$$

which completes the proof. □

**Remark 2.6.** By (30) and Jacobi's triple product identity (6) we have

$$\Omega(-q) = f(-q, -q^5) = (q, q^5, q^6; q^6)_\infty = \frac{(q; q^2)_\infty (q^6; q^6)_\infty}{(q^3; q^6)_\infty}$$

and, hence,

$$\Omega(-q) = \frac{(q; q)_\infty (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty (q^3; q^3)_\infty}. \tag{31}$$

**Lemma 2.7.** *We have*

$$\frac{1}{\phi(-q)} = \frac{\phi(-q^9)}{\phi(-q^3)^4} \left( \phi(-q^9)^2 + 2q\phi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2 \right). \tag{32}$$

**Proof.** A proof of this result can be seen in [13]. □

Putting all these pieces together we obtain the 3-dissection of $\sum_{n\geq 1} v_0(2n)q^n$.

**Theorem 2.8.** *We have*

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}v_0(2n)q^n &= \frac{\phi(-q^9)}{2\phi(-q^3)^4}\left[\phi(q^9)\phi(-q^9)^2\phi(-q^{18})-4q^3\phi(-q^9)^2\Omega(q^3)\Omega(-q^6)\right.\\ &\left.+8q^3\phi(-q^{18})\Omega(q^3)\Omega(-q^3)^2-4q^3\phi(q^9)\phi(-q^9)\Omega(-q^3)\Omega(-q^6)\right]\\ &+q\frac{\phi(-q^9)}{2\phi(-q^3)^4}\left[2\phi(-q^9)^2\phi(-q^{18})\Omega(q^3)+2\phi(q^9)\phi(-q^9)\phi(-q^{18})\Omega(-q^3)\right.\\ &\left.-8q^3\phi(-q^9)\Omega(q^3)\Omega(-q^3)\Omega(-q^6)-8q^3\phi(q^9)\Omega(-q^6)\Omega(-q^3)^2\right]\\ &+q^2\frac{\phi(-q^9)}{2\phi(-q^3)^4}\left[4\phi(q^9)\phi(-q^{18})\Omega(-q^3)^2-16q^3\Omega(q^3)\Omega(-q^6)\Omega(-q^3)^2\right.\\ &\left.+4\phi(-q^9)\phi(-q^{18})\Omega(-q^3)\Omega(q^3)-2\phi(q^9)\phi(-q^9)^2\Omega(-q^6)\right].\end{aligned}$$

The next theorem presents the 2-dissection of $\sum_{n\geq 1}v_0(2n)q^n$.

**Theorem 2.9.** *We have*

$$\frac{1}{2}+\sum_{n=1}^{\infty}v_0(2n)q^n=\frac{1}{2\phi(-q^2)}\left(\phi(q^2)^2+4q\psi(q^4)^2\right).$$

**Proof.** We start by noting that

$$\phi(-q)\phi(q)=\frac{(q^2;q^2)_\infty^4}{(q^4;q^4)_\infty^2}=\phi(-q^2)^2,$$

from which we have

$$\frac{1}{\phi(-q)}=\frac{\phi(q)}{\phi(-q^2)^2}.$$

Then, by (27), it follows that

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}v_0(2n)q^n &= \frac{1}{2}\frac{\phi(q)\phi(-q^2)}{\phi(-q)}\\ &= \frac{1}{2}\frac{\phi(q)^2}{\phi(-q^2)}\\ &= \frac{1}{2\phi(-q^2)}\left(\phi(q^2)^2+4q\psi(q^4)^2\right),\end{aligned}$$

using (11) for the last step. □

We now 2-dissect $\phi(q^3)\Omega(-q)$.

**Lemma 2.10.** *We have*

$$\phi(q^3)\Omega(-q)=f_4^2-q\frac{f_2^2f_{12}^4}{f_4^2f_6^2}.$$

**Proof.** We have

$$\phi(q^3)\Omega(-q) = \sum_{n,m=-\infty}^{\infty} (-1)^m q^{3n^2+3m^2+2m}.$$

When $n + m \equiv 0 \pmod 2$, we write $n = r + s$ and $m = r - s$, and when $n + m \equiv 1 \pmod 2$, we write $n = r + s + 1$ and $m = r - s$. Then,

$$\begin{aligned}\sum_{n,m=-\infty}^{\infty} (-1)^m q^{3n^2+3m^2+2m} &= \sum_{r,s=-\infty}^{\infty} (-1)^{r-s} q^{3(r+s)^2+3(r-s)^2+2(r-s)} \\ &\quad + \sum_{r,s=-\infty}^{\infty} (-1)^{r-s} q^{3(r+s+1)^2+3(r-s)^2+2(r-s)} \\ &= \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{6r^2+6s^2+2r-2s} \\ &\quad + \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{6r^2+6s^2+8r+4s+3}.\end{aligned}$$

By Jacobi's triple product identity (6), we have

$$\begin{aligned}\sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{6r^2+6s^2+2r-2s} &= (q^4, q^8, q^{12}; q^{12})_\infty (q^4, q^8, q^{12}; q^{12})_\infty \\ &= (q^4; q^4)_\infty^2,\end{aligned}$$

and

$$\begin{aligned}\sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{6r^2+6s^2+8r+4s+3} &= q^3(q^{-2}, q^{14}, q^{12}; q^{12})_\infty (q^2, q^{10}, q^{12}; q^{12})_\infty \\ &= q^3(1-q^{-2})(q^{10}, q^{14}, q^{12}; q^{12})_\infty (q^2, q^{10}, q^{12}; q^{12})_\infty \\ &= -q(q^2, q^{10}, q^{12}; q^{12})_\infty^2 \\ &= -q\frac{(q^2, q^6, q^{10}; q^{12})_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^6; q^{12})_\infty^2} \\ &= -q\frac{(q^2; q^4)_\infty^2 (q^{12}; q^{12})_\infty^4}{(q^6; q^6)_\infty^2} \\ &= -q\frac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^4}{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^2}.\end{aligned}$$

Thus $\phi(q^3)\Omega(-q) = (q^4; q^4)_\infty^2 - q\dfrac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^4}{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^2}$. □

We also require the following two identities whose proofs can be found in [12, Section 22.7].

**Lemma 2.11.** *We have*

$$4\frac{f_4^3}{f_{12}} - \frac{f_1^3}{f_3} = 3\frac{f_2^2 f_3^3}{f_1 f_6^2}, \tag{33}$$

$$\frac{f_3^3}{f_1} - 4q\frac{f_{12}^3}{f_4} = \frac{f_1^3 f_6^2}{f_2^2 f_3}. \tag{34}$$

**Proof.** Equation (33) can be found in [12, Eq. (22.7.4)], while (34) can be found in [12, Eq. (22.7.6)]. □

## 3. The parity of $v_0(n)$

A complete characterization of the parity of $v_0(n)$ is presented in the theorem below, which is proven in two different ways. As a corollary, we also obtain infinitely many explicit Ramanujan-like congruences for $v_0(n)$ modulo 2.

**Theorem 3.1.** *For all $n \geq 1$,*

$$v_0(n) \equiv \begin{cases} 1 \pmod 2, & \textit{if } n \textit{ is a perfect square,} \\ 0 \pmod 2, & \textit{otherwise.} \end{cases} \tag{35}$$

**Proof.** Since $q^k \equiv -q^k \pmod 2, \forall k \geq 0$, we have $(-q;q^2)_n \equiv (q;q^2)_n \pmod 2$. Then from (2) it follows that

$$\sum_{n=0}^{\infty} v_0(n)q^n \equiv \sum_{n=0}^{\infty} q^{n^2} \pmod 2,$$

which concludes the proof. □

We now give a combinatorial proof of Theorem 3.1.

