



Strictly substochastic semigroups with application to conservative and shattering solutions to fragmentation equations with mass loss

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Abstract

In the paper we shall present a survey of recent results on substochastic semigroups and provide new methods for determining their honesty. These methods are applied to the fragmentation equation with mass loss, yielding sufficient conditions for the existence of conservative and shattering solutions. Our results provide a mathematical framework that clarifies the discussion of [Phys. Rev. A 43 (1991) 656, Phys. Rev. A 41 (1990) 5755, J. Phys. A 24 (1991) 3967] on shattering fragmentation in such models showing, among others, that there occurs an unexpected mass loss associated with shattering which is not accounted for by the discrete and continuous mass loss, contrary to the conjecture of [Phys. Rev. A 43 (1991) 656, Phys. Rev. A 41 (1990) 5755].

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1. Introduction

Substochastic semigroups are positive strongly continuous semigroups of contractions. They are particularly useful for analysis of deterministic equations related to Markov processes, where they describe the time evolution of the density $u(t, x)$ of some quantity Q , where $x \in \Omega$ is a state variable and Ω is a state space. Equations describing the

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evolution of u are typically constructed by balancing, for any state x , the loss of Q due to the transfer of a part of it to the other states, and the gain due to the transfer of parts of Q from other states to the state x . General form of such equations is

$$\partial_t u = T_0 u + Au + Bu, \quad (1.1)$$

where A is the loss operator, B is the gain operator, and T_0 may describe some transport in the state space (e.g., free streaming or diffusion). The model itself implies that the quantity Q should be preserved. If this is the case, the semigroup describing the evolution is conservative for positive initial data and is called a stochastic semigroup. In many cases, however, the semigroup turns out to be not conservative even if no amount of Q is subtracted from the system during the evolution. Such Markov processes are well-known in the probability theory and are referred to as dishonest [2] or explosive [22]. This phenomenon is much less understood from the functional-analytic point of view and though a number of scattered results, often limited to particular applications, can be found in the literature [1,3,4,15,18–20,23], the systematic study has been initiated quite recently in a series of papers [6,7,9,10,16].

In many cases in the models there is a mechanism that allows Q to decrease. It could be an absorbing or permeable boundary, or some reaction that removes a part of Q from the system. In such a case we say that the semigroup describing the evolution is *strictly substochastic*. In the theory of Markov processes this case is dealt with by introducing an additional state that accounts for the loss, and redefining the process so that the resulting process is Markovian. However, the loss-functional, defining the leakage of Q from the system, carries an important information about the evolution; e.g., it may describe the rate of mass loss due to internal reactions in the fragmentation models. Therefore, it is important not to amalgamate it with the other states so that we can keep track of the rate of Q throughout the evolution. Moreover, also strictly substochastic processes can be dishonest, that is, Q can decrease faster than predicted by the loss-functional; thus it is important to separate these two causes of the loss of Q during the evolution. It turns out that the functional analysis approach is very efficient in both cases, providing necessary and sufficient conditions for honesty of the semigroup and describing the evolution in terms of the loss-functional.

Thus, in the first two sections of the paper we combine and extend some earlier results to provide a complete characterization of honest (and thus also dishonest) strictly substochastic semigroups. In particular, as in the conservative case, a strictly substochastic semigroup is honest if and only if its generator is the closure of the operator $T_0 + A + B$ from its natural domain. As the theorems characterizing honest semigroups are often not easy to use, in Section 3 we provide manageable sufficient conditions for a semigroup to be honest or dishonest. These conditions are based on analysis of extensions of the involved operators to a larger space.

The last two sections are devoted to the application of the general theory developed in the first part of the paper to the fragmentation model with mass loss that was introduced and studied extensively in [11,12,17]; these results present a substantial extension of [8,10]. Fragmentation, and the reverse, coagulation, processes appear in many branches of natural sciences ranging from physics, through chemistry, engineering, physiology to ecology, describing, e.g., polymer degradation, droplets break-up, rock crushing and grinding, solid

drugs break-up in organisms, blood cell aggregation and fragmentation, or animal grouping. In many cases the process is supposed to be mass-conservative, that is, the total mass of the system should be constant in time. However, there are numerous cases when this is not the case: the mass can be consumed by an external reaction causing the surface to recede continuously, finally destroying the bridges between different parts of a particle and triggering the break up, and also, in the case of heterogeneous particles, due to explosive chemical reactions whenever the reactant is exposed either by fragmentation or by surface recession [12,17]. In both, conservative and mass loss, cases it has been observed [11,12,14,17,18,21,24] that the process can be dishonest. In most papers the analysis was carried out only for coefficients of a special form and the phenomenon was termed the “shattering fragmentation” and attributed to the phase transition and formation of a “dust” of zero-size particles (the corresponding “converse” phenomenon in the coagulation processes is called gelation and is attributed to the formation of a gel consisting of infinitely many particles). There are very few attempts to provide a rigorous analysis of the shattering fragmentation, even in the conservative case: we mention [14,18] where purely probabilistic methods are used but still heavy restrictions are imposed on the form of the coefficients. Much more general results in the conservative case, based on the theory of substochastic semigroups, have been obtained recently by one of the authors [10].

In this paper we shall deal with the fragmentation model with mass loss and, using the theory developed in the first two sections, we shall provide sufficient conditions for the solution semigroup to be honest, without placing any assumptions on the form of coefficients. We also give sufficient conditions for dishonesty but here we have to place some regularity assumptions on the fragmentation and continuous mass-loss rates; we also assume that the function describing the distribution of masses of daughter particles depends only on parent/daughter ratio, see [11,12]. In particular, if the coefficients have algebraic singularities or zeros at the origin, then the obtained conditions turn out to be both necessary and sufficient.

We also note that the shattering fragmentation with mass loss is not fully understood even from the physical point of view: in [11,12] the authors argued that shattering should not lead to any additional mass loss that is not accounted for. In this paper we shall show, in particular, that this conjecture is not true.

2. Strictly substochastic semigroups

We will be concerned with the abstract evolution equation of the form (1.1) posed in the space of integrable functions. Hence, let (Ω, μ) be a measure space and let $X = L_1(\Omega, \mu)$. If $Z \subset X$ is a subspace, then Z_+ denotes the cone of non-negative elements of Z and for $f \in X$ symbols f_{\pm} denote the positive and negative part of f , that is, $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$. Let $(G(t))_{t \geq 0}$ be a strongly continuous semigroup on X . We say that $(G(t))_{t \geq 0}$ is a *substochastic semigroup* if for any $t \geq 0$, $G(t)f \geq 0$ for all $f \in X_+$ and $\|G(t)\| \leq 1$, and a *stochastic semigroup* if additionally $\|G(t)f\| = \|f\|$ for $f \in X_+$. Accordingly, we consider linear operators in X : $(T, D(T))$ and $(B, D(B))$, that have the following properties:

- (1) $(T, D(T))$ generates a substochastic semigroup $(G_T(t))_{t \geq 0}$,
- (2) $D(B) \supset D(T)$ and $Bf \geq 0$ for $f \in D(B)_+$,
- (3) for all $f \in D(T)_+$,

$$\int_{\Omega} (Tf + Bf) d\mu \leq 0. \quad (2.1)$$

Theorem 2.1 [6,23]. *Under the above assumptions, there exists a smallest substochastic semigroup $(G_K(t))_{t \geq 0}$ generated by an extension K of the operator $T + B$. This semigroup, for $f \in D(T)$ and $t > 0$, satisfies*

$$\frac{d}{dt} G_K(t)f = K G_K(t)f. \quad (2.2)$$

$(G_K(t))_{t \geq 0}$ can be obtained as a strong limit in X of semigroups $(G_r(t))_{t \geq 0}$ generated by $(T + rB, D(T))$ as $r \nearrow 1^-$; if $f \in X_+$, then the limit is monotonic. Generator K of $(G_K(t))_{t \geq 0}$ is characterized by

$$(\lambda I - K)^{-1} f = \sum_{n=0}^{\infty} (\lambda I - T)^{-1} [B(\lambda I - T)^{-1}]^n f, \quad f \in X, \lambda > 0. \quad (2.3)$$

Equation (2.1) can always be written as

$$\int_{\Omega} (T + B)u d\mu = -c(u), \quad u \in D(T)_+, \quad (2.4)$$

where c is a non-negative (possibly zero) functional defined on $D(T)$. In this paper we shall consider only the situation when c can be written as an integral functional, that is,

$$c(u) = \int_{\Omega} \varsigma(x)u(x) d\mu(x) \quad (2.5)$$

for some positive measurable function ς . We do not assume that c is bounded or closed.

We say that $(G_K(t))_{t \geq 0}$ is *strictly substochastic* if (2.4) holds with $c \neq 0$. For such semigroups we extend the concept of honesty in the following way.

Definition 2.1. We say that $(G_K(t))_{t \geq 0}$ is *honest* if c is finite on $D(K)$ and for any $0 \leq \dot{u} \in D(K)$ the solution $u(t) = G_K(t)\dot{u}$ of (2.2) satisfies

$$\frac{d}{dt} \int_{\Omega} u(t) d\mu = \frac{d}{dt} \|u(t)\| = -c(u(t)). \quad (2.6)$$

Remark 2.1. The question whether it is possible for (2.6) to hold only on some subspaces of X seems to be open. For pure binary fragmentation models with power law fragmentation rate the results of [6] show that if the semigroup is dishonest for one initial condition, it is dishonest for all.

Remark 2.2. In this paper we will be often faced with the situation when we have a subspace $Z \subset X$ such that $Z = RX$, where R is a positive linear operator defined on X . In such a case in general $Z_+ \neq RX_+$ (e.g., for $R = L_\lambda$, its inverse, $\lambda I - T$, may be not a positive operator) and $u \in Z$ does not yield $u_\pm \in Z_+$. However, we can still write $u = \bar{u}_+ - \bar{u}_-$ with $\bar{u}_\pm \in Z_+$ using the following argument. Let, for a given $u \in Z$, $u = Rf$, $f \in X$. Then $f = f_+ - f_-$, $f_+, f_- \in X_+$ and we define

$$\bar{u}_\pm = Rf_\pm \in Z_+, \quad (2.7)$$

since R is a positive operator.

