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A Generalization of the Hyers–Ulam–Rassias Stability of Jensen's Equation

Yang-Hi Lee*

Department of Mathematics Education, Kongju National University of Education, Kongju 314-060, Republic of Korea

and

Kil-Woung Jun†

Department of Mathematics, Chungnam National University, Taejon 305-764, Republic of Korea

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In this paper we prove a generalization of the stability of the Jensen's equation in the spirit of Hyers, Ulam, Rassias, and Găvruta. © 1999 Academic Press

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1. INTRODUCTION

In 1940, Ulam [11] posed the following question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h\colon G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then a homomorphism $H\colon G_1 \to G_2$ exists with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces.

*E-mail: lyhmzi@kongjuw2.kongju-e.ac.kr.
†E-mail: kwjun@math.chungnam.ac.kr.

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Throughout this paper, we denote by X a Banach space. In 1978, Rassias [9] gave a generalization of the Hyers' result in the following way:

Let V be a normed space and let $f: V \to X$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$ (for all $x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique linear mapping $T: V \to X$ such that

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p$$

for all $x \in V$ (for all $x \in V \setminus \{0\}$ if $p < 0$). However, it was shown that a similar result for the case $p = 1$ did not hold (see [10]). Găvruta [1] also obtained a further generalization of the Hyers–Rassias theorem (see also [3, 5, 7]).

According to Theorem 6 in [8], a mapping $f: V \to X$ satisfying $f(0) = 0$ is a solution of the Jensen's functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y),$$

if and only if it satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$.

In 1988, Jung [6] proved the following theorem:

THEOREM A. *Let $p > 0$ be given with $p \neq 1$ and let $\delta \geq 0$ and $\theta \geq 0$ be given. Suppose a mapping $f: V \to X$ satisfies the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \delta + \theta(\|x\|^p + \|y\|^p) \qquad (*)$$

for all $x, y \in V$. Further, assume $f(0) = 0$ and $\delta = 0$ in $(*)$ for the case of $p > 1$. Then there exists a unique additive mapping $F: V \to X$ such that

$$\|f(x) - F(x)\| \leq \delta + \|f(0)\| + \frac{\theta}{2^{1-p} - 1}\|x\|^p \quad \text{for } p < 1,$$

or

$$\|f(x) - F(x)\| \leq \frac{2^{p-1}}{2^{p-1} - 1}\theta\|x\|^p \quad \text{for } p > 1$$

for all $x \in V$.

He noticed that the ideas from the proof of Theorem A cannot be applied to the proof of the stability of $(*)$ for the case $p > 0$. He raised a question whether the Hyers–Ulam–Rassias stability for the case $p < 0$ holds. In this paper, using the ideas from the papers of Hyers [2], Hyers, Isac, and Rassias [3], Rassias [9], and Găvruta [1], we obtain some generalization of Theorem A.

2. STABILITY IN THE CASE $p < 1$

We denote by G an abelian group. Let E be a subset of G such that $nx \in E$ for any integer n and for all $x \in E$. We assume that if $x \in E \setminus \{0\}$ then $2x \neq 0$ and $3x \neq 0$. We also denote by φ: $(E \setminus \{0\}) \times (E \setminus \{0\}) \to [0, \infty)$ a mapping such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty \tag{1}$$

for all $x, y \in E \setminus \{0\}$.

THEOREM 1. *Let f: $E \to X$ be a mapping such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \varphi(x, y) \tag{2}$$

for all $x, y \in E \setminus \{0\}$ with $\frac{x+y}{2} \in E$. Then there exists a unique mapping T: $E \to X$ such that

$$T(x+y) = T(x) + T(y) \quad \textit{for all } x, y \in E \textit{ with } x+y \in E, \tag{3}$$

and

$$\|f(x) - T(x) - f(0)\| \leq 3^{-1}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)) \quad \textit{for all } x \in E \setminus \{0\}. \tag{4}$$

Proof. Let $g(x) = f(x) - f(0)$. Then g satisfies (2). From this, we can assume that $f(0) = 0$ without loss of generality.

Let $x \in E \setminus \{0\}$. For $y = -x$, the inequality (2) implies

$$\| -f(x) - f(-x)\| \leq \varphi(x, -x). \tag{5}$$

Replacing x by $-x$ and y by $3x$, the inequality (2) implies

$$\|2f(x) - f(-x) - f(3x)\| \leq \varphi(-x, 3x). \tag{6}$$

From (5) and (6), we get

$$\|f(x) - 3^{-1}f(3x)\| \leq 3^{-1}(\varphi(x, -x) + \varphi(-x, 3x)). \tag{7}$$

Applying an induction argument to n we obtain

$$\|3^{-n}f(3^n x) - f(x)\| \leq \sum_{k=0}^{n-1} 3^{-k-1}\big(\varphi\big(3^k x, 3^k(-x)\big) + \varphi\big(3^k(-x), 3^k(3x)\big)\big). \tag{8}$$