**Combinatorial proof of Theorem 3.1.** Consider the series

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}. \tag{36}$$

Its general term is

$$\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = \frac{q^{1+3+5+\cdots+(2n-1)}(1+q)(1+q^3)\cdots(1+q^{2n-1})}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}. \tag{37}$$

As the numerator of (37) is the generating function for partitions into odd parts with no gaps, in which each part occurs at most twice and largest part $2n-1$ and

$$\frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$

is the generating function for partitions into odd parts and largest part less than or equal to $2n-1$, we have that (36) is the generating function for pairs $(\lambda, \mu)$ of partitions into odd parts in which

$(i)$ $\lambda$ has no gaps, is nonempty and each part occurs at most twice,
$(ii)$ the largest part of $\mu$ is less than or equal to the largest part of $\lambda$.

Let $\mathcal{V}_n$ be the set of all pairs $(\lambda, \mu)$ satisfying $(i)$ and $(ii)$ above and such that $|\lambda|+|\mu| = n$. In order to give a combinatorial proof of Theorem 3.1, whenever $n$ is not a square, we construct an involution $\psi : \mathcal{V}_n \to \mathcal{V}_n$ such that $\psi(\lambda, \mu) \neq (\lambda, \mu)$, $\forall (\lambda, \mu) \in \mathcal{V}_n$. This proves that $v_0(n)$ is even if $n$ is not a square.

Given $(\lambda, \mu) \in \mathcal{V}_n$, we define $(\lambda', \mu')$ such that $(\lambda', \mu') = \psi(\lambda, \mu)$ in the following way. Let $m \geq 1$ be the smallest part of $\lambda$ that either appears twice in $\lambda$ or once in $\lambda$ and at least once in $\mu$. In the former case, we remove a part $m$ from $\lambda$ and add it to $\mu$; in the last case, we remove a part $m$ from $\mu$ and add it to $\lambda$. We define $(\lambda', \mu')$ as the resulting pair obtained by this procedure. We note that, since $n$ is not a square (which implies that $n$ is not the sum of consecutive and distinct odd numbers), we have $\mu \neq \emptyset$, which assures that such smallest part $m$ does exist.

For example, for $n = 12$ we have:

$$\begin{aligned}
((1),(1,1,1,1,1,1,1,1,1,1,1)) &\longleftrightarrow ((1,1),(1,1,1,1,1,1,1,1,1,1))\\
((1,3),(1,1,1,1,1,1,1,1)) &\longleftrightarrow ((1,1,3),(1,1,1,1,1,1,1))\\
((1,3,3),(1,1,1,1,1)) &\longleftrightarrow ((1,1,3,3),(1,1,1,1))\\
((1,3),(1,1,1,1,1,3)) &\longleftrightarrow ((1,1,3),(1,1,1,1,3))\\
((1,3,3),(1,1,3)) &\longleftrightarrow ((1,1,3,3),(1,3))\\
((1,3),(1,1,3,3)) &\longleftrightarrow ((1,1,3),(1,3,3))\\
((1,3,5),(1,1,1)) &\longleftrightarrow ((1,1,3,5),(1,1))\\
((1,3,5),(3)) &\longleftrightarrow ((1,3,3,5),\emptyset)
\end{aligned}$$

As additional examples of the case $m > 1$, we have

$$\begin{aligned}
\psi((1,3,5,5,7,9,9),(3,3,5,7,7,7)) &= ((1,3,3,5,5,7,9,9),(3,5,7,7,7)),\\
\psi((1,3,5,5,7,9,9),(7,7,9,9)) &= ((1,3,5,7,9,9),(5,7,7,9,9)).
\end{aligned}$$

This construction breaks down if $n$ is a square, $n = k^2$. In this case it is not possible to define $\psi((1,3,5,\ldots,2k-1)),\emptyset)$. But the involution still works in $\mathcal{V}_n$ minus this pair, proving that in this case $v_0(n)$ is odd. $\square$

We close this section by noting that we can deduce from Theorem 3.1 infinitely many Ramanujan-like congruences mod 2 involving $v_0(n)$.

**Corollary 3.2.** *Let $p > 2$ be a prime. It follows that, for all $n \geq 0$,*

$$v_0(pn + r) \equiv 0 \pmod{2},$$

*where $r$ is any quadratic nonresidue modulo $p$.*

This yields an infinite family of results. For each prime $p$ it yields $(p-1)/2$ congruences modulo 2.

## 4. Congruences modulo powers of 2

This section is devoted to presenting identities involving the generating function for $v_0(n)$ which yield a number of congruences modulo certain powers of 2. We begin with a theorem that establishes infinitely many congruences modulo 4.

**Theorem 4.1.** *For all $n \geq 0$,*

$$v_0(4n) \equiv \begin{cases} (-1)^k \pmod{4}, & \text{if } n = k^2,\\ 0 \pmod{4}, & \text{otherwise.} \end{cases} \tag{38}$$

**Proof.** By Theorem 2.9 we have

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty} v_0(2n)q^n &= \frac{1}{2}+\sum_{n=1}^{\infty} v_0(4n)q^{2n}+\sum_{n=1}^{\infty} v_0(4n+2)q^{2n+1}\\ &= \frac{\phi(q^2)^2}{2\phi(-q^2)}+2q\frac{\psi(q^4)^2}{\phi(-q^2)}.\end{aligned}$$

Then, taking the even part on both sides of this equation and using (3) and (8), we obtain

$$1+\sum_{n=1}^{\infty} v_0(4n)q^n = \frac{1}{2}+\frac{1}{2}\frac{f_2^{11}}{f_1^6 f_4^4}. \tag{39}$$

Since $f_m^8 \equiv f_{2m}^4 \pmod 8$, we have

$$\frac{f_2^{11}}{f_1^6 f_4^4} \equiv \frac{f_1^8 f_2^7}{f_1^6 f_2^8} \equiv \frac{f_1^2}{f_2} = \phi(-q) \pmod 8.$$

Thus, by (39),

$$\begin{aligned}1+\sum_{n=1}^{\infty} v_0(4n)q^n &\equiv \frac{1}{2}+\frac{1}{2}\phi(-q) \pmod 4\\ &\equiv 1+\sum_{k=1}^{\infty}(-1)^k q^{k^2} \pmod 4,\end{aligned}$$

from which we obtain

$$\sum_{n=0}^{\infty} v_0(4n)q^n \equiv \sum_{k=0}^{\infty}(-1)^k q^{k^2} \pmod 4,$$

which completes the proof of (38). □

The next result yields an infinite family of congruences modulo 4.

**Corollary 4.2.** *For all primes $p>2$,*

$$v_0(4(pn+r)) \equiv 0 \pmod 4$$

*if $r$ is a quadratic nonresidue modulo $p$.*

### *4.1. Congruences from the odd part of $v_0(n)$*

We now derive some congruences related to identities involving the odd part of $\sum_{n=0}^{\infty} v_0(n)q^n$. In what follows, we denote by $\mathbf{E}(f(q))$ (resp. $\mathbf{O}(f(q))$) the even (resp. odd) part of the function $f(q)$. That is to say, if $f(q)=\sum_{n=0}^{\infty} a(n)q^n$, then

$$\mathbf{E}(f(q)) = \sum_{n=0}^{\infty} a(2n)q^{2n} \quad\text{and}\quad \mathbf{O}(f(q)) = \sum_{n=0}^{\infty} a(2n+1)q^{2n+1}.$$

**Theorem 4.3.** *We have*

$$\sum_{n=0}^{\infty} v_0(8n+5)q^n = 4\frac{f_2^2 f_4^4}{f_1^5}, \tag{40}$$

*from which it follows, for all* $n \geq 0$,

$$v_0(8n+5) \equiv 0 \pmod 4.$$

**Proof.** Taking the even part on both sides of (28) and using (24), we deduce that

$$\sum_{n=0}^{\infty} v_0(4n+1)q^{2n} = \frac{f_4^5}{f_2^4},$$

from which we obtain

$$\sum_{n=0}^{\infty} v_0(4n+1)q^n = \frac{f_2^5}{f_1^4} = \frac{\psi(q)^2}{\phi(-q)}. \tag{41}$$

