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Fixed point theorems for multifunctions with applications to discontinuous operator and differential equations

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Abstract

In this paper we study existence of minimal and maximal fixed points for increasing multivalued functions in ordered topological vector spaces, and apply the obtained results to operator equations and quasilinear elliptic boundary value problems involving discontinuous nonlinearities.
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1. Introduction

In this paper we shall first present fixed point results for multivalued functions defined on a nonempty subset P of an ordered topological vector space X. Our main result implies, for instance, that if X is a lattice-ordered reflexive Banach space with the properties

(W+) if $u_n \rightharpoonup u$ in X, and if (u_n^+) is increasing, then $u_n^+ \rightharpoonup u^+$,

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(N+) $\|u^+\| \leqslant \|u\|$ for each $x \in X$, where $u^+ := \sup\{u, 0\}$,

then a mapping $G : P \to 2^P \setminus \emptyset$, where P is a closed ball in X, has minimal and maximal fixed points if G is increasing in the sense defined later on, and if the values of G are weakly sequentially closed. This result is applied to prove existence results for minimal and maximal solutions of an abstract equation $u = H(u, u)$, where $H : X \times X \to X$.

For instance, the function spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise, possess properties (W+) and (N+). This makes the obtained results applicable, e.g., in the theory of partial differential equations, as we demonstrate in Section 5.

2. Fixed point results

Throughout this section we assume that $X = (X, \leqslant)$ is an ordered topological vector space having the following property:

(C) Each well-ordered chain C of X whose increasing sequences converge contains an increasing sequence which converges to $\sup C$.

We are going to derive conditions which ensure that a multivalued function has a maximal and/or a minimal fixed point, defined as follows.

Definition 2.1. Given a nonempty subset P of X we say that $v \in P$ is a *maximal* fixed point of a mapping $F : P \to 2^P \setminus \emptyset$, if $v \in F(v)$, and if $u = v$ whenever $u \in F(u)$ and $v \leqslant u$. If $u = v$ whenever $u \in F(u)$ and $u \leqslant v$, we say that v is a *minimal* fixed point of F.

A basis of our considerations is the following recursion principle (cf. [4, Lemma 1.1.1]).

Lemma 2.1. *Let $\mathcal{D}$ be a subset of 2^X with $\emptyset \in \mathcal{D}$, and assume that $f : \mathcal{D} \to X$. Then there is a unique well-ordered chain C of X such that*

$$u \in C \quad \text{if and only if} \quad u = f\big(\{v \in C \mid v < u\}\big). \tag{2.1}$$

If $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of C.

As an application of Lemma 2.1 we prove the following fixed point result which is a slight modification to [2, Theorem 2.1].

Lemma 2.2. *Let $F : P \subset X \to 2^P \setminus \emptyset$ satisfy the following hypotheses*:

(F1) *The set $P_0 = \{u \in P \mid u \leqslant v \text{ for some } v \in F(u)\}$ is nonempty.*
(F2) *If $u_n \leqslant v_n \in F(u_n)$, $n \in \mathbb{N}$, and if (v_n) is increasing, then (v_n) has a limit in P_0.*

Then F has a maximal fixed point which is also a maximal element of P_0.

Proof. Denote $\mathcal{D} = \{W \subset P_0 \mid W$ is well ordered and has a strict upper bound in $P_0\}$. Obviously, $\emptyset \in \mathcal{D}$. Let $f : \mathcal{D} \to X$ be a function which assigns to each $W \in \mathcal{D}$ an element $v = f(W) \in F(u)$, $u \leqslant v$, where u is a fixed strict upper bound of W in P_0. By Lemma 2.1 there exists exactly one well-ordered chain C in X satisfying (2.1). If (v_n) is an increasing sequence in C, it follows from the definition of f that $u_n \leqslant v_n \in F(u_n)$ for each $n \in \mathbb{N}$, where (u_n) is a sequence in P_0. Thus (v_n) has a limit in P_0 by (F2). This result and property (C) of X imply that C contains an increasing sequence (v_n) such that $u = \lim_n v_n = \sup C$. Because $u \in P_0$, there exists $v \in F(u)$ such that $u \leqslant v$. Applying (F2) for constant sequences $u_n \equiv u$ and $v_n \equiv v$ we see that $v \in P_0$. Equality $u = v$ must hold, for otherwise $u < v \in P_0$, so that $f(C)$ would exist, and being a strict upper bound of C this would contradict the last conclusion of Lemma 2.1. Consequently, $u = v \in F(u)$, whence u is a fixed point of F.

To prove that u is a maximal fixed point of F, let v be a fixed point of F and satisfy $u \leqslant v$. Since $v \in P_0$, the above reasoning implies that $u = v$, so that u is maximal. The same reasoning shows also that u is a maximal element of P_0. □

To replace the hypothesis (F2) of Lemma 2.2 by more explicit conditions we define monotonicity concepts for multivalued functions.