Let us suppose that some linear relation is defined on Z . Using (2.7), we see that it holds for any $u \in Z_+$ if and only if it holds for any $u \in RX_+$ if and only if it holds for any $u \in Z$. Particularly, (2.4) is equivalent to

$$\int_{\Omega} (T + B)u \, d\mu = -c(u), \quad u \in D(T). \quad (2.8)$$

Lemma 2.1. *If assumptions (1) and (2) hold, then condition (2.4) (and therefore (2.8)) is equivalent to*

$$-c(L_\lambda f) = \lambda \|L_\lambda f\| + \|BL_\lambda f\| - \|f\|, \quad f \in X_+, \quad (2.9)$$

where $L_\lambda = R(\lambda, T) = (\lambda I - T)^{-1}$.

Proof. Firstly, by Remark 2.2, Eq. (2.4) is equivalent to Eq. (2.8), so that, since L_λ is a surjection from X onto $D(T)$, we have

$$-c(u) = - \int_{\Omega} f \, d\mu + \int_{\Omega} BL_\lambda f \, d\mu + \lambda \int_{\Omega} L_\lambda f \, d\mu, \quad (2.10)$$

for any $f = (\lambda I - T)u$, $u \in D(T)$. In particular, this is valid for $f \in X_+$ and, since $L_\lambda X_+ \subset D(T)_+$ and B is a positive operator, we have

$$-c(L_\lambda f) = \lambda \|L_\lambda f\| + \|BL_\lambda f\| - \|f\|, \quad f \in X_+. \quad (2.11)$$

Conversely, let (2.11) be valid for any $f \in X_+$. Writing this in the form (2.10), we obtain its validity for any $u \in L_\lambda X_+$ but by Remark 2.2 it holds also for arbitrary $u \in D(T)$. \square

We assume that the assumptions (1)–(3) and (2.5) hold throughout the rest of this section. The next result is an extension of similar considerations in [4,16] to the strictly substochastic context.

Theorem 2.2. *For any fixed $\lambda > 0$, there is $0 \leq \beta_\lambda \in X^*$ with $\|\beta_\lambda\| \leq 1$ such that for all $f \in X_+$,*

$$\lambda \|R(\lambda, K)f\| = \|f\| - \langle \beta_\lambda, f \rangle - c(R(\lambda, K)f). \quad (2.12)$$

Moreover, c extends to a non-negative continuous linear functional on $D(K)$, given again by (2.5).

Proof. Let us fix $f \in X_+$. From (2.3) and non-negativity we obtain

$$\lambda \|(\lambda I - K)^{-1} f\| = \lim_{N \rightarrow \infty} \sum_{n=0}^N \lambda \|L_\lambda (BL_\lambda)^n f\|.$$

By (2.11), we get

$$\begin{aligned} \sum_{n=0}^N \lambda \|L_\lambda (BL_\lambda)^n f\| &= \sum_{n=0}^N (\|(BL_\lambda)^n f\| - \|(BL_\lambda)^{n+1} f\| - c(L_\lambda (BL_\lambda)^n f)) \\ &= \|f\| - \|(BL_\lambda)^{N+1} f\| - c\left(\sum_{n=0}^N L_\lambda (BL_\lambda)^n f\right). \end{aligned} \quad (2.13)$$

Since the left-hand side is non-negative and c is a non-negative functional, we obtain

$$0 \leq c\left(\sum_{n=0}^N L_\lambda (BL_\lambda)^n f\right) \leq \|f\|, \quad (2.14)$$

and since the series is non-decreasing, the numerical sequence is converging. However, since c is an integral functional with a non-negative kernel, from the monotone convergence theorem we obtain

$$\lim_{N \rightarrow \infty} c\left(\sum_{n=0}^N L_\lambda (BL_\lambda)^n f\right) = c\left(\sum_{n=0}^{\infty} L_\lambda (BL_\lambda)^n f\right) = c(R(\lambda, K)f) < +\infty.$$

Thus, for $f \in X_+$, we have $c(R(\lambda, K)f) \leq \|f\|$. For arbitrary $u = R(\lambda, K)f \in D(K)$ this inequality follows from decomposition (2.7). This shows that c is continuous in the graph topology of $D(K)$.

Returning to (2.13) we see, that then also $\|(BL_\lambda)^{N+1} f\|$ converges to some $\beta_\lambda(f) \geq 0$ and by a similar argument, β_λ extends to a continuous linear functional on X with the norm not exceeding 1. \square

Proposition 2.1. $(G_K(t))_{t \geq 0}$ is honest if and only if for any $f \in X_+$ and $t \geq 0$,

$$\|G_K(t)f\| = \|f\| - c\left(\int_0^t G_K(s)f ds\right). \quad (2.15)$$

Proof. Let $u \in D(K)_+$. Integrating (2.6), we obtain

$$\|G_K(t)u\| = \|u\| - \int_0^t c(G_K(s)u) ds = \|u\| - c\left(\int_0^t G_K(s)u ds\right), \quad (2.16)$$

where we changed the order of integration using Fubini–Tonelli’s theorem. Taking now $f \in X_+$, we can approximate it by the sequence $u_n = n \int_0^{1/n} G_K(s)f ds$, $D(K)_+ \ni u_n \rightarrow f$ in X . Fixing $t > 0$, we see that since $K \int_0^t G_K(s)u_n ds = G_K(t)u_n - u_n$, the

integral $\int_0^t G_K(s)u_n ds$ converges in $D(K)$ to $\int_0^t G_K(s)f ds$. Since c is continuous on $D(K)$, we see that we can extend (2.16) to X_+ . Conversely, if (2.15) is satisfied, then it is satisfied for $f \in D(K)_+$. For such f the function $t \rightarrow G_K(t)f$ is continuous in $D(K)$ and so $t \rightarrow c(G_K(t)f)$ is continuous. Consequently, both $\|G_K(t)f\|$ and $\int_0^t c(G_K(s)u) ds$ are differentiable giving (2.6). \square

Let us introduce the defect function: for $f \in X_+$ and $t \geq 0$ we define

$$\eta_f(t) = \|G_K(t)f\| - \|f\| + \int_0^t c(G_K(s)f) ds. \quad (2.17)$$

Proposition 2.2. *For any $f \in X_+$, η_f is a non-positive and non-increasing function for $t \geq 0$.*

Proof. By Theorem 2.1, $(G_K(t))_{t \geq 0}$ can be obtained by a monotonic strong limit of semi-groups $(G_r(t))_{t \geq 0}$ generated by $(T + rB, D(T))$ as $r \nearrow 1$. For $u \in D(T)_+$ we have

$$\int_{\Omega} (T + rB)u d\mu = \int_{\Omega} (T + B)u d\mu + (r - 1) \int_{\Omega} Bu d\mu \leq -c(u), \quad (2.18)$$

as $B \geq 0$ on $D(T)$ and $r < 1$. Since for $f \in X_+$ and $t_2 > t_1 \geq 0$, $\int_{t_1}^{t_2} G_r(s)f \in D(T)_+$, integrating the equation

$$G_r(t_2)f = G_r(t_1)f + (T + rB) \int_{t_1}^{t_2} G_r(s)f ds$$

over Ω , we obtain

$$\|G_r(t_2)f\| \leq \|G_r(t_1)f\| - c\left(\int_{t_1}^{t_2} G_r(s)f ds\right). \quad (2.19)$$

Since the convergence of $(G_r(t))_{t \geq 0}$ is monotonic and $c \geq 0$, passing to the limit in (2.19), we get

$$\|G_K(t_2)f\| \leq \|G_K(t_1)f\| - c\left(\int_{t_1}^{t_2} G_K(s)f ds\right), \quad (2.20)$$

for any $f \in X_+$ and $t_2 > t_1 \geq 0$ so that η_f is non-positive. Next, for a given $f \in X_+$ we have, as above,

$$\|G_K(t_2)f\| = \|G_K(t_1)f\| + \int_{\Omega} \left(K \int_{t_1}^{t_2} G_K(s)f ds\right) d\mu. \quad (2.21)$$

Subtracting this from (2.20), we obtain

$$\int_{\Omega} K \int_{t_1}^{t_2} G_K(s) f \, ds + c \left(\int_{t_1}^{t_2} G_K(s) f \, ds \right) \leq 0, \quad (2.22)$$

for any $0 \leq t_1 < t_2$ and $f \in X_+$. Thus, by (2.21) and (2.22),

$$\begin{aligned} \eta_f(t_2) - \eta_f(t_1) &= \|G_K(t_2)f\| - \|G_K(t_1)f\| + c \left(\int_{t_1}^{t_2} G_K(s) f \, ds \right) \\ &= \int_{\Omega} K \int_{t_1}^{t_2} G_K(s) f \, ds + c \left(\int_{t_1}^{t_2} G_K(s) f \, ds \right) \leq 0, \end{aligned}$$

which ends the proof. \square

Theorem 2.3. $(G_K(t))_{t \geq 0}$ is honest if and only if $\beta_\lambda \equiv 0$ for any (some) $\lambda > 0$.

