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Analysis of a reaction-diffusion host-pathogen model with horizontal transmission

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In this paper, a diffusive host-pathogen model with horizontal transmission and heterogeneous parameters is proposed and analyzed. We first prove the global existence of solution and a global attractor of the model. We then give the threshold dynamics for extinction and persistence of the disease. Our result suggests that by adding horizontal transmission, even a homogeneous case, the basic reproduction number is larger than the case without horizontal transmission mechanism. This may lead to over-evaluating the threshold role of the basic reproduction number. Finally, we also carry out the bifurcation analysis of steady state solutions by considering disease-induced mortality as the main bifurcation parameter, and such results can help us better understanding how it affects the spatial pattern of the pathogen.

1. Introduction

Since the pioneering work of Anderson and May [1], pathogens have been shown to be an important factor in regulating host behavior and viability, and as a consequence the host-pathogen models have attracted much attention of many researchers. The pathogens may survive in the environment for several decades, a lot of research on host-parasite systems has treated that disease transmission can occur when contagious infection between host and parasite/disease individuals or between members of the same species; and vertically from mother to offsprings. On the other hand, density-dependent host reproduction and host movement behavior can help better understand the mechanisms of spread of infectious diseases. In a recent work, Dwyer [6] considered spatial model with a logistic growth for the hosts, given by the following system

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$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 - ru_1\left(1 - \dfrac{u_1+u_2}{K}\right) - \beta u_1 u_3, & x \in \mathbb{R}, t > 0, \\ \dfrac{\partial u_2}{\partial t} = d\Delta u_2 + \beta u_1 u_3 - \alpha u_2 - r\dfrac{u_1+u_2}{K}, & x \in \mathbb{R}, t > 0, \\ \dfrac{\partial u_3}{\partial t} = -\delta u_3 + \lambda u_2, & x \in \mathbb{R}, t > 0, \end{cases} \tag{1.1}$$

where the pathogen u_3 is assumed to be immobile in the environment, and the host population u_1, u_2 are structured by density-dependent host reproduction (a logistic growth) with one dimensional Laplacian operator accounting for the host movement behavior. r represents the reproductive rate of host population; K is the carrying capacity. Under the assumption that all parameters are constants, Dwyer studied how these parameters affect the spatial spread of the pathogen by considering the existence traveling wave and spreading speed.

However, environment in reality typically varies with respect to space and time, and this heterogeneity may directly have influences on the disease invasion and host persistence. Subsequently, Wang et al. [26] modified the model (1.1) based on the following three facts: i) the population habitat should be a bounded spatial domain with zero-flux boundary condition; ii) space-dependent parameters should be used due to spatial heterogeneity; iii) consumption of pathogen by the hosts must be considered. The model studied in [26] takes the following form,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 - ru_1\left(1 - \dfrac{u_1+u_2}{K(x)}\right) - \beta(x) u_1 u_3, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_2}{\partial t} = d\Delta u_2 + \beta(x) u_1 u_3 - \alpha u_2 - r\dfrac{u_1+u_2}{K(x)}u_2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_3}{\partial t} = -\delta u_3 + \lambda(x) u_2 - \beta(x)(u_1+u_2)u_3, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_i(x,0) = u_i^0(x), & x \in \Omega,\ i = 1,2,3. \end{cases} \tag{1.2}$$

Here spatial region Ω is assumed to be a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$. Ω is isolated from outside for the host, implying the homogeneous Neumann boundary condition; $\frac{\partial}{\partial \nu}$ means the normal derivative along ν to $\partial\Omega$; The positive functions $\beta(x)$ and $\lambda(x)$ represent the rates of disease transmission, and shedding rate at position x, respectively. $K(x)$ is the carrying capacity depending position x. By using theories of monotone dynamical systems and uniform persistence, the authors in [26] studied the disease extinction and persistence. Further, a bifurcation analysis for steady state solutions are performed, which indicates that a backward bifurcation may occur when the parameters are space-dependent.

Due to the mathematical complexity, the model (1.2) assume that both susceptible and infectious hosts have the same diffusion rate. It is critical important to show prove the eventual uniform boundedness of solution of (1.2), which in turn ensures the existence of the global attractor. However, as pointed in Wu and Zou [28], susceptible and infectious hosts may disperse at different rates, and whether some new phenomenon or results in disease spread under different host movement behavior can occur? So considering the fact that susceptible and infectious hosts may disperse at different rates can help better understand the mechanisms of spread of infectious diseases. On the other hand, inspired by [26], Wu and Zou [28] used linear source growth term and bilinear incidence rate for the host (compare to the model (1.2)) to investigate the effect of spatial heterogeneity and distinct diffusion rates on the long term dynamics of diffusive host-pathogen models. They investigated the asymptotic profiles of steady states as one diffusion rate approaches zero and found an interesting phenomenon that the infected hosts will concentrate on certain points which can be characterized as the pathogen's most favored sites, provided that the dispersal rate of infected hosts is very small (see Section 4.2 in Wu and Zou [28]).

However, in the model (1.2) (and also the model in [28]), horizontal transmission is ignored (hence there is no infection pathway mechanism between susceptible and infectious hosts) so that even in the absence of pathogen, the host population would arrive at steady state level. To make things not too complicated, we modify a little bit in model (1.2) by adding horizontal transmission between susceptible and infectious hosts. With these considerations, we consider the following diffusive system

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S + r\left(1 - \dfrac{S+I}{K(x)}\right) S - \beta_1(x) SI - \beta_2(x) SP, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \beta_1(x) SI + \beta_2(x) SP - (b+c)I - r\dfrac{S+I}{K(x)} I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial P}{\partial t} = \lambda(x) I - mP - \beta_3(x)(S+I)P, & x \in \Omega,\ t > 0, \\ S(x,0) = S_0(x) \geq 0, I(x,0) = I_0(x) \geq 0, P(x,0) = P_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.3}$$

with

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0. \tag{1.4}$$

Here $S(x,t)$, $I(x,t)$ stand for the density of susceptible hosts, infective hosts at position x and time t, respectively, while $P(x,t)$ corresponds to the concentration of pathogen particles at position x and time t; $d_S > 0$ and $d_I > 0$ are positive constants measuring the mobility of susceptible and infected hosts, respectively. The horizontal transmission and the pathogen transmission are modeled by the mass action mechanism $\beta_1(x)SI$ and $\beta_2(x)SP$ with transmission rate $\beta_1(x)$ and $\beta_2(x)$. c and m are the natural death rate of infected hosts and the decay rate of pathogen particles. b is the rate of disease-induced mortality. $\lambda(x)$ is the rate of production of pathogen particles by infected hosts. $\beta_3(x)(S+I)P$ is the consumption of the pathogen particles; $K(x)$ is the carrying capacity. All the location-dependent parameters $\beta_1(x)$, $\beta_2(x)$, $\beta_3(x)$, $\lambda(x)$ and $K(x)$ of system (1.3) are continuous, strictly positive and uniformly bounded on $\overline{\Omega}$. The initial conditions, $(S_0(x), I_0(x), P_0(x)), x \in \overline{\Omega}$ are nonnegative continuous functions.

The organization of this paper is as follows. In Section 2, we will firstly summarize the well-posedness of (1.3), such as the existence of a unique mild solution of (1.3) and uniform boundedness of all solutions. In Section 3, we studied the extinction of the disease, the basic reproduction number and principal eigenvalue and prove uniform persistence of system (1.3) by theories of monotone dynamical systems and uniform persistence. We also carry out a bifurcation analysis for steady state solution of the system (1.3) by bifurcation theory. Section 5 is devoted to some detailed conclusions and discussions.

2. Well-posedness of the model

We first set $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^3)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Define its cone by $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^3_+)$, then $(\mathbb{X}, \mathbb{X}^+)$ is a strongly ordered Banach space. In this section, we aim to prove that the solution of the system (1.3) and (1.4) exist globally for $t \in [0, \infty)$ in $\mathbb{X}^+$.

To this end, we take advantage of a semigroup approach. Denote by Γ the Green function associated with $\frac{\partial \nu}{\partial t} = \Delta \nu$ in Ω subject to the Neumann boundary condition. Suppose that $A_1(t),\ A_2(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ are the C_0 semigroups associated with $d_S\Delta$ and $d_I\Delta - (b+c)$ subject to the Neumann boundary condition, respectively. Hence, we obtain that for any $\varphi \in C(\bar{\Omega}, \mathbb{R}),\ t \geq 0$,

$$(A_1(t)\varphi)(x) = \int\limits_{\Omega} \Gamma(d_S t, x, y)\varphi(y)dy,$$

and

$$(A_2(t)\varphi)(x) = e^{-(b+c)} \int_\Omega \Gamma(d_I t, x, y)\varphi(y)dy. \tag{2.1}$$

It then follows from [21, Section 7.1] that for any $t > 0$, $A_i(t) : C(\bar\Omega, \mathbb{R}) \to C(\bar\Omega, \mathbb{R}) (i = 1, 2)$ is strong positive and compact. Denote

$$(A_3(t)\varphi)(x) = e^{-mt}\varphi(x).$$

Hence, $\mathcal{A}(t) := (A_1(t), A_2(t), A_3(t)) : \mathbb{X} \to \mathbb{X},\ t \geq 0$, formulate a C_0 semigroup (see, for example, [19]).

Let $\mathcal{F} = (F_1, F_2, F_3) : \mathbb{X}^+ \to \mathbb{X}$ be defined by

$$\begin{aligned}
F_1(\phi)(x) &= r\left(1 - \frac{\phi_1 + \phi_2}{K(x)}\right)\phi_1 - \beta_1(x)\phi_1\phi_2 - \beta_2(x)\phi_1\phi_3,\\
F_2(\phi)(x) &= \beta_1(x)\phi_1\phi_2 + \beta_2(x)\phi_1\phi_3 - r\frac{\phi_1 + \phi_2}{K(x)}\phi_2,\\
F_3(\phi)(x) &= \lambda(x)\phi_2 - \beta_3(x)(\phi_1 + \phi_2)\phi_3,
\end{aligned}$$

for $x \in \bar\Omega$ and $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+$. It follows that (1.3) can be formulated as the following integral equation

$$u(t) = \mathcal{A}(t)\phi + \int_0^t \mathcal{A}(t-s)\mathcal{F}(u(\cdot, s))ds.$$

The following results concern the local solution of the system (1.3) and (1.4) on $\mathbb{X}^+$.

