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# The Bohr inequality for holomorphic mappings with lacunary series in several complex variables ☆

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ABSTRACT

In this paper, we first give the Bohr inequality of norm type for holomorphic mappings with lacunary series on the unit polydisk in $\mathbb{C}^n$ under some restricted conditions. Next we also establish the Bohr inequality of norm type for holomorphic mappings with lacunary series on the unit ball of complex Banach spaces under some additional conditions, and the Bohr inequality of functional type for holomorphic mappings with lacunary series on the unit ball of complex Banach spaces. Our derived results reduce to the corresponding results in one complex variable.


## 1. Introduction

In one complex variable, it is well-known that the Bohr radius was originally established by Bohr [9] for $1/6$. After that the value $1/6$ was improved to the value $1/3$ by Riesz, Schur and Wiener independently, and the value $1/3$ is optimal. Other new proofs were given by Tomić [24] and Sidon [23].

We also denote by

$$\mathcal{F}_1 = \{f : U \to U | f(z) = \sum_{n=1}^{\infty} a_n z^n\}.$$

Let

$$B_1 = \sup\{r | \sum_{n=1}^{\infty} |a_n| r^n < 1 \text{ for all } f \in \mathcal{F}_1\},$$

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where $r = |z|$.

The result of $B_1 > \frac{1}{2}$ was proved by Tomić [24] (obtained by Landau [19] independently). After that, Ricci [21] proved $\frac{3}{5} < B_1 \le \frac{1}{\sqrt{2}}$. The optimal result $B_1 = \frac{1}{\sqrt{2}}$ was given because of Bombieri [10]. In 2004, Beneteau, Khavinson and Dahlner [6] mainly discussed Hardy space functions that vanished at the origin and obtained an exact positive Bohr radius. After approximately ten years, Abu-Muhanna, Ali [1], Abu-Muhanna, Ali, Ng, Hasni [2] studied the various refined Bohr's theorems for analytic functions, harmonic functions and the hyperbolic metric respectively. In 2017, Kayumov, Ponnusamy [17] gave the sharp Bohr radius for odd analytic functions. One year later, the above authors established the following theorem.

**Theorem A.** *(See [18].) Let $m, k$ be nonnegative numbers, $0 \le m \le k$, $f(z) = z^m \sum\limits_{s=0}^{\infty} a_{sk} z^{sk}$ be analytic on $U$, and $|f(z)| \le 1$ on $U$. Then*

$$|z|^m \sum_{s=0}^{\infty} a_{sk} |z|^{sk} \le 1$$

*for $|z| = r \le r_{k,m}$, where $r_{k,m}$ is the maximal positive root of the equation*

$$-6r^{k-m} + r^{2(k-m)} + 8r^{2k} + 1 = 0\,.$$

*Each $r_{k,m}$ is sharp.*

In several complex variables, there are many significant and attractive results (see [3], [4], [8], [7], [11], [12], [13]). Bayart, Pellegrino, Seoane-Sepúlveda [5] gave the Bohr radius of polydisk behaved asymptotically as $\sqrt{(\log(n))/n}$ in 2014. Unfortunately, the above results can not reduce to the case of one dimension. Concerning the generalizations of the Bohr's theorem in several complex variables for holomorphic mappings by making use of homogeneous expansions, Liu and Wang [20] in 2007 first extended the Bohr's Theorem in one complex variable to holomorphic mappings which map one of the four classical domains $\Omega$ in the sense of Hua [16] into itself, and they proved that Bohr radius 1/3 is sharp. After that Hamada, Honda and Kohr [15] obtained the generalizations of the Bohr's theorem to holomorphic mappings $f : G \to B_Y$, where $G$ is a bounded balanced domain in a complex Banach space $X$ and $B_Y$ is the unit ball in a complex Banach space $Y$, and they showed that the Bohr radius 1/3 is sharp if $B_Y$ is the unit ball of a $J^*$-algebra. Sequently, Roos [22] extended the Bohr's Theorem to all bounded symmetric domains as well, and its proof does not rely on classification. In essence, the above three references all apply holomorphic automorphisms.

In this paper, we will establish the Bohr inequality of norm type for holomorphic mappings with lacunary series on the unit polydisk in $\mathbb{C}^n$ under some restricted conditions. Meanwhile we will give the Bohr inequality of norm type for holomorphic mappings with lacunary series on the unit ball of complex Banach spaces under some additional conditions, and the Bohr inequality of functional type for holomorphic mappings with lacunary series on the unit ball of complex Banach spaces as well. Our derived results reduce to the corresponding results in one complex variable, and further simplify the proof of the related theorem of [18]. Taking into account the difficulty of the Bohr inequality of norm type and functional type for holomorphic mappings with lacunary series concerning holomorphic automorphisms in several complex variables in general. So we do not discuss the corresponding problems with respect to holomorphic automorphisms.

We denote by $X$ a complex Banach space with the norm $\|\,.\,\|$, $X^*$ the dual space of $X$, $B$ the open unit ball in $X$, and $U$ the Euclidean open unit disk in $\mathbb{C}$ respectively. Let $U^n$ be the open unit polydisk in $\mathbb{C}^n$, and let $\mathbb{N}$ and $\mathbb{N}^+$ be the set of all nonnegative integers and the set of all positive integers accordingly. Let $\partial U^n$ denote the boundary of $U^n$, $(\partial U)^n$ be the distinguished boundary of $U^n$. Let the symbol $'$ stand for transpose. For each $x \in X \backslash \{0\}$,

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}$$

is well defined. We write $H(B, X)$ (resp. $H(B, B)$) the set of all holomorphic mappings from $B$ into $X$ (resp. $B$).