Proof. Consider the defect function (2.17),

$$\eta_f(t) = \|G_K(t)f\| - \|f\| + \int_0^t c(G_K(s)f) \, ds$$

for $f \in X_+$. Since

$$K \int_{t_1}^{t_2} G_K(s) f \, ds = G_K(t_2)f - G_K(t_1)f,$$

the function $t \rightarrow \int_0^t G_K(s)f \, ds$ is continuous in the norm $D(K)$, so $t \rightarrow \int_0^t c(G_K(s)f) \, ds$ is continuous by Theorem 2.2. Thus, taking the Laplace transform of η_f , we obtain

$$\int_0^\infty e^{-\lambda t} \eta_f(t) \, dt = \|R(\lambda, K)f\| - \frac{1}{\lambda} \|f\| + \frac{1}{\lambda} c(R(\lambda, K)f) = -\frac{1}{\lambda} \langle \beta_\lambda, f \rangle.$$

If the semigroup is honest, then

$$\|R(\lambda, K)f\| = \frac{1}{\lambda} (\|f\| - c(R(\lambda, K)f)),$$

that is,

$$\langle \beta_\lambda, f \rangle = -\lambda \int_0^\infty e^{-\lambda t} \eta_f(t) \, dt = 0$$

on X_+ and since it is positive, it vanishes identically for all f and $\lambda > 0$. Next, if for some $\lambda_0 > 0$ there is $f \in X$ such that $\langle \beta_{\lambda_0}, f \rangle \neq 0$, then splitting $f = f_+ - f_-$, we have

$\langle \beta_{\lambda_0}, f_+ \rangle \neq \langle \beta_{\lambda_0}, f_- \rangle$ and at least one of these two is strictly positive. Let $\langle \beta_{\lambda_0}, f \rangle > 0$ for some $f \in X_+$. By the uniqueness of the Laplace transform, η_f does not vanish identically, so that $(G_K(t))_{t \geq 0}$ is dishonest. \square

Corollary 2.1. *If $(G_K(t))_{t \geq 0}$ is dishonest, then there is $f \in X_+$ such that*

$$\|G_K(t)f\| < \|f\| - \int_0^t c(G_K(s)f) ds \quad \text{for any } t > 0.$$

Proof. From the definition and Proposition 2.2, there is $\bar{f} \in X_+$ and $t_0 > 0$ such that

$$\|G_K(t)\bar{f}\| < \|\bar{f}\| - \int_0^t c(G_K(s)\bar{f}) ds$$

for all $t > t_0$. Put $t' = \inf\{t > 0: \eta_{\bar{f}}(t) < 0\}$. By continuity and Proposition 2.2, $\eta_{\bar{f}}(t) = 0$ for $t \in [0, t']$. Define $f = G_K(t')\bar{f}$; then for any $t > 0$ we have

$$\begin{aligned} \|G_K(t)f\| &< \|\bar{f}\| - c\left(\int_0^{t+t'} G_K(s)\bar{f} ds\right) \\ &= \|G_K(t')\bar{f}\| + c\left(\int_0^{t'} G_K(s)\bar{f} ds\right) - c\left(\int_0^{t+t'} G_K(s)\bar{f} ds\right) \\ &= \|f\| - c\left(\int_0^t G_K(s)f ds\right). \quad \square \end{aligned}$$

Theorem 2.4. *The semigroup $(G_K(t))_{t \geq 0}$ is honest if and only if $K = \overline{T + B}$.*

Proof. Let us assume first that $K = \overline{T + B}$. If $K = T + B$ with $D(K) = D(T)$, then the statement follows by integrating Eq. (2.2). Let now $T + B \neq K = \overline{T + B}$. By (2.8), we have

$$\int_{\Omega} Ku d\mu = \int_{\Omega} (T + B)u d\mu = -c(u) \quad (2.23)$$

for any $u \in D(T)$. Thus, taking for an arbitrary $u \in D(K)$ a sequence $(u_n)_{n \in \mathbb{N}} \subset D(T)$ converging to u in $D(K)$, we obtain validity of (2.23) for arbitrary $u \in D(K)$ as c is continuous on $D(K)$ by Theorem 2.2. Thus, as before, honesty is obtained by integration of Eq. (2.2).

Conversely, if $(G_K(t))_{t \geq 0}$ is honest, then $\beta_{\lambda} \equiv 0$ for any $\lambda > 0$, which means by Theorem 2.2 that $\lim_{n \rightarrow \infty} (BL_{\lambda})^n f = 0$ for any $f \in X$ and $\lambda > 0$. Let $u = R(\lambda, K)f$ with $f \in X$. By (2.3), we can write $u = \lim_{N \rightarrow \infty} u_N$ in X where $u_N = L_{\lambda} f_N \in D(T)$, where $f_N = \sum_{n=0}^N (BL_{\lambda})^n f$. Moreover, since $D(B) \supset D(T)$,

$$\begin{aligned}(T+B)u_N &= (-(\lambda I - T) + \lambda I + B)u_N = -f_N + \lambda u_N + BL_\lambda f_N \\ &= \lambda u_N - f + (BL_\lambda)^{N+1} f\end{aligned}$$

and $(T+B)u_N$ converges to $\lambda u - f = \lambda u - \lambda u + Ku$. Thus, $D(K) \subset D(\overline{T+B})$ and since K is a closed extension of $T+B$, we obtain $K = \overline{T+B}$. \square

Corollary 2.2. *The semigroup $(G_K(t))_{t \geq 0}$ is honest if and only if for any $u \in D(K)_+$ we have*

$$\int_{\Omega} Ku \, d\mu \geq -c(u). \quad (2.24)$$

The statement holds true also if we replace $D(K)_+$ by $R(\lambda, K)X_+$ for some/any $\lambda > 0$.

Proof. If $(G_K(t))_{t \geq 0}$ is honest, then by Theorem 2.4, $K = \overline{T+B}$ and, as in the first part of the proof of Theorem 2.4, we obtain (2.24) with equality sign. Conversely, if (2.24) holds for any $u \in D(K)_+$, then it holds for $u = R(\lambda, K)f$, $f \in X_+$. Since $Ku = -(\lambda u - Ku) + \lambda u = -f + \lambda R(\lambda, K)f$, we obtain from (2.12)

$$\int_{\Omega} Ku \, d\mu = -\|f\| + \lambda \|R(\lambda, K)f\| = -c(u) - \langle \beta_\lambda, f \rangle, \quad (2.25)$$

and if (2.24) holds, then $\langle \beta_\lambda, f \rangle \leq 0$ for all $f \in X_+$, thus $\beta_\lambda = 0$ and by Theorem 2.3, $(G_K(t))_{t \geq 0}$ is honest. The last statement follows from Remark 2.2. \square

To complete this section we note that an elegant result relating the characterization of the generator K to the properties of 1 as an element of the spectrum or the resolvent set of BL_λ has been recently derived in [16]. Though it pertains to the conservative case $c \equiv 0$ and uses a slightly different set of assumptions, it can be easily modified to suit also the case $c \neq 0$. However, since we shall not be using this result here, we refer the reader to the original paper [16].

3. Extension techniques

The problem with most of the characterization results discussed in Section 2 is that they require the knowledge of the generator itself. We shall present a method of circumventing this difficulty by working with some extension of the operators that appear in the model.

Define by E the set of measurable functions that are defined on Ω and taking values in the extended set of real numbers and by E_f the subspace of E consisting of functions that are finite almost everywhere. E is a lattice with respect to the usual relation: \leq a.e., $X \subset E_f \subset E$ with X and E_f being sublattices of E .

First, we discuss the extension technique introduced in [4], that is an essential technical tool in obtaining several results of this paper.

Let $F \subset E$ be defined by the condition: $f \in F$ if and only if for any non-negative and nondecreasing sequence $(f_n)_{n \in \mathbb{N}}$ satisfying $\sup_n f_n = |f|$ we have $\sup_n (I - T)^{-1} f_n \in X$.

Under some natural assumptions on B (that are satisfied if, e.g., B is an integral operator with non-negative kernel), [4], we construct through B a subset G of E defined as the set of all functions $f \in X$ such that for any non-negative, nondecreasing sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $D(B)$ such that $\sup_n f_n = |f|$, we have $\sup_n Bf_n < +\infty$ a.e. It is easy to check that $D(T) \subseteq G \subseteq X \subseteq F \subseteq E$. We can then define mappings: $B: G_+ \rightarrow E_+$ and $L: F_+ \rightarrow X_+$ by

$$Bf := \sup_n Bf_n, \quad \forall f \in G_+, \quad (3.1)$$

$$Lf := \sup_n (I - T)^{-1} f_n, \quad \forall f \in F_+, \quad (3.2)$$

where $0 \leq f_n \leq f_{n+1}$ for any $n \in \mathbb{N}$, and $\sup_n f_n = f$ [4,5]. We extend the mappings L and B onto F and G , respectively, by linearity. It can be proved [5], that L is a restriction of the so-called Sobolev tower extension of $R(1, T)$ of order -1 , [13], and therefore it is one-to-one. Therefore, we can define the operator T with $D(T) = LF \subset X$ by

$$Tu = u - L^{-1}u, \quad (3.3)$$

so that T is an extension of T . Theorem 1 of [4] characterizes the generator K of the full semigroup $(G_K(t))_{t \geq 0}$ in the following way:

$$Ku = Tu + Bu \quad (3.4)$$

with

$$D(K) = \{u \in D(T) \cap D(B): Tu + Bu \in X, \text{ and } \|(LB)^n u\| \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

If we now consider $u \in D(K)$, then by (3.4) and definition of $D(T)$ we see that $u \in D(T) = LF$, and therefore there exists a unique $f \in F$ satisfying $u = Lf$. For such u we can write $Ku = TLf + BLf$ and, using (3.3), we obtain a representation theorem for Ku ,

$$Ku = Lf - f + BLf. \quad (3.5)$$

In particular, in this case $Lf = u \in D(K)$ is integrable and thus $-f + BLf$ is also integrable. Moreover, if $D(K)_+ \ni u = R(1, K)g$, with $g \in X_+$, then from (3.5) we get

$$f = Lf - Ku + BLf = (I - K)u + Bu \geq 0,$$

that is, $f \in F_+$.

Finally, using Remark 2.2 for any $u \in D(K)$, we can find elements $\bar{u}_\pm \in D(K)_+$ such that $u = \bar{u}_+ - \bar{u}_-$ and

$$K\bar{u}_\pm = L\bar{f}_\pm - \bar{f}_\pm + BL\bar{f}_\pm. \quad (3.6)$$

It turns out in many cases that it is sufficient to use more general operator extensions. In what follows by \mathcal{T} , \mathcal{B} , \mathcal{K} , and \mathcal{L}_λ we shall denote extensions of operators T , B , K , and $R(\lambda, T)$, respectively. By \mathcal{L} we abbreviate \mathcal{L}_1 . At this moment we shall require only that all the extensions have domains and ranges in E_f , \mathcal{B} , \mathcal{L} , and \mathcal{L}_λ are positive operators on their domains and $\mathcal{K} \subset \mathcal{T} + \mathcal{B}$.