Lemma 2.1. *For any initial data $\phi := (\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+$, (1.3) with (1.4) admits a unique solution $u(\cdot, t; \phi) := (S(\cdot, t), I(\cdot, t), P(\cdot, t))$ on $(0, \tau_{max})$ with $u(\cdot, 0; \phi) = \phi$, where $\tau_{max} \leq \infty$. Furthermore, for $t \in (0, \tau_{max})$, $u(\cdot, t; \phi) \in \mathbb{X}^+$.*

Proof. Since $\mathcal{A}$ corresponds to the linear homogeneous part of (1.3) and the domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \{\phi : \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega,\ \mathcal{A}\phi \in C(\bar\Omega, \mathbb{R}^3)\}.$$

It is easy to check that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on $\mathbb{X}$. Let $\bar\beta_1 := \max_{\bar\Omega}\{\beta_1(x)\}$, $\bar\beta_2 := \max_{\bar\Omega}\{\beta_2(x)\}$, $\bar\beta_3 := \max_{\bar\Omega}\{\beta_3(x)\}$ and $\bar K := \min_{\bar\Omega}\{K(x)\}$. We can check that

$$\lim_{h\to 0^+} \operatorname{dist}(\phi + h\mathcal{F}(\phi), \mathbb{X}^+) = 0,\ \forall\ \phi \in \mathbb{X}^+. \tag{2.2}$$

In fact, for any $\phi \in \mathbb{X}$ and $h \geq 0$, we have

$$\begin{aligned}
\phi + h\mathcal{F}(\phi) &= \begin{pmatrix} \phi_1 + h\left(r\left(1 - \frac{\phi_1+\phi_2}{K(x)}\right)\phi_1 - \beta_1(x)\phi_1\phi_2 - \beta_2(x)\phi_1\phi_3\right) \\ \phi_2 + h\left(\beta_1(x)\phi_1\phi_2 + \beta_2(x)\phi_1\phi_3 - r\frac{\phi_1+\phi_2}{K(x)}\phi_2\right) \\ \phi_3 + h\left[\lambda(x)\phi_2 - \beta_3(x)(\phi_1 + \phi_2)\phi_3\right] \end{pmatrix}\\
&\geq \begin{pmatrix} \phi_1\left[1 - h\left(\frac{r}{\bar K}(\phi_1 + \phi_2) + \bar\beta_1\phi_2 + \bar\beta_2\phi_3\right)\right] \\ \phi_2\left[1 - h\frac{r}{\bar K}(\phi_1 + \phi_2)\right] \\ \phi_3[1 - h\bar\beta_3(\phi_1 + \phi_2)] \end{pmatrix}.
\end{aligned}$$

By [21, Corollary 4], (1.3) has a unique positive solution $(S(\cdot,t), I(\cdot,t), P(\cdot,t))$ on $(0, \tau_{max})$, where $0 < \tau_{max} \leq \infty$. □

In what follows, we prove that the local solution can be extended to a global one, that is $\tau_{max} = \infty$. To this end, we only need to prove that the solution is bounded in $\Omega \times (0, \tau_{max})$.

We first give the following lemma, which will be used later.

Lemma 2.2. *[30, Theorem 3.1.5] For any $d_W, r > 0$ and $W^0(x) \not\equiv 0$, the following diffusive logistic equation,*

$$\begin{cases} \dfrac{\partial W}{\partial t} = d_W \Delta W + r\left(1 - \dfrac{W}{K(x)}\right)W, & x \in \Omega,\ t > 0, \\ \dfrac{\partial W}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ W(x,0) = W^0(x), & x \in \Omega, \end{cases} \tag{2.3}$$

admits a unique positive steady state $W^*(x)$, which is globally asymptotically stable in $C(\bar{\Omega}, \mathbb{R})$.

Lemma 2.3. *For any initial data $\phi \in \mathbb{X}^+$, system (1.3) has a unique solution $u(\cdot,t;\phi) := (S(\cdot,t;\phi), I(\cdot,t;\phi), P(\cdot,t;\phi))$ on $[0,\infty)$ with $u(\cdot,t;\phi) = \phi$. The semiflow $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+$ generated by (1.3) is defined by*

$$\Phi(t)\phi = (S(\cdot,t;\phi), I(\cdot,t;\phi), P(\cdot,t;\phi)), \quad \forall x \in \bar{\Omega}, \quad t \geq 0. \tag{2.4}$$

Furthermore, $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+$ is point dissipative.

Proof. First, we prove that $S(x,t)$ is ultimately bounded. It follows from the third equation of (1.3), it is easy to see that $S(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial S}{\partial t} \leq d_S \Delta S + r\left(1 - \dfrac{S}{K(x)}\right)S, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.5}$$

From (2.3), we know (2.5) is bounded and the standard parabolic comparison theorem implies that $S(x,t)$ is uniformly bounded. Further, from Lemma 2.1 and the comparison principle, we have

$$\limsup_{t\to\infty} S(x,t) \leq W^*(x), \text{ uniformly } \text{ for } x \in \bar{\Omega}, \tag{2.6}$$

that is, $S(x,t)$ is ultimately bounded in the sense that $\|S(x,t)\| \leq M_0$ for some positive constant M_0. By using the divergence theorem, we integrate first two equations of (1.3) and adding them up yields

$$\begin{aligned} \frac{\partial}{\partial t}\int_\Omega (S(x,t) + I(x,t))dx &= \int_\Omega rS dx - \int_\Omega r\frac{(S+I)^2}{K(x)}dx - \int_\Omega (b+c)I dx \\ &\leq \int_\Omega r(S+I)dx - \int_\Omega r\frac{(S+I)^2}{K(x)}dx. \end{aligned}$$

It follows that

$$\limsup_{t\to\infty}(\|S(x,t)\|_1 + \|I(x,t)\|_1) \leq M_{11},$$

with $M_{11} = |\Omega|\overline{r}/K_*$. Thus, the solution (S, I) of (1.3) satisfies the L^1 bounded estimate. Then, from the third equation of (1.3), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega Pdx &= \int_\Omega \lambda(x)Idx - \int_\Omega mPdx - \int_\Omega \beta_3(x)(S+I)Pdx \\ &\le \lambda^* M_{11} - m\int_\Omega Pdx.\end{aligned}$$

It follows that

$$\limsup_{t\to\infty}(\|P(x,t)\|_1) \le M_{12},$$

i.e., the solution P of (1.3) satisfies the L^1 bounded estimate.

Through the above, we can get there exists a positive constant M_1, such that

$$\limsup_{t\to\infty}(\|S(x,t)\|_1 + \|I(x,t)\|_1 + \|P(x,t)\|_1) \le M_1. \tag{2.7}$$

Thus, the solution of (1.3) satisfies the L^1 bounded estimate. Recall that $\limsup_{t\to\infty}\|S(\cdot,t)\| \le M_0$. Next we only need to verify the solution (I, P) of (1.3) to be ultimately bounded. To this end, we first verify it satisfies the L^{2^k} bounded estimate, that is, for $k \ge 0$, there exists a positive constant M_{2^k} independent of $u_0 = (S^0(x), I^0(x), P^0(x)) \in \mathbb{X}^+$ such that it satisfies the following estimate

$$\limsup_{t\to\infty}\left(\|I(\cdot,t)\|_{2^k} + \|P(\cdot,t)\|_{2^k}\right) \le M_{2^k}, \quad \forall\ t > T, \tag{2.8}$$

for some large time $T > 0$. We will prove (2.8) holds by the method of induction. The case for $k = 0$ is valid in (2.7). We now assume that (2.8) is true for $k-1$, that is, there exists $M_{2^{k-1}} > 0$ such that

$$\limsup_{t\to\infty}\left(\|I(\cdot,t)\|_{2^{k-1}} + \|P(\cdot,t)\|_{2^{k-1}}\right) \le M_{2^{k-1}}, \forall\ t > T. \tag{2.9}$$

Multiplying the second equation of (1.3) by I^{2^k-1} and integrating over Ω, we get

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}dx \le d_I\int_\Omega I^{2^k-1}\Delta Idx + \int_\Omega \beta_1(x)SI^{2^k}dx + \int_\Omega \beta_2(x)SI^{2^k-1}Pdx - \int_\Omega (b+c)I^{2^k}dx. \tag{2.10}$$

Recall that

$$\begin{aligned}d_I\int_\Omega I^{2^k-1}\triangle Idx \le -d_I\int_\Omega \nabla I\cdot\nabla I^{2^k-1}dx =& -(2^k-1)d_I\int_\Omega(\nabla I\cdot\nabla I)I^{2^k-2}dx \\ =& -\frac{2^k-1}{2^{2k-2}}d_I\int_\Omega |\nabla I^{2^{k-1}}|^2dx.\end{aligned}$$

Hence (2.10) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}dx \le -D_k\int_\Omega |\nabla I^{2^{k-1}}|^2dx + \int_\Omega \beta_1(x)SI^{2^k}dx + \int_\Omega \beta_2(x)SPI^{2^k-1}dx - \int_\Omega (b+c)I^{2^k}dx, \tag{2.11}$$

where $D_k = \frac{2^k-1}{2^{2k-2}}d_I$.

By $\limsup_{t\to\infty} \|S(\cdot,t)\| \le M_0$, there exists $t_0 > 0$ such that

$$\int_\Omega \beta_1(x) S I^{2^k} dx \le \overline{\beta_1}(M_0+1)\int_\Omega I^{2^k} dx, \quad \text{for} \quad t \ge t_0,$$

and

$$\int_\Omega \beta_2(x) S I^{2^k-1} P dx \le \overline{\beta_2}(M_0+1)\int_\Omega P I^{2^k-1} dx, \quad \text{for} \quad t \ge t_0. \tag{2.12}$$

Applying Young's inequality: $ab \le \epsilon a^p + \epsilon^{-\frac{q}{p}} b^q$, where $a, b, \epsilon > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. One can estimate (2.12) by setting $\epsilon_1 = \frac{m}{4\overline{\beta_2}(M_0+1)}$, $p = 2^k$ and $q = 2^k/(2^k-1)$ as follows,

$$\int_\Omega P I^{2^k-1} dx \le \frac{m}{4\overline{\beta_2}(M_0+1)}\int_\Omega P^{2^k} dx + C_{\epsilon_1}\int_\Omega I^{2^k} dx, \quad \text{for} \quad t \ge t_0, \text{where } C_{\epsilon_1} = \epsilon_1^{-\frac{1}{2^k-1}}.$$

Thus, (2.11) can be estimated by

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k} dx \le -D_k\int_\Omega |\nabla I^{2^{k-1}}|^2 dx + \frac{m}{4}\int_\Omega P^{2^k} dx + C_k\int_\Omega I^{2^k} dx, \tag{2.13}$$

where $C_k = \overline{\beta_1}(M_0+1) + \overline{\beta_2}(M_0+1)C_{\epsilon_1}$.