Definition 2.2. We say that $F : P \to 2^P \setminus \emptyset$ is *increasing upwards* if $u, v \in P$, $u \leqslant v$ and $x \in F(u)$ imply an existence of $y \in F(v)$ such that $x \leqslant y$. F is *increasing downwards* if $u, v \in P$, $u \leqslant v$ and $y \in F(v)$ imply that $x \leqslant y$ for some $x \in F(u)$. If F is increasing upwards and downwards we say that F is *increasing*.

As a first application of Lemma 2.2 we prove the following result.

Proposition 2.1. *Let P be a sequentially closed subset of X, and let $F : P \to 2^P \setminus \emptyset$ be increasing upwards and satisfy the hypothesis* (F1) *and the following hypothesis*:

(F3) *The values $F(u)$ of F are sequentially compact, and increasing sequences of the range $F[P] = \bigcup\{F(u) \mid u \in P\}$ of F converge in X.*

Then F has a maximal fixed point.

Proof. It suffices to show that the hypothesis (F2) of Lemma 2.2 holds. Assume that $u_n \leqslant v_n \in F(u_n)$, $n \in \mathbb{N}$, and that (v_n) is increasing. It follows from (F3) that $v = \lim_n v_n = \sup_n v_n$ exists in X. Because P is sequentially closed, then $v \in P$. Since F is increasing upwards, and since $v_n \leqslant v$ for each $n \in \mathbb{N}$, then there is $w_n \in F(v)$ such that $v_n \leqslant w_n$. The sequence (w_n) has by (F3) a subsequence, say (w_k) which has a limit w in $F(v)$. It follows from $v_k \leqslant w_k$ as $k \to \infty$ that $v \leqslant w \in F(v)$. Thus $v \in P_0$, so that (F2) holds. Thus all the hypotheses of Lemma 2.2 hold, which implies the assertion. □

The next result is dual to that of Proposition 2.1.

Proposition 2.2. *Let P be a sequentially closed subset of X, and let $F : P \to 2^P \setminus \emptyset$ be increasing downwards and satisfy the following hypotheses*:

(F4) *The set $P_1 = \{u \in P \mid v \leqslant u \text{ for some } v \in F(u)\}$ is nonempty.*
(F5) *The values $F(u)$ of F are sequentially compact, and decreasing sequences of the range $F[P] = \bigcup\{F(u) \mid u \in P\}$ of F converge in X.*

Then F has a minimal fixed point.

Proof. The given hypotheses ensure that F satisfies the hypotheses of Proposition 2.1 with respect to the dual ordering $\preceq$ of P, defined by $u \preceq v$ if and only if $v \leqslant u$. Thus F has a maximal fixed point u_- in $(P, \preceq)$, and u_- is a minimal fixed point of F in $(P, \leqslant)$. □

The hypothesis (F1) of Lemma 2.2 and the hypothesis (F4) of Proposition 2.2 can be replaced in certain ordered topological vector spaces by properties of P defined as follows.

Definition 2.3. We say that a subset P of X has a *sup-center* (respectively *inf-center*) c if $\sup\{c, u\}$ (respectively $\inf\{c, u\}$) exists and belongs to P for each $u \in P$. If c is both a sup-center and an inf-center of P we say that c is an *order center* of P.

For instance, if X is a lattice-ordered Banach space, and if $\|u^+\| \leqslant \|u\|$ for each $u \in X$, then the center of each closed ball P of X is its order center.

Applying the results of Lemma 2.2 and Propositions 2.1 and 2.2 we are now ready to prove the main fixed point result of this section.

Theorem 2.1. *Let X be a lattice-ordered topological vector space having property* (C) *and the following property*:

(C+) *If $u_n \to u$ in X, and if (u_n^+) is increasing, then $u_n^+ \to u^+$.*

Assume that P is a sequentially closed subset of X, and that $G : P \to 2^P \setminus \emptyset$ is an increasing mapping whose range is relatively sequentially compact and values are sequentially closed.

(a) *If P has a sup-center, then G has a minimal fixed point.*
(b) *If P has an inf-center, then G has a maximal fixed point.*
(c) *If P has an order center, then G has minimal and maximal fixed points.*

Proof. (a) It suffices to consider the case when the zero-vector 0 is a sup-center of P. The relation

$$F(u) = \left\{v^+ \mid v \in G(u)\right\} \tag{2.2}$$

defines an increasing mapping $F : P \to 2^P \setminus \emptyset$. We shall show that F has a maximal fixed point by proving that F satisfies the hypotheses (F1) and (F2) of Lemma 2.2. The hypothesis (F1) holds since $0 \leqslant v$ for each $v \in F(0)$. To prove (F2), assume that $u_n \leqslant v_n \in F(u_n)$,

