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Maximally singular Sobolev functions

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Abstract

It is known that for any Sobolev function in the space $W^{m,p}(\mathbb{R}^N)$, $p \geqslant 1$, $mp \leqslant N$, where m is a nonnegative integer, the set of its singular points has Hausdorff dimension at most $N - mp$. We show that for $p > 1$ this bound can be achieved. This is done by constructing a maximally singular Sobolev function in $W^{m,p}(\mathbb{R}^N)$, that is, such that Hausdorff's dimension of its singular set is equal to $N - mp$. An analogous result holds also for Bessel potential spaces $L^{\alpha,p}(\mathbb{R}^N)$, provided $\alpha p < N$, $\alpha > 0$, and $p > 1$. The existence of maximally singular Sobolev functions has been announced in [Chaos Solitons Fractals 21 (2004), p. 1287].

Keywords: Sobolev function; Bessel potential; Fractal set; Minkowski content; Sierpinski carpet

1. Introduction and the main result

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We say that $a \in \mathbb{R}^N$ is a singular point of u if there exist positive constants R, C and γ such that $u(x) \geqslant C|x - a|^{-\gamma}$ for a.e. x in the open ball $B_R(a)$. The set of all singular points of u is denoted by $\operatorname{Sing} u$. If we consider functions u from the Sobolev space $W^{m,p}(\mathbb{R}^N)$, it is of interest to know how large their singular sets be in the sense of Hausdorff dimension (see, e.g., Falconer [4]), that is, how

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large the value of $\dim_H(\operatorname{Sing} u)$ can be. To this end we define the singular dimension of the Sobolev space by

$$\text{s-dim}\, W^{m,p}(\mathbb{R}^N) := \sup\{\dim_H(\operatorname{Sing} u)\colon\ u \in W^{m,p}(\mathbb{R}^N)\}. \tag{1}$$

It has been shown in [11, Theorem 1] that

$$\text{s-dim}\, W^{m,p}(\mathbb{R}^N) = N - mp, \tag{2}$$

provided $mp \leqslant N$ and $p > 1$. We recall that the inequality $\text{s-dim}\, W^{m,p}(\mathbb{R}^N) \leqslant N - mp$ is known, and follows from Reshetnyak [8, Corollary 2] (or Adams and Hedberg [2, Theorem 5.1.3]), combined with [11, Theorem 4]. For more information about the role of the limiting value $N - mp$ see [11] and references therein. The aim of this article is to show that the supremum in (1) is achieved. Here is the main result.

Theorem 1. *Assume that $mp < N$, where m is a nonnegative integer, and $p > 1$. There exist Sobolev functions $u \in W^{m,p}(\mathbb{R}^N)$ such that*

$$\dim_H(\operatorname{Sing} u) = N - mp. \tag{3}$$

Functions from the space $W^{m,p}(\mathbb{R}^N)$ satisfying condition (3) will be called maximally singular Sobolev functions.

Remark 1. The same result holds for Sobolev spaces $W^{m,p}(\Omega)$, where Ω is an arbitrary open set in $\mathbb{R}^N$. This can be obtained using a slight modification in the proof.

Regarding Theorem 1 it is worth noting that due to the Sobolev imbedding theorem no singularities can occur when $mp > N$, since in this case Sobolev functions possess a Hölder continuous representative, see, e.g., Adams [1] or Gilbarg and Trudinger [5]. Also, no singularities (in the strong sense introduced above) can occur when $mp = N$ either, since in this case $W^{m,p}(\mathbb{R}^N)$ is contained in all Lebesgue spaces $L^q(\mathbb{R}^N)$, $q \geqslant 1$. However, singularities of weaker type (say logarithmic) *are* possible in the latter case.

The existence of maximally singular Lebesgue integrable functions in $L^p(\mathbb{R}^N)$, $1 \leqslant p < \infty$, has been shown in [13, Section 3]. More precisely, there exist functions $u \in L^p(\mathbb{R}^N)$ such that $\dim_H(\operatorname{Sing} u) = N$. The proof was carried out by effective construction. This result is just a special case of Theorem 1 for $m = 0$, except for the case $p = 1$. We do not know if the statement of Theorem 1 holds also when $p = 1$, with arbitrary $m \in \mathbb{N}$. The result about the existence of maximally singular Sobolev functions stated in Theorem 1 has been announced in [13, p. 1287].