Multiplying the third equation of (1.3) by P^{2^k-1} and integrating over Ω, we get

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega P^{2^k} dx \le \int_\Omega \bar{\lambda} P^{2^k-1} I dx - \int_\Omega m P^{2^k} dx. \tag{2.14}$$

Again applying Young's inequality (by setting $\epsilon_2 = \frac{m}{4\bar{\lambda}}$, $p = 2^k/(2^k-1)$ and $q = 2^k$), we have

$$\int_\Omega P^{2^k-1} I dx \le \frac{m}{4\bar{\lambda}}\int_\Omega P^{2^k} dx + C_{\epsilon_2}\int_\Omega I^{2^k} dx, \text{where } C_{\epsilon_2} = \epsilon_2^{1-2^k}.$$

Hence (2.14) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega P^{2^k} dx \le -\frac{3m}{4}\int_\Omega P^{2^k} dx + \bar{\lambda} C_{\epsilon_2}\int_\Omega I^{2^k} dx. \tag{2.15}$$

Consequently, from (2.13) and (2.15), we have

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega (I^{2^k} + P^{2^k}) dx \le -D_k\int_\Omega |\nabla I^{2^{k-1}}|^2 dx + E_k\int_\Omega I^{2^k} dx - \frac{m}{2}\int_\Omega P^{2^k} dx, \tag{2.16}$$

where $E_k = C_k + cC_{\epsilon_2}$.

Applying interpolation inequality: for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\|\xi\|_2^2 \le \epsilon\|\nabla\xi\|_2^2 + C_\epsilon\|\xi\|_1^2, \text{where } \xi \in W^{1,2}(\Omega).$$

Let $\epsilon_3 = D_k/(2E_k)$, $\xi = I^{2^{k-1}}$, then

$$-D_k\int\limits_{\Omega}|\nabla I^{2^{k-1}}|^2dx \le -2E_k\int\limits_{\Omega} I^{2^k}dx + 2E_kC_{\epsilon_3}\left(\int\limits_{\Omega} I^{2^{k-1}}dx\right)^2.$$

Thus (2.16) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int\limits_{\Omega}(I^{2^k}+P^{2^k})dx \le -r_*\int\limits_{\Omega}(I^{2^k}+P^{2^k})dx + 2E_kC_{\epsilon_3}\left(\int\limits_{\Omega} I^{2^{k-1}}dx\right)^2,$$

where $r_* = \min\{E_k, \frac{m}{2}\}$.

It then follows from (2.8) that $\limsup_{t\to\infty}\int\limits_{\Omega} I^{2^{k-1}}dx \le M_{2^{k-1}}^{2^{k-1}}$, which in turn implies that

$$\limsup_{t\to\infty}(\|I(\cdot,t)\|_{2^k}+\|P(\cdot,t)\|_{2^k}) \le M_{2^k}, \text{with } M_{2^k} = \sqrt[2^k]{\frac{2E_kC_{\epsilon_3}}{r_*}}M_{2^{k-1}}.$$

Thus, according to continuous embedding $L^q(\Omega)\subset L^p(\Omega)$, $q\ge p\ge 1$, we can conclude that for any $p>1$, there exists a positive constant M_p, independent of initial conditions, such that

$$\limsup_{t\to\infty}(\|I(\cdot,t)\|_p+\|I(\cdot,t)\|_p)\le M_p.$$

By using the same arguments as those in [28, Lemma 2.4], we have that there exists a positive constant M_∞ such that $\limsup_{t\to\infty}\|I(\cdot,t)\|\le M_\infty$, $\limsup_{t\to\infty}\|P(\cdot,t)\|\le M_\infty$. Thus, the solution exists globally for all $t\in[0,\infty)$, and moreover, $\Phi(t):\mathbb{X}^+\to\mathbb{X}^+$ is point dissipative. □

In what follows, we consider asymptotic smoothness of the solution semiflow $\Phi(t)$, as there is no diffusion term in the third equation in (1.3). To overcome this problem, we introduce the Kuratowski measure of noncompactness, $\kappa(\cdot)$,

$$\kappa(\mathbb{B}) := \inf\{r : \mathbb{B} \text{ has a finite cover of diameter } < r\},$$

for any bounded set $\mathbb{B}$. Then $\mathbb{B}$ is precompact if and only if $\kappa(\mathbb{B}) = 0$. We next claim that $\Phi(t)$ is a κ-contraction in the sense that there exists a continuous function $k(t):\mathbb{R}^+\to\mathbb{R}^+$ with $0\le k(t)<1$ such that for any $t>0$ and bounded set $\mathbb{B}$, $\{\Phi(s)\mathbb{B}, 0\le s\le t\}$ is bounded and $\kappa(\Phi(t)\mathbb{B})\le k(t)\kappa(\mathbb{B})$.

Lemma 2.4. $\Phi(t)$ *is κ-contracting in the sense that*

$$\lim_{t\to\infty}\kappa(\Phi(t)\mathbb{B}) = 0 \quad \textit{for any bounded set } \mathbb{B}\subset\mathbb{X}^+.$$

Proof. Let

$$G(S,I,P) = -mP + \lambda(x)I - \beta_3(x)(S+I)P$$

be the right hand of the third equation of (1.3). Then

$$\frac{\partial G(S,I,P)}{\partial P} = -m - \beta_3(x)(S+I) \le -m, \quad (S,I,P)\in\mathbb{X}^+.$$

Based on this fact, $\Phi(t)$ can be decomposed as $\Phi(t)=\Phi_1(t)+\Phi_2(t),\ t\ge 0$, where

$$\Phi_1(t)u_0 = \left\{ S(\cdot,t;u_0), I(\cdot,t;u_0), \int\limits_0^t e^{\int_s^t (m+S(\cdot,l)+I(\cdot,l))dl} \lambda(x) I(\cdot,s;u_0)ds \right\},\ t \geq 0,$$

and

$$\Phi_2(t)u_0 = \left\{0, 0, e^{-\int_0^t (m+S(\cdot,l)+I(\cdot,l))dl} P^0(x)\right\}, t \geq 0.$$

By Lemma [28, Lemma 2.5], $\Phi_1(t)\mathbb{B}$ is precompact for any $t > 0$. Hence, $\kappa(\Phi_1(t)\mathbb{B}) = 0$. Moreover, the operator norm of $\Phi_2(t)$ can be estimated as

$$\|\Phi_2(t)\| \;=\; \sup_{\psi\in\mathbb{X}} \frac{\|\Phi_2(t)\psi\|_{\mathbb{X}}}{\|\psi\|_{\mathbb{X}}} \;\leq\; \mathrm{e}^{-mt} \sup_{\psi\in\mathbb{X}} \frac{\|\psi\|_{\mathbb{X}}}{\|\psi\|_{\mathbb{X}}} \;=\; \mathrm{e}^{-mt}.$$

It then follows that for $t > 0$,

$$\kappa(\Phi(t)\mathbb{B}) \leq \kappa(\Phi_1(t)\mathbb{B}) + \kappa(\Phi_2(t)\mathbb{B}) \leq 0 + \|\Phi_2(t)\|\, \kappa(\mathbb{B}) \leq \mathrm{e}^{-mt}\kappa(\mathbb{B}).$$

Thus, $\Phi(t)$ is a κ-contraction on $\mathbb{X}^+$ with the contraction function e^{-mt}. □

The following result reveals that solutions of system (1.3) converge to a compact attractor in $\mathbb{X}^+$, which is just a consequence of applying the general results in [7, Theorem 2.4.6].

Theorem 2.1. $\Phi(t)$ *admits a connected global attractor on* $\mathbb{X}^+$.

3. Threshold dynamics

3.1. Extinction

Let $T(t) : C(\bar{\Omega}, \mathbb{R}^2) \to C(\bar{\Omega}, \mathbb{R}^2)$ be the semigroup associated to the following linear problem:

$$\begin{cases} \dfrac{\partial I}{\partial t} = d_I \Delta I + \beta_1(x)S^0(x)I + \beta_2(x)S^0(x)P - (b+c)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial P}{\partial t} = \lambda(x)I - mP, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ I(x,0) = I_0(x), P(x,0) = P_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Since (3.1) is cooperative, $T(t)$ is a positive C_0-semigroup on $C(\bar{\Omega}, \mathbb{R}^2)$. It is easy to see that $T(t)$ has the generator

$$\mathcal{B} = \begin{pmatrix} d_I\Delta + \beta_1(x)S^0(x) - (b+c) & \beta_2(x)S^0(x) \\ \lambda(x) & -m \end{pmatrix}.$$

It then follows from [25, Theorem 3.5] that $\mathcal{B}$ is a closed and resolvent positive operator.

Substituting $I(x,t) = e^{\mu t}\psi_2(x)$ and $P(x,t) = e^{\mu t}\psi_3(x)$ into the first and two equations of (3.1), we can get

$$\begin{cases} \mu\psi_2 = d_I \Delta\psi_2 + \beta_1(x)S^0(x)\psi_2 + \beta_2(x)S^0(x)\psi_3 - (b+c)\psi_2, & x \in \Omega, \\ \mu\psi_3 = \lambda(x)\psi_2 - m\psi_3, & x \in \Omega, \\ \dfrac{\partial \psi_2}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

The existence of the principle eigenvalue of (3.2) is described by the following result.

Lemma 3.1. *Let $s(\mathcal{B}) = \sup\{Re\lambda, \lambda \in \sigma(\mathcal{B})\}$ be the spectral bound of $\mathcal{B}$. Then the following statements hold.*

(i) *$s(\mathcal{B})$ is the principal eigenvalue of the eigenvalue problem (3.2) associated with a strongly positive eigenfunction;*
(ii) *$s(\mathcal{B})$ has the same as ξ_0, where ξ_0 is the principal eigenvalue of the eigenvalue problem*

$$\begin{cases} d_I \Delta\varphi + \left(\beta_1(x)S^0(x) + \dfrac{\beta_2(x)\lambda(x)S^0(x)}{m} - (b+c)\right)\varphi = \xi\varphi, & x \in \Omega, \\ \dfrac{\partial \varphi(x)}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

Proof. Proof of (i). We define an one-parameter family of linear operators on $C(\bar{\Omega}, \mathbb{R})$:

$$L_\mu = d_I \Delta + \beta_1(x)S^0(x) - (b+c) + \frac{\beta_2(x)\lambda(x)S^0(x)}{\mu+m}, \ \mu > -m.$$

Let $C_1 := \min_{x\in\bar{\Omega}}\{\beta_1(x)S^0(x)\} > 0, \ C_2 := \min_{x\in\bar{\Omega}}\{\beta_2(x)\lambda(x)S^0(x)\} > 0$. Recall that the following eigenvalue problem

$$\begin{cases} \hat{\eta}\varphi = d_I \Delta\varphi - (b+c)\varphi, & x \in \Omega, \\ \dfrac{\partial \varphi}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

admits one principle eigenvalue, $\hat{\eta}^0 = -(b+c)$, with an associated eigenvector $\varphi^0 \gg 0$. Denote by $\mu^* = \frac{1}{2}[(\hat{\eta}^0 - m + C_1) + \sqrt{(\hat{\eta}^0 + m + C_1)^2 + 4C_2}$ the larger root of the following algebraic equation

$$\mu^2 + (m - C_1 - \hat{\eta}^0)\mu - (C_2 + m(C_1 + \hat{\eta}^0)) = 0.$$

It follows that $\mu^* > -m$ and

$$\begin{aligned} L_{\mu^*}\varphi^0 &= d_I \Delta\varphi^0 + \beta_1(x)S^0(x)\varphi^0 - (b+c)\varphi^0 + \frac{\beta_2(x)\lambda(x)S^0(x)}{\mu+m}\varphi^0 \\ &\geq \left(\hat{\eta}^0 + C_1 + \frac{C_2}{\mu^*+m}\right)\varphi^0 = \mu^*\varphi^0. \end{aligned}$$

With the aid of [29, Theorem 2.3 (i)], we complete the proof of (i).