$n \in \mathbb{N}$, and that (v_n) is increasing. By the definition (2.2) of F we can choose for each $n \in \mathbb{N}$ a $w_n \in G(u_n)$ such that $w_n^+ = v_n$. Because (w_n) is contained in $G[P]$ which is relatively sequentially compact, then each subsequence of (w_n) has a convergent subsequence. This result and property (C+) of X imply that each subsequence of $(v_n) = (w_n^+)$ has a convergent subsequence. Since (v_n) is increasing, then all these convergent subsequences have the same limit, $v = \sup_n v_n$, whence (v_n) converges to v. Moreover, $v \in P$ because P is sequentially closed. Since G is increasing, $u_n \leqslant v_n \leqslant v$ and $w_n \in G(u_n)$, $n \in \mathbb{N}$, we can choose for each $n \in \mathbb{N}$ a $z_n \in G(v)$ such that $w_n \leqslant z_n$. Because both (w_n) and (z_n) are sequences of $G[P]$ which is relatively sequentially compact, there exist subsequences, say (w_k) and (z_k), such that $w_k \to w$ and $z_k \to z$. Since $G(v)$ is sequentially closed, then $z \in G(v)$. It follows from $w_k \leqslant z_k$ as $k \to \infty$ that $w \leqslant z$. In particular, $w^+ \leqslant z^+$, and since $(v_k) = (w_k^+)$ is increasing, then $v = \lim_k v_k = w^+$ by (C+). Thus $v \leqslant z^+ \in F(v)$, i.e., $v \in P_0$. This proves that the hypothesis (F2) holds.

The above proof shows that F, defined by (2.2), satisfies the hypotheses of Lemma 2.2, whence F has a maximal fixed point u, i.e., $u \in F(u)$. It follows from (2.2) that $u = v^+$ for some $v \in G(u)$, whence u belongs to the set $P_1 = \{u \in P \mid v \leqslant u \text{ for some } v \in G(u)\}$. Thus G satisfies the hypothesis (F4) given for F in Proposition 2.2. Because $G[P]$ is relatively sequentially compact and each $G(u)$ is sequentially compact as a sequentially closed subset of $G[P]$, then also the hypothesis (F5) holds for G, so that G has by Proposition 2.2 a minimal fixed point.

The proof of (b) is dual to the above proof, and (c) follows from (a) and (b). □

Remark 2.1. Each ordered metric space, and thus each ordered normed space has property (C) due to [4, Proposition 1.1.5]. If X is an ordered topological vector space which satisfies the second countability axiom, then each well-ordered chain of X is separable, whence X has property (C) by [4, Lemma 1.1.7]. Another example of an ordered topological vector space with property (C), which is particularly useful in applications, is an ordered Banach space equipped with the weak topology (cf. [1, Appendix, Lemma A.3.1]). As for other results on fixed points of increasing multivalued functions in ordered spaces see, e.g., [2,3,5].

3. Special cases

In this section we consider the special case when X is a lattice-ordered reflexive Banach space equipped with the weak topology. As stated in Remark 2.1, X has property (C). As a consequence of Propositions 2.1 and 2.2 we get the following fixed point result.

Proposition 3.1. *Let P be a bounded and weakly sequentially closed subset of X, and assume that $F : P \to 2^P \setminus \emptyset$ is increasing and has weakly sequentially closed values.*

(a) *If $v \leqslant u \in F(v)$ for some $u, v \in P$, then F has a maximal fixed point.*
(b) *If $v \geqslant u \in F(v)$ for some $u, v \in P$, then F has a minimal fixed point.*

Proof. (a) Because X is reflexive, and P is bounded and weakly sequentially closed, then P is weakly sequentially compact. This result and the hypothesis on the values of F ensure that F satisfies the hypothesis (F3) of Proposition 2.1. The hypothesis (F1) of Lemma 2.2 holds if $v \leqslant u \in F(v)$ for some $u, v \in P$, whence F has a maximal fixed point by Proposition 2.1.

Similar reasoning shows that F satisfies in the case (b) the hypotheses of Proposition 2.2. Thus F has a minimal fixed point. □

As a consequence of Theorem 2.1 we get the following result.

Theorem 3.1. *Let X be a lattice-ordered reflexive Banach space with property* (W+) *given in the Introduction. Assume that P is a bounded and weakly sequentially closed subset of X, and that $G : P \to 2^P \setminus \emptyset$ is an increasing mapping whose values are weakly sequentially closed.*

(a) *If P has a sup-center, then G has a minimal fixed point.*
(b) *If P has an inf-center, then G has a maximal fixed point.*
(c) *If P has an order center, then G has minimal and maximal fixed points.*

Proof. Because X is reflexive and P is bounded, then P is weakly sequentially compact, whence $G[P]$ is weakly relatively sequentially compact. The hypothesis (W+) means that the hypothesis (C+) of Theorem 2.1 holds when X is equipped with the weak topology. Thus the conclusions follow from Theorem 2.1. □

As an application we get the fixed point result stated in the Introduction.

