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Small into isomorphisms on uniformly smooth spaces ☆

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Abstract

Let X be a uniformly smooth infinite dimensional Banach space, and (Ω, Σ, μ) be a σ-finite measure space. Suppose that $T: X \to L^\infty(\Omega, \Sigma, \mu)$ satisfies

$$(1-\varepsilon)\|x\| \leqslant \|Tx\| \leqslant \|x\|, \quad \forall x \in X,$$

for some positive number $\varepsilon < 1/2$ with $\delta_{X^*}(2-2\varepsilon) > 13/14$. Then T is close to an isometry $U: X \to L^\infty(\Omega, \Sigma, \mu)$ such that

$$\|T-U\| \leqslant 16\big(1-\delta_{X^*}(2-2\varepsilon)\big) + \frac{1}{2}\varepsilon,$$

where $\delta_{X^*}(t)$ is the modulus of convexity of the conjugate space X^*.

Keywords: Banach space; Linear operator; Isometry

1. Introduction

Benyamini [2] proved that an isomorphism T of $C(K)$, K compact metric, into some $C(S)$, S compact Hausdorff, is close to an isometry if $\|T\|\|T^{-1}\|$ is close to one. Alspach [1] proved similar results on L^p spaces for $1 \leqslant p < \infty$. For some K being unmetrizable, however, Benyamini [3] gave a counterexample that for an arbitrary $\varepsilon > 0$,

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there is a linear operator $T : C(K) \to C(S)$ satisfying $\|T\|\|T^{-1}\| \leqslant 1 + \varepsilon$, but there is no isometry from $C(K)$ to $C(S)$.

In this paper, we will discuss the small into isomorphisms from an infinite dimensional uniformly smooth Banach space to $L^\infty(\Omega, \Sigma, \mu)$ basing on the following representation theorem for every linear bounded operator $T : X \to L^\infty(\Omega, \Sigma, \mu)$.

Suppose that X is a Banach space and the conjugate space X^* of X has the Radon–Nikodym property and (Ω, Σ, μ) is a σ-finite measure space. For every linear bounded operator $T : X \to L^\infty(\Omega, \Sigma, \mu)$, there is an $h \in L^\infty(\mu, X^*)$ so that for all $x \in X$

$$(Tx)(\omega) = h(\omega)(x) \quad \text{and} \quad \|T\| = \|h\|_\infty. \tag{1.1}$$

Indeed, let S be the restriction of T^* to $L_1(\mu)$. Then S is representable because X^* has RNP, i.e., there is an $h \in L^\infty(\mu, X^*)$ such that $Sf = \int fh\, d\mu$. Thus

$$\int f(\omega)h(\omega)(x)\, d\mu = (Sf)(x) = Tx(f) = \int f(\omega)Tx(\omega)\, d\mu$$

for every $x \in X$ and, since f is arbitrary, it follows that $Tx(\omega) = h(\omega)(x)$ a.e.

2. Small into isomorphisms on uniformly smooth spaces

Let X be a normed vector space. Denote by S_X the set $\{x\colon x \in X,\ \|x\| = 1\}$ and let $B_X = \{x\colon x \in X,\ \|x\| \leqslant 1\}$. If X is a uniformly convex Banach space, and $\varepsilon \in [0, 2]$, the modulus of convexity of X

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\|\colon x, y \in B_X,\ \|x-y\| \geqslant \varepsilon\right\}$$

is positive for all $\varepsilon > 0$ and $\lim_{\varepsilon\to 0} \delta_X(\varepsilon) = 0$, $\lim_{\varepsilon\to 2} \delta_X(\varepsilon) = 1$.

It is well known that if X is a uniformly smooth Banach space then X is reflexive, X and X^* have the RNP, and the conjugate space X^* is uniformly convex.