## 2. Bohr inequality for holomorphic mappings with lacunary series for holomorphic mappings on the unit polydisk in $\mathbb{C}^n$

We first show an example in this section.

**Example 2.1.** Let

$$f(z) = \left( z_1 \frac{z_1 - \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}} z_1}, z_2 \frac{z_2 - \frac{2}{\sqrt{5}}}{1 - \frac{2}{\sqrt{5}} z_2} \right)', z = (z_1, z_2)'.$$

Then $f \in H(U^2, U^2)$. It yields that

$$\sum_{m=1}^{\infty} \frac{\|D^m f(0)(z^m)\|}{m!} > \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} \frac{1 - \left(\frac{1}{\sqrt{2}}\right)^2}{1 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}} \right) = \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} \right) > 1$$

for $z = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})'$ by a simple calculation.

Example 2.1 becomes a counterexample of the following problem which does not hold.

**Problem 2.1.** Let $f \in H(U^n, \overline{U^n})$. If $f(z) = \sum\limits_{m=0}^{\infty} \frac{D^m f(0)(z^m)}{m!}$, then

$$\sum_{m=1}^{\infty} \frac{\|D^m f(0)(z^m)\|}{m!} \leq 1$$

for $\|z\| \leq \frac{1}{\sqrt{2}}$.

We next provide some lemmas below in order to prove the desired results in this section.

**Lemma 2.1.** *Let $p \in \mathbb{N}^+, m \in \mathbb{N}$,*

$$\varphi_1(r) = 2r^p + r - 1, r \in [0, 1),$$
$$\varphi_2(r) = 4r^{2(p-m)} + 4r^{p+1-2m} - 4r^{p-2m} + r^2 - 2r + 1, r \in [0, 1), p > 2m,$$
$$\varphi_3(r) = 4r^p + r^{2+2m-p} - 2r^{1+2m-p} + r^{2m-p} + 4r - 4, r \in [0, 1), m + 1 \leq p \leq 2m.$$

*Then there exists the maximal positive root for each $\varphi_k(r) = 0$ ($k = 1, 2, 3$).*

**Proof.** It is not difficult to verify that there exists the maximal positive root for each $\varphi_k(r) = 0$ ($k = 1, 2, 3$) from the existence theorem of the root and the algebra basic theorem. This completes the proof. □

**Lemma 2.2.** *Let $m, k \in \mathbb{N}, 0 \leq m \leq k$, and the equation*

$$-6r^{k-m} + r^{2(k-m)} + 8r^{2k} + 1 = 0.$$

*Then $2r_{k,m}^{m+k} \leq 1$, where $r_{k,m}$ is the maximal positive root of the above equation.*

**Proof.** Since

$$-6r_{k,m}^{k-m}+r_{k,m}^{2(k-m)}+8r_{k,m}^{2k}+1=0\,,$$

then it is shown that

$$0=-6r_{k,m}^{m+k}+r_{k,m}^{2k}+8r_{k,m}^{2(m+k)}+r_{k,m}^{2m}\geq -4r_{k,m}^{m+k}+8r_{k,m}^{2(m+k)}\,.$$

Thus it follows the result, as desired. This completes the proof. □

**Remark 2.1.** Lemma 2.2 is the same as [18][Lemma 1]. We further simplify the proof here.

In view of the counterexample of Problem 2.1, we need some restricted conditions if we investigate Problem 2.1.

**Theorem 2.1.** *Let* $m\in\mathbb{N}, a=(a_1,a_2,\cdots,a_n)', p\geq m+1, f(z)=(a_1z_1^m+g_1(z), a_2z_2^m+g_2(z),\cdots,a_nz_n^m+g_n(z))'=\frac{D^mf(0)(z^m)}{m!}+\sum\limits_{s=p}^{\infty}\frac{D^sf(0)(z^s)}{s!}\in H(U^n,\mathbb{C}^n)$, $|a_l|=\|a\|=\max\limits_{1\leq l\leq n}\{|a_l|\}, l=1,2,\cdots,n$, *and* $a_jz_j^m+g_j(z)\in H(U^n,\overline{U})$, *where* $\frac{D^mf(0)(z^m)}{m!}=(a_1z_1^m,a_2z_2^m,\cdots,a_nz_n^m)'$, *and* $j$ *satisfies* $|z_j|=\|z\|=\max\limits_{1\leq l\leq n}\{|z_l|\}$. *Then*

$$\frac{\|D^mf(0)(z^m)\|}{m!}+\sum_{s=p}^{\infty}\frac{\|D^sf(0)(z^s)\|}{s!}\leq 1$$

*for* $\|z\|=r\leq r_{p,m}$, *where* $r_{p,m}$ *is the maximal positive root of the equation*

$$\begin{cases}2r^p+r-1=0, & \text{if}\quad m=0,\\ 4r^{2(p-m)}+4r^{p+1-2m}-4r^{p-2m}+r^2-2r+1=0, & \text{if}\quad p>2m, m=1,2,\cdots,\\ 4r^p+r^{2+2m-p}-2r^{1+2m-p}+r^{2m-p}+4r-4=0, & \text{if}\quad m+1\leq p\leq 2m, m=1,2,\cdots.\end{cases}$$

*Especially,* $r_{1,0}=\frac{1}{3}, r_{2,1}=\frac{3}{5}$, *moreover* $r_{1,0}=\frac{1}{3}$ *is optimal.*