Corollary 3.1. *Let X be a lattice-ordered reflexive Banach space with the properties* (W+) *and* (N+) *given in the Introduction. If P is a closed and bounded ball in X, then each increasing mapping $G : P \to 2^P \setminus \emptyset$ with weakly sequentially closed values has minimal and maximal fixed points.*

Proof. The ball P, as a closed and convex set also weakly sequentially closed. Moreover, property (N+) of X ensures that the center of P is its order center. Thus all the hypotheses of Theorem 3.1 hold, which implies the assertion. □

The following result gives examples where Theorem 3.1 and Corollary 3.1 can be applied.

Lemma 3.1. *Each of the following spaces are lattice-ordered and reflexive Banach spaces with properties* (W+) *and* (N+) *when* $1 < p < \infty$:

(a) *$L^p(\Omega)$, ordered a.e. pointwise, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space.*
(b) *$W^{1,p}(\Omega)$, and $W_0^{1,p}(\Omega)$, ordered a.e. pointwise, where Ω is a domain in $\mathbb{R}^N$.*
(c) *l^p, ordered coordinatewise and normed by the p-norm.*
(d) *$\mathbb{R}^N$, ordered coordinatewise and normed by the p-norm.*

Proof. It is well known that the spaces listed in (a)–(d) are reflexive lattice-ordered Banach spaces and posses property (N+). Thus it suffices to show that (W+) holds.

(a) Let (u_n) be a sequence in $X = L^p(\Omega)$, and assume that $u_n \rightharpoonup u$ in X, and that (u_n^+) is increasing. Because (u_n) is bounded and $\|u_n^+\|_p \leqslant \|u_n\|_p$ for all $n \in \mathbb{N}$ by property (N+), then (u_n^+) converges strongly, and hence weakly to a function $v \in X$. Since $u_n^- = u_n^+ - u_n$, $n \in \mathbb{N}$, then (u_n^-) converges weakly to a function $w \in X$. Thus $u_n = u_n^+ - u_n^- \rightharpoonup v - w$, whence $u = v - w$. Because $u_n^+ \geqslant 0$ and $u_n^- \geqslant 0$, $n \in \mathbb{N}$, then $v, w \geqslant 0$, so that $\inf\{v, w\} \geqslant 0$. To show that $v = u^+$ and $w = u^-$ it suffices to show (cf. [7]) that $\inf\{v, w\} = 0$. Assume on the contrary that $\inf\{v, w\} > 0$. Then there exists a set $\Omega_0 \in \mathcal{A}$ with $0 < \mu(\Omega_0) < \infty$ such that $v(x) > 0$ and $w(x) > 0$ for all $x \in \Omega_0$. Because (u_n^+) is increasing we can assume that $(u_n^+(x))$ is increasing for all $x \in \Omega_0$.

Since $\mu(\Omega_0) < \infty$, then the characteristic function χ_{Ω_0} belongs to $L^q(\Omega)$, $q = p/(p-1)$. This result, $u_n^+ \rightharpoonup v$ and $v(x) > 0$ in Ω_0 ensure that

$$\int_{\Omega_0} u_n^+ \, d\mu = \int_{\Omega} u_n^+ \chi_{\Omega_0} \, d\mu \to \int_{\Omega} v \chi_{\Omega_0} \, d\mu = \int_{\Omega_0} v \, d\mu > 0.$$

Thus there exists $j \in \mathbb{N}$ such that $\int_{\Omega_0} u_j^+ \, d\mu > 0$. This implies an existence of a subset Ω_1 of Ω_0 with $\mu(\Omega_1) > 0$ such that $u_j^+(x) > 0$ for all $x \in \Omega_1$. Because $(u_n^+(x))$ is increasing for all $x \in \Omega_1$, then $u_n^+(x) > 0$ for all $n \geqslant j$ and $x \in \Omega_1$.

Since $\mu(\Omega_1) \leqslant \mu(\Omega_0) < \infty$, then the characteristic function χ_{Ω_1} belongs to $L^q(\Omega)$. Because $u_n^- \rightharpoonup w$ and $w(x) > 0$ in Ω_1, then

$$\int_{\Omega_1} u_n^- \, d\mu = \int_{\Omega} u_n^- \chi_{\Omega_1} \, d\mu \to \int_{\Omega} w \chi_{\Omega_1} \, d\mu = \int_{\Omega_1} w \, d\mu > 0.$$

Thus there exists $k \in \mathbb{N}$, $k \geqslant j$, such that $\int_{\Omega_1} u_k^- \, d\mu > 0$. This implies an existence of a subset Ω_2 of Ω_1 with $\mu(\Omega_2) > 0$ such that $u_k^-(x) > 0$ for all $x \in \Omega_2$. But then $u_k^+(x) > 0$ and $u_k^-(x) > 0$ for all $x \in \Omega_2$, a contradiction.

The above proof shows that if $u_n \rightharpoonup u$, and if (u_n^+) is increasing, then $u_n^+ \rightharpoonup u^+$ (in fact, $u_n^+ \to u^+$), so that $X = L^p(\Omega)$ has property (W+).