Taking the odd part in both sides of the first equality in (41), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(8n+5)q^{2n+1} &= f_2^5 \mathbf{O}\left(\frac{1}{f_1^4}\right)\\ &= 4q\frac{f_4^2 f_8^4}{f_2^5}. \quad \text{(by (18))}\end{aligned}$$

Now, dividing by $q$ and replacing $q^2$ by $q$, we have (40). □

**Theorem 4.4.** *We have*

$$\sum_{n=0}^{\infty} v_0(12n+5)q^n = 4\frac{f_2^{10} f_3^2}{f_1^{10} f_6}. \tag{42}$$

*From* (42) *it follows that, for all* $n \geq 0$,

$$v_0(12n+5) \equiv 0 \pmod 4.$$

**Proof.** By (41), using Lemma 2.4 and Lemma 2.7, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(4n+1)q^n &= \frac{\psi(q)^2}{\phi(-q)}\\ &= \frac{\phi(-q^9)}{\phi(-q^3)^4}\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}\right)^2\\ &\quad \times \left(\phi(-q^9)^2 + 2q\phi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right). \end{aligned} \tag{43}$$

Then,

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$$\sum_{n=0}^{\infty} v_0(12n+5)q^{3n+1} = q\frac{\phi(-q^9)}{\phi(-q^3)^4}\left(2\phi(-q^9)\Omega(-q^3)\frac{f_6^2 f_9^4}{f_3^2 f_{18}^2} + 2\phi(-q^9)^2\frac{f_6 f_9 f_{18}}{f_3}\right.$$
$$\left. + 4q^3\Omega(-q^3)^2\frac{f_{18}^4}{f_9^2}\right),$$

from which we obtain

$$\sum_{n=0}^{\infty} v_0(12n+5)q^{n} = 2\frac{\phi(-q^3)}{\phi(-q)^4}$$
$$\times\left(\phi(-q^3)\Omega(-q)\frac{f_2^2 f_3^4}{f_1^2 f_6^2} + \phi(-q^3)^2\frac{f_2 f_3 f_6}{f_1} + 2q\Omega(-q)^2\frac{f_6^4}{f_3^2}\right).$$

Using (8), (9), and (31) this expression becomes

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(12n+5)q^{n} &= 4\frac{\phi(-q^3)}{\phi(-q)^4}\frac{f_1}{f_3}\left(\frac{f_2 f_3^6}{f_1^2 f_6} + q\frac{f_1 f_6^8}{f_2^2 f_3^3}\right)\\ &= 4\frac{\phi(-q^3)}{\phi(-q)^4}\frac{f_1 f_6^2}{f_3}\left(\frac{\phi(-q^3)^3}{\phi(-q)} + q\frac{\psi(q^3)^3}{\psi(q)}\right)\\ &= 4\frac{\phi(-q^3)}{\phi(-q)^4}\frac{f_1 f_6^2}{f_3}\frac{\psi(q)^3}{\psi(q^3)}, \quad \text{(by Lemma 2.1)}\end{aligned}$$

which, after simplification, results in (42). □

**Corollary 4.5.** *We have*

$$\sum_{n=0}^{\infty} v_0(24n+17)q^{n} = 8\frac{f_2^{29} f_3 f_{12}}{f_1^{22} f_4^7 f_6} + 32\frac{f_2^{20} f_6^2}{f_1^{19} f_4 f_{12}} + 128q\frac{f_2^5 f_3^5 f_4^9 f_{12}}{f_1^{14} f_6}, \tag{44}$$

*from which it follows that, for all $n \geq 0$,*

$$v_0(24n+17) \equiv 0 \pmod 8.$$

**Proof.** We recall from [12, Section 30.10] the 2-dissection of $\frac{f_3^2}{f_1^2}$:

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{45}$$

From Theorem 4.4 it follows that

$$\sum_{n=0}^{\infty} v_0(24n+17)q^{2n+1} = 4\frac{f_2^{10}}{f_6}\mathbf{O}\left(\frac{f_3^2}{f_1^2}\frac{1}{f_1^8}\right).$$

By (20) and (45) we can see that

$$\mathbf{O}\left(\frac{f_3^2}{f_1^2}\frac{1}{f_1^8}\right) = 8q\frac{f_4^{20} f_6 f_{12}^2}{f_2^{29} f_8 f_{24}} + 2q\frac{f_4^{29} f_6^2 f_{24}}{f_2^{32} f_8^7 f_{12}} + 32q^3\frac{f_4^5 f_6^2 f_8^9 f_{24}}{f_2^{24} f_{12}}.$$

Then,

$$\sum_{n=0}^{\infty} v_0(24n+17)q^{2n+1} = 8q\frac{f_4^{29}f_6f_{24}}{f_2^{22}f_8^7f_{12}} + 32q\frac{f_4^{20}f_{12}^2}{f_2^{19}f_8f_{24}} + 128q^3\frac{f_4^5f_6f_8^9f_{24}}{f_2^{14}f_{12}}.$$

After dividing both sides of this last expression by $q$ and replacing $q^2$ by $q$, we obtain (44). □

*4.2. Congruences from the even part of $v_0(n)$*

Initially we present the 2-dissection of $\sum_{n=1}^{\infty} v_0(2n)q^n$.

**Theorem 4.6.** *We have*

$$\frac{1}{2} + \sum_{n=1}^{\infty} v_0(4n)q^n = \frac{1}{2}\frac{f_2^{11}}{f_1^6f_4^4}, \tag{46}$$

$$\sum_{n=0}^{\infty} v_0(4n+2)q^n = 2\frac{f_4^4}{f_1^2f_2}. \tag{47}$$

**Proof.** By Theorem 2.9 we have

$$\frac{1}{2} + \sum_{n=1}^{\infty} v_0(2n)q^n = \frac{\phi(q^2)^2}{2\phi(-q^2)} + 2q\frac{\psi(q^4)^2}{\phi(-q^2)},$$

from which we have

$$\frac{1}{2} + \sum_{n=1}^{\infty} v_0(4n)q^n = \frac{\phi(q)^2}{2\phi(-q)} \text{ and } \sum_{n=1}^{\infty} v_0(4n+2)q^n = \frac{2\psi(q^2)^2}{\phi(-q)},$$

which, after simplification, result in (46) and (47), respectively. □

**Corollary 4.7.** *We have*

$$\sum_{n=0}^{\infty} v_0(8n+2)q^n = 2\frac{f_2^4f_4^5}{f_1^6f_8^2}, \tag{48}$$

*from which it follows that, for all $n \geq 0$,*

$$v_0(8n+2) \equiv 0 \pmod 2.$$

**Proof.** Since

$$\frac{f_4^4}{f_2}$$

is an even function, taking the even part on both sides of (47), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(8n+2)q^{2n} &= 2\frac{f_4^4}{f_2}\mathbf{E}\left(\frac{1}{f_1^2}\right)\\ &= 2\frac{f_4^4f_8^5}{f_2^6f_{16}^2}. \quad \text{(by (17))}\end{aligned}$$

Replacing $q^2$ by $q$ we obtain (48). □

**Corollary 4.8.** *We have*

$$\sum_{n=0}^{\infty} v_0(8n+6)q^n = 4\frac{f_2^6 f_8^2}{f_1^6 f_4}, \tag{49}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(8n+6) \equiv 0 \pmod 4.$$

**Proof.** By taking the odd part on both sides of (47), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(8n+6)q^{2n+1} &= 2\frac{f_4^4}{f_2}\mathbf{O}\left(\frac{1}{f_1^2}\right) \\ &= 4q\frac{f_4^6 f_{16}^2}{f_2^6 f_8}. \quad \text{(by (17))}\end{aligned}$$

