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# Mixed means over balls and annuli and lower bounds for operator norms of maximal functions

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**Abstract**

In this paper we prove mixed-means inequalities for integral power means of an arbitrary real order, where one of the means is taken over the ball $B(\mathbf{x}, \delta|\mathbf{x}|)$, centered at $\mathbf{x} \in \mathbb{R}^n$ and of radius $\delta|\mathbf{x}|$, $\delta > 0$. Therefrom we deduce the corresponding Hardy-type inequality, that is, the operator norm of the operator $S_\delta$ which averages $|f| \in L^p(\mathbb{R}^n)$ over $B(\mathbf{x}, \delta|\mathbf{x}|)$, introduced by Christ and Grafakos in Proc. Amer. Math. Soc. 123 (1995) 1687–1693. We also obtain the operator norm of the related limiting geometric mean operator, that is, Carleman or Levin–Cochran–Lee-type inequality. Moreover, we indicate analogous results for annuli and discuss estimations related to the Hardy–Littlewood and spherical maximal functions.


*Keywords:* Mixed means; Integral means; Balls and annuli; Power weights; Hardy's inequality; Hardy–Littlewood maximal function; Spherical maximal function

## 1. Introduction

Generalizing the well-known Hardy's inequality (cf. [7] or [11]) to multidimensional balls, Christ and Grafakos in [2] considered for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\delta > 0$ the following two averaging operators:

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$$(T_\delta f)(\mathbf{x}) = \frac{1}{|B(\delta|\mathbf{x}|)|} \int\limits_{B(\delta|\mathbf{x}|)} \left|f(\mathbf{y})\right| d\mathbf{y}$$

and

$$(S_\delta f)(\mathbf{x}) = \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|} \int\limits_{B(\mathbf{x},\delta|\mathbf{x}|)} \left|f(\mathbf{y})\right| d\mathbf{y},$$

where $B(\mathbf{x}, R)$ is the ball in $\mathbb{R}^n$ centered at $\mathbf{x} \in \mathbb{R}^n$ and of radius $R > 0$, $B(R) = B(\mathbf{0}, R)$, by $|\mathbf{x}|$ we denote the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$, and $|A|$ is the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^n$. They proved that the operator norm of $T_1$ on $L^p(\mathbb{R}^n)$, where $p > 1$, is equal to $p/(p-1)$, which means that it is the same as in the usual one-dimensional case. By using the obvious identity $(T_\delta f)(\mathbf{x}) = (T_1 f)(\delta\mathbf{x})$, it holds directly that the operator norm of $T_\delta$ on $L^p(\mathbb{R}^n)$ is equal to $\frac{p}{p-1}\delta^{-n/p}$.

Christ and Grafakos also solved a *more subtle* problem of deriving the operator norm of $S_\delta$ on $L^p(\mathbb{R}^n)$, which is, in our notation below, equal to $C(n; p; \delta; 0, 0)$ (compare the last theorem in [2] with our relations (14)–(16)). The importance of this result comes from the fact that, in some sense, the operator $S_\delta$ lies between the identity operator and the Hardy–Littlewood maximal function $M$, and that $Mf$ is, in fact, *not much larger than* $f$ (cf. [14, p. 1244]).

Before presenting our idea, let us introduce some necessary notation. Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and let $|S^{n-1}|$ be its area. By $\mathrm{An}(\mathbf{x}; R_1; R_2)$ we denote the annulus in $\mathbb{R}^n$ centered at $\mathbf{x} \in \mathbb{R}^n$ and of inner and outer radii $R_1$ and $R_2$, respectively, $0 \leqslant R_1 < R_2$. If $\omega$ is a weight function on $\mathbb{R}^n$ (a locally integrable non-negative function on $\mathbb{R}^n$) and $A \subseteq \mathbb{R}^n$ is a measurable set, let $|A|_\omega = \int_A \omega(\mathbf{x})\,d\mathbf{x}$. Especially, in the case when $\omega(\mathbf{x}) = |\mathbf{x}|^\alpha$, instead of $|A|_\omega$ we shall write $|A|_\alpha$. Moreover, we shall frequently use the obvious identities

$$\left|B(R)\right|_\alpha = \frac{n}{\alpha+n} R^{\alpha+n}\left|B(1)\right|$$

for $R > 0$ and $\alpha + n > 0$, and

$$\left|B\big(\mathbf{x}, \delta|\mathbf{x}|\big)\right|_\alpha = |\mathbf{x}|^{\alpha+n}\left|B(\mathbf{e}, \delta)\right|_\alpha$$

for $\delta > 0$, $\mathbf{x} \in \mathbb{R}^n$, and an arbitrary vector $\mathbf{e} \in \mathbb{R}^n$, such that $\|\mathbf{e}\| = 1$.

Finally, for $\alpha \in \mathbb{R}$ and a measurable set $A \subseteq \mathbb{R}^n$, such that $|A|_\alpha < \infty$, we define the integral weighted power mean of order $p \neq 0$ of a measurable non-negative (in the case $p < 0$ positive) function $f$ by

$$M_p(f; A, \alpha) = \left(\frac{1}{|A|_\alpha} \int\limits_A f^p(\mathbf{y})|\mathbf{y}|^\alpha\,d\mathbf{y}\right)^{\frac{1}{p}}. \tag{1}$$

It is known (cf. [7] and [11]) that $M_p$ is an increasing function with respect to the variable $p$ and it is also easy to see that

$$M_0(f; A; \alpha) := \lim_{p\to 0} M_p(f; A; \alpha) = \exp\left(\frac{1}{|A|_\alpha} \int\limits_A |\mathbf{y}|^\alpha \log f(\mathbf{y})\,d\mathbf{y}\right)$$

is the related weighted geometric mean $G(f; A; \alpha)$ of a positive measurable function $f$.

