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Effects of depth and evolving rate on phytoplankton growth in a periodically evolving environment ☆

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ABSTRACT

In this paper, we consider a single phytoplankton species which relies on the light for maintaining the metabolism of life in a periodically evolving environment, where the light intensity and the death rate depend on the water column depth triggered by seasonal variation. Based on the basic reproduction number $\mathcal{R}_0$, a threshold type result on the dynamics of the model is established. Especially, various features of $\mathcal{R}_0$ with respect to the vertical turbulent diffusion rate, the buoyant or sinking rate, and the evolution rate of water column depth are derived. Our theoretical results and numerical simulations show that big evolution rate, vertical diffusion rate and water column depth all have an adverse effect on survival of phytoplankton.

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1. Introduction

Phytoplankton populations include a diverse representation of both microscopic eukaryotic algae and prokaryotic cyanobacteria. They are major oxygen producers and also the initial food source for numerous food webs in freshwater, estuarine, and marine waters [22]. Since they transport significant amounts of atmospheric carbon dioxide into the deep oceans, they play a crucial role in the climate dynamics. Its growth and reproduction are not only influenced by its own biological characteristics, but also by substantial environmental factors such that light, water flow, temperature, salinity, nutrient, stress and disturbance, etc.

Influence mechanism of the above factors on phytoplankton growth is an essential and important subject in mathematical ecology. Edwards et al. [9] proposed that light limitation diminishes the temperature sensitivity of bulk phytoplankton growth, even though community structure will be temperature-sensitive. To describe the interactions between photosynthesis and light for phytoplankton, Jassby and Platt [14] gave

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a mathematical formulation, which is most consistent with the data. A mathematical model was proposed by Rhee and Gotham [31] to characterize the growth of two types of phytoplankton, non-nitrogen-fixing and nitrogen-fixing, which both require light in order to grow. Considering the effect of flow velocity, Li et al. [19] showed that phytoplankton biomass and spatial distribution largely depend on the flow condition by the field observation, and turbulent flow has the inhibition effect on phytoplankton biomass, but less impact on composition by enclosure experiments. Chellappa et al. [3] studied a phytoplankton growth with stress and disturbance, which is stimulated by variable hydrology, and indicated that stress and disturbance factors determine phytoplankton diversity, composition and chlorophyll levels.

Temperature, as a physical factor, is of supreme significance to affect the growth of phytoplankton. The earlier work was discussed by Canale and Vogel [2]. Later, Rhee and Gotham [25] investigated the simultaneous effects of nutrient limitation and suboptimal temperature on growth and nutrient uptake kinetics by continuous-culture techniques. Based on patterns observed in nature, Agawin et al. [1] studied the combined effects of both temperature and nutrient concentrations, and tested the relative importance of phytoplankton in warm and nutrient-poor waters. Considering phytoplankton living in freshwater and brackish water, Lionard et al. [20] concluded that increasing or decreasing of salinity brings negative impart on the growth of phytoplankton.

In this paper, we consider the self-shading model with unlimited nutritional environment proposed by Shigesada and Okubo [26], the governing equation is

$$u_t = Du_{xx} - \alpha u_x + F(u) - \lambda u,$$

where $F(u) = \int_0^u f(\frac{I_0}{I_m}e^{-x})dx$, the function f describes the dependence of production rate, and depends on the incident light intensity I_0 and a characteristic light intensity I_m. λ is the ratio of mortality to maximum growth rate. The model has been receiving extensive investigations, see for example, Ebert et al. [8], Kolokolnikov, Ou and Yuan [17], Du and Hsu [6], and Hsu and Lou [12] and the references therein.

In recent years, many mathematical theories have been developed for single-species phytoplankton model. For the question left open in [12], a complete answer has been given by Du and Mei [7], they investigated the global dynamics of the single species model for the case $D = D(x), \alpha = \alpha(x)$, and by applying the comparison principle, Hopf boundary lemma and standard parabolic regularity, the asymptotic profiles of the positive steady states for small or large diffusion and deep water column are given. Because of seasonal changes and diurnal light cycle, Peng and Zhao [23,24] considered the effect of time-periodic light intensity I_0 at the surface. Based on the model in [23,24], Ma and Ou [21] assumed that $D(t)$ are $\alpha(t)$ are time-periodic functions. By the comparison principle, they obtained the globally asymptotic stability of the zero solution when the principal eigenvalue λ_1 is less than zero. For $\lambda_1 > 0$, by transforming the model into a new system, they proved the conjecture in previous studies on the uniqueness and attractivity of the positive periodic solution. A nonlocal reaction-diffusion-advection system modeling the growth of phytoplankton with cell quota structure has been studied by Hsu, Mei and Wang [13], they first established the existence of a critical death rate, and indicated that the phytoplankton survives if and only if its death rate is less than the critical death rate by the strong maximum principle and the standard parabolic regularity theory.

For the above work, what causes our concern is the work of Peng and Zhao in [24]. In order to understand the evolution of a single phytoplankton species which depends merely on light for its metabolism in a eutrophic and vertical water bodies, they developed the model and studied a nonlocal reaction-diffusion-advection model of a single phytoplankton species

$$\begin{cases} u_t = Du_{xx} - \alpha u_x + u[g(I(x,t)) - d(x,t)], & 0 < x < L_0,\ t > 0, \\ Du_x(x,t) - \alpha u(x,t) = 0, & x = 0, L_0,\ t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & 0 < x < L_0,\ t > 0, \end{cases} \tag{1.1}$$

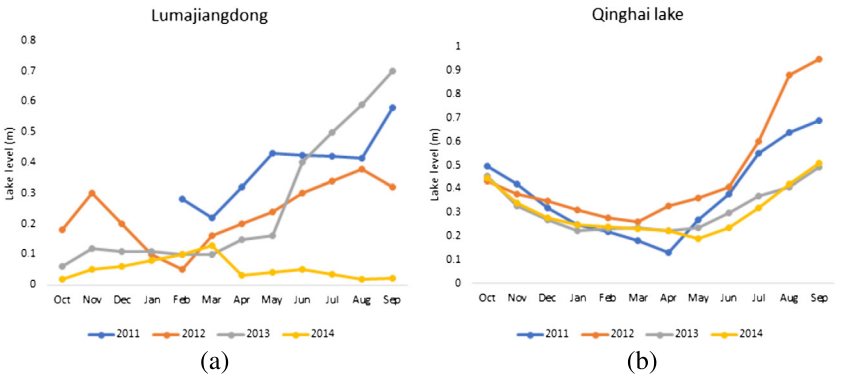


Fig. 1. Figures (a) and (b) describe level changes of Lumajiangdong and Qinghai lake in a hydrological year (October to September) from 2011 to 2014, respectively. These data are from [18] (minor modification). (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

where light intensity $I(x,t) = I_0 e^{-k_0 x - k_1 \int_0^x u(s,t)ds}$, $I_0 = I_0(t) \geq 0, \not\equiv 0$, I_0 is $T-$periodic in t for some $T > 0$ and denotes the incident light intensity, k_0 and k_1 are positive constants which express the background turbidity and the absorption coefficient of the phytoplankton species respectively. Specific growth rate $g(I)$ is a function of the light intensity $I(x,t)$ and g is strictly increasing with $g(0) = 0$. The death rate $d(x,t)$ is a $T-$periodic in t and satisfies that either $d_x(x,t) \geq 0, \not\equiv 0$ on $[0, L_0] \times [0, T]$ and $k \geq 0$, or $d(x,t) \equiv d(t)$ and $k_0 > 0$. x stands for the depth from 0 (the top) to L_0 (the bottom) in a column of water. They investigated persistence and extinction of phytoplankton species in accordance with the basic reproduction number $\mathcal{R}_0$, which depends on the water column depth L, the vertical turbulent diffusion rate D and the settling (or buoyant) rate $\alpha > 0$ (or < 0).

As is well-known, lake depth changes periodically, and the periodic annual changes mainly depend on the replenishment of lake water. Lakes supply comes from precipitation, such as Poyang lake, which is China's largest fresh water lake, has the highest water level in rainy season and the lowest in dry season [32]. Plateau lakes are mainly replenished by melting water from snow and ice, take Lumajiangdong and Qinghai lake for examples. Trends in lake level of lakes across the Tibetan Plateau in a hydrological year are given in [18], it can be easily seen from Fig. 1 that the lakes have striking characteristics, the water level of lakes in summer is significantly higher than that in winter. The rivers, lakes and seas, where phytoplankton species habitats, are evolving, therefore studying the dispersal of phytoplankton in an evolving environment is essential for understanding phytoplankton growth in biological systems.