**Proof.** Let $z\in U^n\setminus\{0\}$ be fixed, and denote $z_0=\frac{z}{\|z\|}$. Letting $h_j(\xi)=f_j(\xi z_0),\xi\in U$, then $h_j\in h(U,\overline{U})$, and

$$h_j(\xi)=a_j\left(\frac{z_j}{\|z\|}\right)^m\xi^m+\sum_{s=p}^{\infty}\frac{D^sf_j(0)(z_0^s)}{s!}\xi^p$$

from the hypothesis of Theorem 2.1, where $j$ satisfies $|z_j|=\|z\|=\max\limits_{1\leq l\leq n}\{|z_l|\}$. We write $b_m=a_j\left(\frac{z_j}{\|z\|}\right)^m, b_s=\frac{D^sf_j(0)(z_0^s)}{s!}, s=p,p+1,\cdots$. Then it is readily shown that $w(\xi)=b_m+\sum\limits_{s=p}^{\infty}b_s\xi^{s-m}\in H(U,\overline{U})$ due to $h_j\in h(U,\overline{U})$. In view of the fact $|b_s|\leq 1-|b_m|^2, s=p,p+1,\cdots$ (see [14]), we conclude that

$$\frac{|D^sf_j(0)(z_0^s)|}{s!}\leq 1-|a_j|^2=1-\|a\|^2\,.$$

Hence

$$\frac{|D^sf_l(0)(z_0^s)|}{s!}\leq 1-\|a\|^2, z_0\in\partial U^n, l=1,2,\cdots,n, s=p,p+1,\cdots$$

if $z_0 \in (\partial U)^n$. Since each $\frac{D^s f_l(0)(z^s)}{s!}(l = 1, 2, \cdots, n)$ is a holomorphic function on $\overline{U^n}$, it yields that

$$\frac{|D^s f_l(0)(z_0^s)|}{s!} \leq 1 - \|a\|^2, z_0 \in \partial U^n, l = 1, 2, \cdots, n, s = p, p+1, \cdots,$$

by the maximum modulus principle. This implies that

$$\frac{\|D^s f(0)(z^s)\|}{s!} \leq (1 - \|a\|^2)\|z\|^s \leq (1 - \|a\|^2) r_{p,m}^s, s = p, p+1, \cdots.$$

Therefore it follows that

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=p}^{\infty} \frac{\|D^s f(0)(z^s)\|}{s!} \leq \|a\| r_{p,m}^m + (1 - \|a\|^2) \frac{r_{p,m}^p}{1 - r_{p,m}}, \|z\| = r \leq r_{p,m}.$$

If $m = 0$, then we show that

$$\|a\| + (1 - \|a\|^2) \frac{r_{p,m}^p}{1 - r_{p,m}} \leq \|a\| + 2(1 - \|a\|) \frac{r_{p,m}^p}{1 - r_{p,m}} = \|a\| + 1 - \|a\| = 1. \tag{2.1}$$

If $p > 2m, m = 1, 2, \cdots$, then it yields that

$$\begin{aligned} &\|a\| r_{p,m}^m + (1 - \|a\|^2) \frac{r_{p,m}^p}{1 - r_{p,m}} \\ &\leq 1 - \frac{r_{p,m}^p}{1 - r_{p,m}} \left( \|a\| - \frac{1 - r_{p,m}}{2 r_{p,m}^{p-m}} \right)^2 \\ &+ \frac{4 r_{p,m}^{2(p-m)} + 4 r_{p,m}^{p+1-2m} - 4 r_{p,m}^{p-2m} + r_{p,m}^2 - 2 r_{p,m} + 1}{4(1 - r_{p,m}) r_{p,m}^{p-2m}} \\ &= 1 - \frac{r_{p,m}^p}{1 - r_{p,m}} \left( \|a\| - \frac{1 - r_{p,m}}{2 r_{p,m}^{p-m}} \right)^2 \leq 1. \end{aligned}$$

This implies that

$$\|a\| r_{p,m}^m + (1 - \|a\|^2) \frac{r_{p,m}^p}{1 - r_{p,m}} \leq 1. \tag{2.2}$$

If $m + 1 \leq p \leq 2m, m = 1, 2, \cdots$, we deduce that

$$\begin{aligned} &\|a\| r_{p,m}^m + (1 - \|a\|^2) \frac{r_{p,m}^p}{1 - r_{p,m}} \\ &\leq 1 - \frac{r_{p,m}^p}{1 - r_{p,m}} \left( \|a\| - \frac{1 - r_{p,m}}{2 r_{p,m}^{p-m}} \right)^2 \\ &+ \frac{4 r_{p,m}^p + r_{p,m}^{2+2m-p} - 2 r_{p,m}^{1+2m-p} + r_{p,m}^{2m-p} + 4 r_{p,m} - 4}{4(1 - r_{p,m})} \\ &= 1 - \frac{r_{p,m}^p}{1 - r_{p,m}} \left( \|a\| - \frac{1 - r_{p,m}}{2 r_{p,m}^{p-m}} \right)^2 \leq 1. \end{aligned}$$

That is

$$\|a\|r_{p,m}^m + (1-\|a\|^2)\frac{r_{p,m}^p}{1-r_{p,m}} \le 1\,. \tag{2.3}$$

In view of Lemma 2.1, (2.1), (2.2) and (2.3), we derive the desired result.