We claim that the sequence $\{3^{-n}f(3^n x)\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned}\|3^{-n}f(3^n x) - 3^{-m}f(3^m x)\| &\leq \sum_{k=m}^{n-1} \|3^{-k-1}f(3^{k+1}x) - 3^{-k}f(3^k x)\| \\ &\leq \sum_{k=m}^{n-1} 3^{-k-1}\big(\varphi\big(3^k x, 3^k(-x)\big) + \varphi\big(3^k(-x), 3^k(3x)\big)\big).\end{aligned}$$

From (1), it follows that

$$\lim_{m\to\infty} \sum_{k=m}^{n-1} 3^{-k-1}\big(\varphi\big(3^k x, 3^k(-x)\big) + \varphi\big(3^k(-x), 3^k(3x)\big)\big) = 0.$$

Because X is a Banach space, it follows that the sequence $\{3^{-n}f(3^n x)\}$ converges. Define

$$T(x) = \lim_{n\to\infty} 3^{-n}f(3^n x)$$

for all x in E. Taking the limit in (8) as $n \to \infty$, we obtain

$$\|T(x) - f(x)\| \leq 3^{-1}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)) \quad \text{for all } x \in E \setminus \{0\}. \tag{4$'$}$$

This computes the proof of the inequality (4). From the definition of T, we get

$$3^n T(x) = T(3^n x) \quad \text{and} \quad T(0) = 0. \tag{9}$$

By (9) and (4′),

$$\begin{aligned}
&\|2T(2x) - 4T(x)\| \\
&\quad = \|2T(2x) - T(3x) - T(x)\| \\
&\quad = 3^{-n}\|2T(3^n \cdot 2x) - T(3^n \cdot 3x) - T(3^n x)\| \\
&\quad \leq 3^{-n}\left(\|2T(3^n \cdot 2x) - 2f(3^n \cdot 2x)\| + \|T(3^n \cdot 3x) - f(3^n \cdot 3x)\|\right) \\
&\qquad + 3^{-n}\|T(3^n x) - f(3^n x)\| \\
&\qquad + 3^{-n}\left\|2f\left(\frac{3^n(3x + x)}{2}\right) - f(3^n \cdot 3x) - f(3^n x)\right\| \\
&\quad \leq 2 \cdot 3^{-n-1}\big(\tilde{\varphi}(3^n 2x, 3^n(-2x)) + \tilde{\varphi}(3^n(-2x), 3^{n+1}2x)\big) \\
&\qquad + 3^{-n-1}\big(\tilde{\varphi}(3^{n+1}x, -3^{n+1}x) + \tilde{\varphi}(3^{n+1}(-x), 3^{n+2}x)\big) \\
&\qquad + 3^{-n-1}\big(\tilde{\varphi}(3^n x, 3^n(-x)) + \tilde{\varphi}(3^n(-x), 3^{n+1}x)\big) \\
&\qquad + 3^{-n}\varphi(3^n x, 3^{n+1}x)
\end{aligned}$$

for all $x \in E \setminus \{0\}$. From this and (9), we obtain

$$2T(x) = T(2x) \quad \text{for all } x \in E. \tag{10}$$

From (1), (2), and (9)

$$\begin{aligned}
&\left\|2T\left(\frac{x + y}{2}\right) - T(x) - T(y)\right\| \\
&\quad = \lim_{n \to \infty} 3^{-n}\left\|2f\left(\frac{3^n x + 3^n y}{2}\right) - f(3^n x) - f(3^n y)\right\| \\
&\quad \leq \lim_{n \to \infty} 3^{-n}\varphi(3^n x, 3^n y) = 0
\end{aligned} \tag{11}$$

for all $x, y \in E \setminus \{0\}$ with $\frac{x+y}{2} \in E$. From (9)–(11),

$$T(x + y) = 2^{-1}(T(2x) + T(2y)) + T(x) + T(y) \qquad \text{for all } x, y \in E \text{ with } x + y \in E.$$

Hence $T\colon E \to X$ is a mapping satisfying (3).

If $F: E \to X$ is an another mapping satisfying (3) and (4), then it follows from (3) and (4) that

$$\begin{aligned}\|T(x) - F(x)\| \\ &= \|3^{-n}T(3^n x) - 3^{-n}F(3^n x)\| \\ &\le \|3^{-n}T(3^n x) - 3^{-n}f(3^n x) - f(0)\| + \|3^{-n}f(3^n x) \\ &\quad + f(0) - 3^{-n}F(3^n x)\| \\ &\le 2 \cdot 3^{-n-1}(\tilde{\varphi}(3^n x, -3^n x) + \tilde{\varphi}(-3^n x, 3 \cdot 3^n x)).\end{aligned}$$

Thus we conclude that

$$T(x) = F(x)$$

for all x in E, which proves the uniqueness of T.