As in [4,5,29], let $(0, L(t))$ be a bounded evolving interval at time $t > 0$ with its shifting depth $L(t)$. For any point $x(t) \in (0, L(t))$, we assume that $u(x(t), t)$ is the population density of the phytoplankton species at position $x(t)$ and time $t > 0$.

Let $x(t) = a(t)$ and $x(t) = b(t)(a(t) < b(t))$ denote arbitrary two depth within the water column where $x(t)$ varies from 0 (the top) to $L(t)$ (the bottom), obviously, $(a(t), b(t)) \subset (0, L(t))$ be a bounded evolving interval at time $t > 0$ with its moving top $x(t) = a(t)$ and bottom $x(t) = b(t)$. According to the law of mass conservation, one sees immediately that

$$\begin{aligned} \frac{d}{dt}\int_{a(t)}^{b(t)} u(x(t),t)dx &= Du_x(b(t),t) - Du_x(a(t),t) \\ &\quad -\alpha \int_{a(t)}^{b(t)} u_x(x(t),t)dx + \int_{a(t)}^{b(t)} f(u(x(t),t))dx \\ &= \int_{a(t)}^{b(t)} (Du_{xx} - \alpha u_x + f(u))dx, \end{aligned} \tag{1.2}$$

where $\frac{d}{dt}\int_{a(t)}^{b(t)} u(x(t),t)dx$ represents the change rate of phytoplankton population density in the interval $(a(t), b(t))$, the term $Du_x(b(t),t) - Du_x(a(t),t)$ denotes the net flow through the boundary (flows in from

boundary $x = a(t)$ and out of boundary $x = b(t)$), $-\alpha \int_{a(t)}^{b(t)} u_x(x(t), t)dx$ is the net increase of the advection term, and $\int_{a(t)}^{b(t)} f(u(x(t), t))dx$ represents the net increase of the reaction term in the interval $(a(t), b(t))$.

Equation (1.2) can be simplified if we suppose that the region evolution is isotropic and uniform. That is, the evolution of the interval takes place at the same proportion as time elapses. Mathematically, $x(t)$ can be described as follows

$$x(t) = \rho(t)y, \quad y \geq 0, \tag{1.3}$$

where the positive continuous function $\rho(t)$ is called evolving rate subject to $\rho(0) = 1$. Let $L_0 = L(0)$, we have evolving interval $(0, L(t)) = (0, \rho(t)L_0)$. Set $a(t) = \rho(t)y_1$ and $b(t) = \rho(t)y_2$, therefore $y_1, y_2 \in (0, L_0)$. Furthermore, if $\rho(t) = \rho(t + T)$ for some $T > 0$, the habitat is periodically evolving, while if $\dot{\rho}(t) \geq 0$, we say it is growing and if $\dot{\rho}(t) \leq 0$, it is called shrinking habitat.

Noticing that $u(x(t), t) = u(\rho(t)y, t)$, we then define $u(x(t), t) = v(y, t)$. From the left side of (1.2), we get

$$\begin{aligned} \tfrac{d}{dt}\int_{a(t)}^{b(t)} u(x(t), t)dx &= \tfrac{d}{dt}\int_{\rho(t)y_1}^{\rho(t)y_2} u(\rho(t)y, t)d(\rho(t)y) \\ &= \tfrac{d}{dt}\int_{y_1}^{y_2} v(y, t)\rho(t)dy \\ &= \int_{y_1}^{y_2} (v_t(y, t)\rho(t) + v(y, t)\dot{\rho}(t))dy. \end{aligned} \tag{1.4}$$

Using (1.3) yields

$$u_{xx} = \frac{1}{\rho^2(t)} v_{yy}, \quad u_x = \frac{1}{\rho(t)} v_y,$$

then the right side of (1.2) becomes

$$\int_{a(t)}^{b(t)} (Du_{xx} - \alpha u_x + f(u))dx = \int_{y_1}^{y_2} (\tfrac{D}{\rho^2(t)} v_{yy} - \tfrac{\alpha}{\rho(t)} v_y + f(v(y, t)))\rho(t)dy. \tag{1.5}$$

We conclude from (1.2), (1.4) and (1.5) that

$$\int_{y_1}^{y_2} (v_t + \frac{\dot{\rho}(t)}{\rho(t)} v)dy = \int_{y_1}^{y_2} (\frac{D}{\rho^2(t)} v_{yy} - \frac{\alpha}{\rho(t)} v_y + f(v, y, t))dy, \quad t > 0. \tag{1.6}$$

According to arbitrariness of $a(t)$ and $b(t)$, (1.6) holds for any $y_1, y_2 \in (0, L_0)$. On account of its boundary condition, we derive the following problem in a fixed interval

$$\begin{cases} v_t = \frac{D}{\rho^2(t)} v_{yy} - \frac{\alpha}{\rho(t)} v_y + [g(I(\rho(t)y, t)) - d(\rho(t)y, t) - \frac{\dot{\rho}(t)}{\rho(t)}]v, & 0 < y < L_0,\ t > 0, \\ Dv_y - \alpha\rho(t)v(y, t) = 0, & y = 0, L_0,\ t > 0 \end{cases} \tag{1.7}$$

with the initial condition

$$v(y, 0) = u_0(y) \geq 0, \not\equiv 0, \quad 0 < y < L_0. \tag{1.8}$$

For later application, we also consider problem (1.7) with the periodic condition

$$v(y, 0) = v(y, T), \quad 0 < y < L_0. \tag{1.9}$$

In order to study the impact of D and α, we make a transformation. Let $z(y,t) = e^{-\frac{\alpha}{D}\rho(t)y}v(y,t)$, (1.7) with (1.8) turns to

$$\begin{cases} z_t = \frac{D}{\rho^2(t)}z_{yy} + \frac{\alpha}{\rho(t)}z_y + [g(I(\rho(t)y,t)) - d(\rho(t)y,t) - \frac{\dot{\rho}(t)}{\rho(t)} - \frac{\alpha y\dot{\rho}(t)}{D}]z, & 0 < y < L_0,\ t > 0, \\ z_y(y,t) = 0, & y = 0, L_0,\ t > 0, \\ z(y,0) = u_0(y) \geq 0, \not\equiv 0, & 0 < y < L_0. \end{cases} \tag{1.10}$$

The present paper is build up as follows. In Section 2, we define the basic reproduction number based on the principal eigenvalue, and a comparison principle for the basic reproduction number is established. Moreover, various features of $\mathcal{R}_0$ with respect to the vertical turbulent diffusion rate and the water column depth are derived. Section 3 is devoted to the asymptotic behavior of solution. The paper ends with some simulations and ecological significance for our analytical findings.

2. The basic reproduction number and comparison principle

In this section, we define the basic reproduction number $\mathcal{R}_0$, and then investigate its threshold-type dynamics for the system (1.7), (1.8) in terms of $\mathcal{R}_0$. To do so, we first consider the corresponding periodic problem.

Obviously, $v^* \equiv 0$ is an equilibrium solution of the periodic problem (1.7), (1.9), a routine computation gives rise to the corresponding linearized system of problem (1.7) at v^*,

$$\begin{cases} w_t = \frac{D}{\rho^2(t)}w_{yy} - \frac{\alpha}{\rho(t)}w_y + g(I_0e^{-k_0\rho(t)y})w - (d(\rho(t)y,t) + \frac{\dot{\rho}(t)}{\rho(t)})w, & 0 < y < L_0,\ t > 0, \\ Dw_y - \alpha\rho(t)w(y,t) = 0, & y = 0, L_0,\ t > 0. \end{cases} \tag{2.1}$$

Let $\eta(t,s)$ be the evolution operator of the problem

$$\begin{cases} w_t = \frac{D}{\rho^2(t)}w_{yy} - \frac{\alpha}{\rho(t)}w_y - (d(\rho(t)y,t) + \frac{\dot{\rho}(t)}{\rho(t)})w, & 0 < y < L_0,\ t > 0, \\ Dw_y - \alpha\rho(t)w(y,t) = 0, & y = 0, L_0,\ t > 0, \end{cases} \tag{2.2}$$

because of $d(\rho(t)y,t) > 0$ for $(y,t) \in [0,L_0] \times [0,T]$, $\int_s^t \frac{\dot{\rho}(z)}{\rho(z)}dz$ is bounded for $\forall\, t \geq s$. On account of the standard semigroup theory, it has become obviously that there exist positive constants K and c_0 such that

$$\|\eta(t,s)\| \leq Ke^{-c_0(t-s)}, \quad \forall\, t \geq s,\ t,s \in \mathbb{R}.$$