The main aim of this paper is to give another approach (analogous to [5]) to the problem of obtaining the operator norm of weighted operators $S_\delta$. Namely, we shall consider the class of operators

$$(S_{\delta,\alpha} f)(\mathbf{x}) = \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_\alpha} \int\limits_{B(\mathbf{x},\delta|\mathbf{x}|)} f(\mathbf{y})|\mathbf{y}|^\alpha \, d\mathbf{y} \tag{2}$$

on weighted $L^p$ spaces (with power weights). Our basic idea is to prove appropriate mixed-means-type inequalities (see Theorems 1 and 2 below) and then apply them to derive the related Hardy and Carleman-type inequalities. Analogous results will be obtained also for annuli. Moreover, in the last section we shall give some estimations of the operator norm of the Hardy–Littlewood and spherical maximal functions on $L^p(\mathbb{R}^n)$.

The analysis used in the proofs is based on Minkowski's integral inequality (cf. [10]), polar coordinates in $\mathbb{R}^n$, and on integral equality

$$\frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}} f(\theta)\, d\theta = \int\limits_{SO(n)} f(\sigma \mathbf{e})\, d\sigma, \tag{3}$$

where $d\sigma$ is the normalized Haar measure on the rotation group $SO(n)$ of $\mathbb{R}^n$, $d\theta$ is induced Lebesgue measure on $S^{n-1}$, $f$ is an integrable function on $(n-1)$-dimensional unit sphere, and the vector $\mathbf{e} \in \mathbb{R}^n$, $\|\mathbf{e}\| = 1$, is arbitrary (cf. [15]). We remark that, due to the compactness of $SO(n)$, the Haar measure is left and right invariant (cf. [8]).

In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

## 2. Mixed-means inequality

We start with the basic inequality.

**Theorem 1.** *Let $r, s, b, \delta, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $r \leqslant s$, $r, s \neq 0$, $b > 0$, $\delta > 0$, and $\alpha_1, \alpha_2 > -n$. If $f$ is a non-negative function on $B((1+\delta)b)$ ($f$ positive in the case when $r < 0$) and $\mathbf{b} = b\mathbf{e}_1 = b(1, 0, \ldots, 0) \in \mathbb{R}^n$, then the inequality*

$$\begin{aligned} &\left[ \frac{1}{|B(b)|_{\alpha_2}} \int\limits_{B(b)} \left( \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_{\alpha_1}} \int\limits_{B(\mathbf{x},\delta|\mathbf{x}|)} f^r(\mathbf{y})|\mathbf{y}|^{\alpha_1}\, d\mathbf{y} \right)^{\frac{s}{r}} |\mathbf{x}|^{\alpha_2}\, d\mathbf{x} \right]^{\frac{1}{s}} \\ &\quad \leqslant \left[ \frac{1}{|B(\mathbf{b}, \delta b)|_{\alpha_1}} \int\limits_{B(\mathbf{b},\delta b)} \left( \frac{1}{|B(|\mathbf{x}|)|_{\alpha_2}} \int\limits_{B(|\mathbf{x}|)} f^s(\mathbf{y})|\mathbf{y}|^{\alpha_2}\, d\mathbf{y} \right)^{\frac{r}{s}} |\mathbf{x}|^{\alpha_1}\, d\mathbf{x} \right]^{\frac{1}{r}} \end{aligned} \tag{4}$$

*holds. Moreover, if $r = s$, then equality in* (4) *holds for all functions $f$. In the case when $r < s$, equality holds in* (4) *for functions $f$ of the form $f(\mathbf{x}) = C|\mathbf{x}|^\lambda$, where $C \geqslant 0$. Finally, for $r > s$ the sign of inequality in* (4) *is reversed.*

**Proof.** Written in terms of characteristic functions, the left-hand side of inequality (4) reads

$$\left[\frac{1}{|B(b)|_{\alpha_2}}\int_{B(b)}\left(\frac{1}{|B(\mathbf{x},\delta|\mathbf{x}|)|_{\alpha_1}}\int_{B((1+\delta)b)} f^r(\mathbf{y})\chi_{B(\mathbf{x},\delta|\mathbf{x}|)}(\mathbf{y})|\mathbf{y}|^{\alpha_1}\,d\mathbf{y}\right)^{\frac{s}{r}}|\mathbf{x}|^{\alpha_2}\,d\mathbf{x}\right]^{\frac{1}{s}}. \tag{5}$$

The next step is to transform (5) to polar coordinates, so let $\mathbf{x}=t\theta$, $\theta\in S^{n-1}$, and $\mathbf{y}=u\phi$, $\phi\in S^{n-1}$. The relation $|\mathbf{y}-\mathbf{x}|\leqslant\delta|\mathbf{x}|$ is then equivalent to $\theta\cdot\phi\geqslant\frac{(1-\delta^2)t^2+u^2}{2ut}$. Note that inequality $\frac{(1-\delta^2)t^2+u^2}{2ut}\leqslant 1$ holds if and only if $(1-\delta)t\leqslant u\leqslant(1+\delta)t$, so in the case $\delta>1$ we have $0\leqslant u\leqslant(1+\delta)t$.

We continue by considering $\delta\leqslant 1$, while for $\delta>1$ the proof follows the same lines. In this setting, (5) is further equal to