Proof of (ii). The result directly follows from [29, Theorem 2.3 (ii)], that is, $s(\mathcal{B})$ has the same sign as $s(L_0)$, where

$$L_0 = d_I \Delta + \beta_1(x)S^0(x) - (b+c) + \frac{\beta_2(x)\lambda(x)S^0(x)}{m}.$$

This completes the proof. $\square$

It is easy to see that system (1.3) has a trivial equilibrium at $M_1(0,0,0)$ and a disease-free equilibrium at $M_2(S^0(x),0,0)$, where $S^0(x) = W^*(x)$ is the unique positive solution of (2.3) which is globally asymptotically stable in $C(\bar{\Omega},\mathbb{X})$ for the dynamics of (2.3).

The following result reveals that $s(\mathcal{B})$ is a threshold for disease extinction.

Theorem 3.1. *If $s(\mathcal{B}) < 0$, then the disease-free equilibrium $(S^0(x),0,0)$ is global attractive for the system (1.3), that is, for any initial data $\phi \in \mathbb{X}^+$, we have*

$$\lim_{t\to\infty} \|(S(x,t),I(x,t),P(x,t)) - (S^0(x),0,0)\| = 0.$$

Proof. We fix $\epsilon_0 > 0$. It then follows from (2.6) that there exists $t_0 > 0$ such that $0 \le S(\cdot,t) \le S^0(x) + \epsilon_0$ for all $t \ge t_0$. It then follows from the comparison principle for cooperative systems (see e.g., [12]) that $(I(x,t),P(x,t)) \le (\hat{I}(x,t),\hat{P}(x,t))$ on $\bar{\Omega} \times [t_0,\infty)$, where $(\hat{I}(x,t),\hat{P}(x,t))$ satisfies

$$\begin{cases} \dfrac{\partial \hat{I}}{\partial t} = d_I\Delta\hat{I} + \beta_1(x)(S^0(x)+\epsilon_0)\hat{I} + \beta_2(x)(S^0(x)+\epsilon_0)\hat{P} - (b+c)\hat{I}, & x\in\Omega, \quad t>t_0,\\ \dfrac{\partial \hat{P}}{\partial t} = \lambda(x)\hat{I} - m\hat{P}, & x\in\Omega, \quad t>t_0,\\ \dfrac{\partial \hat{I}}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>t_0. \end{cases} \tag{3.4}$$

Since $s(\mathcal{B}) < 0$, there exists a small $\varepsilon_0 > 0$ such that $s(\mathcal{B}_{\varepsilon_0}) < 0$ and it corresponded to an associated eigenvector $(\psi_2^{\varepsilon_0}(x),\psi_3^{\varepsilon_0}(x)) \gg 0$. Suppose that for any given $\phi \in \mathbb{X}^+$, there exists some $\alpha > 0$ such that $(I(x,t_0;\phi),P(x,t_0;\phi)) \le \alpha(\psi_2^{\varepsilon_0}(x),\psi_3^{\varepsilon_0}(x))$, for all $x \in \bar{\Omega}$. Recall that (3.4) admits a solution $\alpha e^{s(\mathcal{B}_{\varepsilon_0})(t-t_0)}(\psi_2^{\varepsilon_0}(x),\psi_3^{\varepsilon_0}(x))$, for all $t \ge t_0$. The comparison principle implies that

$$(I(x,t_0;\phi),P(x,t_0;\phi)) \le \alpha e^{s(\mathcal{B}_{\varepsilon_0})(t-t_0)}(\psi_2^{\varepsilon_0}(x),\psi_3^{\varepsilon_0}(x)), \quad t\ge t_0.$$

It follows that $(\hat{I}(x,t),\hat{P}(x,t)) \to (0,0)$ as $t\to\infty$ uniformly for $x\in\overline{\Omega}$. Therefore, we have $(I(x,t),P(x,t)) \to (0,0)$ as $t\to\infty$ uniformly for $x\in\overline{\Omega}$. Moreover, from (2.3), we conclude that $S(x,t)\to S^0(x)$ as $t\to\infty$ uniformly for $x\in\overline{\Omega}$. This completes the proof on the global attractivity of M_2. □

3.2. Basic reproduction number and principal eigenvalue

In what follows, we pay attention to original system (1.3). Linearizing system (1.3) at $(S^0(x),0,0)$, we get

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S\Delta S + r\left(1-\dfrac{2S^0(x)}{K(x)}\right)S - \left(\dfrac{r}{K(x)}+\beta_1(x)\right)S^0(x)I - \beta_2(x)S^0(x)P, & x\in\Omega,\ t>0,\\ \dfrac{\partial I}{\partial t} = d_I\Delta I + \beta_1(x)S^0(x)I + \beta_2(x)S^0(x)P - (b+c)I - r\dfrac{S^0(x)}{K(x)}I, & x\in\Omega,\ t>0,\\ \dfrac{\partial P}{\partial t} = \lambda(x)I - mP - \beta_3(x)S^0(x)P, & x\in\Omega,\ t>0,\\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ S(x,0)=S_0(x), I(x,0)=I_0(x), P(x,0)=P_0(x), & x\in\Omega. \end{cases} \tag{3.5}$$

It is easy to observe that equations for variables I and P in (3.5) are decoupled with the equations of S. Then we first consider the following subsystem (which is cooperative):

$$\begin{cases} \dfrac{\partial I}{\partial t} = d_I \Delta I + \beta_1(x)S^0(x)I + \beta_2(x)S^0(x)P - (b+c)I - r\dfrac{S^0(x)}{K(x)}I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial P}{\partial t} = \lambda(x)I - mP - \beta_3(x)S^0(x)P, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ I(x,0) = I^0(x), I(x,0) = I^0(x), & x \in \Omega. \end{cases} \tag{3.6}$$

Denote by $\Pi(t)$ the solution semiflow of (3.6) on $C(\bar{\Omega}, \mathbb{R}^2)$ with generator

$$\mathcal{B}_{S^0} = \begin{pmatrix} d_I\Delta + \beta_1(x)S^0(x) - (b+c) - r\frac{S^0(x)}{K(x)} & \beta_2(x)S^0(x) \\ \lambda(x) & -m - \beta_3(x)S^0(x) \end{pmatrix} := B + F, \tag{3.7}$$

where

$$B = \begin{pmatrix} d_I\Delta - (b+c) - r\frac{S^0(x)}{K(x)} & 0 \\ \lambda(x) & -m - \beta_3(x)S^0(x) \end{pmatrix}, \ F = \begin{pmatrix} \beta_1(x)S^0(x) & \beta_2(x)S^0(x) \\ 0 & 0 \end{pmatrix}.$$

We can check that $\mathcal{B}_{S^0}$ and B are resolvent-positive operators. Let $\tilde{T}(t) : C(\bar{\Omega}, \mathbb{R}^2) \to C(\bar{\Omega}, \mathbb{R}^2)$ be the C_0-semigroup generated B. It follows that B is cooperative for any $x \in \Omega$, which implies that $\tilde{T}(t)$ is a positive semigroup in the sense that $\tilde{T}(t)C(\bar{\Omega}, \mathbb{R}^2_+) \subseteq C(\bar{\Omega}, \mathbb{R}^2_+)$.

It follows that the next generation operator $\mathcal{L} := -FB^{-1}$ takes the following form

$$\mathcal{L}\phi(x) = \int_0^\infty F(x)\tilde{T}(t)\phi(x)dt = F(x)\int_0^\infty \tilde{T}(t)\phi(x)dt \quad \phi \in C(\overline{\Omega}, \mathbb{R}^2), \quad x \in \overline{\Omega}.$$

Then $\mathcal{L}$ is well-defined, continuous, and positive operator on $C(\bar{\Omega}, \mathbb{R}^2)$, which maps the initial infection distribution ϕ to the distribution of the total new infections produced during the infection period. We then follow the procedure in [29] to define the spectral radius of $\mathcal{L}$ as the basic reproduction number

$$\Re_0 := r(\mathcal{L}) = \sup\{|\lambda|, \lambda \in \sigma(L)\},$$

where $\sigma(\mathcal{L})$ is the spectrum of $\mathcal{L}$. By the general results in [25] and the same arguments as in [29, Lemma 2.2], we have the following result.

Lemma 3.2. *Then $\Re_0 - 1$ has the same sign as $s(\mathcal{B}_{S^0})$.*

Lemma 3.3. *Let $\tilde{\lambda}_0$ be the principal eigenvalue of the problem*

$$\begin{cases} d_I\Delta\varphi - \left(b + c + r\dfrac{S^0(x)}{K(x)}\right)\varphi + \tilde{\lambda}\left(\beta_1(x)S^0(x) + \dfrac{\lambda(x)\beta_2(x)S^0(x)}{m + \beta_3(x)S^0(x)}\right)\varphi = 0, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases}$$

Then $\Re_0 = 1/\tilde{\lambda}_0$.