It is obvious that, if for any such extension \mathcal{K} and any non-negative $u \in D(\mathcal{K})$ such that $\mathcal{K}u$ is integrable and $c(u)$ is finite we have

$$\int_{\Omega} \mathcal{K}u \, d\mu \geq -c(u),$$

then such an inequality holds also for $u \in D(K)_+$ and, by Corollary 2.2, the semigroup is honest. This fact is the basis of the honesty results, Theorems 4.1 and 4.2, through Lemma 4.2 and Corollary 4.1.

On the other hand, by Corollary 2.2, dishonesty is equivalent to the existence of $0 \leq u \in D(K)$ for which

$$\int_{\Omega} Ku \, d\mu < -c(u). \quad (3.7)$$

Though we do not know K , a criterion based on (3.7) can be obtained using a general extension \mathcal{K} .

Theorem 3.1. Assume that there exists $u \in D(\mathcal{K})_+ \cap X$ such that

- (i) $(I - T)u \in D(\mathcal{L})$ and $[\mathcal{L}(I - T)u] = u$,
- (ii) $u(x) - [\mathcal{K}u](x) = g(x) \in X_+$,
- (iii) $c(u)$ is finite and

$$\int_{\Omega} \mathcal{K}u \, d\mu < -c(u), \quad (3.8)$$

then the semigroup $(G_K(t))_{t \geq 0}$ is not honest.

Proof. We shall prove that there exists a non-negative $u_g \in D(K)$ satisfying (3.7). From (ii) we have

$$u(x) - [\mathcal{T}u](x) - [\mathcal{B}u](x) = g(x),$$

where each term is a measurable function that is finite almost everywhere with $g \in X_+$. By (i) we obtain

$$u(x) - [\mathcal{L}\mathcal{B}u](x) = [\mathcal{L}g](x) = [Lg](x), \quad (3.9)$$

where we used the fact that on X the operators \mathcal{L} and $L = R(1, T)$ coincide. From (3.9) we obtain $\mathcal{L}\mathcal{B}u \in X$, thus we can operate with $\mathcal{L}\mathcal{B}$ on both sides of (3.9) and separate terms on the left-hand side writing $\mathcal{B}L = BL$. Repeating this procedure for arbitrary n and summing up the iterates, we obtain

$$u(x) - [(\mathcal{L}\mathcal{B})^{n+1}u](x) = \sum_{i=0}^n [L(BL)^i g](x).$$

From Theorem 2.1 we obtain that the right-hand side converges in norm to a positive element $u_g = (I - K)^{-1}g \in D(K)$. Therefore, the sequence of iterates also converges to a non-negative element $h \in X$. Since $I - \mathcal{K}|_{D(K)} = I - K$, we obtain directly by (ii) that $(I - \mathcal{K})h = 0$. Therefore,

$$\begin{aligned} \int_{\Omega} Ku_g \, d\mu &= \int_{\Omega} \mathcal{K}u \, d\mu - \int_{\Omega} \mathcal{K}h \, d\mu = \int_{\Omega} \mathcal{K}u \, d\mu - \int_{\Omega} h \, d\mu \leq \int_{\Omega} \mathcal{K}u \, d\mu \\ &< -c(u) \leq -c(u_g), \end{aligned} \quad (3.10)$$

where we used the fact that c is a positive linear functional so that $0 \leq c(h) = c(u - u_g) = c(u) - c(u_g)$. \square

4. Fragmentation equation

Our aim is to analyse the solvability of the Cauchy problem

$$\begin{aligned} \partial_t u(x, t) &= \partial_x (r(x)u(x, t)) - a(x)u(x, t) + \int_x^\infty a(y)b(x|y)u(y, t) dy, \\ t > 0, \quad x > 0, \quad u(0, x) &= \dot{u}(x), \end{aligned} \quad (4.1)$$

on $X = L_1([0, \infty[, x dx)$. This equation describes fragmentation of particles with mass loss, see [8,11,12,17] for a more exhaustive discussion of the model. In Eq. (4.1), u is the particle mass distribution function, a is the fragmentation rate, r is the continuous mass loss rate. The non-negative measurable function $b(x|y)$ describes the distribution of particle masses x spawned by the fragmentation of a particle of mass y . It is sufficient if it is defined for $x \leq y$. The continuous mass loss rate r is defined so that $r(m(t)) = -dm/dt$ for a particle of time-dependent mass $m(t)$, while the normalizing condition

$$\int_0^y xb(x|y) dx = y(1 - \lambda(y)), \quad (4.2)$$

where $0 \leq \lambda(y) \leq 1$ for $y \geq 0$, defines the discrete mass loss coefficient that gives the fraction of mass lost during explosive fragmentation [11,12,17]. If $\lambda(y) = 0$ then the sum of masses of all the daughter particles, represented by the integral on the left is equal to the mass of the parent particle y . We assume that

$$\begin{aligned} r &\in C_{\text{loc}}^{0,1}((0, \infty)), \quad r > 0 \quad \text{on } (0, \infty), \quad \text{and} \\ a &\in L_{1,\text{loc}}((0, \infty)), \quad a \geq 0 \quad \text{on } (0, \infty) \text{ a.e.} \end{aligned} \quad (4.3)$$

We do not need any additional assumptions on b when we prove honesty of the semigroup. However, in the section devoted to the dishonesty of the semigroup, some technical calculations will require more information about b and thus, following [11,12], we assume that for some measurable $h \geq 0$, defined on $(0, 1]$,

$$b(x|y) = \frac{1}{y} h\left(\frac{x}{y}\right). \quad (4.4)$$

This, [10,11], corresponds to the assumption that the distribution of daughter particles is determined by the fraction *daughter mass/parent mass* $= x/y$ and not by the masses x and y separately. Substituting this form of $b(x|y)$ into (4.2), we see that due to

$$\frac{1}{y} \int_0^y xb(x|y) dx = \int_0^1 rh(r) dr,$$

(4.2) takes the form

$$\int_0^1 rh(r) dr = 1 - \lambda, \quad (4.5)$$

which forces the discrete mass loss coefficient to be constant.

To apply the theory of Section 2, let $(T_0, D(T_0))$ be defined through the expression $T_0u = (ru)'$ (where “'” denotes the derivative with respect to x) on the domain $D(T_0) = \{u \in X: ru \in AAC \text{ and } (ru)' \in X\}$, where AAC denotes functions that are absolutely continuous on any compact subset of \mathbb{R}_+ , and let A be defined by $Au = -au$ on $D(A) = \{u \in X: au \in X\}$.

Define

$$R(x) = \int \frac{dx}{r(x)}, \quad Q(x) = \int \frac{a(x)}{r(x)} dx \quad \text{on } (0, \infty);$$

we note that both $R, Q \in AAC$.

It can be proved [8], that the operator $(T, D(T))$, being the realization of $(T_0 + A)u = \partial_x(ru) - au$ on

$$D(T) = \begin{cases} D(T_0) \cap D(A), & \text{if } \int_0^\infty \frac{xe^{R(x)+Q(x)}}{r(x)} dx = \infty, \\ \{u \in D(T_0) \cap D(A): \lim_{x \rightarrow \infty} \frac{r(x)u(x)}{e^{R(x)+Q(x)}} = 0\}, & \text{otherwise,} \end{cases}$$

generates a positive semigroup of contractions. We note that the resolvent of T for $\lambda > 0$ is given by

$$R(\lambda, T)f = \frac{e^{\lambda R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(y)-Q(y)} f(y) dy, \quad (4.6)$$

and that obviously $D(T) \subset D(T_0) \cap D(A)$ so that the operators T_0 and A can be separated on $D(T)$.

Finally, defining $(Bu)(x) = \int_x^\infty a(y)b(x|y)u(y)dy$ with $D(B) = D(A)$, one can prove using $D(T) \subset D(T_0) \cap D(A)$ that for any $u \in D(T)_+$,

$$\int_0^\infty (Tu + Bu)x dx = - \int_0^\infty r(x)u(x) dx - \int_0^\infty \lambda(x)a(x)u(x)x dx. \quad (4.7)$$

Therefore the assumptions (1)–(3) of Section 2 are satisfied and Theorem 2.2 gives the existence of a semigroup $(G_K(t))_{t \geq 0}$ generated by an extension $(K, D(K))$ of $(T + B, D(T))$ that solves a realization of (4.1).

To be able to use results of Section 3 for the fragmentation model, we have to specify the extensions of the operators which we will be working with. Possibly the most general choice is as follows. For $u \in \mathcal{D}(T) := \{u \in L_1([0, \infty), x dx): ru \in AAC\}$ we denote

$$[Tu](x) = (r(x)u(x))_x - a(x)u(x); \quad (4.8)$$

thus $T : \mathcal{D}(T) \rightarrow E_f$. By \mathcal{B} we denote the operator defined by the expression

$$[\mathcal{B}u](x) = \int_x^\infty a(y)b(x|y)u(y)dy, \quad (4.9)$$

defined on $\mathcal{D}(\mathcal{B}) = \{u \in L_1([0, \infty[, x dx) : [\mathcal{B}u_+](x) < +\infty, [\mathcal{B}u_-](x) < +\infty \text{ a.e.}\}$. Using these two concepts, we can define an operator that can be thought of as the maximal extension of $T + \mathcal{B}$ in X :

$$[\mathcal{K}u](x) := [Tu](x) + [\mathcal{B}u](x) \quad (4.10)$$

defined on the domain $\mathcal{D}(\mathcal{K}) = \{u \in \mathcal{D}(T) \cap \mathcal{D}(\mathcal{B}) : x \rightarrow [\mathcal{K}u](x) \in L_1([0, \infty[, x dx)\}$. In a similar way we consider an operator \mathcal{L} extending $R(1, A)$, defined by the expression

$$[\mathcal{L}f](x) := \frac{e^{R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-R(y)-Q(y)} f(y) dy, \quad (4.11)$$

that is considered on $\mathcal{D}(\mathcal{L}) = \{u \in E : [\mathcal{L}u_+](x) < +\infty, [\mathcal{L}u_-](x) < +\infty \text{ a.e.}\}$. If we now go back to Arlotti's extension, we see that since $(I - T)^{-1} = R(1, T)$ is an integral operator with a positive kernel, Lebesgue's monotone convergence theorem yields that \mathcal{L} is defined by the same integral expressions as $R(1, T)$ and \mathcal{L} in (4.11) but on the domain consisting of those measurable functions for which the respective integral defines a function in X . Therefore, $\mathcal{L} \subset \mathcal{L}$ (as $\mathcal{L}f$ is not required to belong to X). Similarly, we note that $\mathcal{B} = \mathcal{B}$. At this moment it is not clear whether \mathcal{K} is an extension of K . This is ascertained in the next lemma.