$$\begin{aligned}
&\left[\frac{1}{|B(b)|_{\alpha_2}}\int_{\theta}\int_{t=0}^{b}\left(\frac{1}{|B(t\theta,\delta t)|_{\alpha_1}}\int_{\phi}\int_{u=(1-\delta)t}^{(1+\delta)t} f^r(u\phi)\right.\right.\\
&\quad\left.\left.\times\,\chi_{\phi\cdot\theta\geqslant\frac{(1-\delta^2)t^2+u^2}{2ut}}(u\phi)u^{\alpha_1}u^{n-1}\,du\,d\phi\right)^{\frac{s}{r}}t^{\alpha_2}t^{n-1}\,dt\,d\theta\right]^{\frac{1}{s}}\\
&=\frac{|B(b)|_{\alpha_2}^{-\frac{1}{s}}}{|B(\mathbf{e}_1,\delta)|_{\alpha_1}^{\frac{1}{r}}}\left[\int_{\theta}\int_{t=0}^{b}\left(\int_{\phi}\int_{u=1-\delta}^{u=1+\delta} f^r(ut\phi)\right.\right.\\
&\quad\left.\left.\times\,\chi_{\phi\cdot\theta\geqslant\frac{1}{2}(\frac{1-\delta^2}{u}+u)}(ut\phi)u^{\alpha_1}u^{n-1}\,du\,d\phi\right)^{\frac{s}{r}}t^{\alpha_2}t^{n-1}\,dt\,d\theta\right]^{\frac{1}{s}}\\
&=\frac{|B(b)|_{\alpha_2}^{-\frac{1}{s}}|S^{n-1}|^{\frac{1}{s}+\frac{1}{r}}}{|B(\mathbf{e}_1,\delta)|_{\alpha_1}^{\frac{1}{r}}}\left[\int_{\sigma\in SO(n)}\int_{t=0}^{b}\left(\int_{\sigma'\in SO(n)}\int_{u=1-\delta}^{u=1+\delta} f^r(ut\sigma'\mathbf{e}_1)\right.\right.\\
&\quad\left.\left.\times\,\chi_{\sigma'\mathbf{e}_1\cdot\sigma\mathbf{e}_1\geqslant\frac{1}{2}[\frac{1-\delta^2}{u}+u]}(ut\sigma'\mathbf{e}_1)u^{\alpha_1}u^{n-1}\,du\,d\sigma'\right)^{\frac{s}{r}}t^{\alpha_2}t^{n-1}\,dt\,d\sigma\right]^{\frac{1}{s}},
\end{aligned}\tag{6}$$

where the last equality in (6) is obtained by using (3). Knowing that $SO(n)$ preserves the inner product and using the (right) invariance of the Haar measure, (6) is now equal to

$$\begin{aligned}
&\frac{|B(b)|_{\alpha_2}^{-\frac{1}{s}}|S^{n-1}|^{\frac{1}{s}+\frac{1}{r}}}{|B(\mathbf{e}_1,\delta)|_{\alpha_1}^{\frac{1}{r}}}\left[\int_{\sigma\in SO(n)}\int_{t=0}^{b}\left(\int_{\sigma'\in SO(n)}\int_{u=1-\delta}^{u=1+\delta} f^r(ut\sigma\sigma'\mathbf{e}_1)\right.\right.\\
&\quad\left.\left.\times\,\chi_{\sigma'\mathbf{e}_1\cdot\mathbf{e}_1\geqslant\frac{1}{2}[\frac{1-\delta^2}{u}+u]}(ut\sigma\sigma'\mathbf{e}_1)u^{\alpha_1}u^{n-1}\,du\,d\sigma'\right)^{\frac{s}{r}}t^{\alpha_2}t^{n-1}\,dt\,d\sigma\right]^{\frac{1}{s}},
\end{aligned}\tag{7}$$

while applying Minkowski's inequality and simple transformations, (7) is not greater than

$$\frac{|B(b)|_{\alpha_2}^{-\frac{1}{s}}|S^{n-1}|^{\frac{1}{s}+\frac{1}{r}}}{|B(\mathbf{e}_1,\delta)|_{\alpha_1}^{\frac{1}{r}}}\Bigg[\int\limits_{\sigma'\in SO(n)}\int\limits_{u=1-\delta}^{1+\delta}\Bigg(\int\limits_{\sigma\in SO(n)}\int\limits_{t=0}^{b} f^s(ut\sigma\sigma'\mathbf{e}_1)$$

$$\times\chi_{\sigma'\mathbf{e}_1\cdot\mathbf{e}_1\geqslant\frac{1}{2}[\frac{1-\delta^2}{u}+u]}(ut\sigma\sigma'\mathbf{e}_1)t^{\alpha_2}t^{n-1}\,dt\,d\sigma\Bigg)^{\frac{r}{s}}u^{\alpha_1}u^{n-1}\,dt\,d\sigma'\Bigg]^{\frac{1}{r}}$$

$$=\frac{|B(b)|_{\alpha_2}^{-\frac{1}{s}}|S^{n-1}|^{\frac{1}{s}+\frac{1}{r}}}{|B(\mathbf{b},\delta b)|_{\alpha_1}^{\frac{1}{r}}}\Bigg[\int\limits_{\sigma'}\int\limits_{u=(1-\delta)b}^{(1+\delta)b}\Bigg(\int\limits_{\sigma}\int\limits_{t=0}^{b} f^s\big((ut/b)\sigma\sigma'\mathbf{e}_1\big)$$

$$\times\chi_{\sigma'\mathbf{e}_1\cdot\mathbf{e}_1\geqslant\frac{1}{2}[\frac{(1-\delta^2)b}{u}+\frac{u}{b}]}\big((ut/b)\sigma\sigma'\mathbf{e}_1\big)t^{\alpha_2}t^{n-1}\,dt\,d\sigma\Bigg)^{\frac{r}{s}}u^{\alpha_1}u^{n-1}\,dt\,d\sigma'\Bigg]^{\frac{1}{r}}$$

$$=\frac{|S^{n-1}|^{\frac{1}{s}+\frac{1}{r}}}{|B(\mathbf{b},\delta b)|_{\alpha_1}^{\frac{1}{r}}}\Bigg[\int\limits_{\sigma'}\int\limits_{u=(1-\delta)b}^{(1+\delta)b}\Bigg(\frac{1}{|B(u)|_{\alpha_2}}\int\limits_{\sigma}\int\limits_{t=0}^{u} f^s(t\sigma\sigma'\mathbf{e}_1)$$

$$\times\chi_{\sigma'\mathbf{e}_1\cdot\mathbf{e}_1\geqslant\frac{1}{2}[\frac{(1-\delta^2)b}{u}+\frac{u}{b}]}\big(t\sigma\sigma'\mathbf{e}_1\big)t^{\alpha_2}t^{n-1}\,dt\,d\sigma\Bigg)^{\frac{r}{s}}u^{\alpha_1}u^{n-1}\,dt\,d\sigma'\Bigg]^{\frac{1}{r}}$$

$$=\frac{|S^{n-1}|^{\frac{1}{s}+\frac{1}{r}}}{|B(\mathbf{b},\delta b)|_{\alpha_1}^{\frac{1}{r}}}\Bigg[\int\limits_{\sigma'}\int\limits_{u=(1-\delta)b}^{(1+\delta)b}\chi_{\sigma'\mathbf{e}_1\cdot\mathbf{e}_1\geqslant\frac{1}{2}[\frac{(1-\delta^2)b}{u}+\frac{u}{b}]}(u\sigma'\mathbf{e}_1)$$