Dividing both sides by $q$ and replacing $q^2$ by $q$ we obtain (49). $\square$

**Theorem 4.9.** *We have*

$$\sum_{n=0}^{\infty} v_0(16n+8)q^n = 8\frac{f_2^{17}}{f_1^{14} f_4^2}, \tag{50}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(16n+8) \equiv 0 \pmod 8.$$

**Proof.** Taking the even part on both sides of (46) and using (19) to obtain the even part of $1/(q;q)_\infty^6$, it follows that

$$\begin{aligned}\frac{1}{2} + \sum_{n=1}^{\infty} v_0(8n)q^{2n} &= \frac{1}{2}\frac{f_2^{11}}{f_4^4}\left(\frac{f_4^{14} f_8}{f_2^{19} f_{16}^2} + 8q^2\frac{f_4^4 f_8^3 f_{16}^2}{f_2^{15}}\right) \\ &= \frac{1}{2}\frac{f_4^{10} f_8}{f_2^8 f_{16}^2} + 4q^2\frac{f_8^3 f_{16}^2}{f_2^4},\end{aligned}$$

from which we deduce, by replacing $q^2$ by $q$, that

$$\frac{1}{2} + \sum_{n=1}^{\infty} v_0(8n)q^n = \frac{1}{f_1^4}\left(\frac{1}{2}\frac{f_2^{10} f_4}{f_1^4 f_8^2} + 4q f_4^3 f_8^2\right). \tag{51}$$

Now, taking the odd parts on both sides of this last equation, we have

$$\sum_{n=0}^{\infty} v_0(16n+8)q^{2n+1} = \frac{f_2^{10} f_4}{f_8^2}\mathbf{E}\left(\frac{1}{f_1^4}\right)\mathbf{O}\left(\frac{1}{f_1^4}\right) + 4q f_4^3 f_8^2 \mathbf{E}\left(\frac{1}{f_1^4}\right).$$

Now, by (18), we have

$$\sum_{n=0}^{\infty} v_0(16n+8)q^{2n+1} = \frac{f_4^{14}}{f_2^{14} f_8^4}\left(4q\frac{f_2^{10} f_4}{f_8^2}\frac{f_4^2 f_8^4}{f_2^{10}} + 4q f_4^3 f_8^2\right) = 8q\frac{f_4^{17}}{f_2^{14} f_8^2}.$$

The result follows after dividing both sides of this equation by $q$ and replacing $q^2$ by $q$. $\square$

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**Corollary 4.10.** *We have*

$$\sum_{n=0}^{\infty} v_0(32n+24)q^n = 16\frac{f_2^{42}f_8^2}{f_1^{30}f_4^{13}} + 96\frac{f_2^{28}f_4}{f_1^{26}f_8^2} + 512q\frac{f_2^4f_4^{17}}{f_1^{18}f_8^2} + 768q\frac{f_2^{18}f_4^3f_8^2}{f_1^{22}}, \tag{52}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(32n+24) \equiv 0 \pmod{16}.$$

**Proof.** By (50) we have

$$\sum_{n=0}^{\infty} v_0(32n+24)q^{2n+1} = 8\frac{f_2^{17}}{f_4^2}\mathbf{O}\left(\frac{1}{f_1^{14}}\right).$$

From (22) we see that

$$\mathbf{O}\left(\frac{1}{f_1^{14}}\right) = 2q\frac{f_4^{44}f_{16}^2}{f_2^{47}f_8^{13}} + 12q\frac{f_4^{30}f_8}{f_2^{43}f_{16}^2} + 96q^3\frac{f_4^{20}f_8^3f_{16}^2}{f_2^{39}} + 64q^3\frac{f_4^6f_8^{17}}{f_2^{35}f_{16}^2}.$$

Then,

$$\sum_{n=0}^{\infty} v_0(32n+24)q^{2n+1} = 16q\frac{f_4^{42}f_{16}^2}{f_2^{30}f_8^{13}} + 96q\frac{f_4^{28}f_8}{f_2^{26}f_{16}^2} + 768q^3\frac{f_4^{18}f_8^3f_{16}^2}{f_2^{22}} + 512q^3\frac{f_4^4f_8^{17}}{f_2^{18}f_{16}^2}.$$

After dividing both sides of this last expression by $q$ and replacing $q^2$ by $q$, we obtain (52). □

**Theorem 4.11.** *We have*

$$\sum_{n=0}^{\infty} v_0(16n+12)q^n = 16\frac{f_2^{11}f_4^2}{f_1^{12}}, \tag{53}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(16n+12) \equiv 0 \pmod{16}.$$

**Proof.** Taking the odd part on both sides of (46) and using (19), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(8n+4)q^{2n+1} &= \frac{1}{2}\frac{f_2^{11}}{f_4^4}\left(2q\frac{f_4^{16}f_{16}^2}{f_2^{19}f_8^5} + 4q\frac{f_4^2f_8^9}{f_2^{15}f_{16}^2}\right)\\ &= q\left(\frac{f_4^{12}f_{16}^2}{f_2^8f_8^5} + 2\frac{f_8^9}{f_2^4f_4^2f_{16}^2}\right),\end{aligned}$$

from which we deduce, dividing by $q$ and replacing $q^2$ by $q$, that

$$\sum_{n=0}^{\infty} v_0(8n+4)q^n = \frac{1}{f_1^4}\left(\frac{f_2^{12}f_8^2}{f_1^4f_4^5} + 2\frac{f_4^9}{f_2^2f_8^2}\right). \tag{54}$$

Now, taking the odd parts on both sides of this last equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(16n+12)q^{2n+1} &= 2\frac{f_2^{12}f_8^2}{f_4^5}\mathbf{E}\left(\frac{1}{f_1^4}\right)\mathbf{O}\left(\frac{1}{f_1^4}\right) + 2\frac{f_4^9}{f_2^2f_8^2}\mathbf{O}\left(\frac{1}{f_1^4}\right)\\ &= 8q\frac{f_2^{12}f_8^2}{f_4^5}\frac{f_4^{14}}{f_2^{14}f_8^4}\frac{f_4^2f_8^4}{f_2^{10}} + 8q\frac{f_4^9}{f_2^2f_8^2}\frac{f_4^2f_8^4}{f_2^{10}}\\ &= 16q\frac{f_4^{11}f_8^2}{f_2^{12}},\end{aligned}$$

where we have used (18). The result follows after dividing both sides of this equation by $q$ and replacing $q^2$ by $q$. □

**Corollary 4.12.** *We have*

$$\sum_{n=0}^{\infty} v_0(32n+28)q^n = 192\frac{f_2^{32}}{f_1^{27}f_4^4} + 1024q\frac{f_2^8f_4^{12}}{f_1^{19}}, \tag{55}$$

*from which it follows that, for all $n \geq 0$,*

$$v_0(32n+28) \equiv 0 \pmod{64}.$$

**Proof.** By (53) we have

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(32n+28)q^{2n+1} &= 16f_2^{11}f_4^2\mathbf{O}\left(\frac{1}{f_1^{12}}\right)\\ &= 192q\frac{f_4^{32}}{f_2^{27}f_8^4} + 1024q^3\frac{f_4^8f_4^{12}}{f_2^{19}}. \quad \text{(by (21))}\end{aligned}$$

Now, (55) follows after dividing both sides by $q$ and replacing $q^2$ by $q$. □

**Theorem 4.13.** *We have*

$$\sum_{n=0}^{\infty} v_0(6n+2)q^n = 2\frac{f_2^2f_4^2f_6^2}{f_1^4f_{12}}, \tag{56}$$