Lemma 4.1. $K \subset \mathcal{K}$.

Proof. Let us recall that by (3.4) for every $u \in D(K)$ we have $Ku = Tu + Bu = Tu + \mathcal{B}u$, thus it is sufficient to prove that $T \subset \mathcal{T}$. Since for arbitrary $f \in F$, $\mathcal{L}f$ is defined by $\mathcal{L}f = \mathcal{L}f_+ - \mathcal{L}f_-$, it is enough to consider $f \geq 0$. Let $f \in F_+$ and $u = \mathcal{L}f \in X_+$. Since \mathcal{L} is given by the same expression as \mathcal{L} , we obtain

$$u(x) = \frac{e^{R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-R(y)-Q(y)} f(y) dy \quad \text{a.e.,}$$

where u , being an integrable function, is finite almost everywhere. But this means that $\int_x^\infty e^{-R(y)-Q(y)} f(y) dy$ is finite almost everywhere and therefore $y \rightarrow e^{-R(y)-Q(y)} f(y) \in L_1([\alpha, \infty[, dy)$ for any $\alpha > 0$. But this means that $\int_x^\infty e^{-\lambda R(y)-Q(y)} f(y) dy$ is almost absolutely continuous and since the same is true for the factor $e^{\lambda R(x)+Q(x)}$, that is additionally bounded over compact subsets of $]0, \infty[$, we see that $ru \in AAC$, that is, $u \in \mathcal{D}(T)$. We can thus differentiate ru almost everywhere obtaining $Tu = u - f = \mathcal{T}u$. \square

We shall need the following technical result.

Lemma 4.2. Let \mathcal{B} and \mathcal{L} be the extensions introduced above. If for some $f \in D(\mathcal{L})_+$ both f and $\mathcal{B}\mathcal{L}f$ belong to $L_1([\alpha, N], x dx)$, then

$$\begin{aligned}
& \int_{\alpha}^N (-f(x) + [\mathcal{BL}f](x) + [\mathcal{L}f](x))x \, dx \\
&= -\alpha r(\alpha)[\mathcal{L}f](\alpha) - \int_{\alpha}^N a(y)[\mathcal{L}f](y) \left(\int_0^{\alpha} b(x|y)x \, dx \right) dy \\
&\quad + Nr(N)[\mathcal{L}f](N) + \int_N^{\infty} a(y)[\mathcal{L}f](y) \left(\int_{\alpha}^N b(x|y)x \, dx \right) dy \\
&\quad - \int_{\alpha}^N r(x)[\mathcal{L}f](x) \, dx - \int_{\alpha}^N x\lambda(x)a(x)[\mathcal{L}f](x) \, dy. \tag{4.12}
\end{aligned}$$

Proof. Changing order of integration by Fubini–Tonelli’s theorem, we obtain

$$\begin{aligned}
& \int_{\alpha}^N [\mathcal{BL}f](x)x \, dx \\
&= \int_{\alpha}^N a(y)[\mathcal{L}f](y) \left(\int_{\alpha}^y b(x|y)x \, dx \right) dy + \int_N^{\infty} a(y)[\mathcal{L}f](y) \left(\int_{\alpha}^N b(x|y)x \, dx \right) dy \\
&= - \int_{\alpha}^N y[\mathcal{L}f](y) \, dy + \int_{\alpha}^N y(1+a(y))[\mathcal{L}f](y) \, dy - \int_{\alpha}^N y\lambda(y)a(y)[\mathcal{L}f](y) \, dy \\
&\quad - \int_{\alpha}^N a(y)[\mathcal{L}f](y) \left(\int_0^{\alpha} b(x|y)x \, dx \right) dy \\
&\quad + \int_N^{\infty} a(y)[\mathcal{L}f](y) \left(\int_{\alpha}^N b(x|y)x \, dx \right) dy \\
&= -I_1 + I_2 - I_3 - I_4 + I_5, \tag{4.13}
\end{aligned}$$

where we used (4.2) to get

$$\int_{\alpha}^y b(x|y)x \, dx = \int_0^y b(x|y)x \, dx - \int_0^{\alpha} b(x|y)x \, dx = y - \lambda(y)y - \int_0^{\alpha} b(x|y)x \, dx.$$

Next

$$\begin{aligned}
 I_2 &= \int_{\alpha}^N y(1+a(y)) \left(\frac{e^{R(y)+Q(y)}}{r(y)} \int_y^{\infty} e^{-R(z)-Q(z)} f(z) dz \right) dy \\
 &= \int_{\alpha}^N e^{-R(z)-Q(z)} f(z) \left(\int_{\alpha}^z y \frac{d}{dy} e^{R(y)+Q(y)} dy \right) dz \\
 &\quad + \int_N^{\infty} e^{-R(z)-Q(z)} f(z) \left(\int_{\alpha}^N y \frac{d}{dy} e^{R(y)+Q(y)} dy \right) dz \\
 &= \int_{\alpha}^N f(z) z dz - \alpha r(\alpha) [\mathcal{L}f](\alpha) + Nr(N) [\mathcal{L}f](N) - \int_{\alpha}^N r(x) [\mathcal{L}f](x) dx, \quad (4.14)
 \end{aligned}$$

as

$$\begin{aligned}
 \int_{\alpha}^N r(x) [\mathcal{L}f](x) dx &= \int_{\alpha}^N e^{-R(z)-Q(z)} f(z) \left(\int_{\alpha}^z e^{R(y)+Q(y)} dy \right) dz \\
 &\quad + \int_N^{\infty} e^{-R(z)-Q(z)} f(z) \left(\int_{\alpha}^N e^{R(y)+Q(y)} dy \right) dz.
 \end{aligned}$$

Combining (4.13) with (4.14), we get (4.12). \square

The following auxiliary result can be proved as in [8].

Lemma 4.3. *If $f \in E_+$ is such that $\mathcal{L}f \in X$, then $f \in L_1([\alpha, N], x dx)$ for any $0 < \alpha < N < \infty$.*

Corollary 4.1. *If $u \in D(K)$, then*

$$\begin{aligned}
 \int_0^{\infty} [Ku](x) x dx &= \lim_{\alpha \rightarrow 0^+, N \rightarrow +\infty} \left(-\alpha r(\alpha) u(\alpha) - \int_{\alpha}^N a(y) u(y) \left(\int_0^{\alpha} b(x|y) x dx \right) dy \right. \\
 &\quad \left. + Nr(N) u(N) + \int_N^{\infty} a(y) u(y) \left(\int_{\alpha}^N b(x|y) x dx \right) dy \right) \\
 &\quad - \int_0^{\infty} r(x) u(x) dx - \int_0^{\infty} x \lambda(x) a(x) u(x) dx. \quad (4.15)
 \end{aligned}$$

Proof. Let $u \in D(K)$. Using notation of (3.6), we write $K\bar{u}_{\pm} = \mathbb{L}\bar{f}_{\pm} - \bar{f}_{\pm} + \mathbb{B}\mathbb{L}\bar{f}_{\pm}$ and applying Lemma 4.3, we find that $\bar{f}_{\pm} \in L_1([\alpha, N], x dx)$ for any $0 < \alpha < N < +\infty$, so that we can use Lemma 4.2 for both \bar{f}_{\pm} , hence subtracting and changing $\mathbb{L}f$ into u ,

$$\begin{aligned} \int_{\alpha}^N [Ku](x)x dx &= -\alpha r(\alpha)u(\alpha) - \int_{\alpha}^N a(y)u(y) \left(\int_0^{\alpha} b(x|y)x dx \right) dy + Nr(N)u(N) \\ &\quad + \int_N^{\infty} a(y)u(y) \left(\int_{\alpha}^N b(x|y)x dx \right) dy - \int_{\alpha}^N r(x)u(x) dx \\ &\quad - \int_{\alpha}^N x\lambda(x)a(x)u(x) dx. \end{aligned} \quad (4.16)$$

Since $Ku \in X$, the left-hand side converges to the integral over $[0, \infty)$. Similarly, the last two integrals converge to $c(u)$ by (4.7) and Theorem 2.2, so that (4.15) is proved. \square

4.1. Honesty

Firstly, we shall present the first result on honest solutions to (4.1) obtained in [8]. It can be easily proved using the theory developed earlier in this paper.

Theorem 4.1 [8]. *If, in addition to (4.3), $a \in C^0((0, \eta))$ for some $\eta > 0$ and*

$$\lim_{x \rightarrow 0^+} \left(\frac{r(x)}{x} + a(x) \right) < +\infty, \quad (4.17)$$

then $K = \overline{T + B}$, thus $(G_K(t))_{t \geq 0}$ is honest. Moreover, if there exists $\lambda_0 > 0$ such that $\lambda_0 \leq \lambda(y)$ for all $y \geq 0$, then $K = T + B$, irrespective of (4.17).

The next result shows another class of coefficients for which the semigroup is honest.

Theorem 4.2. *If for any $N < +\infty$ there is $M_N < +\infty$ such that*

$$\sup_{x \in [0, N]} \frac{xa(x)}{r(x)} = M_N, \quad (4.18)$$

then $K = \overline{T + B}$.