Let C_T be the ordered Banach space consisting of all $T-$ periodic and continuous function from $\mathbb{R}$ to $C([0,L_0],\mathbb{R})$ with the maximum norm $\|\cdot\|$ and the positive cone $C_T^+ := \{\mu \in C_T : \sigma(y) \geq 0, \forall\, t \in \mathbb{R}, y \in [0,L_0]\}$. The notation $\sigma(y,t) := \sigma(t)(y)$ will be adopted for any given $\sigma \in C_T$. According to [33], a linear operator A on C_T is introduced

$$A(\sigma)(t) := \int_0^\infty \eta(t,t-s)g(I_0(t-a)e^{-k_0\cdot})\sigma(\cdot,t-s)ds.$$

From the hypothesis of g and I_0, it is quite clear that A is positive, continuous and compact on C_T. We define the spectral radius of A

$$\mathcal{R}_0 = \rho(A)$$

as the basic reproduction number for periodic system (1.7), (1.9). In order to describe the basic reproduction number more clearly, the following two eigenvalue problems are given

$$\begin{cases} \phi_t = \frac{D}{\rho^2(t)}\phi_{yy} - \frac{\alpha}{\rho(t)}\phi_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mu} - d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]\phi, & 0<y<L_0,\ 0<t\le T,\\ D\phi_y - \alpha\rho(t)\phi(y,t) = 0, & y=0,L_0,\ 0<t\le T,\\ \phi(y,0) = \phi(y,T), & 0<y<L_0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \psi_t = \frac{D}{\rho^2(t)}\psi_{yy} - \frac{\alpha}{\rho(t)}\psi_y + [g(I_0(t)e^{-k_0\rho(t)y} \\ \qquad -d(\rho(t)y,t)) - \frac{\dot\rho(t)}{\rho(t)}]\psi + \lambda\psi, & 0<y<L_0,\ 0<t\le T,\\ D\psi_y - \alpha\rho(t)\psi(y,t) = 0, & y=0,L_0,\ 0<t\le T,\\ \psi(y,0) = \psi(y,T), & 0<y<L_0, \end{cases} \tag{2.4}$$

where (μ_0,ϕ) and (λ_0,ψ) are the principal eigen-pairs of eigenvalue problems (2.3) and (2.4), respectively. With above definitions, we have the following result, see similar results in [24].

Lemma 2.1. (i) $\mathcal{R}_0 = \mu_0$. (ii) $\mathrm{sign}(1-\mathcal{R}_0) = \mathrm{sign}\lambda_0$.

Further, based on (1.10), $(\mathcal{R}_0, \varphi(y,t))$ with $\varphi(y,t) = e^{-\frac{\alpha}{D}\rho(t)y}\phi(y,t)$ is the principal eigen-pair of the following periodic-parabolic eigenvalue problem

$$\begin{cases} \varphi_t = \frac{D}{\rho^2(t)}\varphi_{yy} + \frac{\alpha}{\rho(t)}\varphi_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0} \\ \qquad -d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)} - \frac{\alpha y\dot\rho(t)}{D}]\varphi, & 0<y<L_0,\ 0<t\le T,\\ \varphi_y(y,t) = 0, & y=0,L_0,\ 0<t\le T,\\ \varphi(y,0) = \varphi(y,T), & 0<y<L_0, \end{cases} \tag{2.5}$$

which can be written as the following

$$\begin{cases} (e^{\frac{\alpha}{D}\rho(t)y}\varphi)_t = \frac{D}{\rho^2(t)}(e^{\frac{\alpha}{D}\rho(t)y}\varphi_y)_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0} \\ \qquad -d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]e^{\frac{\alpha}{D}\rho(t)y}\varphi, & 0<y<L_0,\ 0<t\le T,\\ \varphi_y(y,t) = 0, & y=0,L_0,\ 0<t\le T,\\ \varphi(y,0) = \varphi(y,T), & 0<y<L_0. \end{cases} \tag{2.6}$$

Lemma 2.2 *(Comparison principle).* (i) *If there is a nonnegative nontrivial function* $\tilde z \in C^{2,1}([0,L_0]\times[0,T])$, *such that*

$$\begin{cases} (e^{\frac{\alpha}{D}\rho(t)y}\tilde z)_t \ge \frac{D}{\rho^2(t)}(e^{\frac{\alpha}{D}\rho(t)y}\tilde z_y)_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_1} \\ \qquad -d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]e^{\frac{\alpha}{D}\rho(t)y}\tilde z, & 0<y<L_0,\ 0<t\le T,\\ \tilde z_y(0,t)\le 0,\ \tilde z_y(L_0,t)\ge 0 & 0<t\le T,\\ \tilde z(y,0)\ge \tilde z(y,T), & 0<y<L_0, \end{cases} \tag{2.7}$$

for some real number $\mathcal{R}_1$, *then* $\mathcal{R}_0\le\mathcal{R}_1$ *and the equality holds only when* $(\mathcal{R}_0,\tilde z)$ *is the principal eigen-pair of eigenvalue problem* (2.5).

(ii) *If there is a nonnegative nontrivial function* $\hat z\in C^{2,1}([0,L_0]\times[0,T])$, *such that*

$$\begin{cases} (e^{\frac{\alpha}{D}\rho(t)y}\hat z)_t \le \frac{D}{\rho^2(t)}(e^{\frac{\alpha}{D}\rho(t)y}\hat z_y)_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_2} \\ \qquad -d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]e^{\frac{\alpha}{D}\rho(t)y}\hat z, & 0<y<L_0,\ 0<t\le T,\\ \hat z_y(0,t)\ge 0,\ \hat z_y(L_0,t)\le 0 & 0<t\le T,\\ \hat z(y,0)\le \hat z(y,T), & 0<y<L_0, \end{cases} \tag{2.8}$$

for some real number $\mathcal{R}_2$, then $\mathcal{R}_0 \geq \mathcal{R}_2$ and the equality holds only when $(\mathcal{R}_0, \hat{z})$ is the principal eigen-pair of eigenvalue problem (2.5).

Proof. We only verify the assertion (i), because (ii) can be proved in the same manner. From [11], one knows that for the adjoint problem of (2.6)

$$\begin{cases} -e^{\frac{\alpha}{D}\rho(t)y}w_t = \frac{D}{\rho^2(t)}(e^{\frac{\alpha}{D}\rho(t)y}w_y)_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0} \\ \qquad\qquad -d(\rho(t)y,t) - \frac{\dot{\rho}(t)}{\rho(t)}]e^{\frac{\alpha}{D}\rho(t)y}w, & 0<y<L_0,\ 0<t\leq T, \\ w_y(y,t) = 0, & y = 0, L_0,\ 0<t\leq T, \\ w(y,0) = w(y,T), & 0<y<L_0, \end{cases} \tag{2.9}$$

which is equivalent to

$$\begin{cases} -w_t = \frac{D}{\rho^2(t)}w_{yy} + \frac{\alpha}{\rho(t)}w_y + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0} \\ \qquad -d(\rho(t)y,t) - \frac{\dot{\rho}(t)}{\rho(t)}]w, & 0<y<L_0,\ 0<t\leq T, \\ w_y(y,t) = 0, & y = 0, L_0,\ 0<t\leq T, \\ w(y,0) = w(y,T), & 0<y<L_0, \end{cases} \tag{2.10}$$

it admits the same principal eigen-pair $(\mathcal{R}_0, w)$.

Now we multiply (2.10) by $e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}$, (2.7) by w, and subtract the resulting equations to obtain

$$\begin{aligned} -[e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}w_t + (e^{\frac{\alpha}{D}\rho(t)y}\tilde{z})_t w] \quad \leq \quad & \frac{D}{\rho^2(t)}[(e^{\frac{\alpha}{D}\rho(t)y}w_y)_y\tilde{z} - (e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}_y)_y w] \\ & + e^{\frac{\alpha}{D}\rho(t)y}[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0} - \frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_1}]w\tilde{z}. \end{aligned} \tag{2.11}$$

Integrating the above inequality over $(0, L_0)\times(0,T)$ and using the initial boundary conditions of (2.7), (2.10), one has

$$\begin{aligned} & \int_0^{L_0}\int_0^T e^{\frac{\alpha}{D}\rho(t)y}[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0} - \frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_1}]w\tilde{z}dtdy \\ \geq\ & -\int_0^{L_0}\int_0^T [e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}w_t + (e^{\frac{\alpha}{D}\rho(t)y}\tilde{z})_t w]dtdy \\ & -\int_0^{L_0}\int_0^T \frac{D}{\rho^2(t)}[(e^{\frac{\alpha}{D}\rho(t)y}w_y)_y\tilde{z} - (e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}_y)_y w]dtdy \\ =\ & -\int_0^{L_0}\int_0^T [e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}w]_t dtdy - \int_0^{L_0}\int_0^T \frac{D}{\rho^2(t)}[(e^{\frac{\alpha}{D}\rho(t)y}w_y)\tilde{z} - (e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}_y)w]_y dtdy \\ =\ & 0 - \int_0^T \frac{D}{\rho^2(t)}[(e^{\frac{\alpha}{D}\rho(t)y}w_y)\tilde{z} - (e^{\frac{\alpha}{D}\rho(t)y}\tilde{z}_y)w]|_0^{L_0}dt \\ =\ & 0, \end{aligned} \tag{2.12}$$

where the periodicity of w and z has been used. Moreover, recalling the hypothesis of g and I yields $\mathcal{R}_0 \leq \mathcal{R}_1$. By [10], the positive eigenfunction corresponding to the principal eigenvalue $\mathcal{R}_0$ is unique (subjects to a constant multiple). Based on the above argument, we are confident that the last claim of lemma can be proved. □