THEOREM 2. *Let V be a vector space and let E be a subset of V satisfying the following conditions*:

(i) *$rx \in E$ for all $x \in E$ and $|r| \ge 1$,*

(ii) *if x is a nonzero element of V, then there exists $n \in N$ such that $nx \in E$,*

(iii) *$0 \in E$.*

Let $f: E \to X$ be a mapping such that

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \varphi(x, y)$$

for all $x, y \in E \setminus \{0\}$ with $\frac{x+y}{2} \in E$. Then there exists a unique additive mapping $T: V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \le 3^{-1}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)) \qquad \text{for all } x \in E \setminus \{0\}.$$

Proof. By the conditions (i) and (iii), we get a unique mapping T': $E \to X$ satisfying (3) and (4) in Theorem 1. If $x \in V$, there exists a $n_x \in N$ such that $n_x x \in E$ by the condition (ii). We can define a mapping T: $V \to X$ by

$$T(x) = \begin{cases} T'(x) & \text{for } x \in E, \\ n_x^{-1} T'(n_x x) & \text{for } x \notin E. \end{cases}$$

If x, $y \in V$, we can choose a $n \in N$ such that nx, ny, and $n(x+y) \in E$ by the conditions (i) and (ii). From this, we get

$$\begin{aligned}\|T(x+y) - T(x) - T(y)\| \\ = n^{-1}\|T'(n(x+y)) - T'(nx) - T'(ny)\| = 0.\end{aligned}$$

This completes the proof.

Applying Theorem 2, we obtain Corollary 3.

COROLLARY 3. *Let V be a normal space and let $E = \{x \in V : \|x\| > a\} \cup \{0\}$ for a fixes $a \geq 0$. Let f: $E \to X$ be a mapping such that*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \varphi(x,y)$$

for all x, $y \in E \setminus \{0\}$ with $\frac{x+y}{2} \in E$. Then there exists a unique additive mapping T: $V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq 3^{-1}(\tilde{\varphi}(x,-x) + \tilde{\varphi}(-x,3x)) \qquad \textit{for all } x \in E \setminus \{0\}.$$

The following corollary is a generalization of Theorem 1 in [4].

COROLLARY 4. *Let V be a normed space. For a fixed a with $0 \leq a < 3$, let ψ: $(a,\infty) \to R^+$ be a function such that*

(i) $\psi(ts) \leq \psi(t)\psi(s)$ *for all t, $s > a$ and*

(ii) $\psi(3)/3 < 1$.

Let f: $V \to X$ be a mapping such that

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \psi(\|x\|) + \psi(\|y\|) \qquad \textit{for all } x, y \textit{ with } \|x\|, \|y\| > a.$$

Then there exists a unique additive mapping T: $V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{3\psi(\|x\|) + \psi(\|3x\|)}{3 - \psi(3)} \qquad \textit{for all } x \in V \textit{ with } \|x\| > a.$$

Proof. Let $E = \{x \in V : \|x\| > a\} \cup \{0\}$ and let $\varphi(x, y) = \psi(\|x\|) + \psi(\|y\|)$ for all $x, y \in \backslash\{0\}$. Then we get

$$\begin{aligned}
\tilde{\varphi}(x, y) &= \sum_{n=0}^{\infty} 3^{-n}\varphi(3^n x, 3^n y) \\
&\leq \sum_{n=0}^{\infty} (\psi(3)/3)^n (\psi(\|x\|) + \psi(\|y\|)) \\
&= \frac{\psi(\|x\|) + \psi(\|y\|)}{1 - \psi(3)/3} < \infty
\end{aligned}$$

from (i) and (ii). Applying Corollary 3, there exists a unique additive mapping $T: V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{3\psi(\|x\|) + \psi(\|3x\|)}{3 - \psi(3)}$$

for all $x \in V$ with $\|x\| > a$.

The following corollary is a generalization of Theorem A.

COROLLARY 5. *Let V be a normed space. Let $p < 1$ and $0 \leq a < 3$. Let $f: V \to X$ be a mapping such that*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in V$ with $\|x\|, \|y\| > a$.

Then there exists a unique additive mapping $T: V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{3 + 3^p}{3 - 3^p}\|x\|^p \quad \textit{for all } x \in V \textit{ with } \|x\| > a.$$

Proof. Define $\psi: (a, \infty) \to R^+$ by $\psi(t) = t^p$ and apply Corollary 4.

Remark. To prove Corollary 5, it is necessary that the range of φ in Theorem 1 does not contain 0. Therefore $\tilde{\varphi}(x, 0)$ cannot be used in the right side of the inequality (4).