$$\times\Bigg(\frac{1}{|B(u)|_{\alpha_2}}\int\limits_{\sigma}\int\limits_{t=0}^{u} f^s(t\sigma\sigma'\mathbf{e}_1)t^{\alpha_2}t^{n-1}\,dt\,d\sigma\Bigg)^{\frac{r}{s}}u^{\alpha_1}u^{n-1}\,dt\,d\sigma'\Bigg]^{\frac{1}{r}}$$

$$=\Bigg[\frac{1}{|B(\mathbf{b},\delta b)|_{\alpha_1}}\int\limits_{B(\mathbf{b},\delta b)}\Bigg(\frac{1}{|B(|\mathbf{x}|)_{\alpha_2}}\int\limits_{B(|\mathbf{x}|)} f^s(\mathbf{y})|\mathbf{y}|^{\alpha_2}\,d\mathbf{y}\Bigg)^{\frac{r}{s}}|\mathbf{x}|^{\alpha_1}\,d\mathbf{x}\Bigg]^{\frac{1}{r}}. \tag{8}$$

The last equality in (8) follows from the invariance of the Haar measure and from relation (3).

Finally, it is straightforward to check that both sides of inequality (4), rewritten with the function $f(\mathbf{x})=|\mathbf{x}|^\lambda$, are equal to

$$b^\lambda M_r\big(|\mathbf{x}|^\lambda;\,B(\mathbf{e}_1,\delta);\,\alpha_1\big)M_s\big(|\mathbf{y}|^\lambda;\,B(1);\,\alpha_2\big),$$

which gives the sharpness of inequality (4). □

**Remark 1.** It is not hard to see that inequality (4) is dilation invariant in the sense that if it holds for $b=1$ and a non-negative function $f:B(1+\delta)\to\mathbb{R}$, then it holds for every $b>0$ and the function $g:B((1+\delta)b)\to\mathbb{R}$ defined by $g(\mathbf{x})=f(\mathbf{x}/b)$. It is also obvious that both sides of inequality (4) are rotation invariant, that is, they will not be changed if a given function $\mathbf{x}\mapsto f(\mathbf{x})$ is replaced by the function $\mathbf{x}\mapsto f(\sigma\mathbf{x})$ for any $\sigma\in SO(n)$.

**Remark 2.** In Theorem 1 it was pointed out that inequality (4) is sharp, owing to the fact that it turns to equality in the case of functions of the form $f(\mathbf{x}) = |\mathbf{x}|^{\lambda}$. In [5, Theorem 5], where integral means were taken over balls in $\mathbb{R}^n$ centered at the origin, this form of functions $f$ was the only possible for achieving equality. The proof presented there was simple since Jensen's inequality was applied to the *angular* part of polar coordinates so radiality of extremal functions was immediate. Moreover, the family of all extremal functions for the related mixed-means inequality (cf. [5, Theorem 7]) was obtained from separability of extremal functions for Minkowski's inequality. The same holds also in our case here, but since the proof is lengthy and quite technical, it will be omitted.

Reformulating the basic inequality in Theorem 1 in terms of integral weighted power means (1) we obtain the related mixed-means inequalities.

**Theorem 2.** *Under the assumptions of Theorem* 1, *the following inequalities hold*:

(1) $M_s(M_r(f; B(\mathbf{x}, \delta|\mathbf{x}|); \alpha_1); B(b); \alpha_2) \leqslant M_r(M_s(f; B(|\mathbf{x}|); \alpha_2); B(\mathbf{b}, \delta b); \alpha_1)$;
(2) $M_s(M_r(f; B(|\mathbf{x}|); \alpha_2); B(\mathbf{b}, \delta b); \alpha_1) \leqslant M_r(M_s(f; B(\mathbf{x}, \delta|\mathbf{x}|); \alpha_1); B(b); \alpha_2)$.

It is important to state the following corollary, especially in view of the fact that the maximal function is not a bounded operator on $L^1$ so the dominate convergence and Lebesgue's differentiation theorem cannot be applied. This result follows directly from Theorem 1 by taking $r = s = 1$.

**Corollary 1.** *Suppose that $b > 0$, $\varepsilon > 0$, and $\alpha_1, \alpha_2 > -n$. If a function $f \in L^1_{\alpha_2}(B((1+\varepsilon)b))$ is non-negative, then*

$$\lim_{\delta \to 0} \int_{B(b)} \left( \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_{\alpha_1}} \int_{B(\mathbf{x},\delta\mathbf{x})} f(\mathbf{y})|\mathbf{y}|^{\alpha_1}\, d\mathbf{y} \right) |\mathbf{x}|^{\alpha_2}\, d\mathbf{x} = \int_{B(b)} f(\mathbf{y})|\mathbf{y}|^{\alpha_2}\, d\mathbf{y}. \tag{9}$$

To conclude this section, we give a generalization of Theorem 1 to annuli in $\mathbb{R}^n$ defined in the Introduction.

**Theorem 3.** *Let $r, s, b, \delta_1, \delta_2, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $r \leqslant s$, $r, s \neq 0$, $b > 0$, $0 \leqslant \delta_1 < \delta_2$, and $\alpha_1, \alpha_2 > -n$. If $f$ is a non-negative function on $B((1+\delta_2)b)$ ($f$ positive in the case $r < 0$) and $\mathbf{b} = b\mathbf{e}_1 = b(1, 0, \ldots, 0) \in \mathbb{R}^n$, then*

$$\begin{aligned} &M_s\bigl(M_r\bigl(f; \mathrm{An}\bigl(\mathbf{x}; \delta_1|\mathbf{x}|; \delta_2|\mathbf{x}|\bigr); \alpha_1\bigr); B(\mathbf{b}); \alpha_2\bigr) \\ &\quad \leqslant M_r\bigl(M_s\bigl(f; B\bigl(|\mathbf{x}|\bigr); \alpha_2\bigr); \mathrm{An}(\mathbf{b}; \delta_1 b; \delta_2 b), \alpha_1\bigr). \end{aligned} \tag{10}$$