Next we consider the general case that $\alpha \neq 0$. Multiplying both sides of (2.5) by $\frac{1}{\varphi}$, and integrating over $(0, L_0)\times(0,T)$ yield

$$
\begin{aligned}
0 &= \int_0^T\int_0^{L_0}[\tfrac{D}{\rho^2(t)}\tfrac{\varphi_{yy}}{\varphi}+\tfrac{\alpha}{\rho(t)}\tfrac{\varphi_y}{\varphi}+\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)]dydt\\
&= \int_0^T\int_0^{L_0}\tfrac{D}{\rho^2(t)}\tfrac{d\varphi_y}{\varphi}dt+\int_0^T\int_0^{L_0}\tfrac{\alpha}{\rho(t)}\tfrac{\varphi_y}{\varphi}dydt\\
&\quad+\int_0^T\int_0^{L_0}[\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)]dydt\\
&= \int_0^T\int_0^{L_0}[\tfrac{D}{\rho^2(t)}\tfrac{\varphi_y^2}{\varphi^2}+\tfrac{\alpha}{\rho(t)}\tfrac{\varphi_y}{\varphi}]dydt\\
&\quad+\int_0^T\int_0^{L_0}[\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)]dydt\\
&= \int_0^T\int_0^{L_0}[D(\tfrac{\varphi_y}{\rho(t)\varphi}+\tfrac{\alpha}{2D})^2-\tfrac{\alpha^2}{4D}]dydt\\
&\quad+\int_0^T\int_0^{L_0}[\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)]dydt\\
&\geq \int_0^T\int_0^{L_0}(-\tfrac{\alpha^2}{4D})dydt+\int_0^T\int_0^{L_0}[\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)]dydt,
\end{aligned}
\tag{2.13}
$$

which derive the following inequality

$$
\mathcal{R}_0\geq\tfrac{\int_0^T\int_0^{L_0}g(I_0(t)e^{-k_0\rho(t)y})dydt}{\int_0^T\int_0^{L_0}[\frac{\alpha^2}{4D}+d(\rho(t)y,t)]dydt}. \tag{2.14}
$$

Remark 2.1. It follows from (2.13) that

$$
\lim_{D\to0,\alpha\to0}\mathcal{R}_0(D,\alpha,L)=\tfrac{\int_0^T\int_0^{L_0}g(I_0(t)e^{-k_0\rho(t)y})dydt}{\int_0^T\int_0^{L_0}d(\rho(t)y,t)dydt}:=\mathcal{R}_0^*. \tag{2.15}
$$

Now we consider the species case $\alpha=0$. Noting that $g(I_0(t)e^{-k_0\rho(t)y})-d(\rho(t)y,t)$ is strictly decreasing with respect to y, which is straightforward to show that

$$
\begin{aligned}
&\tfrac{D}{\rho^2(t)}\varphi_{yy}+[\tfrac{g(I_0(t)e^{-k_0\rho(t)L_0})}{\mathcal{R}_0}-d(\rho(t)L_0,t)-\tfrac{\dot\rho(t)}{\rho(t)}]\varphi\\
\leq\ &\tfrac{D}{\rho^2(t)}\varphi_{yy}+[\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)-\tfrac{\dot\rho(t)}{\rho(t)}]\varphi\\
\leq\ &\tfrac{D}{\rho^2(t)}\varphi_{yy}+[\tfrac{g(I_0(t))}{\mathcal{R}_0}-d(0,t)-\tfrac{\dot\rho(t)}{\rho(t)}]\varphi.
\end{aligned}
\tag{2.16}
$$

We then consider the following auxiliary problem

$$
\begin{cases}
\hat z_t=\tfrac{D}{\rho^2(t)}\hat z_{yy}+[\tfrac{g(I_0(t)e^{-k_0\rho(t)L_0})}{\mathcal{R}_2}-d(\rho(t)L_0,t)-\tfrac{\dot\rho(t)}{\rho(t)}]\hat z, & 0<y<L_0,\ 0<t\leq T,\\
\hat z_y(y,t)=0, & y=0,L_0,\ 0<t\leq T,\\
\hat z(y,0)=\hat z(y,T), & 0<y<L_0.
\end{cases}
\tag{2.17}
$$

Letting $\hat z(y,t)=f(t)$, (2.17) turns to

$$
\begin{cases}
f_t=[\tfrac{g(I_0(t)e^{-k_0\rho(t)L_0})}{\mathcal{R}_2}-d(\rho(t)L_0,t)-\tfrac{\dot\rho(t)}{\rho(t)}]f(t), & 0<t\leq T,\\
f(0)=f(T),
\end{cases}
\tag{2.18}
$$

integrating (2.18) over $(0,T)$ with a view to periodicity of $\rho(t)$ yields

$$
\mathcal{R}_2=\tfrac{\int_0^T g(I_0(t))e^{-k_0\rho(t)L_0})dt}{\int_0^T d(\rho(t)L_0,t)dt}. \tag{2.19}
$$

It is accessible to obtain from (2.17)

$$
\begin{cases}
\hat z_t\leq\tfrac{D}{\rho^2(t)}\hat z_{yy}+[\tfrac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_2}-d(\rho(t)y,t)-\tfrac{\dot\rho(t)}{\rho(t)}]\hat z, & 0<y<L_0,\ 0<t\leq T,\\
\hat z_y(y,t)=0, & y=0,L_0,\ 0<t\leq T,\\
\hat z(y,0)=\hat z(y,T), & 0<y<L_0,
\end{cases}
\tag{2.20}
$$

which together with Lemma 2.2 gives that $\mathcal{R}_2 \le \mathcal{R}_0$.

Similarly, we continue in this fashion to obtain $\mathcal{R}_1 \ge \mathcal{R}_0$ with

$$\mathcal{R}_1 = \frac{\int_0^T g(I_0(t))dt}{\int_0^T d(0,t)dt}. \tag{2.21}$$

We then conclude that

$$\frac{\int_0^T g(I_0(t)e^{-k_0\rho(t)L_0})dt}{\int_0^T d(\rho(t)L_0,t)dt} \le \mathcal{R}_0(D,0,L) \le \frac{\int_0^T g(I_0(t))dt}{\int_0^T d(0,t)dt}. \tag{2.22}$$

More specifically, assume that most of the light is absorbed dependent of chlorophyll concentration [17,26] (i.e. $k_0 = 0$) and $d(\rho(t)y,t) \equiv d(t)$, then $\mathcal{R}_0(D,0,L) = \frac{\int_0^T g(I_0(t))dt}{\int_0^T d(t)dt}$.

Now we are more interested in the relationship of $\mathcal{R}_0$ with respect to the water column depth L and the vertical turbulent diffusion rate D. We first present the following monotonicity, whose proof is the same as that of [23] and so is omitted.

Lemma 2.3. *Assume that $\alpha = 0$, for any given $D, L > 0$, $\varphi_y(y,t) < 0$ in $(0,L)$ for all t, where $\varphi(y,t)$ is the nonnegative eigenvalue function corresponding to the principal eigenvalue of problem (2.5).*

Based on the lemma, we have the monotonicity of $\mathcal{R}_0$ with respect to the water column depth L.