In general it is impossible to approximate a small into isomorphism from a finite dimensional uniformly smooth Banach space to $L^\infty(\Omega, \Sigma, \mu)$ by an isometry. For example (see [5]), let $X = \mathbf{R}^n$ be Euclidean space with $1 < n < \infty$. For each $\varepsilon > 0$, there exists a finite ε-net $\{g_1, g_2, \ldots, g_n\}$ of the unit ball B_{X^*}. Define

$$f_k = \begin{cases} g_k, & k = 1, 2, \ldots, n, \\ 0, & k = n+1, \ldots. \end{cases}$$

It is easy to verify that the operator $T : X \to \ell^\infty$ defined by $Tx = (f_1(x), f_2(x), \ldots)$ satisfies $(1-\varepsilon)\|x\| \leqslant \|Tx\| \leqslant \|x\|$. But any isometry $U : X \to \ell^\infty$ can be written as

$$(Ux)(k) = h_k(x), \quad h_k \in B_{X^*},\ k = 1, 2, \ldots,$$

for some sequence $\{h_k\}_{k=1}^\infty$ which is dense in S_{X^*}. Hence

$$\|T - U\| = \sup_k \|f_k - h_k\| \geqslant \sup\{\|h_k\|\colon k = n+1, \ldots\} = 1.$$

The infinite dimensionality of X will be used through the following lemma:

Lemma 2.1 [5]. *Let X be an infinite dimensional space. Then for any $x \in S_X$ and $\varepsilon \in (0,1)$ there is an set $\{x_n\}_{n=1}^{\infty} \subseteq S_X$ such that*

(i) $\|x - x_n\| = \varepsilon$ *for all* $n = 1, 2, \ldots,$ *and*
(ii) $\|x_i - a x_j\| > \varepsilon/3$ *for all* $|a| = 1$ *and* $i \neq j$.

Theorem 2.2. *Suppose that X is an infinite dimensional uniformly smooth Banach space, (Ω, Σ, μ) is a σ-finite measure space, and $T : X \to L^{\infty}(\Omega, \Sigma, \mu)$ is a linear bounded operator satisfying $(1-\varepsilon)\|x\| \leqslant \|Tx\| \leqslant \|x\|$, $\forall x \in X$, for some positive number $\varepsilon < 1/2$ with $\delta_{X^*}(2 - 2\varepsilon) > 13/14$. Then T is close to an isometry $U : X \to L^{\infty}(\Omega, \Sigma, \mu)$ such that*

$$\|T - U\| \leqslant 16\bigl(1 - \delta_{X^*}(2 - 2\varepsilon)\bigr) + \frac{1}{2}\varepsilon.$$

Proof. We may assume that μ is a finite measure. Since X is reflexive, TB_X is weakly compact. By a theorem of Rosenthal [4], it is norm separable, and so is X since T is an isomorphism.

Since X^* has RNP, there exists an $h \in L^{\infty}(\mu, X^*)$ such that

$$(Tx)(\omega) = h(\omega)(x) \quad \text{and} \quad \|T\| = \|h\|_{\infty}.$$

For h there exists a countable valued mapping

$$h_1 = \sum_{i=1}^{\infty} f_i \chi_{E_i}, \quad f_i \in X^*,\ E_i \cap E_j = \emptyset\ (i \neq j),\ \mu(E_i) > 0,\ \bigcup_{i=1}^{\infty} E_i = \Omega, \tag{2.1}$$

such that

$$\|h - h_1\|_{\infty} \leqslant \frac{1}{2}\varepsilon \quad \text{and} \quad \|h_1\|_{\infty} \leqslant 1.$$

The operator T_1 given by $(T_1 x)(\omega) = h_1(\omega)(x)$ satisfies

$$\|T - T_1\| = \|h - h_1\|_{\infty} \leqslant \frac{1}{2}\varepsilon \quad \text{and} \quad \left(1 - \frac{3}{2}\varepsilon\right)\|x\| \leqslant \|T_1 x\| \leqslant \|x\|, \quad \forall x \in X.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence in S_X and choose $\{g_n\}_{n=1}^{\infty} \subset S_{X^*}$ such that $g_j(x_j) = 1$ for all j. For each $x \in X$, by (2.1), we have that $\forall E \in \Sigma$

$$\int_E (T_1 x)(\omega)\, d\mu = \int_E h_1(\omega)(x)\, d\mu = \sum_{i=1}^{\infty} \mu(E \cap E_i) f_i(x).$$

And for each $x \in X$

$$\begin{aligned} \|T_1 x\| &= \sup_{\mu(E)>0} \frac{\left|\int_E T_1 x(\omega)\, d\mu\right|}{\mu(E)} = \sup_{\mu(E)>0} \left| \sum_{i=1}^{\infty} \frac{\mu(E \cap E_i)}{\mu(E)} f_i(x) \right| \\ &\geqslant \left(1 - \frac{3}{2}\varepsilon\right)\|x\|. \end{aligned} \tag{2.2}$$