(b) Let X be any of the spaces $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$. To prove that (W+) holds, let (u_n) be a sequence in X, and assume that $u_n \rightharpoonup u$. Since X is compactly embedded in $L^p(\Omega)$, then $u_n \to u$ in $L^p(\Omega)$. The mapping $u \mapsto u^+$ is continuous in $L^p(\Omega)$, whence $u_n^+ \to u^+$ in $L^p(\Omega)$. Let (u_k) be a subsequence of (u_n). Because (u_k) is bounded in X, it follows from (N+) that (u_k^+) is bounded in X. Thus there exists a subsequence (u_j) of (u_k) such that $u_j^+ \rightharpoonup v$ in X, so that $u_j^+ \to v$ in $L^p(\Omega)$ due to the compact embedding $X \subset L^p(\Omega)$. Because the entire sequence (u_n^+) converges to u^+ in $L^p(\Omega)$, we infer that $v = u^+$. Consequently, each subsequence of (u_n^+) has a subsequence which converges weakly to u^+, which implies that the entire sequence (u_n^+) converges weakly in X to u^+.

The above proof shows that if $u_n \rightharpoonup u$ in X, then $u_n^+ \rightharpoonup u^+$ in X, so that (W+) holds.

The proof that the spaces given in (c) and (d) posses property (W+) is trivial. □

4. Application to an operator equation

In this section we assume that X is a separable and reflexive lattice-ordered Banach space. We are going to derive existence results for minimal and maximal solutions of the equation

$$u = H(u,u), \tag{4.1}$$

where $H : X \times X \to X$.

In the proofs we need the following auxiliary results.

Lemma 4.1. *Assume that $H : X \times X \to X$ satisfies the following growth condition*:

(H0) $\|H(u,v)\| \leqslant M + a\|u\|^{\alpha} + b\|v\|^{\beta}$ *for all $u, v \in X$, where $M, a, b \geqslant 0$, and either*

$$\text{(i)} \quad 0 \leqslant \alpha, \beta < 1 \quad \text{or} \quad \text{(ii)} \quad \alpha = \beta = 1 \quad \text{and} \quad a + b < 1.$$

Then there exists a ball $P = \{u \in X \mid \|u\| \leqslant R\}$ such that

(a) *if u is a solution of* (4.1)*, then $u \in P$*;
(b) *$H(u,v) \in P$ for all $u, v \in P$.*

Proof. Assume first that (H0)(i) holds. Then the function

$$f(z) = \frac{M}{z} + \frac{a}{z^{1-\alpha}} + \frac{b}{z^{1-\beta}}, \quad z > 0,$$

is strictly decreasing and there exists exactly one $R > 0$ such that $f(R) = 1$. Choosing this R as the radius of the ball P it is easy to see that (a) and (b) hold.

If (H0)(ii) holds, then (a) and (b) are valid when $R = M/(1 - (a + b))$. □

Lemma 4.2. *Let the hypothesis* (H0) *hold, let P be as in Lemma* 4.1*, and assume that*

(H1) *$H(\cdot, v)$ is weakly sequentially continuous for each $v \in P$.*

Then for each $v \in P$ the solutions of the equation

$$u = H(u,v) \tag{4.2}$$

form a nonempty and weakly sequentially closed subset of P.

Proof. Let $v \in P$ be given. The choice of P and the hypothesis (H0) ensure that the relation

$$T(u) = H(u,v), \quad u \in P,$$

defines a mapping $T : P \to P$, which is weakly sequentially continuous by the hypothesis (H1). Moreover P is a nonempty, closed, bounded and convex subset of a separable and reflexive Banach space. Thus T has a fixed point by second Schauder fixed point theorem

(cf. [8, Corollary 9.7]). To prove that the set of all fixed points of T is weakly sequentially closed, let (u_n) be a sequence of fixed points of T, and assume that $u_n \rightharpoonup u$. The limit u belongs to P and, because T is weakly sequentially continuous, then $u_n = T(u_n) \rightharpoonup T(u)$, whence $u = T(u)$.

The above proof shows that the set of all fixed points of T is a nonempty and weakly sequentially closed subset of P. Applying the definition of T, the hypothesis (H0) and the choice of P it is easy to show that u is a solution of (4.2) if and only if u is a fixed point of T. This concludes the proof. □

Our first existence results for (4.1) are applications of Proposition 3.1.