Lemma 2.4. *Assume that $\alpha = 0$, for any given $D > 0$, $\mathcal{R}_0(D,0,L)$ is strictly monotone decreasing in L.*

Proof. For ease of notations, we write $\mathcal{R}_0(L) = \mathcal{R}_0(D,0,L)$. Given $0 < L_0 < L_1$, our goal is to show that $\mathcal{R}_0(L_0) > \mathcal{R}_0(L_1)$. We denote by $(\mathcal{R}_0(L_1),\varphi)$ and $(\mathcal{R}_0(L_0),w)$ are the principal eigen-pairs of (2.6) and (2.10), respectively. By virtue of our notations, we have

$$\begin{cases} \varphi_t = \frac{D}{\rho^2(t)}\varphi_{yy} + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0(L_1)} - d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]\varphi, & 0 < y < L_1,\ 0 < t \le T, \\ \varphi_y(y,t) = 0, & y = 0, L_1,\ 0 < t \le T, \\ \varphi(y,0) = \varphi(y,T), & 0 < y < L_1, \end{cases} \tag{2.23}$$

and

$$\begin{cases} -w_t = \frac{D}{\rho^2(t)}w_{yy} + [\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0(L_0)} - d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]w, & 0 < y < L_0,\ 0 < t \le T, \\ w_y(y,t) = 0, & y = 0, L_0,\ 0 < t \le T, \\ w(y,0) = w(y,T), & 0 < y < L_0. \end{cases} \tag{2.24}$$

Now multiplying (2.23) by w, and (2.24) by φ, then subtracting the resulting equations to obtain

$$-[\varphi w_t + \varphi_t w] = \frac{D}{\rho^2(t)}[w_{yy}\varphi - \varphi_{yy}w] + \varphi w[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0(L_0)} - \frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0(L_1)}]. \tag{2.25}$$

Integrating the above equation over $(0,L_0)\times(0,T)$ and using the initial boundary conditions of (2.23), (2.24), one has

$$\begin{aligned} & \int_0^{L_0}\int_0^T \varphi w[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0(L_0)} - \frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0(L_1)}]dtdy \\ = & -\int_0^{L_0}\int_0^T [\varphi w_t + \varphi_t w]dtdy - \int_0^T\int_0^{L_0} \frac{D}{\rho^2(t)}[w_y\varphi - \varphi_y w]_y dydt \\ = & \int_0^T \frac{D}{\rho^2(t)} w_y\varphi_y(L_0,t)w(L_0,t)dt \\ < & 0, \end{aligned} \tag{2.26}$$

in which we have used the fact that $\varphi_y(y,t)<0$ for $0<y<L_1, 0<t<T$ due to Lemma 2.3, we then have $\mathcal{R}_0(L_0)>\mathcal{R}_0(L_1)$. □

Lemma 2.5. *Assume that $\alpha=0$, for any given $D>0$, $\mathcal{R}_0(D,0,L)$ is strictly monotone decreasing in D.*

Proof. Let $\mathcal{R}_0=\mathcal{R}_0(D,0,L)$ and $(\mathcal{R}_0,\varphi)$ is the principal eigen-pair of (2.5). Clearly, $\mathcal{R}_0(D,0,L)$ and φ are C^1-function of D. For simplicity of presentation, we denote $\frac{\partial\varphi}{\partial D}$ by φ' and $\frac{\partial\mathcal{R}_0}{\partial D}$ by $\mathcal{R}_0'$. Letting $\alpha=0$, by assumption, and differentiating (2.5) with respect to D yield

$$\begin{cases} \varphi_t'=\frac{D}{\rho^2(t)}\varphi_{yy}'+\frac{1}{\rho^2(t)}\varphi_{yy}+[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)-\frac{\dot\rho(t)}{\rho(t)}]\varphi' \\ \qquad -(\frac{1}{\mathcal{R}_0})'g(I_0(t)e^{-k_0\rho(t)y})\varphi, & 0<y<L_0,\ 0<t\le T, \\ \varphi_y'(y,t)=0, & y=0,L_0,\ 0<t\le T, \\ \varphi'(y,0)=\varphi'(y,T), & 0<y<L_0. \end{cases} \tag{2.27}$$

Let w be the principal eigenfunction corresponding to $\mathcal{R}_0$ which satisfies (2.10). Multiplying (2.27) by w and integrating the equality over $(0,L_0)\times(0,T)$, by integration by parts, yield

$$\begin{cases} -\int_0^{L_0}\int_0^T\varphi' w_t dtdy=-D\int_0^{L_0}\int_0^T\frac{1}{\rho^2(t)}\varphi_y' w_y dtdy \\ \quad+\int_0^{L_0}\int_0^T w\Big\{\frac{1}{\rho^2(t)}\varphi_{yy}+[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)-\frac{\dot\rho(t)}{\rho(t)}]\varphi' \\ \quad-(\frac{1}{\mathcal{R}_0})'g(I_0(t)e^{-k_0\rho(t)y})\varphi\Big\}dtdy, & 0<y<L_0,\ 0<t\le T, \\ \varphi_y'(y,t)=0, & y=0,L_0,\ 0<t\le T, \\ \varphi'(y,0)=\varphi'(y,T), & 0<y<L_0. \end{cases} \tag{2.28}$$

By substituting $-w_t=\frac{D}{\rho^2(t)}w_{yy}+[\frac{g(I_0(t)e^{-k_0\rho(t)y})}{\mathcal{R}_0}-d(\rho(t)y,t)-\frac{\dot\rho(t)}{\rho(t)}]w$ into (2.28), we get

$$(\frac{1}{\mathcal{R}_0})'=-\frac{\int_0^{L_0}\int_0^T\frac{1}{\rho^2(t)}\varphi_{yy}w dtdy}{\int_0^{L_0}\int_0^T g(I_0(t)e^{-k_0\rho(t)y})\varphi w dtdy}=\frac{\int_0^{L_0}\int_0^T\frac{e^{\frac{\alpha}{D}\rho(t)y}}{\rho^2(t)}\varphi_y w_y dtdy}{\int_0^{L_0}\int_0^T g(I_0(t)e^{-k_0\rho(t)y})\varphi w dtdy}. \tag{2.29}$$

Considering (2.10), set $\xi(y,t)=w(y,T-t)$, we have

$$\begin{cases} \xi_t=\frac{D}{\rho^2(T-t)}\xi_{yy}+[\frac{g(I_0(T-t)e^{-k_0\rho(T-t)y})}{\mathcal{R}_0}-d(\rho(T-t)y,t)+\frac{\dot\rho(T-t)}{\rho(T-t)}]\xi, & 0<y<L_0,\ 0<t\le T, \\ \xi_y(y,t)=0, & y=0,L_0,\ 0<t\le T, \\ \xi(y,0)=\xi(y,T), & 0<y<L_0. \end{cases} \tag{2.30}$$

In accordance to $d(\rho(t)y,t)$ is strictly increasing with respect to y, using the same argument as in the proof of Lemma 2.3, we can easily carry out

$$\xi_y(y,t)=w_y(y,T-t)<0,\qquad 0<y<L_0,\ 0<t\le T. \tag{2.31}$$

So (2.29) implies $(\frac{1}{\mathcal{R}_0})'>0$. □

3. The threshold-type dynamics

The following result shows the long time behavior of solution depends on the basic reproduction number $\mathcal{R}_0$.

Theorem 3.1. *The following statements are valid:*

(i) If $\mathcal{R}_0 \le 1$, then for any solution $v(y,t)$ of system (1.7), (1.8), we can deduce that $\lim_{t\to\infty} v(y,t) = 0$ for $y \in [0, L_0]$, that is, the phytoplankton species dies out in a long run.

(ii) If $\mathcal{R}_0 > 1$, then problem (1.7), (1.9) admits at least one positive periodic solution. Moreover, there exists $\delta_0 > 0$ such that any positive solution $v(y,t)$ of system (1.7), (1.8) satisfies $\limsup_{t\to\infty} v(y,t) \ge \delta_0$ uniformly for $y \in [0, L_0]$, that is, the phytoplankton species persists uniformly.