We shall prove a more general result than in Theorem 1, involving Bessel potential spaces

$$L^{\alpha,p}(\mathbb{R}^N) := \{G_\alpha * f\colon\ f \in L^p(\mathbb{R}^N)\}, \tag{4}$$

see Adams and Hedberg [2], Ziemer [10], or Stein [9] for definition of the Bessel potential kernel G_α and properties of these spaces.

Theorem 2. *Assume that $\alpha p < N$, where $\alpha > 0$, and $p > 1$. There exist Sobolev functions $u \in L^{\alpha,p}(\mathbb{R}^N)$ such that*

$$\dim_H(\operatorname{Sing} u) = N - \alpha p. \tag{5}$$

Theorem 1 is indeed a special case of Theorem 2 due to the following important result of Calderón (see, e.g., Ziemer [10, Theorem 2.6.1], or the original paper of Calderón [3]): if m is a positive integer and $1 < p < \infty$, then $W^{m,p}(\mathbb{R}^N) = L^{m,p}(\mathbb{R}^N)$. Calderón's theorem does not hold for $p = 1$.

Proof of Theorem 2. Since s-dim $L^{\alpha,p}(\mathbb{R}^N) = N - \alpha p$, see [11, Theorem 2], there exists an increasing sequence of positive real numbers s_k such that $s_k \to N - \alpha p$ as $k \to \infty$, and a sequence of nonnegative functions $f_k \in L^p(\mathbb{R}^N)$ such that for $u_k := G_\alpha * f_k \in L^{\alpha,p}(\mathbb{R}^N)$ we have

$$\dim_H(\operatorname{Sing} u_k) \geqslant s_k. \tag{6}$$

Now define a function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$u(x) := \sum_{k=1}^{\infty} c_k \|f_k\|_p^{-1} u_k(x), \tag{7}$$

where by $\|f_k\|_p$ we denoted L^p-norm of f_k. Let us choose a sequence of positive numbers c_k such that $\sum_k c_k < \infty$. Then

$$u = \sum_{k=1}^{\infty} c_k \|f_k\|_p^{-1} (G_\alpha * f_k) = G_\alpha * \left(\sum_{k=1}^{\infty} c_k \|f_k\|_p^{-1} f_k \right), \tag{8}$$

that is, $u = G_\alpha * f$, where

$$f(x) := \sum_{k=1}^{\infty} c_k \|f_k\|_p^{-1} f_k(x). \tag{9}$$

To show that $u \in L^{\alpha,p}(\mathbb{R}^N)$, by Calderón's theorem it suffices to check that $f \in L^p(\mathbb{R}^N)$:

$$\|f\|_p \leqslant \sum_{k=1}^{\infty} c_k < \infty.$$

Since all functions in the series (7) of u are nonnegative (recall that $f_k \geqslant 0$), it is clear that $\bigcup_k \operatorname{Sing} u_k \subseteq \operatorname{Sing} u$. Using countable stability of Hausdorff's dimension, see Falconer [4, p. 29], and (6), we obtain that

$$\begin{aligned} \dim_H(\operatorname{Sing} u) &\geqslant \dim_H\left(\bigcup_k \operatorname{Sing} u_k \right) = \sup_k \bigl(\dim_H(\operatorname{Sing} u_k)\bigr) \geqslant \sup_k s_k = \lim_k s_k \\ &= N - \alpha p. \end{aligned}$$

This together with $\dim_H(\operatorname{Sing} u) \leqslant N - \alpha p$ (see Reshetnyak [8, Corollary 21] or Adams and Hedberg [2, Theorem 5.1.13], combined with [11, Theorem 4]) implies that $\dim_H(\operatorname{Sing} u) = N - \alpha p$. $\square$

Remark 2. We do not know if Theorems 1 and 2 hold for $p = 1$ as well.

2. Construction of maximally singular Sobolev functions

In the proof of Theorem 2 we have the liberty of choosing nonnegative functions f_k, see the definition of maximally singular Sobolev function u in (7), with the sole requirement that $f_k \in L^p(\mathbb{R}^N)$ and $\dim_H(\mathrm{Sing}(G_\alpha * f_k)) \to N - \alpha p$. We first recall the construction of the sequence f_k involving generalized Cantor sets, following the idea of the proof of [11, Theorem 2], and then discuss some other possibilities.