Proof. Firstly, we consider functions $u \in D(K)_+$ with bounded support, for which (4.15) takes the form

$$\begin{aligned} \int_0^{\infty} [Ku](x)x dx &= \lim_{\alpha \rightarrow 0^+} \left(-\alpha r(\alpha)u(\alpha) - \int_{\alpha}^{N_0} a(y)u(y) \left(\int_0^{\alpha} b(x|y)x dx \right) dy \right) \\ &\quad - \int_0^{\infty} r(x)u(x) dx - \int_0^{\infty} x\lambda(x)a(x)u(x) dx \end{aligned} \quad (4.19)$$

for some fixed N_0 . Since $u \in D(K)$, $ru \in L_1([0, N_0])$. Writing for arbitrary $g \in L_1([0, N_0])$,

$$I_{\alpha, N_0}(g) = \int_{\alpha}^{N_0} \frac{ya(y)}{r(y)} \left(\frac{1}{y} \int_0^{\alpha} b(x|y)x dx \right) g(y) dy$$

we have, by $\int_0^{\alpha} b(x|y)x dx \leq \int_0^y b(x|y)x dx = y(1 - \lambda(y)) \leq y$,

$$|I_{\alpha, N_0}(g)| \leq M_{N_0} \int_0^{N_0} |g(x)| dx,$$

so that $(I_{\alpha, N_0})_{0 < \alpha < N_0}$ is a family of linear functionals from $L_1([0, N_0])$ to \mathbb{R} , uniformly bounded with respect to α . Let us take g_0 with $\text{supp } g_0 \subset [\alpha_0, N_0]$ for some $\alpha_0 > 0$; then

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} I_{\alpha, N_0}(g_0) &= \lim_{\alpha \rightarrow 0^+} \int_{\alpha}^{N_0} \frac{a(y)}{r(y)} \left(\int_0^{\alpha} b(x|y)x dx \right) g_0(y) dy \\ &= \lim_{\alpha \rightarrow 0^+} \int_{\alpha_0}^{N_0} \frac{a(y)}{r(y)} \left(\int_0^{\alpha} b(x|y)x dx \right) g_0(y) dy = 0, \end{aligned}$$

by Lebesgue dominated convergence theorem, as $\int_0^{\alpha} b(x|y)x dx$ tends to 0 and is dominated by y . Since the set of compactly supported functions is dense in $L_1([0, N_0])$, by Banach–Steinhaus theorem we see that $I_{\alpha, N_0}(g)$ converges for any $g \in L_1([0, N_0])$ and, again by density, the limit is equal to zero for any $g \in L_1([0, N_0])$.

Returning to (4.19), we see that the above result yields also the existence of $\lim_{\alpha \rightarrow 0^+} \alpha r(\alpha)u(\alpha) = l \geq 0$. If $l \neq 0$, then $r(\alpha)u(\alpha) \geq c/\alpha$ for some $c > 0$ as $\alpha \rightarrow 0$, which contradicts $ru \in L_1([0, \infty))$. Thus $l = 0$ and

$$\int_0^{\infty} [Ku](x)x dx = - \int_0^{\infty} r(x)u(x) dx - \int_0^{\infty} x\lambda(x)a(x)u(x) dx \quad (4.20)$$

for any $u \in D(K_+)$ with bounded support. By Corollary 2.2, it is enough to show that (4.20) is valid for arbitrary $u \in R(1, K)X_+$. Let then $u = R(1, K)f \in X_+$; we take sequence $(f_N)_{N \in \mathbb{N}} = (\chi_N f)_{N \in \mathbb{N}}$, where χ_N is the characteristic function of $[0, N]$, and define through (2.3), elements of $D(K)_+$ by

$$u_N = R(1, K)f_N = \sum_{n=0}^{\infty} L(BL)^n f_N.$$

Due to the definitions of B and L , we see that if $f_N \in X$ vanishes for $x > N$, then the same holds for both Bf_N and Lf_N , so by induction all the partial sums above have support in $[0, N]$. The series converges monotonically in X , thus it converges almost everywhere, and therefore u_N has a bounded support. Clearly, as $(f_N)_{N \in \mathbb{N}}$ converges in X , $(u_N)_{N \in \mathbb{N}}$ converges to u in $D(K)$ and since the functional c , given here by the left-hand side

of (4.20), is continuous in $D(K)$ norm, we see that (4.20) holds for any $u \in R(1, K)X_+$. Therefore, by Corollary 2.2, $K = \overline{T + B}$. \square

4.2. Dishonesty

In this subsection we shall consider only b given by (4.4): $b(x | y) = y^{-1}h(x/y)$, and satisfying

$$-\int_0^1 zh(z) \ln z \, dz < +\infty. \quad (4.21)$$

Since in this case λ is a constant by (4.5), and for a constant nonzero λ the semigroup is honest by Theorem 4.1, we can confine our analysis to the case $\lambda = 0$. To be able to use the results of the previous subsection, we need some additional regularity in a neighbourhood of 0 so that, in addition to (4.3), we assume that there is $\eta > 0$ for which the following properties hold:

$$a, r \in C^1((0, \eta]), \quad a, r > 0 \quad \text{on } (0, \eta], \quad \text{and} \quad \frac{1}{xa(x)} \in L_1([0, \eta]). \quad (4.22)$$

Next, denote $\phi(x) = r(x)/xa(x)$. By Theorem 4.2, if $1/\phi$ is bounded at 0, then the semigroup is honest. Thus, we assume here that

$$\lim_{x \rightarrow 0^+} \phi(x) = 0. \quad (4.23)$$

The above assumptions are natural, ruling out (with some safety margin) the honesty. The next assumption is of a technical character. We suppose that

$$\lim_{x \rightarrow 0^+} \frac{x\phi'(x)}{\phi(x)} = L < +\infty. \quad (4.24)$$

Note that $L \geq 0$. In fact, since for any $0 < \delta < x < \eta$ we have $\phi(x) - \phi(\delta) = \int_\delta^x \phi'(s) \, ds$, by (4.23) we obtain $\phi(x) = \int_0^x \phi'(s) \, ds$, and if $L < 0$, then on some interval ϕ' would be strictly negative, giving negative ϕ .

A more intuitive interpretation of (4.24) is given in the proposition below.

Proposition 4.1. *If the limit (4.24) exists, then*

$$L = \sup \left\{ l \geq 0 : \lim_{x \rightarrow 0^+} \frac{\phi(x)}{x^l} = 0 \right\} = \inf \left\{ l \geq 0 : \lim_{x \rightarrow 0^+} \frac{\phi(x)}{x^l} = +\infty \right\}. \quad (4.25)$$

Proof. Since

$$\left(\frac{\phi(x)}{x^l} \right)' = \frac{\phi(x)}{x^{l+1}} \left(\frac{x\phi'(x)}{\phi(x)} - l \right), \quad (4.26)$$

we see that if $l < L$, then $\phi(x)/x^l$ is increasing and if $l > L$, then it is decreasing, so in both cases $\lim_{x \rightarrow 0^+} \phi(x)/x^l = \rho_l$ exists. Let first $l < L$; then $0 \leq \rho_l < +\infty$. If we assume that $\rho_l > 0$, then taking $l < l' < L$, we have $\phi(x)/x^{l'} = x^{l-l'}\phi(x)/x^l \rightarrow \infty$ so that $\rho_{l'} = +\infty$

which is a contradiction. Hence, $\rho_l = 0$ for all $l < L$. Taking now $l > L$ and denoting for a moment $f(x) = \phi(x)/x^l$, (4.26) yields $f'(x)/f(x) = g(x)/x$ for some g which satisfies $g(x) \leq -c < 0$ over some interval $(0, \delta)$. Thus

$$\frac{\phi(x)}{x^l} = f(x) = C \exp\left(-\int_x^\delta \frac{g(s)}{s} ds\right) \geq C\left(\frac{\delta}{x}\right)^c$$

for some constant C . Since $c > 0$, we see that $\rho_l = +\infty$ if $l > L$. \square

Remark 4.1. The converse of this proposition is not true. In fact, taking $\phi(x) = x(1 + x \sin x^{-1})$, we see that L defined by (4.25) is equal to 1 but the limit (4.24) does not exist. There are also functions with $L = \infty$, e.g., $\phi(x) = \exp(-1/x)$.

Let us then write

$$\phi(x) = x^L g(x). \quad (4.27)$$

Lemma 4.4. Let g be the function defined by (4.27) and $g_\delta(x) = x^\delta g(x)$. Then for any $\delta > 0$,

$$\lim_{x \rightarrow 0^+} g_\delta(x) = 0, \quad (4.28)$$

and g_δ is strictly increasing in some interval $(0, \eta)$.

Proof. Equation (4.28) follows from Proposition 4.1 by $\phi(x)/x^{L-\delta} = x^\delta g(x)$. Next, by (4.27) and (4.24), we have

$$\lim_{x \rightarrow 0^+} \frac{xg'(x)}{g(x)} = 0,$$

so that by

$$(x^\delta g(x))' = x^\delta g'(x) + \delta x^{\delta-1} g(x) = x^{\delta-1} g(x) \left(\frac{xg'(x)}{g(x)} + \delta \right), \quad (4.29)$$

the function $g_\delta(x)$ is strictly increasing in a neighbourhood of 0. \square

Further, we assume that, if $L = 0$, then

$$\frac{g(x)}{x} \in L_1([0, \eta]), \quad (4.30)$$

otherwise we do not impose any additional condition on g .

Theorem 4.3. Let the coefficients a and r of the problem (4.1) satisfy (4.3), (4.22)–(4.24), and, if $L = 0$, (4.30), and let b be of the form (4.4) and satisfy (4.21). Then the semigroup $(G_K(t))_{t \geq 0}$ is not honest.