*from which it follows that, for all $n \geq 0$,*

$$v_0(6n+2) \equiv 0 \pmod{2}.$$

**Proof.** By Theorem 2.8 we have

$$\begin{aligned}\sum_{n=1}^{\infty} v_0(6n+2)q^{3n+1} = q\frac{\phi(-q^9)}{2\phi(-q^3)^4}&\left(2\phi(-q^9)^2\phi(-q^{18})\Omega(q^3)\right.\\ &+ 2\phi(q^9)\phi(-q^9)\phi(-q^{18})\Omega(-q^3)\\ &- 8q^3\phi(-q^9)\Omega(q^3)\Omega(-q^3)\Omega(-q^6)\\ &\left.-8q^3\phi(q^9)\Omega(-q^6)\Omega(-q^3)^2\right),\end{aligned}$$

from which

$$\begin{aligned}\sum_{n=1}^{\infty} v_0(6n+2)q^n &= \frac{\phi(-q^3)}{\phi(-q)^4}\left(\phi(-q^3)^2\phi(-q^6)\Omega(q)+\phi(q^3)\phi(-q^3)\phi(-q^6)\Omega(-q)\right.\\ &\quad \left.-4q\phi(-q^3)\Omega(q)\Omega(-q)\Omega(-q^2)-4q\phi(q^3)\Omega(-q^2)\Omega(-q)^2\right)\\ &= \frac{\phi(-q^3)}{\phi(-q)^4}\left(\phi(-q^3)\Omega(q)+\phi(q^3)\Omega(-q)\right)\\ &\quad \times\left(\phi(-q^3)\phi(-q^6)-4q\Omega(-q)\Omega(-q^2)\right).\end{aligned}$$

By Lemma 2.10 it follows that

$$\phi(-q^3)\Omega(q)+\phi(q^3)\Omega(-q)=2(q^4;q^4)_\infty^2.$$

Then

$$\sum_{n=1}^{\infty} v_0(6n+2)q^n = 2\frac{\phi(-q^3)}{\phi(-q)^4}(q^4;q^4)_\infty^2\left(\phi(-q^3)\phi(-q^6)-4q\Omega(-q)\Omega(-q^2)\right).$$

Using (31) we have that

$$\begin{aligned}\phi(-q^3)\phi(-q^6)-4q\Omega(-q)\Omega(-q^2) &= \frac{f_3^2f_6^2}{f_6f_{12}}-4q\frac{f_1f_6^2}{f_2f_3}\frac{f_2f_{12}^2}{f_4f_6}\\ &= \frac{f_1f_6}{f_3f_{12}}\left(\frac{f_3^3}{f_1}-4q\frac{f_{12}^3}{f_4}\right)\\ &= \frac{f_1^4f_6^3}{f_2^2f_3^2f_{12}}. \quad \text{(by (34))}\end{aligned}$$

Finally,

$$\sum_{n=1}^{\infty} v_0(6n+2)q^n = 2\frac{\phi(-q^3)}{\phi(-q)^4}f_4^2\frac{f_1^4f_6^3}{f_2^2f_3^2f_{12}} = 2\frac{f_3^2f_2^4}{f_1^8f_6}f_4^2\frac{f_1^4f_6^3}{f_2^2f_3^2f_{12}} = 2\frac{f_2^2f_4^2f_6^2}{f_1^4f_{12}}. \quad \square$$

**Corollary 4.14.** *We have*

$$\sum_{n=0}^{\infty} v_0(12n+2)q^n = 2\frac{f_2^{16}f_3^2}{f_1^{12}f_4^4f_6}, \tag{57}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(12n+2)\equiv 0 \pmod 2.$$

**Proof.** Since

$$\frac{f_2^2f_4^2f_6^2}{f_{12}}$$

is an even function, taking the even part on both sides of (56) we obtain

$$\sum_{n=0}^{\infty} v_0(12n+2)q^{2n} = 2\,\frac{f_2^2f_4^2f_6^2}{f_{12}}\mathbf{E}\left(\frac{1}{f_1^4}\right).$$

Please cite this article in press as: E.H.M. Brietzke et al., Congruences related to an eighth order mock theta function of Gordon and McIntosh, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.06.016

Using (18) it follows that

$$\sum_{n=0}^{\infty} v_0(12n+2)q^{2n} = 2\frac{f_2^2 f_4^2 f_6^2}{f_{12}} \frac{f_4^{14}}{f_2^{14} f_8^4}.$$

Replacing $q^2$ by $q$, (57) follows. □

**Corollary 4.15.** *We have*

$$\sum_{n=0}^{\infty} v_0(12n+8)q^{n} = 8\frac{f_2^4 f_3^2 f_4^4}{f_1^8 f_6}, \tag{58}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(12n+8) \equiv 0 \pmod{8}.$$

**Proof.** Taking the odd part on both sides of (56) we obtain

$$\sum_{n=0}^{\infty} v_0(12n+8)q^{2n+1} = 2\frac{f_2^2 f_4^2 f_6^2}{f_{12}} \mathbf{O}\left(\frac{1}{f_1^4}\right).$$

Using (18) it follows that

$$\sum_{n=0}^{\infty} v_0(12n+8)q^{2n+1} = 2\frac{f_2^2 f_4^2 f_6^2}{f_{12}} \frac{4q\, f_4^2 f_8^4}{f_2^{10}}.$$

Dividing by $q$ we obtain

$$\sum_{n=0}^{\infty} v_0(12n+8)q^{2n} = 8\frac{f_4^4 f_6^2 f_8^4}{f_2^8 f_{12}}.$$

Now replacing $q^2$ by $q$, we obtain (58). □

## 5. Congruences modulo powers of 3

In this section, we present some congruences for $v_0(n)$ modulo powers of 3. In order to do so, many additional identities involving the generating function for $v_0(n)$ are derived.

**Theorem 5.1.** *We have*

$$\sum_{n=0}^{\infty} v_0(6n+4)q^{n} = 3\frac{f_2^3 f_6^3}{f_1^4 f_4}, \tag{59}$$

*from which it follows that, for all* $n \geq 0$,

$$v_0(6n+4) \equiv 0 \pmod{3}.$$

**Proof.** By Theorem 2.8 we have

$$\begin{aligned}\sum_{n=1}^{\infty} v_0(6n+4)q^{3n+2} = q^2\frac{\phi(-q^9)}{2\phi(-q^3)^4}\big(&4\phi(q^9)\phi(-q^{18})\Omega(-q^3)^2\\ &-16q^3\Omega(q^3)\Omega(-q^6)\Omega(-q^3)^2\\ &+4\phi(-q^9)\phi(-q^{18})\Omega(-q^3)\Omega(q^3)\\ &-2\phi(q^9)\phi(-q^9)^2\Omega(-q^6)\big),\end{aligned}$$

from which we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} v_0(6n+4)q^n &= \frac{\phi(-q^3)}{\phi(-q)^4}\big(2\phi(q^3)\phi(-q^6)\Omega(-q)^2 - 8q\Omega(q)\Omega(-q^2)\Omega(-q)^2\\ &\quad + 2\phi(-q^3)\phi(-q^6)\Omega(-q)\Omega(q) - \phi(q^3)\phi(-q^3)^2\Omega(-q^2)\big)\\ &= \frac{\phi(-q^3)}{\phi(-q)^4}\big(2\phi(-q^6)\Omega(-q)\left(\phi(q^3)\Omega(-q)+\phi(-q^3)\Omega(q)\right)\\ &\quad - 8q\Omega(q)\Omega(-q^2)\Omega(-q)^2 - \phi(q^3)\phi(-q^3)^2\Omega(-q^2)\big)\\ &= \frac{\phi(-q^3)}{\phi(-q)^4}\big(4(q^4;q^4)_\infty^2\phi(-q^6)\Omega(-q) \qquad \text{(by Lemma 2.10)}\\ &\quad - \Omega(-q^2)\left(8q\Omega(q)\Omega(-q)^2 + \phi(q^3)\phi(-q^3)^2\right)\big).\end{aligned}$$