Proof. In fact, the operator $-FB^{-1}$ can be computed as

$$-FB^{-1}\psi = -\begin{pmatrix} \beta_1(x)S^0(x) & \beta_2 S^0(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (d_I\Delta - (b+c) - r\frac{S^0(x)}{K(x)})^{-1} & 0 \\ \frac{\lambda(x)(d_I\Delta - (b+c) - r\frac{S^0(x)}{K(x)})^{-1}}{m + \beta_3(x)S^0(x)} & -\frac{1}{m+\beta_3(x)S^0(x)} \end{pmatrix}\psi$$

$$= \begin{pmatrix} -\left(\beta_1(x)S^0(x) + \frac{\lambda(x)\beta_2 S^0(x)}{m+\beta_3(x)S^0(x)}\right)\left(d_I\Delta - (b+c) - r\frac{S^0(x)}{K(x)}\right)^{-1} & \frac{\beta_2(x)S^0(x)}{m+\beta_3(x)S^0(x)} \\ 0 & 0 \end{pmatrix}\psi.$$

So $\Re_0 := r(\mathcal{L}) = r(-(\beta_1(x)S^0(x) + \frac{\lambda(x)\beta_2(x)S^0(x)}{m+\beta_3(x)S^0(x)})(d_I\Delta - (b+c) - r\frac{S^0(x)}{K(x)})^{-1})$. Therefore, $\Re_0$ is the principle eigenvalue of

$$-\left(\beta_1(x)S^0(x) + \frac{\lambda(x)\beta_2(x)S^0(x)}{m+\beta_3(x)S^0(x)}\right)\left(d_I\Delta - (b+c) - r\frac{S^0(x)}{K(x)}\right)^{-1}\varphi = \Re_0\varphi,\ \varphi \in C^2(\overline{\Omega})$$

that is,

$$d_I\Delta\varphi - \left(b+c+r\frac{S^0(x)}{K(x)}\right)\varphi + \left(\beta_1(x)S^0(x) + \frac{\lambda(x)\beta_2(x)S^0(x)}{m+\beta_3(x)S^0(x)}\right)\frac{1}{\Re_0}\varphi = 0,\ \varphi \in C^2(\overline{\Omega}),$$

which completes the proof. $\square$

Remark 3.1. From Lemma 3.3, $\Re_0$ have the following variational formula:

$$\Re_0 = \frac{1}{\tilde{\lambda}_0} = \sup_{\phi \in H^1(\Omega),\ \phi \neq 0} \left\{ \frac{\int_\Omega (\beta_1(x)S^0(x) + \frac{\lambda(x)\beta_2(x)S^0(x)}{m+\beta_3(x)S^0(x)})\phi^2 \mathrm{d}x}{\int_\Omega \left(d_I |\nabla\phi|^2 + (b+c+r\frac{S^0(x)}{K(x)})\phi^2\right)\mathrm{d}x} \right\}. \tag{3.8}$$

From (3.8), we can easily get the information that how $\Re_0$ depends on the diffusion coefficient d_I.

Remark 3.2. When all parameters in (1.3) are constant, one can easily see that $S^0(x) = K$, and $\Re_0$ can be reduced to

$$\Re_0^{const.} = \frac{1}{\tilde{\lambda}_0} = \left(\beta_1 K + \frac{\beta_2\lambda K}{m+\beta_3 K}\right)/(b+c+r). \tag{3.9}$$

Substituting $I(x,t) = e^{\Lambda t}\psi_2(x),\ P(x,t) = e^{\Lambda t}\psi_3(x)$, into (3.6), we obtain

$$\begin{cases} \Lambda\psi_2 = d_I\Delta\psi_2 + \beta_1(x)S^0(x)\psi_2 + \beta_2(x)S^0(x)\psi_3 - (b+c)\psi_2 - r\dfrac{S^0(x)}{K(x)}\psi_2, & x \in \Omega, \\ \Lambda\psi_3 = \lambda(x)\psi_2 - m\varphi_3 - \beta_3(x)S^0(x)\psi_3, & x \in \Omega, \\ \dfrac{\partial\psi_2}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

The existence of the principal eigenvalue of (3.10) are stated in the following result.

Lemma 3.4. *Let $\mathcal{B}_{S^0}$ be defined in (3.7) and $s(\mathcal{B}_{S^0})$ be the spectral bound. If $s(\mathcal{B}_{S^0}) \geq 0$, then $s(\mathcal{B}_{S^0})$ is the principal eigenvalue of eigenvalue problem (3.10) associated with a strongly positive eigenfunction.*

Proof. It follows from (3.6) that

$$\begin{cases} I(\cdot,t,\phi) = A_2(t)\phi_2 + \displaystyle\int_0^t A_2(t-s)g(I(\cdot,s,\phi),P(\cdot,s,\phi))ds, \\ P(\cdot,t,\phi) = \tilde{A}_3(t)\phi_3 + \displaystyle\int_0^t \tilde{A}_3(t)(t-s)[\lambda(\cdot)I(\cdot,s,\phi)]ds, \end{cases}$$

where $g(I,P) = \left(-\frac{r}{K(x)}S^0(\cdot) + \beta_1(\cdot)S^0(\cdot)\right)I + \beta_2(\cdot)S^0(\cdot)P$, and $\tilde{A}_3(t)\phi_3 = e^{-(m+\beta_3(\cdot)S^0(\cdot))t}\phi_3$ for $\phi_3 \in C(\bar{\Omega}, \mathbb{R})$.

We define a linear operator

$$\Pi_2(t)\phi = (0, \tilde{A}_3(t)\phi_3), \quad \phi = (\phi_2, \phi_3) \in C(\bar{\Omega}, \mathbb{R}^2), \tag{3.11}$$

and a nonlinear operator

$$\Pi_3(t)\phi = \left(I(\cdot, t, \phi), \int_0^t \tilde{A}_3(t)(t-s)[\lambda(\cdot)I(\cdot, s, \phi)]ds \right), \quad \phi = (\phi_2, \phi_3) \in C(\bar{\Omega}, \mathbb{R}^2).$$

It is easy to see that $\Pi(t) = \Pi_2(t) + \Pi_3(t)$. Similar to Lemma 2.4, we know that $\Pi_3(t)$ is compact. Hence, from (3.11), we have

$$\sup_{\phi \in C(\bar{\Omega}, \mathbb{R}^2), \|\phi\| \neq 0} \frac{\|\Pi_2(t)\phi\|}{\|\phi\|} \leq \sup_{\phi \in C(\bar{\Omega}, \mathbb{R}^2), \|\phi\| \neq 0} \frac{\|e^{-(m+\beta_3(\cdot)S_0(\cdot))t\phi_3}\|}{\|\phi\|} \leq \sup_{\phi \in C(\bar{\Omega}, \mathbb{R}^2), \|\phi\| \neq 0} \frac{\|e^{-mt\phi_3}\|}{\|\phi\|} \leq e^{-mt},$$

and hence $\|\Pi_2(t)\| \leq e^{-mt}$.

It follows that for any bounded set $\mathbb{B}$ in $C(\bar{\Omega}, \mathbb{R}^2)$, there holds

$$\kappa(\Pi(t)\mathbb{B}) \ \leq \ \kappa(\Pi_2(t)\mathbb{B}) + \kappa(\Pi_3(t)\mathbb{B}) \ \leq \ \|\Pi_2(t)\|\kappa(\mathbb{B}) \ \leq \ e^{-mt}\kappa(\mathbb{B}), \quad t > 0.$$

Thus, $\Pi(t)$ is a κ-contraction on $C(\bar{\Omega}, \mathbb{R}^2)$ with a contracting function e^{-mt}, that is, the essential spectra radius, $\omega_{ess}(\Pi(t)) \leq -m$. Here $\omega_{ess}(\Pi(t)) := \lim_{t\to\infty} \frac{\alpha(\Pi(t))}{t}$. Here $\alpha(\cdot)$ is the measure of non-compactness. Recall that

$$\omega_{S^0} = \max\{s(\mathbb{B}_{S^0}), \omega_{ess}(\Pi(t))\},$$

where ω_{S^0}, defined as $\omega_{S^0} := \lim_{t\to\infty} \frac{\ln \|\Pi(t)\|}{t}$, is the exponential growth bound of $\Pi(t)$ such that

$$\|\Pi(t)\| \leq Me^{\omega_{S^0}t}, \text{for some } M > 0.$$

On the other hand, the spectral radius $r(\Pi(t))$ of $\Pi(t)$ satisfies

$$r(\Pi_t) \ = \ e^{s(\mathcal{B}_{S^0})t} \ \geq \ 1, \ \text{when } s(\mathcal{B}_{S^0}) \geq 0, \ t > 0.$$

This implies that $\omega_{ess}(\Pi(t)) < r(\Pi_t)$ for any $t > 0$. The result directly follows from a generalized Krein-Rutman Theorem [13]. □

3.3. Persistence

The following result concerns the disease persistence when $s(\mathcal{B}_{S^0}) > 0$. To this end, we set

$$\mathbb{X}_0 := \left\{\phi \in \mathbb{X}^+ : \ \phi_1(\cdot) \not\equiv 0 \ \text{ and } \ \phi_2(\cdot) \not\equiv 0\right\},$$

and

$$\partial\mathbb{X}_0 := \mathbb{X}^+ \backslash \mathbb{X}_0 = \left\{\phi \in \mathbb{X}^+ : \ \phi_1(\cdot) \equiv 0 \ \text{ or } \ \phi_2(\cdot) \equiv 0\right\}.$$

Then $\mathbb{X} = \mathbb{X}_0 \cup \partial\mathbb{X}_0$, $\mathbb{X}_0$ is relatively open with $\overline{\mathbb{X}}_0 = \mathbb{X}$, and $\partial\mathbb{X}_0$ is relatively closed in $\mathbb{X}$. Let $\Phi(t)$ be defined by (2.4). Set $M_\partial = \{\phi \in \partial\mathbb{X}_0 : \Phi(t)\phi \in \partial\mathbb{X}_0, \forall\ t \geq 0\}$ and let $\omega(\phi)$ be the omega limit set of the forward orbit $\gamma^+(\phi) := \{\Phi(t)(\phi) : t \geq 0\}$.

Theorem 3.2. *If $s(\mathcal{B}_{S^0}) > 0$ (or $\Re_0 > 1$), then (1.3) is uniformly persistent in the sense there exists $\delta > 0$ such that for any $\phi \in \mathbb{X}^+$ with $\phi_i \not\equiv 0,\ i = 1, 2,$*

$$\liminf_{t\to\infty} S(x,t;\phi) \geq \delta,\ \liminf_{t\to\infty} I(x,t;\phi) \geq \delta,\ \liminf_{t\to\infty} P(x,t;\phi) \geq \delta, \text{uniform for all} \quad x \in \bar{\Omega}.$$

Furthermore, system (1.3) and (1.4) admits at least one positive steady state.

Proof. We prove the following claims.