It is not difficult to check that

$$f(z) = \left(\frac{\lambda - z_1}{1-\lambda z_1}, \lambda, \cdots, \lambda\right)', z = (z_1, z_2, \cdots, z_n)' \in U^n$$

($\lambda \in [0,1]$, $l = 1, 2, \cdots, m$) satisfies the condition of Theorem 2.1. Putting $z = (r, 0, \cdots, 0)'$ $(0 \le r < 1)$, it is shown that

$$\|f(0)\| + \sum_{m=1}^{\infty} \frac{\|D^m f(0)(z)^m)\|}{m\,!} = \lambda + \sum_{m=1}^{\infty}(\lambda^{m-1} - \lambda^{m+1})r^m = \lambda + \frac{(1-\lambda^2)r}{1-\lambda r}$$

by a simple calculation. Therefore we show that $\lambda + \frac{(1-\lambda^2)r}{1-\lambda r} > 1$ if and only if

$$r > \frac{1}{1+2\lambda} = \frac{1}{1+2\|f(0)\|}\,.$$

Note that $\frac{1}{1+2\|f(0)\|} \to \frac{1}{3}$ as $\|f(0)\|_m \to 1-$. Then we see that s $r_{1,0} = \frac{1}{3}$ is optimal. This completes the proof. □

**Theorem 2.2.** *Let $m \in \mathbb{N}, p \ge m+1, f(z) = \frac{D^m f(0)(z^m)}{m!} + \sum\limits_{s=p}^{\infty} \frac{D^s f(0)(z^s)}{s!} \in H(U^n, \overline{U^n})$. If $\frac{|D^m f_l(0)(z^m)|}{m!} = \frac{\|D^m f(0)(z^m)\|}{m!}, l = 1, 2, \cdots, n$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=p}^{\infty} \frac{\|D^s f(0)(z^s)\|}{s!} \le 1$$

*for $\|z\| = r \le r_{p,m}$, where $r_{p,m}$ is the same as Theorem 2.1.*

**Proof.** Fix $z \in U^n \setminus \{0\}$, and denote $z_0 = \frac{z}{\|z\|}$. Defining $h_l(\xi) = f_l(\xi z_0), \xi \in U, l = 1, 2, \cdots, n$, then $h_l \in h(U, \overline{U})$, and

$$h_l(\xi) = \frac{D^m f_l(0)(z_0^m)}{m!}\xi^m + \sum_{s=p}^{\infty} \frac{D^s f_l(0)(z_0^s)}{s!}\xi^s$$

from the condition of Theorem 2.2. Hence we easily deduce that $w(\xi) = b_m + \sum\limits_{s=p}^{\infty} b_s \xi^{s-m} \in H(U, \overline{U})$ due to $h_l \in h(U, \overline{U})$. Note that $|b_s| \le 1 - |b_m|^2, s = p, p+1, \cdots$ (see [14]). It follows that

$$\frac{|D^s f_l(0)(z_0^s)|}{s!} \le 1 - \left(\frac{|D^m f_l(0)(z_0^m)|}{m!}\right)^2 = 1 - \left(\frac{\|D^m f(0)(z_0^m)\|}{m!}\right)^2, l = 1, 2, \cdots, n\,.$$

That is

$$\frac{\|D^s f(0)(z^s)\|}{s!} \le \left(1 - \left(\frac{\|D^m f(0)(z_0^m)\|}{m!}\right)^2\right)\|z\|^s \le \left(1 - \left(\frac{\|D^m f(0)(z_0^m)\|}{m!}\right)^2\right) r_{p,m}^s, s = p, p+1, \cdots.$$

Therefore it yields that

$$\frac{\|D^m f(0)(z^m)\|}{m!}+\sum_{s=p}^{\infty}\frac{\|D^s f(0)(z^s)\|}{s!}\le \|a\|r_{p,m}^m+(1-\|a\|^2)\frac{r_{p,m}^p}{1-r_{p,m}},\|z\|=r\le r_{p,m}\,,$$

where $a=\frac{\|D^m f(0)(z_0^m)\|}{m!}$. With the analogous arguments in the proof of Theorem 2.1, we derive the desired results. This completes the proof. □

**Theorem 2.3.** *Let $m,k\in\mathbb{N},0\le m\le k, f(z)=\frac{D^m f(0)(z^m)}{m!}+\sum_{s=1}^{\infty}\frac{D^{sk+m}f(0)(z^{sk+m})}{(sk+m)!}\in H(U^n,\overline{U^n})$. If $\frac{|D^m f_l(0)(z^m)|}{m!}=\frac{\|D^m f(0)(z^m)\|}{m!}, l=1,2,\cdots,n$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!}+\sum_{s=1}^{\infty}\frac{\|D^{sk+m}f(0)(z^{sk+m})\|}{(sk+m)!}\le 1$$

*for $\|z\|=r\le r_{k,m}$, where $r_{k,m}$ is the maximal positive root of the equation*

$$-6r^{k-m}+r^{2(k-m)}+8r^{2k}+1=0\,.$$

*Each $r_{k,m}$ is sharp.*

**Proof.** Fix $z\in U^n\setminus\{0\}$, and set $z_0=\frac{z}{\|z\|}$. Letting $h_l(\xi)=f_l(\xi z_0),\xi\in U,l=1,2,\cdots,n$, then $h_l\in H(U,\overline{U})$, and

$$h_l(\xi)=b_m\xi^m+\sum_{s=1}^{\infty}b_{sk+m}\xi^{sk+m}=\frac{D^m f_l(0)(z_0^m)}{m!}\xi^m+\sum_{s=1}^{\infty}\frac{D^{sk+m}f_l(0)(z_0^{sk+m})}{(sk+m)!}\xi^{sk+m}$$

from the hypothesis of Theorem 2.3. We write $\eta=\xi^k$, then it easily yields that $w(\eta)=c_0+\sum_{s=1}^{\infty}c_s\eta^s\in H(U,\overline{U})$ due to $h_l\in H(U,\overline{U})$, here $c_s=b_{sk+m}=\frac{D^{sk+m}f_l(0)(z_0^{sk+m})}{(sk+m)!}, s=0,1,\cdots$. Inspired by the idea from [18], we choose arbitrary $\rho>1$ which satisfies $\rho\cdot r_{k,m}\le 1$. Hence we show that