Using (3), (4), (8), and (30) we have

$$\begin{aligned}&4f_4^2\phi(-q^6)\Omega(-q) - \Omega(-q^2)\left(8q\Omega(q)\Omega(-q)^2 + \phi(q^3)\phi(-q^3)^2\right)\\ &= 4f_4^2\frac{f_6^2}{f_{12}}\frac{f_1f_6^2}{f_2f_3} - \frac{f_2f_{12}^2}{f_4f_6}\left(8q\frac{f_2^2f_3f_{12}}{f_1f_4f_6}\frac{f_1^2f_6^4}{f_2^2f_3^2} + \frac{f_6^5}{f_3^2f_{12}^2}\frac{f_3^4}{f_6^2}\right)\\ &= 4\frac{f_1f_4^2f_6^4}{f_2f_3f_{12}} - \frac{f_2f_3^2f_6^2}{f_4} + \frac{f_1f_2f_6^2}{f_3f_4}\left(-8q\frac{f_{12}^3}{f_4}\right)\\ &= -3\frac{f_2f_3^2f_6^2}{f_4} + \frac{f_1f_6^4}{f_2f_3f_4}\left(4\frac{f_4^3}{f_{12}} + 2\frac{f_1^3}{f_3}\right). \qquad \text{(by (34))}\end{aligned}$$

Now, by (33) it follows that

$$\begin{aligned}&4f_4^2\phi(-q^6)\Omega(-q) - \Omega(-q^2)\left(8q\Omega(q)\Omega(-q)^2 + \phi(q^3)\phi(-q^3)^2\right)\\ &= -3\frac{f_2f_3^2f_6^2}{f_4} + \frac{f_1f_6^4}{f_2f_3f_4}\left(3\frac{f_2^2f_3^3}{f_1f_6^2} + 3\frac{f_1^3}{f_3}\right)\\ &= 3\frac{f_1^4f_6^4}{f_2f_3^2f_4}.\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} v_0(6n+4)q^n = 3\frac{\phi(-q^3)}{\phi(-q)^4}\frac{f_1^4f_6^4}{f_2f_3^2f_4} = 3\frac{f_2^3f_6^3}{f_1^4f_4},$$

completing the proof. □

**Corollary 5.2.** *We have*

$$\sum_{n=0}^{\infty} v_0(12n+4)q^n = 3\frac{f_2^{13}f_3^3}{f_1^{11}f_4^4}, \tag{60}$$

*from which it follows that, for all $n \geq 0$,*

$$v_0(12n+4) \equiv 0 \pmod 3.$$

**Proof.** Since

$$\frac{f_2^3 f_6^3}{f_4}$$

is an even function, taking the even part on both sides of (59) we obtain

$$\sum_{n=0}^{\infty} v_0(12n+4)q^{2n} = 3\frac{f_2^3f_6^3}{f_4}\mathbf{E}\left(\frac{1}{f_1^4}\right).$$

Using (18) it follows that

$$\sum_{n=0}^{\infty} v_0(12n+4)q^{2n} = 3\frac{f_2^3f_6^3}{f_4}\frac{f_4^{14}}{f_2^{14}f_8^4}.$$

Replacing $q^2$ by $q$, (60) follows. □

**Theorem 5.3.** *We have*

$$\sum_{n=0}^{\infty} v_0(12n+9)q^n = 9\frac{f_2^4f_3^4f_6}{f_1^8}. \tag{61}$$

*From (61) it follows that, for all $n \geq 0$,*

$$v_0(12n+9) \equiv 0 \pmod 9.$$

**Proof.** By (43) we have

$$\sum_{n=0}^{\infty} v_0(12n+9)q^{3n+2} = q^2\frac{\phi(-q^9)}{\phi(-q^3)^4}\left(\phi(-q^9)^2\frac{f_{18}^4}{f_9^2} + 4\phi(-q^9)\Omega(-q^3)\frac{f_6f_9f_{18}}{f_3}\right.$$
$$\left. +4\Omega(-q^3)^2\frac{f_6^2f_9^4}{f_3^2f_{18}^2}\right).$$

This means

$$\sum_{n=0}^{\infty} v_0(12n+9)q^n = \frac{\phi(-q^3)}{\phi(-q)^4}\left(\phi(-q^3)^2\frac{f_6^4}{f_3^2} + 4\phi(-q^3)\Omega(-q)\frac{f_2f_3f_6}{f_1}\right.$$
$$\left. +4\Omega(-q)^2\frac{f_2^2f_3^4}{f_1^2f_6^2}\right)$$
$$= 9\frac{\phi(-q^3)}{\phi(-q)^4}f_3^2f_6^2, \quad \text{(by (8) and (31))}$$

which, after simplification, gives us (61). □

## 6. Congruences involving other moduli

In this section, we prove a number of congruences moduli certain numbers of the form $2^\alpha 3^\beta 5^\gamma$. In some of the following proofs we make use of the well-known results stated in the two lemmas below.

**Lemma 6.1.** *Given a prime $p$ and positive integers $k$ and $m$, we have*

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod 4, \tag{62}$$

$$f_k^{pm} \equiv f_{pk}^{m} \pmod p. \tag{63}$$

**Proof.** These two congruences follow directly from the binomial theorem. □

**Lemma 6.2.** *The following identities hold true:*

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}, \tag{64}$$

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{65}$$

**Proof.** Equation (64) is Euler's Pentagonal Theorem given in (7). Equation (65) is Jacobi's identity (see [5, Eq. (1.3.24)]). □

**Theorem 6.3.** *We have*

$$\sum_{n=0}^{\infty} v_0(12n+10) q^n = 12 \frac{f_2 f_3^3 f_4^4}{f_1^7}. \tag{66}$$

*From (66) it follows that, for all $n \geq 0$,*

$$v_0(12n+10) \equiv 0 \pmod{12}.$$

**Proof.** Taking the odd part on both sides of (59) we obtain

$$\sum_{n=0}^{\infty} v_0(12n+10) q^{2n+1} = 3 \frac{f_2^3 f_6^3}{f_4} \mathbf{O}\left(\frac{1}{f_1^4}\right).$$

Using (18) it follows that

$$\sum_{n=0}^{\infty} v_0(12n+10) q^{2n+1} = 3 \frac{f_2^3 f_6^3}{f_4} 4q \frac{f_4^2 f_8^4}{f_2^{10}}.$$

Dividing by $q$ and replacing $q^2$ by $q$ (66) follows. □

**Theorem 6.4.** *We have*

$$\sum_{n=0}^{\infty} v_0(24n+21) q^n = 36 \frac{f_2^{10} f_3^3 f_4^2 f_6^4}{f_1^{16} f_{12}^2} + 36 \frac{f_2^{19} f_3^4 f_6}{f_1^{19} f_4^4} + 144q \frac{f_2^4 f_3^5 f_4^6 f_{12}^2}{f_1^{14} f_6^2}. \tag{67}$$

*From* (67) *it follows that, for all* $n \geq 0$*,*

$$v_0(24n+21) \equiv 0 \pmod{36}.$$

**Proof.** By Theorem 5.3 we have

$$\sum_{n=0}^{\infty} v_0(24n+21)q^{2n+1} = 9f_2^4 f_6 \mathbf{O}\left(\frac{f_3^4}{f_1^4}\frac{1}{f_1^4}\right).$$

By (18) and (45) it follows that

$$\mathbf{O}\left(\frac{f_3^4}{f_1^4}\frac{1}{f_1^4}\right) = 4q\frac{f_4^{10}f_6^2f_8^2f_{12}^4}{f_2^{20}f_{24}^2} + 4q\frac{f_4^{19}f_6^3f_{12}}{f_2^{23}f_8^4} + 16q^3\frac{f_4^4f_6^4f_8^6f_{24}^2}{f_2^{18}f_{12}^2}.$$