Claim 1. *$\mathbb{X}_0$ is positively invariant with respect to $\Phi(t)$, that is, $\Phi(t)\mathbb{X}_0 \subseteq \mathbb{X}_0$, for all $t \geq 0$.*

Suppose that $(S(\cdot,t;\phi), I(\cdot,t;\phi), P(\cdot,t;\phi))$ is the solution (1.3) with $\phi \in \mathbb{X}^+$. It follows from Lemma 2.1 that $S(x,t)$ satisfies

$$d_S\Delta S - \frac{\partial S}{\partial t} + h_1(x,t)S = -rS \leq 0,\ x \in \Omega,\ t > 0,$$

where $h_1(x,t) = -\frac{r}{K(x)}[S(x,t) + I(x,t)] - \beta_1(x)I(x,t) - \beta_2(x)P(x,t) \leq 0$. Then $S(x,t;\phi) > 0$ directly follows from the strong maximum principle and the Hopf boundary lemma. From the second equation of (2.1), we have $\frac{\partial I}{\partial t} \geq d_I\Delta I - (b+c)I - r\frac{S+I}{K(x)}I$, I is the upper solution of the problem

$$\begin{cases} \dfrac{\partial \hat{I}}{\partial \hat{t}} = d_I\Delta\hat{I} - (b+c)\hat{I} - r\dfrac{S+I}{K(x)}\hat{I}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \hat{I}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ I(x,0) = \hat{I}_0(x) = I^0, & x \in \Omega. \end{cases}$$

By the maximum principle and $I^0 \not\equiv 0$, we have $\hat{I}(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. So $I(x,t) > \hat{I}(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$ directly follows from the comparison principle. From the third equation of (1.3), we get

$$P(x,t) = e^{-\int_0^t (m+S(\cdot,l)+I(\cdot,l))dl}P^0(x) + \int_0^t e^{-\int_s^t (m+S(\cdot,l)+I(\cdot,l))dl}\lambda(x)I(\cdot,s;u_0)ds,$$

which in turn implies that $P(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. This proves Claim 1.

Claim 2. *$\omega(\psi) = M_1 \bigcup M_2$, $\forall\ \psi \in M_\partial$, where $M_1 = \{(0,0,0)\}$ and $M_2 = \{(S^0,0,0,)\}$.*

If $\psi \in M_\partial$, then we have $\Phi(t)\psi \in M_\partial$, $\forall\ t \geq 0$. It follows that $S(x,t;\psi) \equiv 0$ or $I(x,t,\psi) \equiv 0$. Suppose that $I(x,t;\psi) \equiv 0$, $\forall\ t \geq 0$. From the third equation of system (1.3), we have $\lim_{t\to\infty} P(x,t;\psi) = 0$ uniformly for $x \in \bar{\Omega}$. Thus, it follows from the first equation of system (1.3) and Lemma 2.2 that $\lim_{t\to\infty} S(x,t;\psi) = 0$ or $\lim_{t\to\infty} S(x,t;\psi) = S^0$ uniformly for $x \in \overline{\Omega}$. Suppose that $I(x,\tilde{t}_0,\psi) \not\equiv 0$, for some $\tilde{t}_0 \geq 0$, Claim 1 implies that $I(x,t,\psi) > 0$, $\forall\ x \in \bar{\Omega}, t > \tilde{t}_0$. Hence, the case for $S(x,\tilde{t},\psi) \equiv 0$, $\forall\ t > \tilde{t}_0$, holds. It follows that the second equation of (1.3) becomes $\frac{\partial I}{\partial t} = d_I\Delta I + \beta_1(x)SI + \beta_2(x)SP - (b+c)I - r\frac{S+I}{K(x)}I, x \in \Omega,\ t > \tilde{t}_0$, which implies that $\lim_{t\to\infty} I(x,t;\psi) = 0$. Further, from the third equation of (1.3), we have $\lim_{t\to\infty} P(x,t;\psi) = 0$ uniformly for $x \in \overline{\Omega}$. Hence $\omega(\psi) = M_1 \bigcup M_2$, $\forall\ \psi \in M_\partial$. This proves Claim 2.

Claim 3. *For any* $\phi \in \mathbb{X}_0$, $\limsup_{t\to\infty} \|\Phi(t)\phi - M_i\| \geq \frac{\sigma_0}{2}, \forall\ \phi \in \mathbb{X}_0,\ \forall\ i = 1, 2.$

It follows from Lemma 3.4, $s(\mathcal{B}_{S^0}) > 0$ is the principal eigenvalue of eigenvalue problem (3.10) associated with a strongly positive eigenfunction. For convenience of later discussion, we suppose that there is a small $\sigma_0 > 0$ such that $s(\mathcal{B}_{S^0}^{\sigma_0}) > 0$ is still the principle eigenvalue problem (3.10), where

$$\mathcal{B}_{S^0}^{\sigma_0} = \begin{pmatrix} d_I\Delta + \beta_1(x)(S^0(x) - \sigma_0) - (b+c) - r\frac{(S^0(x)+\sigma_0)}{K(x)} & \beta_2(x)(S^0 - \sigma_0) \\ \lambda(x) & -m - \beta_3(x)(S^0(x) + \sigma_0)(x) \end{pmatrix}.$$

Let $\tilde{\psi} = (\tilde{\psi}_2, \tilde{\psi}_3)$ be the strongly positive eigenfunction corresponding $s(\mathcal{B}_{S^0}^{\sigma_0})$. Without loss of generality, we further assume σ_0 is sufficiently small such that

$$1 - \sigma_0 \max_{x\in\bar{\Omega}} \left(\frac{1}{K(x)} + \frac{\beta_1(x)}{r} + \frac{\beta_2(x)}{r} \right) > 0.$$

Assume for the contrary that there exists $\phi_0 \in \mathbb{X}_0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)(\phi_0) - M_1\| < \frac{\sigma_0}{2} \text{ or } \limsup_{t\to\infty} \|\Phi(t)(\phi_0) - M_2\| < \frac{\sigma_0}{2}.$$

We consider the first case that $\limsup_{t\to\infty} \|\Phi(t)(\phi_0) - M_2\| < \frac{\sigma_0}{2}$. It follows that there exists $t_1 > 0$ such that $S^0(x) - \frac{\sigma_0}{2} < S(x,t,\phi_0) < S^0(x) + \frac{\sigma_0}{2}$, $I(x,t,\phi_0) < \frac{\sigma_0}{2}$, $P(x,t,\phi_0) < \frac{\sigma_0}{2}$, $\forall\ t \geq t_1,\ x \in \bar{\Omega}$. Thus $I(x,t,\phi_0)$ and $P(x,t,\phi_0)$ satisfies

$$\begin{cases} \dfrac{\partial I}{\partial t} \geq d\Delta I + \beta_1(x)(S^0(x) - \sigma_0)I + \beta_2(x)(S^0(x) - \sigma_0)P \\ \qquad - (b+c)I - r\dfrac{S^0(x) + \sigma_0}{K(x)}I, & x \in \Omega,\ t > t_1, \\ \dfrac{\partial P}{\partial t} \geq -mP + \lambda(x)I - \beta_3(x)(S^0(x) + \sigma_0)P, & x \in \Omega,\ t > t_1, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, t > t_1. \end{cases}$$

It follows that $I(x,t,\phi_0) > 0, P(x,t,\phi_0) > 0, \forall\ x \in \bar{\Omega},\ t > 0$. Recall that the linear system

$$\begin{cases} \dfrac{\partial \check{I}}{\partial t} = d\Delta \check{I} + \beta_1(x)(S^0(x) - \sigma_0)\check{I} + \beta_2(x)(S^0(x) - \sigma_0)\check{I} \\ \qquad - (b+c)\check{I} - r\dfrac{S^0(x) + \sigma_0}{K(x)}\check{I}, & x \in \Omega,\ t > t_1, \\ \dfrac{\partial \check{P}}{\partial t} = -m\check{P} + \lambda(x)\check{I} - \beta_3(x)(S^0(x) + \sigma_0)\check{P}, & x \in \Omega,\ t > t_1, \\ \dfrac{\partial \check{I}}{\partial \nu} = 0, & x \in \partial\Omega, t > t_1, \end{cases}$$

admits a solution $\varepsilon_0 e^{s(\mathcal{B}_{S^0})(t-t_1)}\tilde{\psi}$ for some positive constant ε_0. From the standard comparison principle, we have

$$(I(x,t,\phi_0), P(x,t,\phi_0)) \geq \varepsilon_0 e^{s(\mathcal{B}_{S^0})(t-t_1)}\tilde{\psi}, \quad \forall\ t > t_1,\ x \in \bar{\Omega}.$$

which in turn implies that $I(x,t,\phi_0)$ and $P(x,t,\phi_0)$ is unbounded as $s(\mathcal{B}_{S^0}) > 0$. This leads to a contradiction.

We next consider the case that $\limsup_{t\to\infty} \|\Phi(t)(\phi_0) - M_1\| < \frac{\sigma_0}{2}$. It follows that there exists $t_2 > 0$ such that $S^0(x) < \frac{\sigma_0}{2}$, $I(x,t,\phi_0) < \frac{\sigma_0}{2}$ and $P(x,t,\phi_0) < \frac{\sigma_0}{2}$, $\forall\ t \geq t_2,\ x \in \bar{\Omega}$. From the first equation of (1.3), we get

$$\begin{cases} \dfrac{\partial S}{\partial t} \geq d\Delta S + r(1-\sigma_0\theta_1)S, & x \in \Omega, t > t_2, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where $\theta_1 := \max_{x\in\bar{\Omega}} \left(\frac{1}{K(x)} + \frac{\beta_1(x)}{r} + \frac{\beta_2(x)}{r}\right)$.

Recall that

$$\begin{cases} \dfrac{\partial \tilde{S}}{\partial t} = d\Delta \tilde{S} + r(1-\sigma_0\theta_1)\tilde{S}, & x \in \Omega, t > t_2, \\ \dfrac{\partial \tilde{S}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

admits a solution $\tau e^{r(1-\sigma_0\theta_1)(t-t_2)}\tilde{\xi}$ for some positive constant τ, where $\tilde{\xi}$ is strongly positive eigenfunction corresponding $r(1-\sigma_0\theta_1)$. From the standard comparison principle, we have

$$S(x,t,\phi_0) \geq e^{r(1-\sigma_0\theta_1)(t-t_2)}\tilde{\xi},\ \forall\ t > t_2,\ x \in \bar{\Omega}.$$

Hence $S(x,t,\phi_0)$ is unbounded as $(1-\sigma_0\theta_1) > 0$, which leads to a contradiction. This proves Claim 3.