$$\begin{aligned}\sum_{s=1}^{\infty}|b_{sk+m}||\xi|^{sk}&=\sum_{s=1}^{\infty}|c_s||\xi|^{sk}\le\sum_{s=1}^{\infty}|c_s|r_{k,m}^{sk}\\&\le\sqrt{\sum_{s=1}^{\infty}|c_s|^2\rho^{sk}r_{k,m}^{sk}}\sqrt{\sum_{s=1}^{\infty}\rho^{-sk}r_{k,m}^{sk}}\\&\le\sqrt{r_{k,m}^k\rho^k\frac{(1-|c_0|^2)^2}{1-|c_0|^2r_{k,m}^k\rho^k}}\sqrt{\frac{r_{k,m}^k\rho^{-k}}{1-r_{k,m}^k\rho^{-k}}}\\&=\frac{r_{k,m}^k(1-|c_0|^2)}{\sqrt{1-|c_0|^2r_{k,m}^k\rho^k}}\frac{1}{\sqrt{1-r_{k,m}^k\rho^{-k}}}\end{aligned}\tag{2.4}$$

for $|\xi|\le r_{k,m}$ from [18], where $|c_0|=\frac{\|D^m f(0)(z_0^m)\|}{m!}$.

Also we conclude that

$$|\xi|^m\sum_{s=1}^{\infty}\frac{|D^{sk+m}f_l(0)(z_0^{sk+m})|}{(sk+m)!}|\xi|^{sk}\le r_{k,m}^m\frac{r_{k,m}^k(1-|c_0|^2)}{\sqrt{1-|c_0|^2r_{k,m}^k\rho^k}}\frac{1}{\sqrt{1-r_{k,m}^k\rho^{-k}}},l=1,2,\cdots,n$$

from (2.4). Consequently,

$$|\xi|^m \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z_0^{sk+m})\|}{(sk+m)!} |\xi|^{sk} \le r_{k,m}^m \frac{r_{k,m}^k (1-|c_0|^2)}{\sqrt{1-|c_0|^2 r_{k,m}^k \rho^k}} \frac{1}{\sqrt{1-r_{k,m}^k \rho^{-k}}} .$$

Taking $\xi = \|z\|$, it yields that

$$\sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z^{sk+m})\|}{(sk+m)!} \le r_{k,m}^m \frac{r_{k,m}^k (1-|c_0|^2)}{\sqrt{1-|c_0|^2 r_{k,m}^k \rho^k}} \frac{1}{\sqrt{1-r_{k,m}^k \rho^{-k}}} . \tag{2.5}$$

Now it suffices to discuss the two cases $|c_0| \ge r_{k,m}^k$ and $|c_0| < r_{k,m}^k$ respectively.

Case A: $|c_0| \ge r_{k,m}^k$. Putting $\rho = \frac{1}{\sqrt[k]{|c_0|}}$ in this case from (2.5), we deduce that

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z^{sk+m})\|}{(sk+m)!} \le r_{k,m}^m \left( |c_0| + \frac{r_{k,m}^k (1-|c_0|^2)}{1-|c_0| r_{k,m}^k} \right) .$$

A direct calculation shows that

$$\begin{aligned} & r_{k,m}^m \left( |c_0| + \frac{r_{k,m}^k (1-|c_0|^2)}{1-|c_0| r_{k,m}^k} \right) \\ &= \frac{r_{k,m}^m |c_0| - 2 r_{k,m}^{k+m} |c_0|^2 + r_{k,m}^{k+m}}{1-|c_0| r_{k,m}^k} \\ &= 1 + \frac{-1 + (r_{k,m}^m + r_{k,m}^k)|c_0| - 2 r_{k,m}^{k+m} |c_0|^2 + r_{k,m}^{k+m}}{1-|c_0| r_{k,m}^k} \\ &= 1 + \frac{1}{1-|c_0| r_{k,m}^k} \left( -2 r_{k,m}^{k+m} \left( |c_0| - \frac{r_{k,m}^m + r_{k,m}^k}{4 r_{k,m}^{k+m}} \right)^2 + \frac{-6 r_{k,m}^{k-m} + r_{k,m}^{2(k-m)} + 8 r_{k,m}^{2k} + 1}{8 r_{k,m}^{k-m}} \right) \\ &= 1 - \frac{2 r_{k,m}^{k+m}}{1-|c_0| r_{k,m}^k} \left( |c_0| - \frac{r_{k,m}^m + r_{k,m}^k}{4 r_{k,m}^{k+m}} \right)^2 \le 1 . \end{aligned}$$

That is

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z^{sk+m})\|}{(sk+m)!} \le 1, \|z\| \le r_{k,m} . \tag{2.6}$$

Case B: $|c_0| < r_{k,m}^k$. Taking $\rho = \frac{1}{r_{k,m}}$ in this case from (2.5), we conclude that

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z^{sk+m})\|}{(sk+m)!} \le r_{k,m}^m \left( |c_0| + \frac{r_{k,m}^k \sqrt{1-|c_0|^2}}{\sqrt{1-r_{k,m}^{2k}}} \right) .$$

Considering the function $H(t) = t + \frac{r_{k,m}^k \sqrt{1-t^2}}{\sqrt{1-r_{k,m}^{2k}}}, t \in [0, r_{k,m}^k]$, it is not difficult to verify that $H(t)$ is an increasing function on the interval $[0, r_{k,m}^k]$ because of the fact $H'(t) \ge 0, t \in [0, r_{k,m}^k]$. Therefore

$$r_{k,m}^m \left( |c_0| + \frac{r_{k,m}^k \sqrt{1-|c_0|^2}}{\sqrt{1-r_{k,m}^{2k}}} \right) \le r_{k,m}^m H(r_{k,m}^k) = 2 r_{k,m}^{k+m} \le 1$$

from Lemma 2.2. This implies that

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z^{sk+m})\|}{(sk+m)!} \le 1, \|z\| \le r_{k,m}\,. \tag{2.7}$$

Hence it follows the desired result from (2.6) and (2.7).