Example 1. We can define functions $f_k : \mathbb{R}^N \to \bar{\mathbb{R}}$ using fractal sets A_k (to be specified shortly) by

$$f_k(x) := d(x, A_k)^{-\gamma_k} \quad \text{for } x \in A_k(R_k) \text{ and } f(x) = 0 \text{ otherwise,} \tag{10}$$

where $d(x, A_k)$ denotes the Euclidean distance from x to the set A_k, and $A_k(R_k)$ is R_k-neighbourhood of A_k (the Minkowski sausage of radius $R_k > 0$ around A_k).

Let s_k be an increasing sequence of positive real numbers converging to $N - \alpha p$. We choose sets A_k so that

$$\overline{\dim}_B A_k = \dim_H A_k = s_k. \tag{11}$$

We can define A_k to be for example of the form of Cantor's grill:

$$A_k := C^{(a_k)} \times [0, 1]^{m_k}, \tag{12}$$

where $C^{(a_k)}$ is a generalized, uniform Cantor set with the parameter $a_k \in (0, 1/2)$ chosen so that $\dim_B C^{(a_k)} = \dim_H C^{(a_k)} = s_k - m_k$, with $m_k := \lfloor s_k \rfloor$, see Falconer [4, Example 4.5 with $m = 2$ and $\lambda = a_k$], that is $a_k = 2^{m_k - s_k}$ (we may assume without loss of generality that $s_k \notin \mathbb{Z}$ for all k). Here we have used the fact that $\dim_B C^{(a_k)} = \dim_H C^{(a_k)} = \log 2 / \log(1/a_k)$.

Also, we choose the exponents γ_k so that

$$\alpha < \gamma_k < \frac{N - s_k}{p}.$$

Since $p\gamma_k < N - \overline{\dim}_B A_k$, we can use the Harvey–Polking lemma, see [13, Lemma 1], to obtain that $f_k \in L^p(\mathbb{R}^N)$.

From the proof of [11, Theorem 4, see (12)] we know that $A_k \subseteq \mathrm{Sing}(G_\alpha * f_k)$. Moreover, $G_\alpha * f_k$ has the order of singularity equal at least to $\gamma_k - \alpha$ on A_k, that is, $(G_\alpha * f_k)(x) \geqslant C_k / d(x, A_k)^{\gamma_k - \alpha}$ for all $x \in A_k(R_k) \setminus A_k$, where $C_k > 0$. Hence, $\dim_H(\mathrm{Sing}(G_\alpha * f_k)) \geqslant s_k$, so that (6) is fulfilled.

Remark 3. We can easily achieve that the singular set of u in Theorem 2 is even dense in $\mathbb{R}^N$. It suffices to take $\{x_k\}$ to be a countable, dense set in $\mathbb{R}^N$, and then to choose fractal sets A_k as above so that $x_k \in A_k$. Hence, $\{x_k\}$ is contained in $\mathrm{Sing}\, u$.

Remark 4. It is clear that we can choose the sets A_k and the values $R_k > 0$ in the above construction so that the family of Minkowski sausages $A_k(R_k)$, $k \in \mathbb{N}$, is disjoint. This implies that the supports of functions f_k appearing in the series (9) are disjoint. Hence, assuming that $p \in \mathbb{N}$, $p \geqslant 2$ we conclude that $\|f\|_p = \sum_k c_k$. Using the fact that the $W^{m,p}(\mathbb{R}^N)$-norm of $u := G_m * f$, $f \in L^p(\mathbb{R}^N)$, defined by $\|f\|_p$, is equivalent with the usual one, see [9], we obtain that $\|u\|_{m,p} = \sum_k c_k$.

Remark 5. Using scaling and translation we can easily achieve that the family of sets $A_k(R_k)$, $k \in \mathbb{N}$, is bounded, contained in a prescribed open ball having arbitrarily small measure. Hence, there exist Sobolev functions that are maximally singular on a subset of a prescribed open ball having arbitrarily small volume.