*For $r = s$ we have equality in* (10). *If $r < s$, then equality holds in* (10) *for functions $f$ of the form $f(\mathbf{x}) = C|\mathbf{x}|^{\lambda}$, where $C \geqslant 0$. In the case when $r > s$, the sign of inequality in* (10) *is reversed.*

**Proof.** Since the proofs of Theorems 1 and 3 follow the same line, we omit technical details here. The only difference appears when using the characteristic function of annuli in polar coordinates $\mathbf{x} = t\theta$, $\mathbf{y} = u\phi$ we get the inequalities $\frac{1}{2ut}[u^2 + (1-\delta_2^2)t^2] \leqslant \phi \cdot \theta \leqslant \frac{1}{2ut}[u^2 + (1-\delta_1^2)t^2]$. $\square$

**Remark 3.** Although Theorem 1 is an obvious consequence of Theorem 3, we chose to give the complete proof of Theorem 1 and a sketch of the proof of Theorem 3 only to avoid more complex and awkward notation in the case of annuli.

## 3. Hardy and Carleman-type inequalities

The mixed means and related inequalities can be used in proving different integral inequalities. Analogously to the procedure established in [4,5], in this section we apply the previously obtained mixed-means inequality (4) to deduce the Hardy-type inequality for the operator $S_\delta$.

**Theorem 4.** *Let $p > 1$, $0 < b \leqslant \infty$, $\alpha_1, \alpha_2 \in \mathbb{R}$, and $\delta > 0$ be such that $\alpha_1, \alpha_2 > -n$, and let $p > (\alpha_2 + n)/(\alpha_1 + n)$ if $\delta \geqslant 1$. If $f \in L^p_{\alpha_2}(B((1+\delta)b))$ is a non-negative function, then $S_{\delta,\alpha_1}(f) \in L^p_{\alpha_2}(B(b))$ and the inequality*

$$\left[\int_{B(b)} \left(\frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_{\alpha_1}} \int_{B(\mathbf{x},\delta|\mathbf{x}|)} f(\mathbf{y})|\mathbf{y}|^{\alpha_1}\, d\mathbf{y}\right)^p |\mathbf{x}|^{\alpha_2}\, d\mathbf{x}\right]^{\frac{1}{p}} \leqslant C(n; p; \delta; \alpha_1, \alpha_2)\left(\int_{B((1+\delta)b)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2}\, d\mathbf{y}\right)^{\frac{1}{p}} \tag{11}$$

*holds, where*

$$C(n; p; \delta; \alpha_1, \alpha_2) = \frac{1}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \int_{B(\mathbf{e_1},\delta)} |\mathbf{x}|^{-\frac{\alpha_2+n}{p}} |\mathbf{x}|^{\alpha_1}\, d\mathbf{x} \tag{12}$$

*is the best possible constant. The same holds for $p \geqslant 1$ if $0 < \delta < 1$.*

**Proof.** Let $0 < b < \infty$. Inequality (4) for $r = 1$ and $s = p$ can be written in the form

$$\left[\int_{B(b)} \left(\frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_{\alpha_1}} \int_{B(\mathbf{x},\delta|\mathbf{x}|)} f(\mathbf{y})|\mathbf{y}|^{\alpha_1}\, d\mathbf{y}\right)^p |\mathbf{x}|^{\alpha_2}\, d\mathbf{x}\right]^{\frac{1}{p}} \leqslant \frac{|B(b)|_{\alpha_2}^{\frac{1}{p}}}{|B(\mathbf{b}, \delta b)|_{\alpha_1}} \int_{B(\mathbf{b},\delta b)} \left(\frac{1}{|B(|\mathbf{x}|)|_{\alpha_2}} \int_{B(|\mathbf{x}|)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2}\, d\mathbf{y}\right)^{\frac{1}{p}} |\mathbf{x}|^{\alpha_1}\, d\mathbf{x}. \tag{13}$$

Since $\int_{B(|\mathbf{x}|)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2}\,d\mathbf{y} \leqslant \int_{B((1+\delta)b)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2}\,d\mathbf{y}$ holds obviously for every $\mathbf{x} \in B(\mathbf{b}, (1+\delta)b)$, inequality (11) follows by using a simple substitution $\mathbf{x} = b\mathbf{x}'$ in

$$\frac{|B(b)|_{\alpha_2}^{\frac{1}{p}}}{|B(\mathbf{b}, \delta b)|_{\alpha_1}} \int_{B(\mathbf{b},\delta b)} |\mathbf{x}|^{-\frac{\alpha_2+n}{p}} |\mathbf{x}|^{\alpha_1}\,d\mathbf{x}.$$

To prove that the constant $C(n;\, p;\, \delta;\, \alpha_1, \alpha_2)$ is the best possible for (11), we consider the functions $f_\varepsilon(\mathbf{x}) = |\mathbf{x}|^{-(\alpha_2+n)/p+\varepsilon}$, $\varepsilon > 0$. Now, rewrite inequality (11) for $f_\varepsilon$ and denote the integrals on its left-hand and right-hand sides by $I_l(f_\varepsilon)$ and $I_r(f_\varepsilon)$, respectively. It is easy to see that

$$I_l(f_\varepsilon) = |S^{n-1}|^{\frac{1}{p}} (p\varepsilon)^{-\frac{1}{p}} b^\varepsilon \frac{1}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \int_{B(\mathbf{e_1},\delta)} |\mathbf{x}|^{-\frac{\alpha_2+n}{p}+\varepsilon+\alpha_1}\,d\mathbf{x}$$

and

$$I_r(f_\varepsilon) = |S^{n-1}|^{\frac{1}{p}} (p\varepsilon)^{-\frac{1}{p}} (1+\delta)^\varepsilon b^\varepsilon,$$

which obviously gives

$$\lim_{\varepsilon\to 0} \frac{I_l(f_\varepsilon)}{C(p;\, n;\, \delta;\, \alpha_1, \alpha_2) I_r(f_\varepsilon)} = 1.$$

This implies that for any $\varepsilon_1 > 0$ there exists $\varepsilon > 0$ such that the inequality $I_l(f_\varepsilon) > (1 - \varepsilon_1) C(p;\, n;\, \delta;\, \alpha_1, \alpha_2) I_r(f_\varepsilon)$ holds, so we obtained that the constant $C(n;\, p;\, \delta;\, \alpha_1, \alpha_2)$ is the best possible for inequality (11).