Proposition 4.1. *Let the hypotheses* (H0) *and* (H1) *hold, let P be as in Lemma* 4.1 *and assume also that*

(H2) *If $u_1, v_1, v_2 \in P$, $v_1 \leqslant v_2$ and $u_1 = H(u_1, v_1)$, there exists $u_2 \in P$ such that $u_1 \leqslant u_2$ and $u_2 = H(u_2, v_2)$.*
(H3) *If $u_2, v_1, v_2 \in P$, $v_1 \leqslant v_2$ and $H(u_2, v_2) = u_2$, there exists $u_1 \in P$ such that $u_1 \leqslant u_2$ and $u_1 = H(u_1, v_1)$.*

(a) *If $v \leqslant u = H(u, v)$ for some $u, v \in P$, then* (4.1) *has a maximal solution.*
(b) *If $H(u, v) = u \leqslant v$ for some $u, v \in P$, then* (4.1) *has a minimal solution.*

Proof. As a bounded and closed ball of X the set P is bounded and weakly sequentially closed. It follows from Lemmas 4.1 and 4.2 that the relation

$$G(v) = \{u \in P \mid u = H(u, v)\} \tag{4.3}$$

defines a mapping $G : P \to 2^P \setminus \emptyset$ whose values are weakly sequentially closed. The hypotheses (H2) and (H3) mean that G is increasing.

(a) If $v \leqslant u = H(u, v)$ for some $u, v \in P$, then the hypothesis of Proposition 3.1(a) holds for $F = G$. Thus G has a maximal fixed point v. This result and the definition (4.3) of G imply that v is a maximal solution of (4.1) in P. Since all the solutions of (4.1) belong to P by Lemma 4.1, then v is a maximal solution of (4.1) in the whole X. This proves (a).

Similar reasoning shows that $F = G$ satisfies in the case (b) the hypotheses of Proposition 2.2. Thus G has a minimal fixed point, which is a minimal solution of (4.1). □

As a consequence of Corollary 3.1 we prove the following result.

Theorem 4.1. *Let X be a separable and reflexive lattice-ordered Banach space with the properties* (W+) *and* (N+). *Assume that $H : X \times X \to X$ satisfies the hypotheses* (H0)–(H3). *Then Eq.* (4.1) *possesses minimal and maximal solutions.*

Proof. Let the ball P be as in Lemma 4.1. By the proof of Proposition 4.1 the relation (4.3) defines an increasing mapping $G : P \to 2^P \setminus \emptyset$ whose values are weakly sequentially closed. Thus G has by Corollary 3.1 minimal and maximal fixed points. In view of (4.3) and Lemma 4.1 these fixed points are minimal and maximal solutions of (4.1) in X. □

Remark 4.1. All the spaces listed in Lemma 3.1 are also separable, whence they posses all the properties assumed for X in Theorem 4.1.

Instead of the hypotheses (H0) and (H1) it suffices to assume that the results of Lemmas 4.1 and 4.2 hold.

5. Discontinuous quasilinear elliptic BVP

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and denote by $X := W_0^{1,p}(\Omega)$, $1 < p < \infty$, the usual Sobolev space, whose elements have generalized homogeneous boundary values. Assuming that X is equipped with the usual norm and a.e. pointwise ordering it follows from Lemma 3.1 that X is a lattice-ordered reflexive Banach space possessing the properties (W+) and (N+) given in the Introduction. As an example of a quasilinear discontinuous boundary value problem (BVP for short) we consider the following functional Dirichlet BVP:

$$u \in X: \quad -\Delta_p u = f\big(\cdot, u(\cdot), u\big) \quad \text{in } X^*, \tag{5.1}$$

where X^* denotes the dual space of X, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < \infty$ is the p-Laplacian. According to (5.1) we are seeking functions $u \in X$ that satisfy the following relation:

$$\sum_{i=1}^{N} \int_\Omega |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega f\big(x, u(x), u\big)\varphi(x)\, dx, \quad \forall \varphi \in X.$$

For the nonlinearity $f : \Omega \times \mathbb{R} \times X \to \mathbb{R}$ we make the following assumptions:

(f1) $f(\cdot, s, v)$ is measurable, $f(x, \cdot, v)$ is continuous and $f(x, s, \cdot)$ is monotone nondecreasing for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and $v \in X$.

(f2) f satisfies the growth condition

$$\big|f(x, s, v)\big| \leqslant k(x) + \mu|s|^{p-1} \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R} \text{ and } v \in X \tag{5.2}$$

with $k \in L^q_+(\Omega)$, $q = p/(p-1)$, and $0 \leqslant \mu < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of Δ_p.

It is known that the variational characterization of the first eigenvalue λ_1 of Δ_p which is positive is given by

$$\lambda_1 = \inf_{0 \neq u \in X} \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx} \tag{5.3}$$

(see [6]). Note that the function $(x, v) \mapsto f(x, s, v)$ in the right-hand side of (5.1) is, in general, nonlinear and discontinuous.

Let us denote the norms in X and $L^p(\Omega)$ by $\|\cdot\|_X$ and $\|\cdot\|_p$, respectively. By applying the general theory developed in the preceding sections we are going to prove the following existence result.

Theorem 5.1. *Let hypotheses* (f1) *and* (f2) *be satisfied. Then the BVP* (5.1) *has minimal and maximal solutions in* X.

Proof. The proof will be given in two steps.

(a) A priori bound.