Proof. Our strategy is to use Theorem 3.1 so that we invoke the operator extensions introduced at the beginning of this section and construct $u \in \mathcal{D}(\mathcal{K})_+$ satisfying the assumptions

of this theorem. If u is such a function with a bounded support, then we can write (4.12) as

$$\begin{aligned} & \int_0^\infty [\mathcal{K}u](x)x \, dx \\ &= \lim_{\alpha \rightarrow 0^+} \left(-\alpha r(\alpha)u(\alpha) - \int_\alpha^\infty a(y)u(y) \left(\int_0^\alpha b(x|y)x \, dx \right) dy - \int_\alpha^\infty r(y)u(y) \, dy \right) \\ &= \lim_{\alpha \rightarrow 0^+} (-e_{1,\alpha}(u) - e_{2,\alpha}(u) - c_\alpha(u)). \end{aligned} \quad (4.31)$$

Let us start with assumption (iii) of Theorem 3.1. Assuming for a moment that c_α has a finite limit, we look for a function for which $\lim_{\alpha \rightarrow 0^+} (e_{1,\alpha}(u) + e_{2,\alpha}(u)) > 0$. A heuristic argument indicates that a good choice is

$$u_\eta(x) = \begin{cases} \frac{1}{xr(x) + \theta x^2 a(x)}, & \text{for } 0 < x < \eta, \\ \psi(x), & \text{for } \eta \leq x < \xi, \\ 0, & \text{for } x \geq \xi, \end{cases} \quad (4.32)$$

where η, ξ are positive numbers and ψ is a positive function joining $(\eta r(\eta) + \theta \eta^2 a(\eta))^{-1}$ with 0 in a sufficiently regular way. Both η and ξ as well as the function ψ will be determined later. We also introduced an arbitrary constant $\theta > 0$ to have a better flexibility in the sequel. Let us fix an arbitrary set of parameters. Since $\int_0^\alpha b(x|y)x \, dx \rightarrow 0$ as $\alpha \rightarrow 0$ in a dominated way over each bounded interval $[\eta, \xi] \subset (0, +\infty)$, we see that

$$\begin{aligned} & \lim_{\alpha \rightarrow 0^+} \int_\alpha^\infty a(y)u_\eta(y) \left(\int_0^\alpha b(x|y)x \, dx \right) dy \\ &= \lim_{\alpha \rightarrow 0^+} \int_\alpha^\eta a(y)u_\eta(y) \left(\int_0^\alpha b(x|y)x \, dx \right) dy. \end{aligned}$$

Since on $(0, \eta]$ we have

$$u_\eta(x) = \frac{1}{xr(x) + \theta x^2 a(x)} = \frac{1}{x^2 a(x)} \frac{1}{\phi(x) + \theta},$$

using $b(x|y) = h(x/y)/y$, we have

$$\int_\alpha^\eta \left(\int_0^\alpha b(x|y)x \, dx \right) \frac{1}{y^2(\phi(y) + \theta)} dy = \int_{\alpha/\eta}^1 \left(\frac{1}{r} \int_0^r zh(z) \, dz \right) \frac{1}{\phi(\alpha/r) + \theta} dr.$$

Since, by (4.21),

$$-\int_0^1 zh(z) \ln z \, dz = \int_0^1 \left(\int_z^1 \frac{dr}{r} \right) zh(z) \, dz = \int_0^1 \left(\int_0^r zh(z) \, dz \right) \frac{1}{r} dr < +\infty,$$

we have

$$\left| \int_0^{\alpha/\eta} \left(\frac{1}{r} \int_0^r zh(z) dz \right) \frac{1}{\phi(\alpha/r) + \theta} dr \right| \leq \frac{1}{\theta} \left| \int_0^{\alpha/\eta} \left(\frac{1}{r} \int_0^r zh(z) dz \right) dr \right| \rightarrow 0,$$

as $\alpha \rightarrow 0$,

and finally,

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} \int_{\alpha}^{\eta} a(y) u_{\eta}(y) \left(\int_0^{\alpha} b(x|y) x dx \right) dy &= \frac{1}{\theta} \int_0^1 \left(\frac{1}{r} \int_0^r zh(z) dz \right) dr \\ &= -\frac{1}{\theta} \int_0^1 zh(z) \ln z dz > 0, \end{aligned}$$

by (4.23) and the Lebesgue dominated convergence theorem.

Next, by (4.23) and (4.30), we see that

$$\int_0^{\eta} \frac{r(x)}{xr(x) + \theta x^2 a(x)} dx = \int_0^{\eta} \frac{1}{x} \frac{\phi(x)}{\phi(x) + \theta} dx < +\infty,$$

so that $c(u_{\eta})$ exists. Moreover, by (4.22),

$$\int_0^{\eta} u_{\eta}(x) x dx = \int_0^{\eta} \frac{x dx}{xr(x) + \theta x^2 a(x)} = \int_0^{\eta} \frac{1}{xa(x)} \frac{1}{\phi(x) + \theta} dx < +\infty$$

as $1/(\phi(x) + \theta)$ is bounded, thus $u_{\eta} \in X$. Next, by (4.23) the limits $e_{1,\alpha}$ and $e_{2,\alpha}$ can be separated, giving

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} (e_{1,\alpha}(u) + e_{2,\alpha}(u)) &= \lim_{\alpha \rightarrow 0^+} \frac{\phi(\alpha)}{\phi(\alpha) + \theta} - \frac{1}{\theta} \int_0^1 zh(z) \ln z dz \\ &= -\frac{1}{\theta} \int_0^1 zh(z) \ln z dz > 0. \end{aligned}$$

In the next step we shall deal with assumption (ii). Firstly, let us consider the cut-off of the operator $-\mathcal{K}$:

$$[K_{\eta} f](x) = -[r(x)f(x)]' + a(x)f(x) - \int_x^{\eta} \frac{1}{y} a(y) h\left(\frac{x}{y}\right) f(y) dy,$$

for $0 < x \leq \eta$. By (4.32) we obtain for $x \in (0, \eta]$,

$$\begin{aligned}
-[r(x)u_\eta(x)]' + a(x)u_\eta(x) &= -\left(\frac{\phi(x)}{x(\phi(x) + \theta)}\right)' + \frac{1}{x^2} \frac{1}{\phi(x) + \theta} \\
&= \frac{1}{x^2} \frac{\phi(x) + 1}{\phi(x) + \theta} - \frac{\theta\phi'(x)}{x(\phi(x) + \theta)^2}.
\end{aligned} \tag{4.33}$$

We also have

$$\begin{aligned}
\int_x^\eta \frac{1}{y} a(y) h\left(\frac{x}{y}\right) u_\eta(y) dy &= \int_x^\eta h\left(\frac{x}{y}\right) \frac{1}{y^3(\phi(y) + \theta)} dy \\
&= \frac{1}{\theta} \int_x^\eta h\left(\frac{x}{y}\right) \frac{1}{y^3} dy - \frac{1}{\theta} \int_x^\eta h\left(\frac{x}{y}\right) \frac{\phi(y)}{y^3(\phi(y) + \theta)} dy.
\end{aligned}$$

The first integral is easily calculated to be

$$\frac{1}{\theta} \int_x^\eta h\left(\frac{x}{y}\right) \frac{1}{y^3} dy = \frac{1}{\theta} \frac{1}{x^2} \int_{x/\eta}^1 zh(z) dz = \frac{1}{\theta} \frac{1}{x^2} - \frac{1}{\theta} \frac{1}{x^2} \int_0^{x/\eta} zh(z) dz,$$

where we used (4.5) with $\lambda = 0$. Thus

$$\begin{aligned}
[K_\eta u_\eta](x) &= \frac{\phi(x)}{\theta x^2} \left(\frac{\theta - 1}{\phi(x) + \theta} - \frac{\theta^2 x \phi'(x)}{\phi(x)(\phi(x) + \theta)^2} + \frac{\int_x^\eta h\left(\frac{x}{y}\right) \frac{\phi(y)}{y^3(\phi(y) + \theta)} dy}{\phi(x)/x^2} \right) \\
&\quad + \frac{1}{\theta} \frac{1}{x^2} \int_0^{x/\eta} zh(z) dz \\
&= \frac{\phi(x)}{\theta x^2} F_\eta(x) + G_\eta(x),
\end{aligned} \tag{4.34}$$

where G_η is strictly positive for $x > 0$. Let us denote

$$I_\eta(x) = \int_x^\eta h\left(\frac{x}{y}\right) \frac{\phi(y)}{y^3(\phi(y) + \theta)} dy,$$

and observe that for $0 < x < \eta_0$, where $\eta_0 < \eta$, we have $I_\eta(x) \geq I_{\eta_0}(x)$. Thus, trying to bound away $I_\eta(x)$ from zero, we can focus on $I_{\eta_0}(x)$ with arbitrarily small η_0 . Hence, by (4.23), for any $\varepsilon > 0$ we can find η_0 such that $1/(\theta + \phi(x)) \geq 1/(\theta + \varepsilon)$ for $x \in (0, \eta_0]$. Writing now $\phi(x) = x^{L-\delta} x^\delta g(x) = x^{L\delta} g_\delta(x)$, by Lemma 4.4 we obtain $\lim_{x \rightarrow 0^+} g_\delta(x) = 0$ and g_δ is increasing. Thus, $\inf_{y \in [x, \eta_0]} g_\delta(y) = g_\delta(x)$ and

$$\frac{x^2 I_\eta(x)}{\phi(x)} \geq \frac{1}{\theta + \varepsilon} \frac{\int_x^{\eta_0} h(x/y) y^{L\delta-3} g_\delta(y) dy}{x^{L\delta-2} g_\delta(x)} \geq \frac{1}{\theta + \varepsilon} \int_{x/\eta_0}^1 h(z) z^{1-L\delta} dz, \tag{4.35}$$

yielding

$$\liminf_{x \rightarrow 0^+} \frac{x^2 I_\eta(x)}{\phi(x)} \geq \frac{1}{\theta + \varepsilon} \liminf_{x \rightarrow 0^+} \int_{x/\eta_0}^1 h(z) z^{1-L_\delta} dz = \frac{1}{\theta + \varepsilon} \int_0^1 h(z) z^{1-L_\delta} dz \quad (4.36)$$

as the last limit exists (possibly infinite). Let us define $H(\lambda) = \int_0^1 h(z) z^{1-\lambda} dz$. Using (4.5) with $\lambda = 0$, we have $H(0) = 1$ and, by easy calculation, $H(1) > 1$. Moreover, by $z^\alpha \leq z^\beta$ for $0 \leq z \leq 1$ and $\alpha \geq \beta$, $H(\lambda)$ is a non-decreasing function and therefore, by the dominated convergence theorem, it is continuous wherever it is finite (and left-continuous at the right endpoint of the domain if it is finite here).