The case $b = \infty$ follows from the finite case by taking $\lim_{b\to\infty}$. The statement that the same constant is the best possible also in this case can be proved easily by taking the characteristic function of the set $B((1+\delta)b)$ for some finite $b > 0$. $\square$

Using a method of calculating integrals over balls $B(\mathbf{e}_1, \delta)$ described in [2], that is, the crucial formula

$$\int_{S^{n-1}} \chi_{\theta\cdot\mathbf{e}_1 \geqslant t}\,d\theta = |S^{n-2}| \int_t^1 (1-s^2)^{(n-3)/2}\,ds$$

for $n \geqslant 2$, we obtain the following:

(i) If $0 < \delta < 1$, $\alpha_1, \alpha_2 > -n$, $p \geqslant 1$, and $p \neq (\alpha_2+n)/(\alpha_1+n)$, then

$$\begin{aligned} C(n;\, p;\, \delta;\, \alpha_1, \alpha_2) = {} & \frac{p|S^{n-2}|}{\alpha_1 p - \alpha_2 + n(p-1)} \frac{1}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \\ & \times \int_{\sqrt{1-\delta^2}}^1 (1-s^2)^{\frac{n-3}{2}} \Big[\big(s + \sqrt{s^2+\delta^2-1}\big)^{\alpha_1 - \frac{\alpha_2}{p} + \frac{n}{p'}} \\ & - \big(s - \sqrt{s^2+\delta^2-1}\big)^{\alpha_1 - \frac{\alpha_2}{p} + \frac{n}{p'}}\Big]\,ds; \end{aligned} \tag{14}$$

(ii) If $0 < \delta < 1$, $\alpha_1, \alpha_2 > -n$, and $p = (\alpha_2 + n)/(\alpha_1 + n) \geqslant 1$, then

$$C(n; p; \delta; \alpha_1, \alpha_2) = \frac{|S^{n-2}|}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \int\limits_{\sqrt{1-\delta^2}}^{1} (1 - s^2)^{\frac{n-3}{2}} \log \frac{s + \sqrt{s^2 + \delta^2 - 1}}{s - \sqrt{s^2 + \delta^2 - 1}} \, ds; \tag{15}$$

(iii) If $\delta \geqslant 1$, $\alpha_1, \alpha_2 > -n$, $p > 1$, and $p > (\alpha_2 + n)/(\alpha_1 + n)$, then

$$\begin{aligned} C(n; p; \delta; \alpha_1, \alpha_2) = {} & \frac{p|S^{n-2}|}{\alpha_1 p - \alpha_2 + n(p-1)} \frac{1}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \\ & \times \int\limits_{-1}^{1} (1 - s^2)^{\frac{n-3}{2}} \left(s + \sqrt{s^2 + \delta^2 - 1}\right)^{\alpha_1 - \frac{\alpha_2}{p} + \frac{n}{p'}} ds. \end{aligned} \tag{16}$$

Note that $p' = p/(p-1)$ is the usual conjugate exponent of $p$. The case $n = 1$ is elementary.

Similarly, from Theorem 1 we deduce also the boundedness of the operator

$$(M_{p,\delta,\alpha} f)(\mathbf{x}) = M_p\big(f; B\big(\mathbf{x}, \delta|\mathbf{x}|\big); \alpha\big) \tag{17}$$

on $L^1$ spaces for $0 \neq p < 1$.

**Theorem 5.** *Suppose that $0 \neq p < 1$, $0 < b \leqslant \infty$, $\delta > 0$, $\alpha_1, \alpha_2 > -n$, and $p < (\alpha_2 + n)/(\alpha_1 + n)$ if $\delta \geqslant 1$. If the function $f \in L^1_{\alpha_2}(B((1+\delta)b))$ is non-negative, then $M_{p,\delta,\alpha_1}(f) \in L^1_{\alpha_2}(B(b))$ and the inequality*

$$\begin{aligned} & \int\limits_{B(b)} \left( \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_{\alpha_1}} \int\limits_{B(\mathbf{x}, \delta|\mathbf{x}|)} f^p(\mathbf{y}) |\mathbf{y}|^{\alpha_1} \, dy \right)^{\frac{1}{p}} |\mathbf{x}|^{\alpha_2} \, d\mathbf{x} \\ & \quad \leqslant C(n; p; \delta; \alpha_1, \alpha_2) \int\limits_{B((1+\delta)b)} f(\mathbf{y}) |\mathbf{y}|^{\alpha_2} \, d\mathbf{y} \end{aligned} \tag{18}$$

*holds, where*

$$C(n; p; \delta; \alpha_1, \alpha_2) = \left( \frac{1}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \int\limits_{B(\mathbf{e_1}, \delta)} |\mathbf{y}|^{-p(\alpha_2+n)} |\mathbf{y}|^{\alpha_1} \, d\mathbf{y} \right)^{\frac{1}{p}}$$

*is the best possible constant.*

**Proof.** To prove that the constant $C(n; p; \delta; \alpha_1, \alpha_2)$ is the best possible for inequality (18) one can use the same procedure as in Theorem 4, with the *extremal almost divergent* functions $f_\varepsilon(\mathbf{y}) = |\mathbf{y}|^{-p(\alpha_2+n)+\varepsilon}$, $\varepsilon > 0$. □

Finally, we give the related Carleman or Levin–Cochran–Lee-type inequality for geometric mean (cf. [3,6,9]).