Consider the mapping

$$f(z) = \left(z_1^m \frac{z_1^k - a}{1 - az_1^k}, z_2^m \frac{z_2^k - a}{1 - az_2^k}, \cdots, z_n^m \frac{z_n^k - a}{1 - az_n^k}\right)', z \in U^n$$

with $a = r^{-k}\left(1 - \frac{\sqrt{1-r^{2k}}}{\sqrt{2}}\right)$, where $r = r_{k,m}$. Then $f$ satisfies the hypothesis of Theorem 2.3. Setting $|z_1| = |z_2| = \cdots = |z_2| = r$, it is shown that the number $r = r_{k,m}$ are sharp by the same way in the proof of [18][Theorem 1]. This completes the proof. □

Setting $\frac{D^m f_l(0)(z^m)}{m!} = a_l z_l^m, l = 1, 2, \cdots, n$, Theorem 2.3 gives the following Corollary readily.

**Corollary 2.1.** *Let $m, k \in \mathbb{N}, a = (a_1, a_2, \cdots, a_n)', 0 \le m \le k, f(z) = (a_1 z_1^m + g_1(z), a_2 z_2^m + g_2(z), \cdots, a_n z_n^m + g_n(z))' = \frac{D^m f(0)(z^m)}{m!} + \sum\limits_{s=1}^{\infty} \frac{D^{sk+m} f(0)(z^{sk+m})}{(sk+m)!} \in H(U^n, \overline{U^n})$, $|a_1 z_1^m| = \cdots = |a_n z_n^m|$, where $\frac{D^m f(0)(z^m)}{m!} = (a_1 z_1^m, a_2 z_2^m, \cdots, a_n z_n^m)'$. Then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} + \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(z^{sk+m})\|}{(sk+m)!} \le 1$$

*for $\|z\| \le r_{k,m}$, where $r_{k,m}$ is the maximal positive root of the equation*

$$-6r^{k-m} + r^{2(k-m)} + 8r^{2k} + 1 = 0\,.$$

*Each $r_{k,m}$ is sharp.*

## 3. Bohr inequality for holomorphic mappings with lacunary series for holomorphic mappings on the unit ball of complex Banach spaces

**Theorem 3.1.** *Let $m \in \mathbb{N}, p \ge m+1, f(x) = xg(x) = \frac{D^m f(0)(x^m)}{m!} + \sum\limits_{s=p}^{\infty} \frac{D^s f(0)(x^s)}{s!} \in H(B, \overline{B})$, where $g \in H(B, \mathbb{C})$. Then*

$$\frac{\|D^m f(0)(x^m)\|}{m!} + \sum_{s=p}^{\infty} \frac{\|D^s f(0)(x^s)\|}{s!} \le 1$$

*for $\|x\| = r \le r_{p,m}$, where $r_{p,m}$ is the same as Theorem 2.1.*

**Proof.** Fix $x \in B \setminus \{0\}$, and set $x_0 = \frac{x}{\|x\|}$. We define $h(\xi) = T_x(f(\xi x_0)), \xi \in U$. Then $h \in H(U, \overline{U})$,

$$h(\xi) = \xi^m \frac{D^{m-1} g(0)(x_0^{m-1})}{(m-1)!} + \sum_{s=p}^{\infty} \frac{D^{s-1} g(0)(x_0^{s-1})}{(s-1)!} \xi^s$$

by the condition of Theorem 3.1. Then it easily yields that $w(\xi) = b_m + \sum\limits_{s=p}^{\infty} b_s \xi^{s-m} \in H(U, \overline{U})$ because of $h \in H(U, \overline{U})$, where $b_m = \frac{D^{m-1}g(0)(x_0^{m-1})}{(m-1)!}, b_s = \frac{D^{s-1}g(0)(x_0^{s-1})}{(s-1)!}, s = p, p+1, \cdots$. Also $|b_s| \le 1 - |b_m|^2, s = p, p+1, \cdots$ (see [14]). It follows that

$$\frac{|D^{m-1}g(0)(x^{m-1})|}{(m-1)!} \le 1$$

and

$$\frac{|D^{s-1}g(0)(x^{s-1})|}{(s-1)!} \le \left(1 - \left(\frac{|D^{m-1}g(0)(x_0^{m-1})|}{(m-1)!}\right)^2\right) r_{p,m}^{s-1}, \|x\| = r \le r_{p,m}, s = p, p+1, \cdots .$$

We mention that

$$\frac{\|D^s f(0)(x^s)\|}{s!} = \frac{|D^{s-1}g(0)(x^{s-1})|}{(s-1)!}\|x\|, x \in B, s = m, p, p+1, \cdots$$

Therefore,

$$\frac{\|D^m f(0)(x^m)\|}{m!} + \sum_{s=p}^{\infty} \frac{\|D^s f(0)(x^s)\|}{s!} \le \frac{|D^{m-1}g(0)(x_0^{m-1})|}{(m-1)!} r_{p,m}^m + \left(1 - \left(\frac{|D^{m-1}g(0)(x_0^{m-1})|}{(m-1)!}\right)^2\right) \frac{r_{p,m}^p}{1 - r_{p,m}} .$$

Similar to that in the proof of Theorem 2.1, it follows the desired result. This completes the proof. □