Proof. *(i)* When $\mathcal{R}_0 < 1$, problem (2.3) admits an eigen-pair $(\mathcal{R}_0; \phi)$ such that $\phi(y,t) > 0$ for $(y,t) \in [0, L_0]\times[0,T]$. Letting $\tilde{v}(y,t) = Me^{-\lambda t}\phi(y,t)$, where $0 < \lambda \le g(I_0 e^{-k_0\rho(t)y})(\frac{1}{\mathcal{R}_0} - 1)$ for $(y,t) \in [0, L_0]\times[0,T]$, from the hypothesis of g and mathematical expression of I, we then have

$$\begin{aligned}
& \tilde{v}_t - \tfrac{D}{\rho^2(t)}\tilde{v}_{yy} + \tfrac{\alpha}{\rho(t)}\tilde{v}_y - [g(I(y,t)) - d(y,t) - \tfrac{\dot{\rho}(t)}{\rho(t)}]\tilde{v} \\
\ge\ & \tilde{v}_t - \tfrac{D}{\rho^2(t)}\tilde{v}_{yy} + \tfrac{\alpha}{\rho(t)}\tilde{v}_y - [g(I_0 e^{-k_0\rho(t)y}) - d(y,t) - \tfrac{\dot{\rho}(t)}{\rho(t)}]\tilde{v} \\
=\ & Me^{-\lambda t}\phi_t - \lambda M e^{-\lambda t}\phi - Me^{-\lambda t}\tfrac{D}{\rho^2(t)}\phi_{yy} + Me^{-\lambda t}\tfrac{\alpha}{\rho(t)}\phi_y \\
& -Me^{-\lambda t}\phi[g(I_0 e^{-k_0\rho(t)y}) - d(y,t)] + Me^{-\lambda t}\tfrac{\dot{\rho}(t)}{\rho(t)}\phi \\
=\ & \tilde{v}\{-\lambda + \tfrac{g(I_0 e^{-k_0\rho(t)y})}{\mathcal{R}_0} - g(I_0 e^{-k_0\rho(t)y})\} \\
\ge\ & 0,
\end{aligned}$$

for $(y,t) \in (0, L_0)\times(0,\infty)$ and

$$\begin{aligned}
& D\tilde{v}_y - \alpha\rho(t)\tilde{v}(y,t) = 0,\ y = 0, L_0,\ t > 0, \\
& \tilde{v}(y,0) = u_0(y) \ge 0, \not\equiv 0,\ 0 < y < L_0,
\end{aligned}$$

therefore $\tilde{v}$ is the upper solution of (1.7), (1.8) if M is large enough. Since $\lim_{t\to+\infty} \tilde{v}(y,t) = 0$, then $\lim_{t\to+\infty} v(y,t) = 0$ uniformly for $y \in [0, L_0]$.

Next, we claim that (1.7), (1.9) has no positive solution if $\mathcal{R}_0 \le 1$. Assume that $v^*(y,t)$ is a positive solution of (1.7), (1.9). By the assumption that g is strictly increasing with respect to I and $I = I_0 e^{-k_0\rho(t)y - k_1\int_0^{\rho(t)y} v^*(s,t)ds}$, we obtained

$$\begin{cases}
v_t^* - \tfrac{D}{\rho^2(t)}v_{yy}^* + \tfrac{\alpha}{\rho(t)}v_y^* - [g(I_0(t)e^{-k_0\rho(t)y}) & \\
\qquad -d(\rho(t)y,t) - \tfrac{\dot{\rho}(t)}{\rho(t)}]v^* \le, \not\equiv 0, & 0 < y < L_0,\ 0 < t \le T, \\
Dv_y^* - \alpha\rho(t)v^*(y,t) = 0, & y = 0, L_0,\ 0 < t \le T, \\
v^*(y,0) = v^*(y,T), & 0 < y < L_0.
\end{cases} \tag{3.1}$$

In a similar way, let $z(y,t) = e^{-\frac{\alpha}{D}\rho(t)y}v^*(y,t)$, thus $z > 0$ and z satisfies

$$\begin{cases}
z_t - \tfrac{D}{\rho^2(t)}z_{yy} - \tfrac{\alpha}{\rho(t)}z_y - [g(I_0(t)e^{-k_0\rho(t)y}) & \\
\qquad -d(\rho(t)y,t) - \tfrac{\dot{\rho}(t)}{\rho(t)}]z \le, \not\equiv 0 & 0 < y < L_0,\ 0 < t \le T, \\
z_y(y,t) = 0, & y = 0, L_0,\ 0 < t \le T, \\
z(y,0) = z(y,T), & 0 < y < L_0.
\end{cases} \tag{3.2}$$

In connection with [24, Proposition 5.2(ii)], which deserves $\lambda_0 < 0$. In addition, we deduce from Lemma 2.1*(ii)* that $\mathcal{R}_0 > 1$, which contradicts $\mathcal{R}_0 \le 1$. So we admit that (1.7), (1.9) does not have any positive solution provided that $\mathcal{R}_0 \le 1$.

For $\mathcal{R}_0 = 1$, then $\lambda_0 = 0$, (1.7), (1.9) has no positive solution, it is pretty easy for us to verify conclusion (i) by making use of [6, Theorem 3.4] and [24, Lemma 2.4].

(ii) When $\mathcal{R}_0 > 1$, assume, for the sake of contradiction, that for any δ there exists a positive solution $v(y,t)$ of problem (1.7) and (1.8) such that

$$\limsup_{t\to\infty} v(y,t) \le \frac{\delta}{2}. \tag{3.3}$$

For the above given δ, there exists T_δ such that

$$0 \le v(y,t) \le \delta \text{ for } (y,t) \in [0, L_0] \times [T_\delta, +\infty).$$

Since $I(\rho(t)y,t) = I_0 e^{-k_0\rho(t)y - k_1\int_0^{\rho(t)y} v(s,t)ds}$, then

$$I_0 e^{-(k_0+k_1\delta)\rho(t)y} \le I(\rho(t)y,t) \le I_0 e^{-k_0\rho(t)y}.$$

Then we have

$$\begin{aligned} v_t - \tfrac{D}{\rho^2(t)}v_{yy} + \tfrac{\alpha}{\rho(t)}v_y &= [g(I(\rho(t)y,t)) - d(\rho(t)y,t) - \tfrac{\dot\rho(t)}{\rho(t)}]v \\ &\ge [g(I_0 e^{-(k_0+k_1\delta)\rho(t)y}) - d(\rho(t)y,t) - \tfrac{\dot\rho(t)}{\rho(t)}]v \end{aligned} \tag{3.4}$$

for $(y,t) \in [0, L_0] \times [T_\delta, +\infty)$. We now choose a sufficiently small number $\eta > 0$ such that

$$v(y, T_\delta) \ge \eta\phi(y, T_\delta), \tag{3.5}$$

where $\phi(y,t) > 0$ for $(y,t) \in [0, L_0] \times [T_\delta, +\infty)$ satisfies (2.3) with $\mathcal{R}_0 > 1$. Set $0 < \lambda_0 \le (1 - \frac{1}{\mathcal{R}_0})g(I_0 e^{-k_0\rho(t)y})$ for $y \in (0, L_0)$, and direct calculations show that, $\underline{v}(y,t) = \eta e^{\lambda_0(t-T_\delta)}\phi(y,t)$ satisfies

$$\begin{cases} \underline{v}_t \le \frac{D}{\rho^2(t)}\underline{v}_{yy} - \frac{\alpha}{\rho(t)}\underline{v}_y + [g(I_0 e^{-(k_0+k_1\delta)\rho(t)y}) \\ \qquad -d(\rho(t)y,t) - \frac{\dot\rho(t)}{\rho(t)}]\underline{v}, & 0 < y < L_0,\ 0 < t \le T_\delta, \\ D\underline{v}_y - \alpha\rho(t)\underline{v}(y,t) = 0, & y = 0, L_0,\ 0 < t \le T_\delta, \\ \underline{v}(y, T_\delta) = \eta\phi(y, T_\delta) \le v(y, T_\delta), & 0 < y < L_0. \end{cases} \tag{3.6}$$

It follows from (3.4) and the comparison principle that

$$v(y,t) \ge \underline{v}(y,t) = \eta e^{\lambda_0(t-T_\delta)}\phi(y,t) \text{ for } y \in [0, \mathrm{L}_0],\ t \ge \mathrm{T}_\delta,$$

therefore, $v(y,t) \to \infty$ as $t \to \infty$, which contradicts (3.3). This proves statement (ii). □

4. Simulation and discussion

In this section, we first carry out numerical simulations for problem (1.7), (1.8) to illustrate the theoretical results. The term $g(I(\rho(t)y,t))$ is the specific growth rate of phytoplankton as a function of light intensity $I(\rho(t)y,t)$. A standard model that incorporates saturation is $g(I) = \frac{aI}{1+bI}$ (see [17]). Let us fix some coefficients. Assume that

$$D = 0.001,\ \alpha = 0.001,\ \ a = 3,\ b = 2,\ I = I_0 e^{-k_0\rho(t)y},\ I_0 = 0.1,\ k_0 = 0.2,$$
$$L_0 = 1,\ z_0(y) = 4 + 2\cos(\pi y) + \cos(2\pi y),$$

then the asymptotic behaviors of the solution to problem are shown by choosing different $\rho(t)$ and $d(\rho(t)y,t)$.