**Theorem 6.** *Suppose that $\alpha_1, \alpha_2 > -n$, $0 < b \leqslant \infty$, and $\delta > 0$. If the function $f \in L^1_{\alpha_2}(B(1+\delta)b))$ is positive, then $G(f, B(\mathbf{x}, \delta|\mathbf{x}|), \alpha_1) \in L^1_{\alpha_2}(B(b))$ and the inequality*

$$\int\limits_{B(b)} \exp\left[\frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|} \int\limits_{B(\mathbf{x},\delta|\mathbf{x}|)} |\mathbf{y}|^{\alpha_1} \log f(\mathbf{y})\, d\mathbf{y}\right] |\mathbf{x}|^{\alpha_2}\, d\mathbf{x} \leqslant C(n; \delta; \alpha_1, \alpha_2) \int\limits_{B((1+\delta)b)} f(\mathbf{y}) |\mathbf{y}|^{\alpha_2}\, d\mathbf{y} \tag{19}$$

*holds, where*

$$C(n; \delta; \alpha_1, \alpha_2) = \exp\left[\frac{\alpha_2 + n}{|B(\mathbf{e_1}, \delta)|_{\alpha_1}} \int\limits_{B(\mathbf{e_1},\delta)} |\mathbf{x}|^{\alpha_1} \log \frac{1}{|\mathbf{x}|}\, d\mathbf{x}\right] \tag{20}$$

*is the best possible constant.*

**Proof.** Inequality (19) follows from inequality (18) by taking the limiting procedure $\lim_{p\to 0}$. The proof that the constant $C(n; \delta; \alpha_1, \alpha_2)$ is the best possible for (19) is analogous to the corresponding one in Theorem 4 by using the functions $f_\varepsilon(\mathbf{y}) = |\mathbf{y}|^{-n-\alpha_2+\varepsilon}$, $\varepsilon > 0$. $\square$

The versions of Theorems 4–6 for the case of annuli are obvious, so we omit to state them explicitly. For the sake of discussion in the final section, here we just mention that the best possible constant in the annuli-version of Theorem 4 (see also the assumptions of Theorem 3) is equal to

$$C(n; p; \delta_1, \delta_2; \alpha_1, \alpha_2) = \frac{1}{|An(\mathbf{e}_1; \delta_1, \delta_2)|_{\alpha_1}} \int\limits_{An(\mathbf{e}_1;\delta_1,\delta_2)} |\mathbf{x}|^{-\frac{\alpha_2+n}{p}} |\mathbf{x}|^{\alpha_1}\, d\mathbf{x}. \tag{21}$$

## 4. Concluding remarks

First, we give several remarks on the constant $C(n; p; \delta; \alpha_1, \alpha_2)$ from (12) for the case when $\alpha_1 = \alpha_2 = 0$. In this setting it is equal to

$$C(n; p; \delta) := C(n; p; \delta; 0, 0) = \frac{1}{|B(\mathbf{e_1}, \delta)|} \int\limits_{B(\mathbf{e_1},\delta)} |\mathbf{x}|^{-\frac{n}{p}}\, d\mathbf{x}.$$

Using [10, Theorem 3.4] we have

$$\begin{aligned} \|S_\delta\|_{L^p} &= \frac{1}{|B(\mathbf{e}_1, \delta)|} \int\limits_{B(\mathbf{e}_1,\delta)} |\mathbf{x}|^{-\frac{n}{p}}\, d\mathbf{x} = \frac{1}{|B(\mathbf{e}_1, \delta)|} \int\limits_{\mathbb{R}^n} |\mathbf{x}|^{-\frac{n}{p}} \chi_{B(\mathbf{e}_1,\delta)}\, d\mathbf{x} \\ &< \frac{1}{|B(\mathbf{e}_1, \delta)|} \int\limits_{\mathbb{R}^n} |\mathbf{x}|^{-\frac{n}{p}} \chi_{B(\delta)}\, d\mathbf{x} = \frac{p}{p-1} \delta^{-\frac{n}{p}} = \|T_\delta\|_{L^p}. \end{aligned}$$

Suppose that $p > 1$ and $f \in L^p(\mathbb{R}^n)$ is a non-negative function. Obviously, $(S_\delta f)(\mathbf{x}) \leqslant (Mf)(\mathbf{x})$ (a.e.) holds for every $\delta > 0$, where

$$(Mf)(\mathbf{x}) = \sup_{r>0} \frac{1}{|B(\mathbf{x}, r)|} \int\limits_{B(\mathbf{x},r)} f(\mathbf{y})\, d\mathbf{y}$$

is the usual Hardy–Littlewood maximal function. Therefore, for every $\delta > 0$ the norm of the operator $S_\delta$ on $L^p(\mathbb{R}^n)$, that is, $C(n;\, p;\, \delta)$, is a lower bound of the norm of the operator $M$ on $L^p(\mathbb{R}^n)$. We know that $\sup_{\delta>0} C(n;\, p;\, \delta)$ is attained for some $1 < \delta < 2$ (see, e.g., [2, Section 3]).

Now, consider the harmonic case of the function $\mathbf{x} \mapsto |\mathbf{x}|^{-n/p}$, that is, the case when $p = n/(n-2)$ and $n \geqslant 3$. For $0 < \delta < 1$ we have $C(n;\, p;\, \delta) = 1$, while for $\delta \geqslant 1$ using (16) we obtain

$$\begin{aligned}
C(n;\, p;\, \delta) &= \frac{n}{2} \frac{|S^{n-2}|}{|S^{n-1}|} \frac{1}{\delta^n} \int\limits_{-1}^{1} (1-s^2)^{\frac{n-3}{2}} \bigl(s + \sqrt{s^2 + \delta^2 - 1}\bigr)^2\, ds \\
&= n \frac{\Gamma(n/2)}{\sqrt{\pi}\, \Gamma((n-1)/2)} \Biggl[ 2 \int\limits_0^1 (1-s^2)^{\frac{n-3}{2}} s^2\, ds \\
&\qquad + (\delta^2 - 1) \int\limits_0^1 (1-s^2)^{\frac{n-3}{2}}\, ds \Biggr] \\
&= \frac{1}{2} \biggl( \frac{n}{\delta^{n-2}} - \frac{n-2}{\delta^n} \biggr),
\end{aligned} \tag{22}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ is the usual Gamma function. It is obvious from (22) that $C(n;\, p;\, 1) = 1$ and that $C(n;\, p;\, \delta)$ is strictly decreasing with respect to $\delta \geqslant 1$. Analogously, in the case $p = n/(n-4)$, $n \geqslant 5$ (super-harmonic case), for $\delta \geqslant 1$ we obtain $C(n;\, p;\, \delta) = \frac{1}{2\delta^n} \bigl[ \frac{12}{n+2} + 4(\delta^2 - 1) + \frac{n}{2}(\delta^2 - 1)^2 \bigr]$, so we see that $C(n;\, p;\, 1) = 6/(n+2)$ and that $C(n;\, p;\, \delta)$ is strictly decreasing for $\delta \geqslant 1$. Thus, $C(n;\, p;\, \delta) < 1 = \lim_{\delta \to 0} C(n;\, p;\, \delta)$.