Define a continuous function $p : \mathbb{X}^+ \to [0,\infty)$ by

$$p(\phi) = \min\{\min_{x\in\overline{\Omega}} \phi_1(x), \min_{x\in\overline{\Omega}} \phi_2(x)\},\ \forall\ \phi \in \mathbb{X}^+.$$

Obviously, $p^{-1}(0,\infty) \subseteq \mathbb{X}_0$, and p has the property that if either $p(\phi) = 0$ with $\phi \in \mathbb{X}_0$ or $p(\phi) > 0$, then $p(\Phi(t)\phi) > 0,\ \forall\ t > 0$. Thus p is a generalized distance function for the semiflow $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+$ (see e.g. [20]). Further, $W^s(M_i) \bigcap \mathbb{X}_0 = \emptyset, \forall i = 1,2$, where $W^s(M_i)$ is the stable subset of $M_i,\ i = 1,2$. It is also confirmed that there is no cycle in M_∂ from $M_1 \bigcup M_2$ to $M_1 \bigcup M_2$. Hence, from [20, Theorem 3] and similar arguments in [8, Theorem 3.4], we arrive at the conclusion that there exists a $\delta > 0$ such that

$$\liminf_{t\to\infty} p(\Phi_t(\psi)) > \delta,\ \forall\ \psi \in \mathbb{X}_0,$$

which implies that

$$\liminf_{t\to\infty} S(x,t;\phi) \geq \delta\ \ \liminf_{t\to\infty} I(x,t;\phi) \geq \delta,\ \text{and}\ \liminf_{t\to\infty} P(x,t;\phi) \geq \delta, \forall\ \phi \in \mathbb{X}_0.$$

Therefore, $\Phi(t)$ is uniformly persistent with respect to $(\mathbb{X}_0, \partial\mathbb{X}_0)$. It follows from [11, Theorem 4.7] that system (1.3) admits at least one steady state in $\mathbb{X}_0$ (see, e.g. the proof of [26, Theorem 2.3]), which is a positive steady state. This completes the proof. □

4. Bifurcation analysis

Recall that system (1.3) is uniformly persistent when $\Re_0 > 1$ (see Theorem 3.2), thus (1.3) admits at least one positive steady state. In this section, we consider disease-induced mortality b as the main bifurcation parameter to do some bifurcation analysis on steady state solutions.

A steady state of (1.3) is a solution of the elliptic system

$$\begin{cases} d_S\Delta S + r\left(1-\dfrac{S+I}{K(x)}\right)S - \beta_1(x)SI - \beta_2(x)SP = 0, & x\in\Omega,\\ d_I\Delta I + \beta_1(x)SI + \beta_2(x)SP - (b+c)I - r\dfrac{S+I}{K(x)}I = 0, & x\in\Omega,\\ \lambda(x)I - mP - \beta_3(x)(S+I)P = 0, & x\in\Omega,\\ \dfrac{\partial S}{\partial\nu} = \dfrac{\partial I}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{4.1}$$

By (4.1), we see that (S, I, P) is a PSS of (1.3) if and only if (S, I) is a positive solution of the problem

$$\begin{cases} d_S\Delta S + r\left(1-\dfrac{S+I}{K(x)}\right)S - \beta_1(x)SI - \dfrac{\lambda(x)\beta_2(x)SI}{m+\beta_3(x)(S+I)} = 0, & x\in\Omega,\\ d_I\Delta I + \beta_1(x)SI + \dfrac{\lambda(x)\beta_2(x)SI}{m+\beta_3(x)(S+I)} - (b+c)I - r\dfrac{S+I}{K(x)}I = 0, & x\in\Omega,\\ \dfrac{\partial S}{\partial\nu} = \dfrac{\partial I}{\partial\nu} = 0, & x\in\partial\Omega, \end{cases} \tag{4.2}$$

and P satisfies

$$P(x) = \frac{\lambda(x)I(x)}{m+\beta_3(x)(S(x)+I(x))}.$$

Hence in the sequel, we will focus on (4.2) instead of (4.1).

It is easy to see that $(S^0(x), 0)$ is a semi-trivial steady state solution of (4.2), where $S^0(x)$ is described in (2.3). Denote by b^0 the principle eigenvalue of the following eigenvalue problem:

$$\begin{cases} d_I\Delta\psi + \left[\beta_1(x)S^0(x) + \dfrac{\beta_2(x)\lambda(x)S^0(x)}{m+\beta_3(x)S^0(x)} - \dfrac{r}{K(x)}S^0(x) - c\right]\psi = b\psi, & x\in\Omega,\\ \dfrac{\partial\psi}{\partial\nu} = 0, & x\in\partial\Omega, \end{cases} \tag{4.3}$$

associated with positive eigenfunction $\psi_0(x)$ (determined by the normalization $\max_{x\in\bar\Omega}\psi_0(x) = 1$). Notice that $b = b^0$ is equivalent to $\tilde\lambda_0 = 1$ or $\Re_0 = 1$.

Define a function

$$H(x) = \left[\beta_1(x)S^0(x) + \frac{\beta_2(x)\lambda(x)S^0(x)}{m+\beta_3(x)S^0(x)} - \frac{r}{K(x)}S^0(x) - c\right]. \tag{4.4}$$

It follows that $b^0 = H$ if $H(x) = H$ is a constant.

We next consider the case when $H(x) \not\equiv$ constant and it could change sign in Ω. It is well-known from [14, Theorem 4.2] that the following eigenvalue problem with indefinite weight:

$$\begin{cases} \Delta\varphi(x) + \Lambda H(x)\varphi = 0, & x\in\Omega,\\ \dfrac{\partial\varphi}{\partial\nu} = 0, & x\in\partial\Omega, \end{cases} \tag{4.5}$$

admits a nonzero principal eigenvalue $\Lambda_0 = \Lambda_0(H)$ if and only if $H(x)$ charges sign in Ω and $\int_\Omega H(x)dx \neq 0$.

It then follows from [14, Proposition 4.4] and also in [26, Lemma 3.1], the sign of the principal eigenvalue b^0 of the problem (4.3) are described by the following result.

Lemma 4.1. *The following statements hold.*

(i) *If* $\int_\Omega H(x)dx \geq 0$*, then* $b_0 > 0$ *for all* $d_I > 0$*;*
(ii) *If* $\int_\Omega H(x)dx < 0$*, then*

$$\begin{cases} b^0 > 0 & \textit{for all } d_I < \frac{1}{\Lambda_0(H)}. \\ b^0 < 0 & \textit{for all } d_I > \frac{1}{\Lambda_0(H)}. \end{cases}$$

Remark 4.1. Suppose that the coefficients of (4.1) are all constants. It follows that $S^0(x) \equiv K$ and

$$H(x) \equiv H = \left[\beta_1 K + \frac{\beta_2 \lambda K}{m + \beta_3 K} - r - c\right] = \frac{(m + \beta_3 K)(\beta_1 K - r - c) + \beta_2 \lambda K}{m + \beta_3 K}. \tag{4.6}$$

Hence $b^0 = H > 0$ if $\beta_1 K \geq r + c$.

We next regard b as a bifurcation parameter and investigate that a local branch (and also a global continuum) of positive solution of (4.2) bifurcated from $\{(b, S^0(x), 0) : b > 0\}$. To this end, we rewrite (4.2) by $u = S$ and $w = I$ as

$$\begin{cases} d_u \Delta u + \dfrac{r}{K(x)}(K(x) - u - w)u - \beta_1(x)uw - \dfrac{\beta_2(x)\lambda(x)uw}{m + \beta_3(x)(u + w)} = 0, & x \in \Omega, \\ d_w \Delta w + \beta_1(x)uw + \dfrac{\beta_2(x)\lambda(x)uw}{m + \beta_3(x)(u + w)} - (b + c)w - \dfrac{r}{K(x)}(u + w)w = 0, & x \in \Omega, \\ \dfrac{\partial u(x)}{\partial \nu} = \dfrac{\partial w(x)}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.7}$$

We now give the result on the set of steady state solution of (4.2).

Theorem 4.1. *Let* b^0 *be the principle eigenvalue problem (4.3). Let*

$$\Sigma = \left\{(b, u, w) \in R^+ \times X \times X : (b, u, w) \textit{ is a positive solution of } (4.2)\right\}, \tag{4.8}$$

where $X = \left\{u \in W^{2,p}(\Omega) : \frac{\partial u(x)}{\partial \nu} = 0,\ x \in \partial\Omega\right\}$*. The following statements hold:*

(i) *There is a connected component* Σ_1 *of* $\bar{\Sigma}$ *containing* $(b^0, S^0, 0)$*, and the projection* $proj_b\Sigma_1$ *of* Σ_1 *into the b-axis satisfies* $(0, b^0] \subset proj_b\Sigma_1 \subset (0, M]$ *for*

$$M = \max_{x \in \bar{\Omega}} \left[\beta_1(x)S^0(x) + \frac{\beta_2(x)\lambda(x)S^0(x)}{m} - c\right]. \tag{4.9}$$

In particular, (4.2) admits at least one positive steady state solution for $0 < b < b^0$*.*

(ii) *Near* $b = b^0$*,* Σ_1 *is a smooth curve*

$$C_1 = \{(b(s), u(s), w(s)) : s \in (0, \varepsilon)\}, \tag{4.10}$$

where $u(s) = S^0(\cdot) + s\phi_0(\cdot) + o(s)$*,* $w(s) = s\psi_0(s) + o(s)$ *where* $\psi_0(x) > 0$ *is the principle eigenvalue of (4.3), and* $\phi_0(x) < 0$ *satisfies*

$$\begin{cases} -d_u \Delta\phi_0(x) + \dfrac{r}{K(x)}(K(x) - 2S^0(x))\phi_0(x) = -q(x)\psi_0(x), & x \in \Omega, \\ \dfrac{\partial\phi_0(x)}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.11}$$

where

$$q(x)=\frac{r}{K(x)}S^0(x)+\beta_1(x)S^0(x)+\frac{\beta_2(x)\lambda(x)S^0(x)}{m+\beta_3(x)S^0(x)}.$$

Further, $b'(0)$ can be calculated by

$$b'(0)=\frac{L}{\int_\Omega \psi_0^2(x)dx}, \tag{4.12}$$

where

$$L=\int_\Omega \frac{\beta_2(x)\lambda(x)m\phi_0(x)(\psi_0(x))^2}{[m+\beta_3(x)S^0(x)]^2}-\int_\Omega \frac{r\left[\phi_0(x)(\psi_0(x))^2+(\psi_0(x))^3\right]}{K(x)}-\int_\Omega \frac{\lambda(x)\beta_2(x)\beta_3(x)S^0(x)(\psi_0(x))^3}{[m+\beta_3(x)S^0(x)]^2}. \tag{4.13}$$

Proof. We then follow the procedure in [22] to consider the solution of (4.2).