Let u be any solution of (5.1). Then by using the special test function $\varphi = u$, the growth condition (5.2), the characterization (5.3) of the eigenvalue λ_1 and Young's inequality we obtain

$$\|\nabla u\|_p^p \leqslant \int_\Omega \big|k(x)u(x)\big|\,dx + \mu \int_\Omega \big|u(x)\big|^p\,dx \leqslant C(\varepsilon) + \frac{\varepsilon+\mu}{\lambda_1}\|\nabla u\|_p^p$$

for any $\varepsilon > 0$. In view of (f2) and selecting ε sufficiently small the last inequality implies the existence of a positive constant R such that

$$\|u\|_X \leqslant R.$$

(b) Existence of minimal and maximal solutions of (5.1).

We define a mapping $H : X \times X \to X$ as follows: $(w, v) \mapsto H(w, v) =: u$, where u is the unique solution of the BVP

$$u \in X: \quad -\Delta_p u = f\big(\cdot, w(\cdot), v\big) \quad \text{in } X^*. \tag{5.4}$$

Due to the growth condition (5.2) the right-hand side of (5.4) belongs to $L^q(\Omega) \subset X^*$ with $q = p/(p-1)$. Since $-\Delta_p : X \to X^*$ is strictly monotone, bounded, continuous and coercive, the existence of a unique solution of (5.4) follows from the main theorem on monotone operators, cf., e.g., [9, Theorem 26.A]. Thus the mapping H is well defined. Moreover, any solution of the equation

$$u = H(u, u) \tag{5.5}$$

is a solution of the BVP (5.1) and vice versa, and thus the BVP (5.1) is given in the abstract form of the preceding section. We are going to apply Theorem 4.1 to ensure the existence of minimal and maximal solutions of (5.5).

In view of (f2) we see in just the same way as in part (a) that there is a ball B of radius R, i.e., $B = \{u \in X \mid \|u\|_X \leqslant R\}$ such that if u satisfies (5.5), then $u \in B$, and $H(w, v) \in B$ for all $w, v \in B$. Thus the result of Lemma 4.1 holds. Now for $v \in B$ fixed we consider the equation

$$u = H(u, v). \tag{5.6}$$

The solutions of (5.6) are given by the solutions of the following BVP:

$$u \in X: \quad -\Delta_p u = f\big(\cdot, u(\cdot), v\big) \quad \text{in } X^*. \tag{5.7}$$

Next we are going to prove that the solution set of (5.6) is a nonempty and weakly sequentially closed subset of B, so that the result of Lemma 4.2 holds. Denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X^* and define the operator $F_v : X \to X^*$ by

$$\big\langle F_v(u), \varphi\big\rangle = \int_\Omega f\big(x, u(x), v\big)\varphi(x)\,dx, \quad \varphi \in X.$$

Because the mapping $f(\cdot,\cdot,v):\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function for all v satisfying the growth condition (5.2), and since the embedding $X\subset L^p(\Omega)$ is compact, we see that $F_v:X\to X^*$ is well defined and compact. The operator $-\Delta_p:X\to X^*$ is strictly monotone, continuous and bounded. Finally, by means of (f2) one can show that $-\Delta_p-F_v:X\to X^*$ is also coercive. Since the compact perturbation of a monotone operator is pseudomonotone we may apply the main theorem on pseudomonotone operators (cf. [9, Theorem 27.A] to deduce that (5.7) possesses solutions, and due to the property of the mapping H all solutions belong to B. Denote the solution set of (5.7) by $G(v)$. To show that $G(v)$ is weakly sequentially closed, assume that (u_n) is a sequence in $G(v)$, and that $u_n\rightharpoonup u$. Since B is weakly sequentially closed, then $u\in B$. We only need to verify that $u\in G(v)$, which means that u satisfies

$$-\Delta_p u-F_v(u)=0\quad\text{in } X^*. \tag{5.8}$$

By definition the functions u_n satisfy

$$-\Delta_p u_n-F_v(u_n)=0\quad\text{in } X^*. \tag{5.9}$$

From (5.9) we see that

$$\langle -\Delta_p u_n-F_v(u_n),u_n-u\rangle\to 0\quad\text{as } n\to\infty,$$

which implies in view of the pseudomonotonicity of $-\Delta_p-F_v:X\to X^*$ (cf. [1, Appendix, Definition D.1.2]) the following convergence property:

$$-\Delta_p u_n-F_v(u_n)\rightharpoonup -\Delta_p u-F_v(u).$$

Thus the passage to the limit in (5.9) yields that (5.8) holds, whence $u\in G(v)$.