Similarly, we can obtain lower bounds for the operator norm of the spherical maximal function $(SMf)(\mathbf{x}) = \sup_{r>0} \frac{1}{|S^{n-1}(\mathbf{x}, r)|} \int_{S^{n-1}(\mathbf{x}, r)} f(\theta)\, d\theta$ on $L^p(\mathbb{R}^n)$ for $p > n/(n-1)$. This can be made by using annuli-version of Theorem 4, the boundedness of $SM$ on $L^p(\mathbb{R}^n)$ for $p > n/(n-1)$, the dominated convergence theorem, and the relation (cf. [13])

$$\lim_{\varepsilon \to 0} C(n;\, p;\, \delta - \varepsilon, \delta + \varepsilon) = \frac{1}{|S^{n-1}(\mathbf{e}_1, \delta)|} \int\limits_{S^{n-1}(\mathbf{e}_1, \delta)} |\theta|^{-\frac{n}{p}}\, d\theta,$$

where $C(n;\, p;\, \delta_1, \delta_2)$ is the constant in (21) for $\alpha_1 = \alpha_2 = 0$. Note that

$$\frac{1}{|\operatorname{An}(\mathbf{x}, (\delta - \varepsilon)|\mathbf{x}|, (\delta + \varepsilon)|\mathbf{x}|)|} \int\limits_{\operatorname{An}(\mathbf{x}, (\delta-\varepsilon)|\mathbf{x}|, (\delta+\varepsilon)|\mathbf{x}|)} f(\mathbf{y})\, d\mathbf{y}$$

$$= \frac{1}{|\operatorname{An}(\mathbf{0}, \delta-\varepsilon, \delta+\varepsilon)|} \int\limits_{\operatorname{An}(\mathbf{0},\delta-\varepsilon,\delta+\varepsilon)} f\big(\mathbf{x} - |\mathbf{x}|\mathbf{y}\big)\, d\mathbf{y}$$

$$= \frac{1}{|\operatorname{An}(\mathbf{0}, \delta-\varepsilon, \delta+\varepsilon)|} \int\limits_{t=\delta-\varepsilon}^{\delta+\varepsilon} \int\limits_{\theta} f\big(\mathbf{x} - |\mathbf{x}|t\theta\big) t^{n-1}\, dt\, d\theta$$

$$= \frac{n}{(\delta+\varepsilon)^n - (\delta-\varepsilon)^n} \int\limits_{t=\delta-\varepsilon}^{\delta+\varepsilon} t^{n-1} \left( \frac{1}{|S^{n-1}|} \int\limits_{\theta} f(\mathbf{x} - |\mathbf{x}|t\theta)\, d\theta \right) dt$$

$$\leqslant (SMf)(\mathbf{x}).$$

The concluding result is related to [1, Lemma 1] and to the failure of the $L^p$-boundedness of the spherical maximal operator for $p = n/(n-1)$ (see also [12]). In what follows, by $C(n; p; 1-\varepsilon, 1+\varepsilon)$ we denote the constant $C(n; p; \delta_1, \delta_2; \alpha_1, \alpha_2)$ in (21) for the case $\alpha_1 = \alpha_2 = 0$ and $0 < \varepsilon < 1$.

**Theorem 7.** *For $n \geqslant 2$ and a small enough $\varepsilon > 0$ there exists a constant $K > 0$ (independent on $\varepsilon$), such that*

$$C\big(n; n/(n-1); 1-\varepsilon, 1+\varepsilon\big) \geqslant K \log \frac{1}{\varepsilon}. \tag{23}$$

**Proof.** Using (14) and (15) we have

$$C\big(n; n/(n-1); 1-\varepsilon; 1+\varepsilon\big)$$

$$= \frac{|S^{n-2}|}{|S^{n-1}|} \frac{2n}{(1+\varepsilon)^n - (1-\varepsilon)^n} \left[ \int\limits_0^1 (1-s^2)^{\frac{n-3}{2}} \big(s^2 + (1+\varepsilon)^2 - 1\big)^{\frac{1}{2}}\, ds \right.$$

$$\left. - \int\limits_{\sqrt{1-(1-\varepsilon)^2}}^1 (1-s^2)^{\frac{n-3}{2}} \big(s^2 + (1-\varepsilon)^2 - 1\big)^{\frac{1}{2}}\, ds \right]$$

$$\geqslant \frac{|S^{n-2}|}{|S^{n-1}|} \frac{2n}{(1+\varepsilon)^n - (1-\varepsilon)^n} \int\limits_0^1 (1-s^2)^{\frac{n-3}{2}} \big[\big(s^2 + (1+\varepsilon)^2 - 1\big)^{\frac{1}{2}} - s\big]\, ds$$

$$\geqslant \frac{|S^{n-2}|}{|S^{n-1}|} \frac{2n}{(1+\varepsilon)^n - (1-\varepsilon)^n} \int\limits_0^{1/2} (1-s^2)^{\frac{n-3}{2}} \big[\big(s^2 + (1+\varepsilon)^2 - 1\big)^{\frac{1}{2}} - s\big]\, ds$$

$$\geqslant K_1 \frac{1}{\varepsilon} \int\limits_0^{1/2} \big[\big(s^2 + (1+\varepsilon)^2 - 1\big)^{\frac{1}{2}} - s\big]\, ds \geqslant K \log \frac{1}{\varepsilon},$$

where the constant $K_1$ is independent on $\varepsilon$ and it is obtained for a small enough $\varepsilon$. □

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