Returning to (4.36), we see that if $H(L_\delta) = \infty$, then also $(x^2 I_\eta(x))/\phi(x)$ is unbounded at 0, and if $H(L_\delta)$ is finite, then, since ε is arbitrary,

$$\liminf_{x \rightarrow 0^+} \frac{x^2 I_\eta(x)}{\phi(x)} \geq \frac{1}{\theta} H(L_\delta).$$

Since the first two terms of F_η in (4.34) have (finite) limits, we can write

$$\begin{aligned} \liminf_{x \rightarrow 0^+} F_\eta(x) &= \lim_{x \rightarrow 0^+} \frac{\theta - 1}{\phi(x) + \theta} - \lim_{x \rightarrow 0^+} \frac{\theta^2 x \phi'(x)}{\phi(x)(\phi(x) + \theta)^2} + \liminf_{x \rightarrow 0^+} \frac{x^2 I_\eta(x)}{\phi(x)} \\ &\geq 1 - L + \frac{1}{\theta} (H(L_\delta) - 1). \end{aligned}$$

Obviously, we can assume that $H(L_\delta) < +\infty$. Denote $\mathcal{F}(L, \theta, \delta) = 1 - L + \frac{1}{\theta} (H(L_\delta) - 1)$. If $0 \leq L < 1$, then $1 - L > 0$ and since $H(L_\delta) \neq -\infty$, we can always make $\mathcal{F}(L, \theta, \delta)$ positive by taking sufficiently large θ . If $L > 1$, then we can take δ sufficiently small for $L_\delta > 1$. Then, $H(L_\delta) \geq H(1) > 1$ and $H(L_\delta) - 1 > 0$ so that $\mathcal{F}(L, \theta, \delta) > 0$ if θ is sufficiently small. Finally, let $L = 1$ so that $\mathcal{F}(1, \theta, \delta) = \frac{1}{\theta} (H(1 - \delta) - 1)$ and the sign of \mathcal{F} is the same as of $H(1 - \delta) - 1$. If $H(1) = \infty$, then either $H(\lambda) = \infty$ in some neighbourhood of 1, in which case by taking sufficiently small $\delta > 0$, we get also $H(1 - \delta) = \infty$, or $H(\lambda) < +\infty$ on $(-\infty, 1)$, in which case $H(1 - \delta)$ can be made arbitrarily large (by the monotonic convergence theorem), and thus larger than 1. On the other hand, if $H(1) < +\infty$, then it is continuous from the left, and since $H(1) > 1$, there is $\delta > 0$ such that $H(1 - \delta) > 1$. Hence, in any case we can find $\delta > 0$ and θ for which $\liminf_{x \rightarrow 0^+} F_\eta(x) \geq c > 0$ for some constant c , and therefore $F_\eta(x) > 0$ on some interval $(0, \eta_1]$.

Now we prove that $[K_\eta u_\eta](x) > 0$ close to zero yields $u_\eta(x) - [K u_\eta](x) \geq 0$ on $(0, \infty)$. Firstly, note that if $\eta_2 < \eta_1$, then for $x \in (0, \eta_2]$ we have $u_{\eta_2}(x) = u_{\eta_1}(x)$ and $[K_{\eta_2} u_{\eta_2}](x) = [K_{\eta_2} u_{\eta_1}](x)$ with

$$[K_{\eta_2} u_{\eta_2}](x) = [K_{\eta_1} u_{\eta_1}](x) + \int_{\eta_2}^{\eta_1} a(y) b(x | y) u_{\eta_1}(y) dy. \quad (4.37)$$

From the previous considerations, $[K_\eta u_\eta](x) > 0$ for $x \in (0, \eta_1]$ for some $\eta_1 > 0$ and by (4.37), $[K_{\eta_i} u_{\eta_i}](x) > 0$ on $(0, \eta_i]$, $i = 1, 2$. Hence, we have $[K_{\eta_1} u_{\eta_1}](x) > 0$ on $(0, \eta_1]$ for some fixed θ . Let us fix this θ , take some $\eta_2 < \eta_1$ and consider the function u_{η_2} of (4.32)

with $\psi(x) = (\varepsilon^{-1}(-x + \eta_2) + r(\eta_2)u_{\eta_2}(\eta_2))/r(x)$ and $\xi = \eta_2 + \varepsilon r(\eta_2)u_{\eta_2}(\eta_2)$, where $u_{\eta_2}(\eta_2) = (\eta_2 r(\eta_2) + \theta \eta_2^2 a(\eta_2))^{-1}$ and ε is still to be chosen. At this moment we require that, as defined, $\xi \leq \eta_1$. We have $\psi(\eta_2) = u_{\eta_2}(\eta_2)$ and $\psi(\xi) = 0$ so that u_{η_2} is a Lipschitz continuous function on $(0, \infty)$. Moreover, $(r(x)\psi(x))' = -\varepsilon^{-1}$ on (η_2, ξ) . Since $\xi \leq \eta_1$, $\inf_{\eta_2 < x < \xi} r(x) \geq \inf_{\eta_2 < x < \eta_1} r(x) = r_0$ and thus $\psi(x) \leq r(\eta_2)u_{\eta_2}(\eta_2)/r_0$ on any interval $[\eta_2, \xi]$ independently of ε . For $x \in (0, \eta_2]$ we have

$$\begin{aligned} u_{\eta_2}(x) - [\mathcal{K}u_{\eta_2}](x) &= u_{\eta_2}(x) + [K_{\eta_2}u_{\eta_2}](x) - \int_{\eta_2}^{\xi} a(y)b(x|y)\psi(y)dy \\ &= u_{\eta_2}(x) + [K_{\eta_1}u_{\eta_1}](x) + \int_{\xi}^{\eta_1} a(y)b(x|y)u_{\eta_1}(y)dy \\ &\quad + \int_{\eta_2}^{\xi} a(y)b(x|y)(u_{\eta_1}(y) - \psi(y))dy. \end{aligned}$$

Next, let $\vartheta = \inf_{x \in [\eta_2, \eta_1]} u'_{\eta_1}(x)$. We have

$$\psi'(x) = -\frac{1}{\varepsilon r(x)} - \frac{r'(x)}{r(x)}\psi(x)$$

and since r is a differentiable function on $(0, \infty)$ and bounded away from zero on each compact interval, we have $r_0 \leq r(x) \leq r_1$ and $|r'(x)| \leq R$ on $[\eta_2, \eta_1]$, so that

$$\sup_{x \in [\eta_2, \xi]} \psi'(x) \leq -(r_1 \varepsilon)^{-1} + r_0^{-2} R r(\eta_2) u_{\eta_2}(\eta_2).$$

Therefore, we can find ε for which $\vartheta > \sup_{x \in [\eta_2, \xi]} \psi'(x)$, yielding $u_{\eta_1}(y) - \psi(y) \geq 0$ on $[\eta_2, \xi]$ and $u_{\eta_2}(x) - [\mathcal{K}u_{\eta_2}](x) \geq 0$ on $(0, \eta_2]$.

Since $\psi(x) \geq 0$ on $[\eta_2, \xi]$, putting $M = \sup_{x \in [\eta_2, \eta_1]} |a(x)x^2\psi(x)|$, we obtain

$$\begin{aligned} \psi(x) - (r(x)\psi(x))' + a(x)\psi(x) &= \int_x^{\xi} a(y)b(x|y)\psi(y)dy \\ &\geq \frac{1}{\varepsilon} - M \int_x^{\xi} \frac{1}{y^3} h\left(\frac{x}{y}\right) dy \geq \frac{1}{\varepsilon} - \frac{M}{\eta_2^2} \int_0^1 zh(z)dz = \frac{1}{\varepsilon} - \frac{M}{\eta_2^2}, \end{aligned}$$

and taking sufficiently small ε , we make this term also non-negative.

It remains to prove (i). Since all the functions are almost absolutely continuous, integrating by parts, we get

$$[\mathcal{L}((ru_{\eta_2})')](x) = \frac{e^{R(x)+Q(x)}}{r(x)} \int_x^{\infty} e^{-R(y)-Q(y)} \partial_y (r(y)u_{\eta_2}(y)) dy$$

$$\begin{aligned}
&= \frac{e^{R(x)+Q(x)}}{r(x)} \lim_{y \rightarrow \infty} \frac{r(y)}{e^{R(y)+Q(y)}} u_{\eta_2}(y) \\
&\quad - u_{\eta_2}(x) + \frac{e^{R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-R(y)-Q(y)} (1+a(y)) u_{\eta_2}(y) dy \\
&= -u_{\eta_2}(x) + [\mathcal{L}((1+a)u_{\eta_2})](x)
\end{aligned}$$

on account of $u_{\tilde{\eta}}$ having bounded support. Thus u_{η} satisfies assumption (i) and the theorem is proved. \square

5. Example

In the series of papers [11,12,17] the authors have developed a theory of the fragmentation model (4.1) with power law rates $r(x) = x^\gamma$, $a(x) = x^\alpha$ and $b(x|y)$ given either by (4.4) or by the power law $b(x|y) = (\nu+2)x^\nu/y^{\nu+1}$, presenting in [11,12] formal arguments to support the claim that for $\alpha < 0$ and $\gamma \geq \alpha + 1$ there is a runaway fragmentation, that is, a cascade of fragmentation events that reduce finite-mass particles to infinite numbers of zero-mass particles in a finite time. However, they have conjectured that, unlike in formally mass-conserving fragmentation models, the discrete and continuous mass loss account for all mass loss and preclude the unexpected mass loss normally associated with shattering.

For such coefficients our theory gives the following results. The function $r(x)/x + a(x) = x^{\gamma-1} + x^\alpha$ is finite at 0 if and only if $\gamma - 1 \geq 0$ and $\alpha \geq 0$ and the semigroup is honest by Theorem 4.1. Also $xa(x)/r(x) = x^{\alpha+1-\gamma}$ is bounded at 0 if and only if $\alpha + 1 - \gamma \geq 0$ so that the semigroup is also honest by Theorem 4.2.

Otherwise, we are in the open sector $\alpha < 0$ and $\gamma > \alpha + 1$. In such a case we have $\phi(x) = x^{\gamma-\alpha-1}$ and assumption (4.23) is satisfied (meaning that we are in “fragmentation regime,” as defined by [12]). Further, we see that (4.22) is satisfied since $\alpha < 0$ and (4.24) is automatically satisfied as $x\phi'(x)/\phi(x) = \gamma - \alpha - 1 = L > 0$. Thus, provided h satisfies assumptions (4.21), e.g., if h is given by the power law, then in the sector $\alpha < 0$ and $\gamma > \alpha + 1$ there occurs shattering transformation with unaccounted mass loss due to (2.1), contrary to the conjecture of [11,12].

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