**Theorem 3.2.** *Let $m \in \mathbb{N}, p \ge m+1, f(x) = \frac{D^m f(0)(x^m)}{m!} + \sum\limits_{s=p}^{\infty} \frac{D^s f(0)(x^s)}{s!} \in H(B, \overline{B})$. Then*

$$\frac{|T_u(D^m f(0)(x^m))|}{m!} + \sum_{s=p}^{\infty} \frac{|T_u(D^s f(0)(x^s))|}{s!} \le 1$$

*for $\|x\| = r \le r_{p,m}$, where $r_{p,m}$ is the same as Theorem 2.1, $u$ is fixed, and $\|u\| = 1$.*

**Proof.** Let $x \in B \setminus \{0\}$ be fixed, and we write $x_0 = \frac{x}{\|x\|}$. Define $h(\xi) = T_u(f(\xi x_0)), \xi \in U$, where $u$ is fixed, and $\|u\| = 1$. Then $h \in H(U, \overline{U})$,

$$h(\xi) = \xi^m \frac{T_u(D^m f(0)(x_0^m))}{m!} + \sum_{s=p}^{\infty} \frac{T_u(D^s f(0)(x_0^s))}{s!} \xi^s$$

from the hypothesis of Theorem 3.2. Then we conclude that $w(\xi) = b_m + \sum\limits_{s=p}^{\infty} b_s \xi^{s-m} \in H(U, \overline{U})$ due to $h \in H(U, \overline{U})$, where $b_m = \frac{T_u(D^m f(0)(x_0^m))}{m!}, b_s = \frac{T_u(D^s f(0)(x_0^s))}{s!}, s = p, p+1, \cdots$. Also $|b_s| \le 1 - |b_m|^2, s = p, p+1, \cdots$ (see [14]). It yields that

$$\frac{|T_u(D^m f(0)(x_0^m))|}{m!} \le 1$$

and

$$\frac{|T_u(D^s f(0)(x^s))|}{s!} \le \left(1 - \left(\frac{|T_u(D^m f(0)(x_0^m))|}{m!}\right)^2\right) r_{p,m}^s, \|x\| = r \le r_{p,m}, s = p, p+1, \cdots .$$

Consequently,

$$\frac{|T_u(D^m f(0)(x^m))|}{m!}+\sum_{s=p}^{\infty}\frac{|T_u(D^s f(0)(x^s))|}{s!}$$
$$\le \frac{|T_u(D^m f(0)(x_0^m))|}{m!}r_{p,m}^m+\left(1-\left(\frac{|T_u(D^m f(0)(x_0^m))|}{m!}\right)^2\right)\frac{r_{p,m}^p}{1-r_{p,m}}.$$

With analogous arguments in the proof of Theorem 2.1, we drive the desired result, as follows. This completes the proof. □

**Theorem 3.3.** *Let $m,k\in\mathbb{N}, 0\le m\le k, f(x)=xg(x)=\frac{D^m f(0)(x^m)}{m!}+\sum_{s=1}^{\infty}\frac{D^{sk+m}f(0)(x^{sk+m})}{(sk+m)!}\in H(B,\overline{B})$, where $g\in H(B,\mathbb{C})$. Then*

$$\frac{\|D^m f(0)(x^m)\|}{m!}+\sum_{s=1}^{\infty}\frac{\|D^{sk+m}f(0)(x^{sk+m})\|}{(sk+m)!}\le 1$$

*for $\|x\|\le r_{k,m}$, where $r_{k,m}$ is the same as Theorem 2.3.*

**Proof.** Let $x\in B\setminus\{0\}$ be fixed, and put $x_0=\frac{x}{\|x\|}$. We define $h(\xi)=T_x(f(\xi x_0)), \xi\in U$. Then $h\in H(U,\overline{U})$,

$$h(\xi)=\xi^m\frac{D^{m-1}g(0)(x_0^{m-1})}{(m-1)!}+\sum_{s=1}^{\infty}\frac{D^{sk+m-1}g(0)(x_0^{sk+m-1})}{(sk+m-1)!}\xi^{sk+m}$$

by the condition of Theorem 3.3. Then it readily yields that $w(\xi)=b_m+\sum_{s=1}^{\infty}b_{sk+m}\xi^{sk}\in H(U,\overline{U})$ because of $h\in H(U,\overline{U})$, where $b_m=\frac{D^{m-1}g(0)(x_0^{m-1})}{(m-1)!}, b_{sk+m}=\frac{D^{sk+m-1}g(0)(x_0^{sk+m-1})}{(sk+m-1)!}, s=1,2,\cdots$. Denote $\eta=\xi^s$. Then it is shown that $w(\eta)=c_0+\sum_{s=1}^{\infty}c_s\eta^s\in H(U,\overline{U})$ due to $h\in H(U,\overline{U})$, here $c_s=b_{sk+m}, s=0,1,\cdots$. Choose arbitrary $\rho>1$ which satisfies $\rho\cdot r_{k,m}\le 1$. Hence it follows that

$$\sum_{s=1}^{\infty}|b_{sk+m}||\xi|^{sk}=\sum_{s=1}^{\infty}|c_s||\xi|^{sk}\le\sum_{s=1}^{\infty}|c_s|r_{k,m}^{sk}\le\frac{r_{k,m}^k(1-|c_0|^2)}{\sqrt{1-|c_0|^2r_{k,m}^k\rho^k}}\frac{1}{\sqrt{1-r_{k,m}^k\rho^{-k}}}\tag{3.1}$$

for $|\xi|\le r_{k,m}$ by the same way in the proof of Theorem 3.1.