3. STABILITY IN THE CASE $p > 1$

For the case $p > 1$ let $\phi: (E \setminus \{0\}) \times (E \setminus \{0\}) \to [0, \infty)$ be a mapping such that

$$\tilde{\phi}(x, y) = \sum_{k=0}^{\infty} 3^k \phi(3^{-k}x, 3^{-k}y) < \infty.$$

Then we follow a similar approach as the above arguments and obtain the results from Theorem 6 to Corollary 9.

THEOREM 6. *Let V be a vector space and let E be a subset of V satisfying the following conditions*:

(i) *$rx \in E$ for all $x \in E$ and $|r| \le 1$,*

(ii) *if x is a nonzero element of V, there exists $n \in N$ such that $n^{-1}x \in E$.*

Let f: $E \to X$ be a mapping such that

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \phi(x, y)$$

for all x, $y \in E \setminus \{0\}$ with $\frac{x+y}{2} \in E$. Then there exists a unique additive mapping T: $V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \le \tilde{\phi}\big(3^{-1}x, 3^{-1}(-x)\big) + \tilde{\phi}\big(3^{-1}(-x), x\big) \quad (12)$$

for all $x \in E \setminus \{0\}$.

Proof. We can assume that $f(0) = 0$ without loss of generality.

We obtain the sequence $\{3^n f(3^{-n}x)\}$ is a Cauchy sequence. Denote

$$T'(x) = \lim_{n \to \infty} 3^n(3^{-n}x)$$

for all x in E. We can easily show that T': $E \to X$ is a unique mapping satisfying (3) and (12). If $x \in V$, there exists a $n_x \in N$ such that $n_x^{-1}x \in E$ by the condition (ii). We can define a mapping T: $V \to X$ by

$$T(x) = \begin{cases} n_x T'\big(n_x^{-1}x\big) & \text{for } x \notin E, \\ T'(x) & \text{for } x \in E. \end{cases}$$

If x, $y \in V$, we can choose a $n \in N$ such that $n^{-1}(x+y), n^{-1}x, n^{-1}y \in E$ by the conditions (i) and (ii). From this, we get

$$\begin{aligned} &\|T(x+y) - T(x) - T(y)\| \\ &\qquad = n\|T'\big(n^{-1}(x+y)\big) - T'(n^{-1}x) - T'\big(n^{-1}y\big)\| = 0. \end{aligned}$$

COROLLARY 7. *Let V be a normed space and let $E = \{x \in V : \|x\| < a\}$ for a fixed $a > 0$. Let $f\colon E \to X$ be a mapping such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \phi(x, y)$$

for all x, $y \in \setminus\{0\}$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \le \tilde{\phi}(3^{-1}x, 3^{-1}(-x)) + \tilde{\phi}(3^{-1}(-x), x) \quad \text{for all } x \in E \setminus \{0\}.$$

COROLLARY 8. *Let V be a normed space and let $E = \{x \in V : \|x\| < a\}$ for a fixed $x > 3$. Let a function $\psi\colon (0, a) \to R^+$ satisfy*

(i) $\psi(ts) \ge \psi(t)\psi(s)$ *for all* $0 < t$, $s < a$ *and*

(ii) $\psi(3)/3 > 1$.

Let $f\colon E \to X$ be a mapping such that

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \psi(\|x\|) + \psi(\|y\|) \quad \text{for all } x, y \in E \setminus \{0\}.$$

Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \le \frac{3\psi(\|3^{-1}x\|) + \psi(\|x\|)}{1 - 3/\psi(3)} \quad \text{for all } x \in E \setminus \{0\}.$$

Proof. Let $\phi(x, y) = \psi(\|x\|) + \psi(\|y\|)$ for all x, $y \in E \setminus \{0\}$. We get

$$\begin{aligned}
\tilde{\phi}(x, y) &= \sum_{n=0}^{\infty} 3^n\, \phi(3^{-n}x, 3^{-n}y) \\
&= \sum_{n=0}^{\infty} 3^n (\psi(\|3^{-n}x\|) + \psi(\|3^{-n}y\|)) \\
&\le \sum_{n=0}^{\infty} (3/\psi(3))^n (\psi(\|x\|) + \psi(\|y\|)) \\
&= \frac{\psi(\|x\|) + \psi(\|y\|)}{1 - 3/\psi(3)} < \infty,
\end{aligned}$$

from (i) and (ii). Applying Corollary 7, the proof is completed.

COROLLARY 9. *Let V be a normed space. Let $p > 1$ and $a > 3$. Let $f\colon V \to X$ be a mapping such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \|x\|^p + \|y\|^p$$

for all x, y with $0 < \|x\|, \|y\| < a$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \le \frac{3^p + 3}{3^p - 3}\|x\|^p \quad \text{for all } x \text{ with } 0 \le \|x\| < a.$$

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