Next we are going to show that the multivalued mapping $G:B\to 2^B\setminus\emptyset$ defined above is increasing in the sense of Definition 2.2. To this end let $v_1,v_2\in B$ be given with $v_1\leqslant v_2$, and let $u_1\in G(v_1)$, i.e., $u_1=H(u_1,v_1)$ which means

$$u_1\in X:\quad -\Delta_p u_1=f\big(\cdot,u_1(\cdot),v_1\big)\quad\text{in } X^*. \tag{5.10}$$

We need to show the existence of $u_2\in G(v_2)$ such that $u_1\leqslant u_2$ holds. Consider the auxiliary BVP

$$u\in X:\quad -\Delta_p u=\hat f\big(\cdot,u(\cdot),v_2\big)\quad\text{in } X^*, \tag{5.11}$$

where $\hat f$ is defined as follows:

$$\hat f(x,s,v)=\begin{cases} f(x,s,v) & \text{if } s>u_1(x),\\ f(x,u_1(x),v) & \text{if } s\leqslant u_1(x).\end{cases}$$

Since $\hat f$ possesses the same regularity and growth conditions as f, the existence of solutions of (5.11) can be shown in the same way as for (5.7). Let u_2 be any solution of (5.11). Then subtracting (5.11) from (5.10) and taking into account the monotonicity of $v\mapsto f(x,s,v)$ we obtain the inequality

$$-(\Delta_p u_1-\Delta_p u_2)\leqslant f\big(\cdot,u_1(\cdot),v_2\big)-\hat f\big(\cdot,u_2(\cdot),v_2\big)\quad\text{in } X^*, \tag{5.12}$$

which yields with the nonnegative test function $\varphi=(u_1-u_2)^+$ the inequality

$$\sum_{i=1}^{N}\int_{\Omega}\left(|\nabla u_1|^{p-2}\frac{\partial u_1}{\partial x_i}-|\nabla u_2|^{p-2}\frac{\partial u_2}{\partial x_i}\right)\frac{\partial(u_1-u_2)^+}{\partial x_i}\,dx$$
$$\leqslant\int_{\Omega}\big(f\big(x,u_1(x),v_2\big)-\hat{f}\big(x,u_2(x),v_2\big)\big)(u_1-u_2)^+(x)\,dx. \tag{5.13}$$

Applying the definition of $\hat{f}$ one readily sees that the right-hand side of (5.13) is zero. This yields

$$0\leqslant\sum_{i=1}^{N}\int_{\{u_1\geqslant u_2\}}\left(|\nabla u_1|^{p-2}\frac{\partial u_1}{\partial x_i}-|\nabla u_2|^{p-2}\frac{\partial u_2}{\partial x_i}\right)\frac{\partial(u_1-u_2)}{\partial x_i}\,dx\leqslant 0,$$

which implies $\nabla(u_1-u_2)^+=0$, and thus $(u_1-u_2)^+=0$, i.e., $u_1\leqslant u_2$. But then we have $\hat{f}(\cdot,u_2(\cdot),v_2)=f(\cdot,u_2(\cdot),v_2)$, and therefore the solution u_2 is a solution of the BVP

$$u_2\in X:\quad -\Delta_p u_2=f\big(\cdot,u_2(\cdot),v_2\big)\quad\text{in } X^*,$$

which means $u_2\in G(v_2)$. In a similar way one can also prove that for any $u_2\in G(v_2)$ there exists $u_1\in G(v_1)$ such that $u_1\leqslant u_2$. This completes the proof for the multifunction G to be increasing, or equivalently, that the hypotheses (H2) and (H3) of Proposition 4.1 hold for H defined above. Now we are able to apply Theorem 4.1 (instead of assuming (H0) and (H1) we have proved that the results of Lemmas 4.1 and 4.2 hold), which ensures the existence of minimal and maximal solutions of Eq. (5.5) and thus of the BVP (5.1). □

Replacing $f(x,s,v)$ by $g(x,s,v(x))$ and μ by $\mu_1+\mu_2$ in the proof of Theorem 5.1 we get the following existence theorem for the BVP

$$u\in X:\quad -\Delta_p u=g\big(\cdot,u(\cdot),u(\cdot)\big)\quad\text{in } X^*. \tag{5.14}$$

Proposition 5.1. *Let the function $g:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the following hypotheses*:

(g1) *$g(\cdot,s,\cdot):\Omega\times\mathbb{R}\to\mathbb{R}$ is superpositionally measurable, $s\mapsto g(x,s,r)$ is continuous and $r\mapsto g(x,s,r)$ is monotone nondecreasing for a.e. $x\in\Omega$ and for all $s,r\in\mathbb{R}$.*
(g2) *$|g(x,s,r)|\leqslant k(x)+\mu_1|s|^{p-1}+\mu_2|r|^{p-1}$ for a.e. $x\in\Omega$ and all $s,r\in\mathbb{R}$, with $k\in L^q_+(\Omega)$, $q=p/(p-1)$, $\mu_1\geqslant 0$ and $\mu_2\geqslant 0$.*
(g3) *$\mu_1+\mu_2<\lambda_1$, where λ_1 is the first Dirichlet eigenvalue of Δ_p.*

Then the BVP (5.14) *has minimal and maximal solutions in X.*

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