Then,

$$\sum_{n=0}^{\infty} v_0(24n+21)q^{2n+1} = 36q\frac{f_4^{10}f_6^3f_8^2f_{12}^4}{f_2^{16}f_{24}^2} + 36q\frac{f_4^{19}f_6^4f_{12}}{f_2^{19}f_8^4} + 144q^3\frac{f_4^4f_6^5f_8^6f_{24}^2}{f_2^{14}f_{12}^2}.$$

After dividing both sides of this last expression by $q$ and replacing $q^2$ by $q$, we obtain (67). $\square$

**Theorem 6.5.** *We have*

$$v_0(20n+t) \equiv 0 \pmod{20}, \text{ for } t \in \{13, 17\}, \tag{68}$$

$$v_0(60n+t) \equiv 0 \pmod{180}, \text{ for } t \in \{33, 57\}. \tag{69}$$

**Proof.** From (41), it follows that

$$\sum_{n=0}^{\infty} v_0(4n+1)q^n = \frac{f_2^5}{f_1^5}f_1 = \frac{f_2^5}{f_1^5}\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}.$$

Since $f_k^5 = f_{5k} \pmod 5$, there is a power series $k(q^5) \in \mathbb{Z}[[q^5]]$ such that

$$\frac{f_2^5}{f_1^5} \equiv \frac{f_{10}}{f_5} \equiv k(q^5) \pmod 5.$$

Thus

$$\sum_{n=0}^{\infty} v_0(4n+1)q^n \equiv k(q^5)\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2} \pmod 5.$$

Since $(3n^2-n)/2 \not\equiv 3, 4 \pmod 5$ it follows that the coefficients of $q^{5n+3}$ and $q^{5n+4}$ in $\sum_{n=0}^{\infty} v_0(4n+1)q^n$ are congruent to $0 \pmod 5$, which proves that $v_0(20n+t) \equiv 0 \pmod 5$, for $t \in \{13, 17\}$.

In order to complete the proof of (68), we take identity (41) modulo 4 to produce

$$\sum_{n=0}^{\infty} v_0(4n+1)q^n \equiv f_2^3 \equiv \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)} \pmod 4,$$

using (62) and (65). Since $n(n+1) \not\equiv 3, 4 \pmod 5$ it follows that the coefficients of $q^{5n+3}$ and $q^{5n+4}$ in $\sum_{n=0}^{\infty} v_0(4n+1)q^n$ are congruent to $0 \pmod 4$, which completes the proof of (68).

To prove (69) we use (61), from which it follows that

$$\sum_{n=0}^{\infty} v_0(12n+9)q^n = 9h(q)\frac{f_1^2 f_6}{f_2 f_3}, \tag{70}$$

where

$$h(q) = \frac{f_2^5 f_3^5}{f_1^{10}}.$$

There is a power series $g(q^5) \in \mathbb{Z}[[q^5]]$ such that $h(q) \equiv g(q^5) \pmod 5$, since $f_1^5 \equiv f_5 \pmod 5$. Note that

$$\begin{aligned}
\frac{(q\,;q)_\infty^2(q^6\,;q^6)_\infty}{(q^2\,;q^2)_\infty(q^3\,;q^3)_\infty} &= \frac{(q\,;q)_\infty^2(q^6\,;q^6)_\infty}{(q^2\,;q^6)_\infty(q^4\,;q^6)_\infty(q^6\,;q^6)_\infty(q^3\,;q^3)_\infty} \\
&= \frac{(q\,;q)_\infty(q\,;q^3)_\infty(q^2\,;q^3)_\infty(q^3\,;q^3)_\infty}{(q^2\,;q^6)_\infty(q^4\,;q^6)_\infty(q^3\,;q^3)_\infty} \\
&= \frac{(q\,;q)_\infty(q\,;q^3)_\infty(q^2\,;q^3)_\infty(q^3\,;q^3)_\infty}{(q\,;q^3)_\infty(-q\,;q^3)_\infty(q^2\,;q^3)_\infty(-q^2\,;q^3)_\infty(q^3\,;q^3)_\infty} \\
&= \frac{(q\,;q)_\infty(q^3\,;q^3)_\infty}{(-q\,;q^3)_\infty(-q^2\,;q^3)_\infty(q^3\,;q^3)_\infty}.
\end{aligned}$$

Using (6) and (7), it follows that

$$\frac{f_1^2 f_6}{f_2 f_3} = \frac{f(-q,-q^2)f(-q^3,-q^6)}{f(q^2,q)}.$$

On the other hand, taking $x = -q$ and $\lambda = -1$ in (12), we get

$$f(q^6,q^3) - qf(q^9,1) = \frac{f(-q^3,-q^6)f(-q,-q^2)}{f(q^2,q)}.$$

Combining this with (70) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} v_0(12n+9)q^n &= 9h(q)(f(q^6,q^3) - qf(q^9,1)) \\
&\equiv 9g(q^5)(f(q^6,q^3) - qf(q^9,1)) \pmod 5.
\end{aligned}$$

Therefore, to prove that $v_0(60n+33) \equiv v_0(60n+57) \equiv 0 \pmod{45}$, it suffices to prove that both series $f(q^6,q^3)$ and $qf(q^9,1)$ contain only powers of $q$ with exponents congruent to 0, 1, and 3 modulo 5. Note that

$$f(q^6,q^3) = \sum_{n=-\infty}^{\infty} q^{\frac{6n(n+1)}{2}} q^{\frac{3n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} q^{\frac{9n^2+3n}{2}}$$

and

$$\frac{9n^2+3n}{2} \equiv 2n^2 - n \equiv 0,1,3 \pmod 5, \qquad \forall n \in \mathbb{Z}.$$

Also, by (5),

$$qf(q^9,1) = 2q\psi(q^9) = 2\sum_{n=0}^{\infty} q^{\frac{9n(n+1)}{2}+1}$$

and

$$\frac{9n(n+1)}{2} + 1 \equiv 2n(n+1) + 1 \equiv 0, 1, 3 \pmod 5, \qquad \forall n \in \mathbb{N}.$$

Therefore

$$\sum_{n=0}^{\infty} v_0(12(5n+j)+9)q^n \equiv 0 \pmod{45}, \quad \text{for } j = 2, 4. \tag{71}$$

To finish the proof of (69) it suffices to show that $v_0(12(5n+j)+9) \equiv 0 \pmod 4$, $j = 2, 4$.

Consider again the right hand side of (61),

$$9\frac{f_2^4 f_3^4 f_6}{f_1^8} = 9\frac{f_2^4}{f_1^8}\frac{f_3^4}{f_6^2} f_6^3 = 9f_6^3 \frac{1}{\phi(-q)^4}\phi(-q^3)^2.$$

We have $\phi(-q) = 1 + 2u(q)$, where $u(q) = \sum_{k=1}^{\infty}(-1)^k q^{k^2} \in \mathbb{Z}[[q]]$ and, hence, $\phi(-q)^2 \equiv 1 \pmod 4$. Likewise, $\phi(-q^3)^2 \equiv 1 \pmod 4$. Therefore,

$$9\frac{f_2^4 f_3^4 f_6}{f_1^8} \equiv f_6^3 \pmod 4.$$

Replacing $q$ by $q^6$ in Jacobi's identity (65) we get

$$9\frac{f_2^4 f_3^4 f_6}{f_1^8} \equiv \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{6n(n+1)}{2}} \pmod 4. \tag{72}$$

Since all the exponents in the right hand side of (72) are congruent to 0, 1, or 3 modulo 5, it follows that $v_0(60n+33) \equiv v_0(60n+57) \equiv 0 \pmod 4$, which completes the proof of (69). □