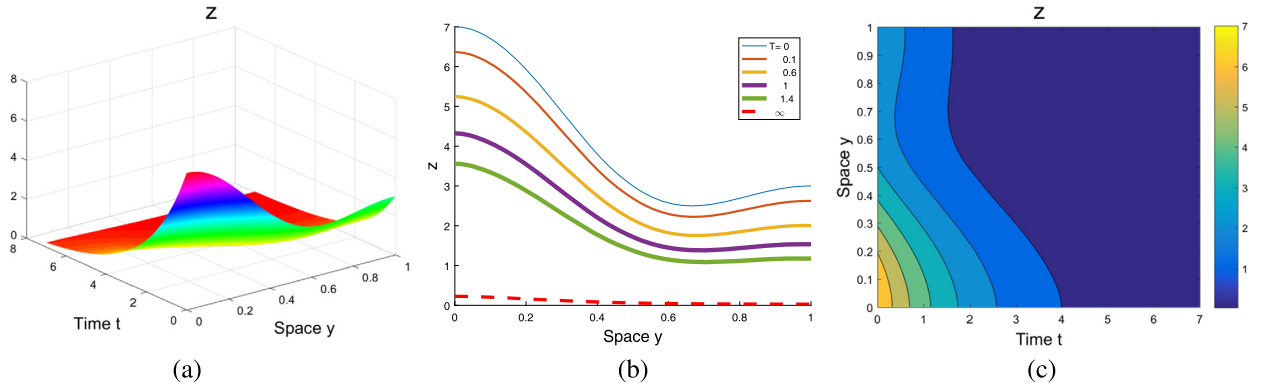


Fig. 2. $\rho_1(t) \equiv 1$. The interval is fixed and $\mathcal{R}_0 < 1$. Graphs (a)–(c) showed that z decay to 0. Graphs (b) and (c) are the cross-sectional view and contour map, respectively.

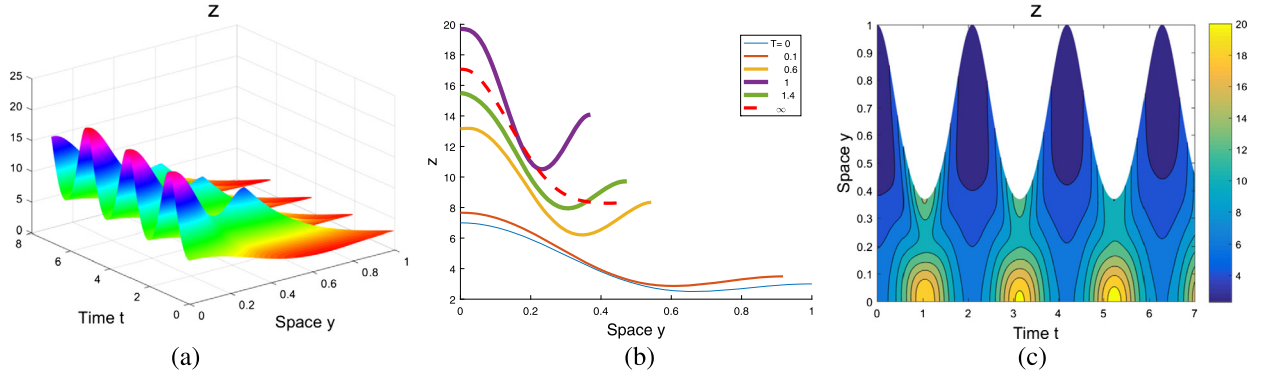


Fig. 3. $\rho_2(t) = e^{-0.5(1-\cos(3t))}$. The interval is evolving with a larger evolution rate $\rho_2(t)$ and $\mathcal{R}_0 > 1$. Graph (a) shows that z stabilized to a positive periodic steady state. Graphs (b) and (c), which are the cross-sectional view and contour map respectively, present the periodic evolution of the interval.

Example 4.1. Fix $d(\rho(t)y, t) = 0.2 + 0.1\rho(t)y$. We first choose $\rho_1(t) \equiv 1$, which means that the habitat is fixed. From (2.15), direct calculations show that

$$\mathcal{R}_0^*(\rho_1) = \frac{\int_0^T \int_0^{L_0} g(I_0(t)e^{-k_0\rho(t)y})dydt}{\int_0^T \int_0^{L_0} d(\rho(t)y, t)dydt} \approx \frac{0.4819}{0.5236} < 1.$$

It is easily seen from Fig. 2 that the phytoplankton species z decays to zero.

Now we choose $\rho_2(t) = e^{-0.5(1-\cos(3t))}$, it follows from (2.15) that

$$\mathcal{R}_0^*(\rho_2) = \frac{\int_0^T \int_0^{L_0} g(I_0(t)e^{-k_0\rho(t)y})dydt}{\int_0^T \int_0^{L_0} d(\rho(t)y, t)dydt} \approx \frac{0.4963}{0.4864} > 1.$$

Easily to see from Fig. 3 that the phytoplankton species z stabilizes to a positive periodic steady state.

One can see from the example that the phytoplankton vanishes in a fixed interval, but persists in a periodically evolving interval.

Example 4.2. Fix $d(\rho(t)y, t) = 0.1 + 0.2\rho(t)y$. We first choose $\rho_3(t) \equiv 1$, which means that the interval is fixed. Direct calculations show that

$$\mathcal{R}_0^*(\rho_3) = \frac{\int_0^T \int_0^{L_0} g(I_0(t)e^{-k_0\rho(t)y})dydt}{\int_0^T \int_0^{L_0} d(\rho(t)y, t)dydt} \approx \frac{0.4819}{0.4189} > 1.$$

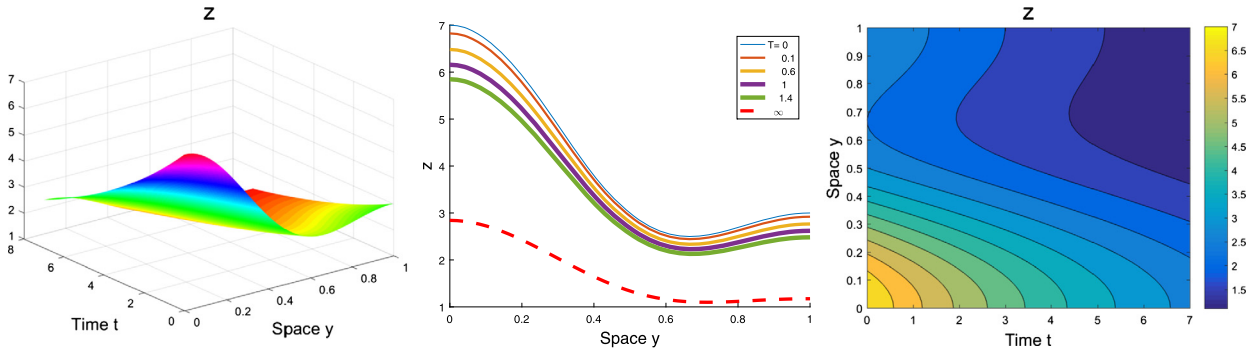


Fig. 4. $\rho_3(t) \equiv 1$. The interval is fixed and $\mathcal{R}_0 > 1$. Graphs (a)–(c) showed that z decays to 0. Graphs (b) and (c) are the cross-sectional view and contour map respectively.

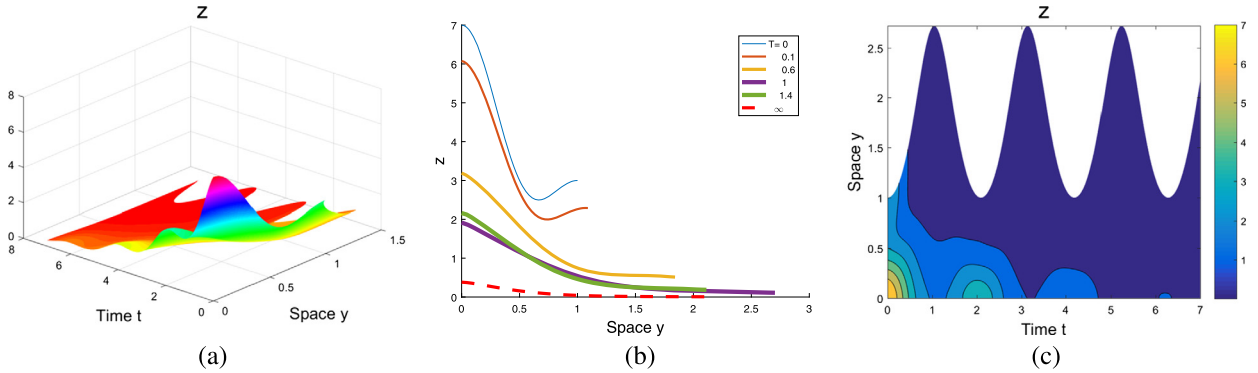


Fig. 5. $\rho_4(t) = e^{-0.5(1-\cos(3t))}$. The interval is evolving with a larger evolution rate $\rho_4(t)$ and $\mathcal{R}_0 < 1$. Graph (a) shows that z is stable. Graphs (b) and (c), which are the cross-sectional view and contour map respectively, present the periodic evolution of the interval.