In particular, for each x_j, $j = 1, 2, \dots$, by (2.2), there is an i_j such that

$$\left|f_{i_j}(x_j)\right| > 1 - 2\varepsilon. \tag{2.3}$$

Let $|\lambda_j| = 1$ satisfy $\lambda_j f_{i_j}(x_j) > 1 - 2\varepsilon$. Then $2 - 2\varepsilon \leqslant \lambda_j f_{i_j}(x_j) + g_j(x_j)$ and $\|f_{i_j} + g_j\| \geqslant 2 - 2\varepsilon$. So

$$\|g_j - \lambda_j f_{i_j}\| \leqslant \delta(\varepsilon) := 2\bigl(1 - \delta_{X^*}(2 - 2\varepsilon)\bigr) \quad \text{and} \quad \lim_{\varepsilon \to 0} \delta(\varepsilon) = 0. \tag{2.4}$$

We assert that there exists a set $\{t_n\colon t_n \in \mathbf{N},\ t_i \neq t_j\ (i \neq j)\}_{n=1}^{\infty}$ such that $\|g_n - a_n f_{t_n}\| \leqslant 8\delta(\varepsilon)$ for some a_n with $|a_n| = 1$, $n = 1, 2, \dots$, and $f_{t_i} \neq f_{t_j}$ $(t_i \neq t_j)$. If this assertion is true, then define $\bar{g}(\omega) = \sum_{n=1}^{\infty} \bar{g}_k \chi_{E_k}$ and $\bar{g}_k$ is defined as follows: for $k = 1, 2, \dots$,

$$\bar{g}_k = \begin{cases} a_n^{-1} g_n, & \text{if } k = t_n \in \{t_1, t_2, \dots\}, \\ f_k, & \text{otherwise}. \end{cases}$$

It is obvious that $\bar{g} \in L^{\infty}(\mu, X^*)$, $\|\bar{g}\|_{\infty} \leqslant 1$, $\|h_1 - \bar{g}\|_{\infty} \leqslant 8\delta(\varepsilon)$ and on E_{i_j}, $|\bar{g}(\omega)(x_j)| = |g_j(x_j)| = 1$.

Define $U : X \to L^{\infty}(\mu)$ as follows:

$$(Ux)(\omega) = \bar{g}(\omega)(x) = \sum_{n=1}^{\infty} \bar{g}_k(x)\chi_{E_k}.$$

Then $\|Ux_j\| = 1$ for all j and $\|U - T_1\| = \|\bar{g} - h_1\|_{\infty}$. Since $\{x_j\}_{j=1}^{\infty}$ is dense in S_X, then for all $x \in S_X$, $\|Ux\| = 1$. Hence U is a linear isometry satisfying

$$\|U - T\| \leqslant \|U - T_1\| + \|T_1 - T\| \leqslant 8\delta(\varepsilon) + \frac{1}{2}\varepsilon \tag{2.5}$$

and $\lim_{\varepsilon \to 0}[8\delta(\varepsilon) + \varepsilon/2] = 0$.

We will prove the above assertion by induction following [5].

For $n = 1$, by (2.4), there is an a_1 with $|a_1| = 1$ and $t_1 \in \mathbf{N}$ such that $\|g_1 - a_1 f_{t_1}\| \leqslant 8\delta(\varepsilon)$. Suppose that for the set $A = \{1, 2, \dots, n-1\}$ and $\{g_1, g_2, \dots, g_{n-1}\}$ there exists a set $\{f_{t_1}, f_{t_2}, \dots, f_{t_{n-1}}\}$ such that

$$\|g_s - a_s f_{t_s}\| \leqslant 8\delta(\varepsilon) \quad \text{and} \quad f_{t_s} \neq f_{t_{s'}} \quad \text{for } s \in A,\ s \neq s'.$$

For each $x \in S_X$, let g_x be the supporting functional at x satisfying $g_x(x) = 1$. And by (2.2) there exists an i_j such that

$$\left|f_{i_j}(x)\right| > 1 - 2\varepsilon.$$

So there is a λ_{i_j} with $|\lambda_{i_j}| = 1$ such that

$$\|g_x - \lambda_{i_j}\hat{f}_{i_j}\| \leqslant \delta(\varepsilon). \tag{2.6}$$

Since X^* is uniformly convex, then $S_{X^*} = \{g_x\colon x \in S_X\}$ and $\{af_n\colon n = 1, 2, \dots,\ a \in \mathbf{K}, |a| = 1\}$ is a $\delta(\varepsilon)$-net of S_{X^*}.