Also according to (3.1), we deduce that

$$|\xi|^m\sum_{s=1}^{\infty}\frac{|D^{sk+m-1}g(0)(x_0^{sk+m-1})|}{(sk+m-1)!}|\xi|^{sk}\le r_{k,m}^m\frac{r_{k,m}^k(1-|c_0|^2)}{\sqrt{1-|c_0|^2r_{k,m}^k\rho^k}}\frac{1}{\sqrt{1-r_{k,m}^k\rho^{-k}}}.$$

Note that

$$\frac{\|D^{sk+m}f(0)(x^{sk+m})\|}{(sk+m)!}=\frac{|D^{sk+m-1}g(0)(x^{sk+m-1})|}{(sk+m-1)!}\|x\|, x\in B, s=0,1,2,\cdots.$$

Thus

$$|\xi|^m\sum_{s=1}^{\infty}\frac{\|D^{sk+m}f(0)(x_0^{sk+m})|}{(sk+m)!}|\xi|^{sk}\le r_{k,m}^m\frac{r_{k,m}^k(1-|c_0|^2)}{\sqrt{1-|c_0|^2r_{k,m}^k\rho^k}}\frac{1}{\sqrt{1-r_{k,m}^k\rho^{-k}}}.$$

Set $\xi = \|x\|$. Then it is shown that

$$\frac{\|D^m f(0)(x^m)\|}{m!} + \sum_{s=1}^{\infty} \frac{\|D^{sk+m} f(0)(x^{sk+m})\|}{(sk+m)!} \leq 1$$

for $\|x\| \leq r_{k,m}$ by the same method in the proof of [18], where $r_{k,m}$ is the same as Theorem 3.1.

Consider the mapping

$$f(x) = x T_u^{m-1}(x) \frac{T_u^k(x) - a}{1 - a T_u^k(x)}, x \in B$$

with $a = r^{-k}\left(1 - \frac{\sqrt{1-r^{2k}}}{\sqrt{2}}\right)$, where $r = r_{k,m}$, $u$ is fixed, and $\|u\| = 1$. Putting $x = ru$, it follows that the number $r = r_{k,m}$ are sharp by the same approach in the proof of [18][Theorem 1]. This completes the proof. $\square$

**Theorem 3.4.** *Let $m, k \in \mathbb{N}, 0 \leq m \leq k, f(x) = \frac{D^m f(0)(x^m)}{m!} + \sum_{s=1}^{\infty} \frac{D^{sk+m} f(0)(x^{sk+m})}{(sk+m)!} \in H(B, \overline{B})$. Then*

$$\frac{|T_u(D^m f(0)(x^m))|}{m!} + \sum_{s=1}^{\infty} \frac{|T_u(D^{sk+m} f(0)(x^{sk+m}))|}{(sk+m)!} \leq 1$$

*for $\|x\| \leq r_{k,m}$, where $r_{k,m}$ is the same as Theorem 2.3, $u$ is fixed, and $\|u\| = 1$.*

**Proof.** Let $x \in B \setminus \{0\}$ be fixed, and we write $x_0 = \frac{x}{\|x\|}$. Define $h(\xi) = T_u(f(\xi x_0)), \xi \in U$, where $u$ is fixed, and $\|u\| = 1$. Then $h \in H(U, \overline{U})$, and

$$h(\xi) = \xi^m \frac{T_u(D^m f(0)(x_0^m))}{m!} + \sum_{s=1}^{\infty} \frac{T_u(D^{sk+m} f(0)(x_0^{sk+m}))}{(sk+m)!} \xi^{sk+m}$$

from the hypothesis of Theorem 3.4. Then we conclude that $w(\xi) = b_m + \sum_{s=1}^{\infty} b_{sk+m} \xi^{sk} \in H(U, \overline{U})$ due to $h \in H(U, \overline{U})$, where $b_m = \frac{T_u(D^m f(0)(x_0^m))}{m!}, b_{sk+m} = \frac{T_u(D^{sk+m} f(0)(x_0^{sk+m}))}{(sk+m)!}, s = 1, 2, \cdots$. Set $\eta = \xi^k$. Then it is easily known that $v(\eta) = c_0 + \sum_{s=1}^{\infty} c_s \eta^s \in H(U, \overline{U})$ because of $h \in H(U, \overline{U})$, here $c_s = b_{sk+m}, s = 0, 1, \cdots$. Choose any $\rho > 1$ which satisfies $\rho \cdot r_{k,m} \leq 1$. Then we conclude that

$$\sum_{s=1}^{\infty} |b_{sk+m}||\xi|^{sk} = \sum_{s=1}^{\infty} |c_s||\xi|^{sk} \leq \sum_{s=1}^{\infty} |c_s| r_{k,m}^{sk} \leq \frac{r_{k,m}^k (1 - |c_0|^2)}{\sqrt{1 - |c_0|^2 r_{k,m}^k \rho^k}} \frac{1}{\sqrt{1 - r_{k,m}^k \rho^{-k}}} \tag{3.2}$$

for $|\xi| \leq r_{k,m}$ from [18].

Also from (3.2), it yields that

$$|\xi|^m \sum_{s=1}^{\infty} \frac{|T_u(D^{sk+m} f(0)(z_0^{sk+m}))|}{(sk+m)!} |\xi|^{sk} \leq \frac{r_{k,m}^k (1 - |c_0|^2)}{\sqrt{1 - |c_0|^2 r_{k,m}^k \rho^k}} \frac{1}{\sqrt{1 - r_{k,m}^k \rho^{-k}}}.$$

Putting $\xi = \|x\|$, then it yields that

$$\frac{|T_u(D^m f(0)(x^m))|}{m!} + \sum_{s=1}^{\infty} \frac{|T_u(D^{sk+m} f(0)(x^{sk+m}))|}{(sk+m)!} \leq 1$$

for $\|x\| \leq r_{k,m}$ by the same arguments in the proof of Theorem 2.3. The arguments concerning sharpness of the number $r = r_{k,m}$ are the same as Theorem 3.3. This completes the proof. $\square$

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