**Theorem 6.6.** *For all $n \geq 0$,*

$$v_0(36n+t) \equiv 0 \pmod{36}, \textit{ for } t \in \{21, 33\}.$$

**Proof.** By Theorem 5.3, we know that $v_0(36n+t) \equiv 0 \pmod 9$, for $t \in \{21, 33\}$. Thus, we shall prove that $v_0(36n+t) \equiv 0 \pmod 4$, for $t \in \{21, 33\}$. Taking (61) modulo 4 we obtain

$$\sum_{n=0}^{\infty} v_0(12n+9)q^n = \frac{f_2^4 f_3^4 f_6}{f_1^8} \equiv f_3^4 f_6 \pmod 4,$$

from which it follows that the coefficients of $q^{3n+1}$ and $q^{3n+2}$ in $\sum_{n=0}^{\infty} v_0(12n+9)q^n$ are congruent to 0 modulo 4, completing the proof. □

**Theorem 6.7.** *For all $n \geq 0$,*

$$v_0(48n+28) \equiv 0 \pmod{48}, \tag{73}$$

$$v_0(48n+40) \equiv 0 \pmod{48}. \tag{74}$$

**Proof.** By (53) and (4), it follows that

$$\sum_{n=0}^{\infty} v_0(16n+12)q^n = 16\frac{f_2^{11}f_4^2}{f_1^{12}} = 16\frac{f_2^{12}}{f_1^{12}}\psi(q^2) = 16\frac{f_2^{12}}{f_1^{12}}\sum_{n=0}^{\infty} q^{n(n+1)}.$$

Since $n(n+1) \not\equiv 1 \pmod 3, \forall n \geq 0$, and

$$\sum_{n=0}^{\infty} v_0(16n+12)q^n \equiv 16\frac{f_6^4}{f_3^4}\sum_{n=0}^{\infty} q^{n(n+1)} \pmod 3,$$

we deduce that the coefficient of $q^{3n+1}$ in this last expression is congruent to $0 \pmod 3$, which completes the proof of (73).

By (50), we have

$$\sum_{n=0}^{\infty} v_0(16n+8)q^n = 8\frac{f_2^{17}}{f_1^{14}f_4^2} = 8\frac{f_2^{12}}{f_1^{12}}\frac{f_2^5}{f_1^2f_4^2} \equiv 8\frac{f_6^4}{f_3^4}\phi(q) \pmod 3.$$

Since $\phi(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$ and $n^2 \not\equiv 2 \pmod 3$ for any $n$, we have that the coefficient of $q^{3n+2}$ in $\frac{f_2^{17}}{f_1^{14}f_4^2}$ is a multiple of 6. Hence (74) follows. □

**Theorem 6.8.** *For all $n \geq 0$,*

$$v_0(96n+28) \equiv 0 \pmod{192}. \tag{75}$$

**Proof.** By Corollary 4.12, we already know that $v_0(32n+28) \equiv 0 \pmod{64}$. Using (55) and (4) it follows that

$$\sum_{n=0}^{\infty} v_0(32n+28)q^n \equiv q\frac{f_2^8f_4^{12}}{f_1^{19}} \equiv q\frac{f_2^6f_4^{12}}{f_1^{18}}\psi(q) \equiv q\frac{f_6^2f_{12}^4}{f_3^6}\psi(q) \pmod 3. \tag{76}$$

By (4), we have $\psi(q) = \sum_{n\geq 0} q^{n(n+1)/2}$. Since $n(n+1)/2 \not\equiv 2 \pmod 3, \forall n \geq 0$, we see that the coefficient of $q^{3n}$ on both sides of (76) is congruent to $0 \pmod 3$, which completes the proof of (75). □

**Theorem 6.9.** *For all $n \geq 0$,*

$$v_0(80n+t) \equiv 0 \pmod 5, \ \textit{for } t \in \{32, 48, 52, 68\}.$$

**Proof.** Taking the even part on both sides of (51) using (18) and (20) and replacing $q^2$ by $q$, we obtain

$$\begin{aligned}\frac{1}{2} + \sum_{n=1}^{\infty} v_0(16n)q^n &= \frac{1}{2}\frac{f_2^{29}}{f_1^{18}f_4^{10}} + 24q\frac{f_2^5f_4^6}{f_1^{10}} \\ &= \frac{1}{2}\frac{f_2^{30}}{f_1^{20}f_4^{10}}\frac{f_1^2}{f_2} + 24q\frac{f_2^5f_4^5}{f_1^{10}}f_4.\end{aligned}$$

By (3) and (8) we have

$$\frac{f_1^2}{f_2} = \phi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

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Then, by (64), it follows that

$$\frac{1}{2}+\sum_{n=1}^{\infty} v_0(16n)q^n = \frac{1}{2}\frac{f_2^{30}}{f_1^{20}f_4^{10}}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} + 24q\frac{f_2^5 f_4^5}{f_1^{10}}\sum_{n=-\infty}^{\infty}(-1)^n q^{2(3n^2-n)}.$$

Using (63), it follows that

$$1+\sum_{n=1}^{\infty} v_0(16n)q^n \equiv \frac{1}{2}+\frac{1}{2}\frac{f_{10}^6}{f_5^4 f_{20}^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} + 24q\frac{f_{10}f_{20}}{f_5^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{2(3n^2-n)} \pmod 5.$$

Since $n^2 \not\equiv 2,3 \pmod 5$ and $2(3n^2-n) \not\equiv 1,2 \pmod 5$, then the coefficients of $q^{5n+2}$ and $q^{5n+3}$ in $\sum_{n=0}^{\infty} v_0(16n)q^n$ are congruent to $0 \pmod 5$, which implies that

$$v_0(80n+32) \equiv v_0(80n+48) \equiv 0 \pmod 5.$$

In order to prove that $v_0(80n+t) \equiv 0 \pmod 5$, for $t \in \{52, 68\}$, we consider (54), from which, after taking the even parts on both sides, we are left with

$$\sum_{n=0}^{\infty} v_0(16n+4)q^n = 3\frac{f_2^{23}}{f_1^{16}f_4^6} + 16q\frac{f_4^{10}}{f_1^8 f_2}.$$

Recalling that $\phi(q)^4 - \phi(-q)^4 = 16q\psi(q^2)^4$ (see Entry 25 (vii) in [4, p. 40]) it follows that

$$\phi(-q)^4 \equiv \phi(q)^4 + 4q\psi(q^2)^4 \pmod 5.$$

Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} v_0(16n+4)q^n &= 3\frac{f_2^3 f_4^2}{f_1^8}\left(\frac{f_2^{20}}{f_1^8 f_4^8} + 4q\frac{f_4^8}{f_2^4}\right) + 4q\frac{f_4^{10}}{f_1^8 f_2} \\
&= 3\frac{f_2^3 f_4^2}{f_1^8}\left(\phi(q)^4 + 4q\psi(q^2)^4\right) + 4q\frac{f_4^{10}}{f_1^8 f_2} \\
&\equiv 3\frac{f_2^3 f_4^2}{f_1^8}\phi(-q)^4 + 4q\frac{f_4^{10}}{f_1^8 f_2} \pmod 5 \\
&\equiv 3\frac{f_4^2}{f_2} + 4q\frac{f_4^{10}}{f_1^{10}}\frac{f_1^2}{f_2} \pmod 5 \\
&\equiv 3\psi(q^2) + 4q\frac{f_{20}^2}{f_5^2}\phi(-q) \pmod 5 \\
&\equiv 3\sum_{n=0}^{\infty} q^{n(n+1)} + 4q\frac{f_{20}^2}{f_5^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} \pmod 5.
\end{aligned}$$

Since $n(n+1) \not\equiv 3,4 \pmod 5$ and $n^2 \not\equiv 2,3 \pmod 5$, then the coefficients of $q^{5n+3}$ and $q^{5n+4}$ in $\sum_{n=0}^{\infty} v_0(16n+4)q^n$ are congruent to $0 \pmod 5$, which completes the proof. □

## 7. Concluding remarks

While we have provided elementary proofs of numerous congruences satisfied by $v_0(n)$, computational data seems to indicate that other congruences may exist. We leave such investigations to the interested reader.

## Acknowledgment

The second author was supported by FAPESP (grant no. 2016/14057-2).

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