By Lemma 2.1, for g_n there is a set $\{x_n^*\}_{n=1}^{\infty} \subset S_{X^*}$ such that

$$\|g_n - x_i^*\| = 7\delta(\varepsilon), \quad i = 1, 2, \dots, \tag{2.7}$$

and

$$\|x_i^* - ax_j^*\| > \frac{7}{3}\delta(\varepsilon) \quad (i \neq j) \text{ for all } |a| = 1.$$

Hence for each x_i^* $(i = 1, 2, \ldots)$, there is an f_{s_i} such that

$$\|x_i^* - b_i f_{s_i}\| \leqslant \delta(\varepsilon) \tag{2.8}$$

and for $s_i \neq s_j$

$$\begin{aligned}\|f_{s_i} - f_{s_j}\| &\geqslant \left\|b_i^{-1}x_i^* - b_j^{-1}x_j^*\right\| - \left\|f_{s_i} - b_i^{-1}x_i^*\right\| - \left\|f_{s_j} - b_i^{-1}x_i^*\right\| \\ &\geqslant \frac{7}{3}\delta(\varepsilon) - \delta(\varepsilon) - \delta(\varepsilon) > 0.\end{aligned}$$

So if $s_i \neq s_j$ then $f_{s_i} \neq f_{s_j}$.

Due to the infinity of the elements in $\{f_{s_n}\}_{n=1}^{\infty}$ and (2.7) and (2.8), for g_n we can select an f_{s_n} denoted by f_{t_n} with $s_n \neq t_i$ $(i = 1, 2, \ldots, n-1)$ such that

$$\|g_n - a_n f_{t_n}\| \leqslant \|g_n - x_{t_n}^*\| + \|x_{t_n}^* - a_n f_{t_n}\| \leqslant 8\delta(\varepsilon).$$

Thus for the set $\{g_1, g_2, \ldots, g_n\}$ there are $\{f_{t_1}, f_{t_2}, \ldots, f_{t_n}\}$ such that $\|g_i - a_i f_{t_i}\| \leqslant 8\delta(\varepsilon)$. By induction, the assertion is true. □

Corollary 2.3. *Let X be a separable uniformly smooth Banach space with infinite dimensions and $T : X \to C[0,1]$ satisfy $(1-\varepsilon)\|x\| \leqslant \|Tx\| \leqslant \|x\|$, $\forall x \in X$, for some sufficiently small positive number ε. Then there exist into isometries $U \in B(C[0,1], C[0,1])$ and $V \in B(X, C[0,1])$ such that*

$$\|U \circ Tx - Vx\| \leqslant \eta(\varepsilon)\|x\|, \quad \forall x \in X,$$

where

$$\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0.$$

Proof. Let $I : C[0,1] \to L^{\infty}[0,1]$ be the identity. Then I is a linear isometry and $I \circ T : X \to L^{\infty}[0,1]$ satisfies

$$(1-\varepsilon)\|x\| \leqslant \|I \circ Tx\| \leqslant \|x\|, \quad \forall x \in X.$$

According to Theorem 2.2, there is an isometry $V_0 \in B(X, L^{\infty}[0,1])$ such that

$$\|I \circ T - V_0\| \leqslant \eta(\varepsilon), \qquad \lim_{\varepsilon \to 0} \eta(\varepsilon) = 0.$$

Let Y_1 be the image space $V_0 X$, Y_2 be the image space $I(C[0,1])$ and $Y = Y_1 + Y_2$. Since X and $C[0,1]$ are separable, then Y_1, Y_2 and Y are also separable Banach spaces. Since $C[0,1]$ is universal for all separable normed linear spaces, there is a linear isometry $U_1 : Y \to C[0,1]$. Define $U = U_1 \circ I$ and $V = U_1 \circ V_0$. Then U and V are isometries satisfying

$$\|U \circ Tx - Vx\| \leqslant \eta(\varepsilon)\|x\|, \quad \forall x \in X, \quad \text{and} \quad \lim_{\varepsilon \to 0} \eta(\varepsilon) = 0. \quad \square$$

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