Define $\mathcal{F}:\mathbb{R}\times X\times X\to Y\times Y$ by

$$\mathcal{F}(b,u,w)=\begin{pmatrix} d_u\Delta u+\frac{r}{K(x)}(K(x)-u-w)u-\beta_1(x)uw-p(u,w)\\ d_w\Delta w+\beta_1(x)uw+p(u,w)-(b+c)w-\frac{r}{K(x)}(u+w)w\end{pmatrix},$$

where $p(u,w)=\frac{\beta_2(x)\lambda(x)uw}{m+\beta_3(x)(u+w)}$. Direct calculations give

$$\begin{aligned}\mathcal{F}_{(u,w)}(b,u,w)[\phi,\psi]&=\begin{pmatrix} d_u\Delta\phi\\ d_w\Delta\psi\end{pmatrix}\\ &+\begin{pmatrix}\frac{r}{K(x)}(K(x)-2u-w)-\beta_1(x)w-p_u & -\frac{r}{K(x)}u-\beta_1(x)u-p_w\\ \beta_1(x)w+p_u-\frac{r}{K(x)}w & \beta_1(x)u+p_w-(b+c)-\frac{r}{K(x)}(u+2w)\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix},\end{aligned}$$

where

$$p_u:=p_u(u,w)=\frac{\beta_2(x)\lambda(x)(m+\beta_3(x)w)w}{(m+\beta_3(x)(u+w))^2},\quad p_w:=p_w(u,w)=\frac{\beta_2(x)\lambda(x)(m+\beta_3(x)u)u}{(m+\beta_3(x)(u+w))^2}.$$

We can check that

$$p_u(S^0,0)=0,\quad p_w(S^0,0)=\frac{\beta_2(x)\lambda(x)S^0(x)}{m+\beta_3(x)S^0(x)}.$$

Furthermore, we can compute

$$\mathcal{F}_b(b,u,w)=\begin{pmatrix}0\\ -w\end{pmatrix},\quad \mathcal{F}_b(b,u,w)[\phi,\psi]=\begin{pmatrix}0\\ -\psi\end{pmatrix},$$

and

$$\mathcal{F}_{(u,w),(u,w)}(b,u,w)[\phi,\psi]^2=\begin{pmatrix}-\left(\frac{2r}{K(x)}+p_{uu}\right)\phi^2-2\left(\frac{r}{K(x)}+\beta_1(x)+p_{uw}\right)\phi\psi-p_{ww}\psi^2\\ p_{uu}\phi^2+2\left(p_{uw}+\beta_1(x)-\frac{r}{K(x)}\right)\phi\psi+\left(p_{ww}-\frac{2r}{K(x)}\right)\psi^2\end{pmatrix},$$

where

$$\begin{aligned}
p_{uu} &:= p_{uu}(u,w) = \tfrac{-2\lambda(x)\beta_2(x)\beta_3(x)(m+\beta_3(x)w)w}{[m+\beta_3(u+w)]^3},\\
p_{uw} &:= p_{uw}(u,w) = \tfrac{\beta_2(x)\lambda(x)(m^2+\beta_3(x)mu+\beta_3(x)mw+2(\beta_3(x))^2uw)}{[m+\beta_3(u+w)]^3},\\
p_{ww} &:= p_{ww}(u,w) = \tfrac{-2\lambda(x)\beta_2(x)\beta_3(x)(m+\beta_3(x)u)u}{[m+\beta_3(u+w)]^3}.
\end{aligned}$$

We can check that

$$p_{uu}(S^0,0)=0,\ p_{uw}(S^0,0)=\frac{\beta_2(x)\lambda(x)m}{[m+\beta_3(x)S^0(x)]^2},\ p_{ww}(S^0,0)=\frac{-2\lambda(x)\beta_2(x)\beta_3(x)S^0(x)}{[m+\beta_3(x)S^0(x)]^2}. \tag{4.14}$$

In particular,

$$\mathcal{F}_{(u,w)}(b^0,S^0,0)[\phi,\psi]=\begin{pmatrix} d_u\Delta\phi+\frac{r}{K(x)}(K(x)-2S^0(x))\phi-q(x)\psi\\ d_w\Delta\psi-b^0\psi+H(x)\psi\end{pmatrix},$$

where $H(x)$ is defined as in (4.4) and

$$q(x)=\tfrac{r}{K(x)}S^0(x)+\beta_1(x)S^0(x)+\tfrac{\beta_2(x)\lambda(x)S^0(x)}{m+\beta_3(x)S^0(x)}. \tag{4.15}$$

It follows that the kernel $N(F_{(u,w)}(b^0,S^0,0))=span(\phi_0,\psi_0)$, where ψ_0 is the positive eigenfunction of (4.3) and ϕ_0 satisfies (4.11). Recall that, from Lemma 2.2, $S^0(x)$ is globally asymptotically stable in $C(\bar\Omega,\mathbb{R})$. It follows that

$$\left[d_u\Delta+\frac{r}{K(x)}(K(x)-2S^0(x))\right]^{-1},$$

exists and it is a positive operator. Thus, $\phi_0(x)<0$ for $x\in\Omega$.

Let $Y=L^p(\Omega)$. We next consider the range

$$R(F_{u,w}(b^0,S^0,0))=\{(h_1,h_2)\in Y^2:\int_\Omega h_2(x)\psi_0(x)dx=0\}. \tag{4.16}$$

It is easy to see that $(h_1,h_2)\in R(F_{u,w}(b^0,S^0,0))$ if and only if there exists $(\phi,\psi)\in X\times X$ such that

$$\begin{aligned}
h_1&=d_u\Delta\phi+\tfrac{r}{K(x)}(K(x)-2S^0(x))\phi-q(x)\psi,\\
h_2&=d_w\Delta\psi-b^0\psi+H(x)\psi,
\end{aligned}$$

where $q(x)$ and $H(x)$ are defined as in (4.15) and (4.4). Hence,

$$\int_\Omega h_2(x)\psi_0(x)dx=d\int_\Omega\Delta\psi(x)\psi_0(x)dx+\int_\Omega[-b^0\psi_0(x)+H(x)\psi_0(x)]\psi(x)dx. \tag{4.17}$$

It then follows from integration by parts and the boundary condition of ψ and ψ_0 that

$$\int_\Omega\Delta\psi(x)\psi_0(x)dx=\int_\Omega\Delta\psi_0(x)\psi(x)dx. \tag{4.18}$$

With the help of (4.3), (4.18) and (4.17), we have $\int_\Omega h_2(x)\psi_0(x)dx=0$, which in turn implies that (4.16) is valid. Since

$$F_{b,(u,w)}(b^0,S^0,0)[\phi_0,\psi_0]=(0,-\psi_0), \tag{4.19}$$

and $\int_\Omega[-\psi_0(x)]\psi_0(x)dx<0$. It follows that

$$F_{b,(u,w)}(b^0, S^0, 0)[\phi_0, \psi_0] \notin R(F_{u,w}(b^0, S^0, 0)).$$

Follow the theorem of bifurcation from a simple eigenvalue (see e.g. Crandall and Rabinowitz [23], we can conclude that the set of positive solution to (4.2) near $(b^0, S^0(x), 0)$ is a curve in from (4.10), with $(u'(0), w'(0)) = (\phi_0, \psi_0)$.

Further, $b'(0)$ can be calculated as follows

$$b'(0) = -\frac{\langle l, F_{(u,w),(u,w)}(b^0, S^0, 0)[\phi_0, \psi_0]^2\rangle}{2\langle l, F_{b,(u,w)}(b^0, S^0, 0)[\phi_0, \psi_0]\rangle},$$

where l is a linear function on Y^2 defined as $\langle l, [h_1, h_2]\rangle = \int_\Omega h_2(x)\psi_0(x)dx$. Note that the second component of $F_{(u,w),(u,w)}(b^0, S^0, 0)[\phi_0, \psi_0]^2$ takes the form

$$G(x) := 2\left(p_{uw}(S^0(x), 0) - \frac{r}{K(x)}\right)\phi_0(x)\psi_0(x) + \left(p_{ww}(S^0(x), 0) - \frac{2r}{K(x)}\right)(\psi_0(x))^2,$$

where $p_{uw}(S^0(x), 0)$ and $p_{ww}(S^0(x), 0)$ are defined in (4.14). Thus

$$b'(0) = -\frac{\int_\Omega G(x)\phi_0 dx}{2\int_\Omega \psi_0^2(x)dx} := \frac{L}{\int_\Omega \psi_0^2(x)dx}, \tag{4.20}$$

where L is defined as in (4.13). Using the similar arguments as in [26, Theorem 3.1], we could end with the proof. □

5. Conclusion and discussion

In this paper, we formulate and analyze a diffusive host-pathogen model with horizontal transmission mechanism and heterogeneous coefficients. In the model, we assume that susceptible and infective hosts may disperse at different rates, there is no diffusion term in the pathogen equation. This assumption brings some difficulties in estimating the ultimate boundedness of the solution. We overcome this problem by using the method of induction and continuous embedding theorem. We then consider asymptotic smoothness of the solution semiflow by introducing the Kuratowski measure of noncompactness to identify the existence of a connected global attractor.

We identify the threshold behavior of model in a bounded habitat of general spatial dimension, which is determined in the sense that: If $s(\mathcal{B}) < 0$, then the disease-free equilibrium $(S^0(x), 0, 0)$ is global attractive (see Theorem 3.1); If $s(\mathcal{B}_{S^0}) > 0$ (or $\Re_0 > 1$), then (1.3) is uniformly persistent and (1.3) admits a positive steady state, representing the persistence of pathogen. $\Re_0$ is mathematically defined as the spectral radius of the next generation operator, then it can be calculated as in Lemma 3.3 and (3.8) (see also a homogeneous case as in (3.9)). While we can not determine the case $\Re_0 < 1$ for the extinction of the disease.

We extend the original model [26] by including one realistic complication, horizontal transmission mechanism and thus strengthens the original conclusion. Compared to the formula as in (2.29) and (2.30) of [26], we can conclude that the principal eigenvalue of the associated eigenvalue problem and the long-time behavior remain similar as those in [26]. However, by adding horizontal transmission, even a homogeneous case as in (3.9), the basic reproduction number is larger than the case without horizontal transmission mechanism. This may lead to over-evaluating the threshold role of the basic reproduction number. We also explored the bifurcation analysis of steady state solutions by considering disease-induced mortality b as the main bifurcation parameter, and such results can help us better understanding how it affects the spatial pattern of the pathogen.

As like some recent works on asymptotical profiles of the positive steady state for large and small diffusion rates, we refer interested readers to [2,9,10,15–18,24,26–28] and the references therein. Would considering asymptotic profiles of the positive steady state as the dispersal rate of susceptible or infected hosts tends to zero may help us to understand how host's mobility affect spatial pattern of the pathogen. On the other hand, some recent work investigated the effect of the spatial heterogeneity of environment on basic reproduction number and disease dynamics (see e.g. [3–5]). Would the spatial heterogeneity can enhance the infectious risk of disease? Thus, we have to leave this interesting problem for further investigation.

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