It is easily seen from Fig. 4 that the phytoplankton species z stabilizes to a positive periodic steady state.

Now we choose $\rho_4(t) = e^{0.5(1-\cos(3t))}$, it follows from (2.15) that

$$\mathcal{R}_0(\rho_4) = \frac{\int_0^T \int_0^{L_0} g(I_0(t)e^{-k_0\rho(t)y})dydt}{\int_0^T \int_0^{L_0} d(\rho(t)y,t)dydt} \approx \frac{0.4536}{0.5767} < 1.$$

Easily to see from Fig. 5 that the phytoplankton species z decays to zero.

Next, we verify the monotonicity of $\mathcal{R}_0$ with respect to L. Assume that

$$D = 0.001,\ \alpha = 0,\quad a = 1,\ b = 3,\ g = \frac{aI}{1+bI} = \frac{aI_0e^{-k_0\rho(t)y}}{1+bI_0e^{-k_0\rho(t)y}},$$
$$I_0 = 2,\ k_0 = 0.02,\ \rho_5(t) = e^{-0.5(1-\cos(3t))},$$
$$z_0(y) = 4 + 2\cos(\pi y) + \cos(2\pi y).$$

Example 4.3. We first choose $d(\rho_5(t)y,t) = 0.29 + 3\rho_5(t)y$ and $L_0 = 3$. From (2.23), direct calculations show that

$$\mathcal{R}_0(\rho_5) \le \frac{\int_0^T g(I_0(t))dt}{\int_0^T d(0,t)dt} \approx \frac{0.5984}{0.6074} < 1.$$

It is easily seen from Fig. 6 that phytoplankton species z decays to zero.

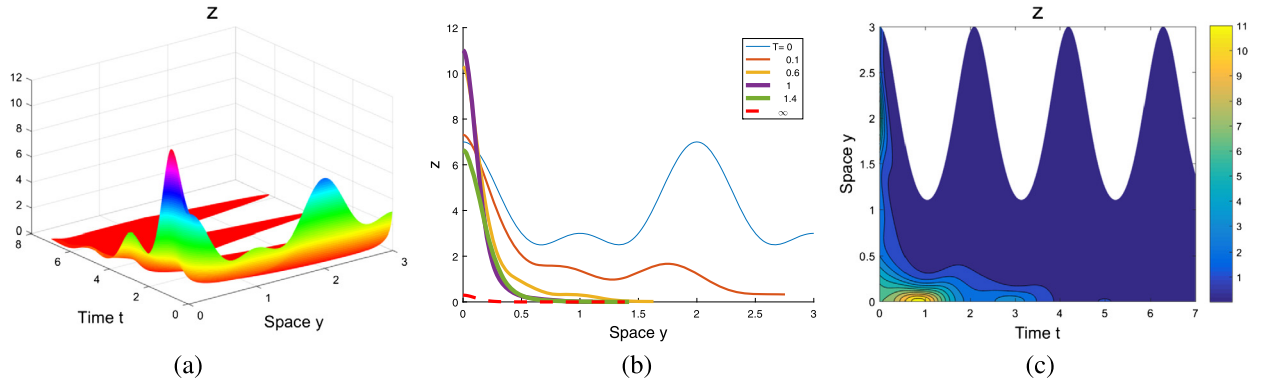


Fig. 6. $d(\rho(t)y,t) = 0.29 + 3\rho(t)y$, $L_0 = 3$ such that $\mathcal{R}_0 < 1$. Graphs (a)–(c) showed that z decays to 0.

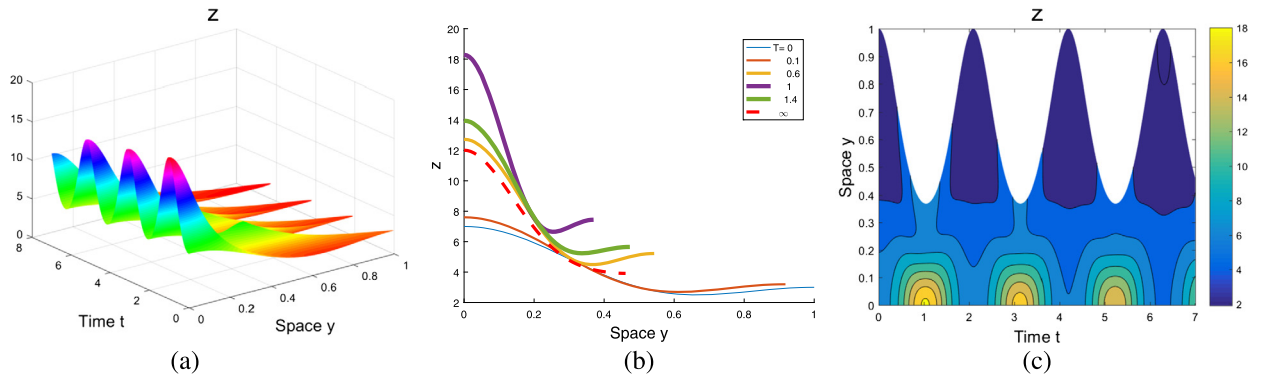


Fig. 7. $d(\rho(t)y,t) = 0.29 + 0.1\rho(t)y$, $L_0 = 1$ such that $\mathcal{R}_0 > 1$. Graphs (a)–(c) showed that z is stable.

Now we choose $d(\rho_5(t)y,t) = 0.29 + 0.1\rho_5(t)y$ and $L_0 = 1$, it follows from (2.23) that

$$\mathcal{R}_0(\rho_5) \geq \frac{\int_0^T g(I_0(t)e^{-k_0\rho(t)L_0})dt}{\int_0^T d(\rho(t)L_0,t)dt} \approx \frac{0.5973}{0.579} > 1.$$

Easily to get from Fig. 7 that z stabilizes to a positive periodic steady state.

Recently, lots of models in evolving domains have been theoretically studied, for instance, models in periodically evolving domains [16,27], in growing domains [28,29], and in shrinking domains [30]. In this paper, to understand the impacts of the evolving rate and depth on the dynamics of the single phytoplankton species, which depends upon the light for keeping the metabolism working constantly, a mathematical model in a periodically evolving environment has been established and studied.

The main conclusions are as follows: firstly, we introduce the basic reproduction number $\mathcal{R}_0$, which is used to describe threshold-type dynamics of our model, and give comparison principle in order to investigate the connection of $\mathcal{R}_0$ with regard to evolving rate $\rho(t)$, diffusion rate D, the sinking or buoyant rate α and the depth of water L_0. It is shown that $\mathcal{R}_0(D,0,L)$ is strictly monotone decreasing in L and D respectively if $\alpha = 0$. Especially, for enough small D and α, the limitation of $\mathcal{R}_0(D,\alpha,L)$ (denote $\mathcal{R}_0^*$) relies on the domain evolution $\rho(t)$ and depth L. A point that should be stressed is that $\mathcal{R}_0^*$ is strictly monotone decreasing with respect to $\rho(t)$. Secondly, it is proved in Theorem 3.1 that if $\mathcal{R}_0 \leq 1$, the phytoplankton species dies out in a long run, while if $\mathcal{R}_0 > 1$, then problem (1.7), (1.9) admits at least one positive periodic solution, and there exists $\delta_0 > 0$ such that $\limsup_{t\to\infty} v(y,t) \geq \delta_0$ uniformly for $y \in [0,L_0]$, which means the phytoplankton species persists uniformly.

Furthermore, our numerical simulations indicate that the periodical domain evolution with small evolution rate has a positive effect on the survival of phytoplankton (see Figs. 2 and 3), on the contrary, with large evolution rate has a negative effect on the survival of phytoplankton (see Figs. 4 and 5). Meanwhile, large depth of water has an adverse effect on the survival of phytoplankton (see Figs. 6 and 7), which matches closely our theoretical analysis.

Very recently a nonlocal two-species phytoplankton model in a eutrophic environment has been studied in [15]. Mathematically, by introducing a special cone, they presented a generalized notion of super- and subsolutions of the nonlocal competition system, established the comparison principle and derived the monotonicity of the underlying semiflow with respect to the cone. Biologically, they showed that there are three situations in which a competitor has competitive advantages: (i) smaller advection rate; (ii) faster diffusion rate if the sinking rate is large; (iii) slower advection rate if the phytoplankton species are buoyant.

In our paper, we focus on the impact of evolving domain on survival of phytoplankton. To be specific, phytoplankton is better for survival in the evolving domain with small evolution rate than in one with large evolution rate, that is to say, smaller evolution rate should be selected. However, in view of the complexity of the model, caused by $\rho(t)$ and α, the mathematical expression of $\mathcal{R}_0$ has not yet given if $\alpha \neq 0$, we leave this for future work.

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