Of course, the sets A_k in the above construction can be defined in many other ways instead of (12). In order to be able to construct various examples of Sobolev functions in the form $u = G_\alpha * f$, see (4), it is of interest to know examples of Lebesgue integrable functions of the form $f(x) := d(x, A)^{-\gamma}$ defined on a neighbourhood of A, for various sets $A \subset \mathbb{R}^N$ and $\gamma > 0$. The upper and lower d-dimensional Minkowski contents of A defined by (see, for example, [6])

$$\mathcal{M}^{*d}(A) := \limsup_{r\to 0} \frac{|A(r)|}{r^{N-d}}, \qquad \mathcal{M}^d_*(A) := \liminf_{r\to 0} \frac{|A(r)|}{r^{N-d}}, \tag{13}$$

are important tools to achieve this. By $|A(r)|$ we denote N-dimensional Lebesgue measure of the Minkowski sausage $A(r)$. We shall need the following result, see [13, Theorem 2], in which a nondegeneracy condition of Minkowski contents is crucial. It provides necessary and sufficient conditions for the Lebesgue integrability of $d(x, A)^{-\gamma}$ on a neighbourhood of A. See also [12] for related results.

Theorem 3. *Let A be a bounded set in $\mathbb{R}^N$, and $d \in [0, N)$. If the lower and upper d-dimensional Minkowski contents are nondegenerate, that is,*

$$0 < \mathcal{M}^d_*(A) \leqslant \mathcal{M}^{*d}(A) < \infty, \tag{14}$$

(*hence* $d = \dim_B A$), *then for any* $r > 0$,

$$\int_{A(r)} d(x, A)^{-\gamma}\, dx < \infty \quad \Leftrightarrow \quad \gamma < N - \dim_B A. \tag{15}$$

In the rest of this paper we discuss integrability of the singular function $d(x, S)^{-\gamma}$ generated by the Sierpinski carpet S, in order to illustrate the complexity of possible Sobolev functions in $L^{\alpha,p}(\mathbb{R}^N)$ when $\alpha p < N$, or in $W^{k,p}(\mathbb{R}^N)$ when $kp < N$.

Theorem 4. *Let S be the Sierpinski carpet.*

(a) *Then*

$$d := \dim_H S = \dim_B S = \frac{\log 8}{\log 3},$$

and d-dimensional Minkowski contents of S are nondegenerate (*i.e., both are different from* 0 *and* ∞). *Moreover, the values $\mathcal{M}_*^d(S)$ and $\mathcal{M}^{*d}(S)$ can be expressed explicitly, see* (17) *and* (18), *and*

$$\mathcal{M}_*^d(S) \approx 1.3506702, \qquad \mathcal{M}^{*d}(S) \approx 1.3556171. \tag{16}$$

(b) *Singular integral $I_r := \int_{S(r)} d(x,S)^{-\gamma}\,dx$, $\gamma > 0$, is finite if and only if $\gamma < 2 - d$. Furthermore, we have the following asymptotic behaviour of I_r as $r \to 0$:*

$$\frac{2-d}{2-d-\gamma}\mathcal{M}_*^d(S) \leqslant \liminf_{r\to 0} \frac{I_r}{r^{N-d-\gamma}} \leqslant \limsup_{r\to 0} \frac{I_r}{r^{N-d-\gamma}} \leqslant \frac{2-d}{2-d-\gamma}\mathcal{M}^{*d}(S).$$

(c) *Let $f : \mathbb{R}^N \to \bar{\mathbb{R}}$ be the function defined by $f(x) = d(x,S)^{-\gamma}$ for $x \in S(R)$, and $f(x) = 0$ otherwise, where $R > 0$ is fixed. If $0 < \alpha < \gamma < \frac{1}{p}(N-d)$ then $f \in L^p(\mathbb{R}^N)$, $G_\alpha * f \in L^{\alpha,p}(\mathbb{R}^N)$, and $A \subseteq \mathrm{Sing}(G_\alpha * f)$. Moreover, the Sobolev function $G_\alpha * f$ has the order of singularity at least $\gamma - \alpha$ on S, that is,*

$$(G_\alpha * f)(x) \geqslant \frac{C}{d(x,A)^{\gamma-\alpha}} \quad \text{for a.e. } x \in S(R),$$

where C is a positive constant.

Proof. (a) The result about Hausdorff dimension is well known, and follows easily from Falconer [4, Theorem 9.3].

Now we compute the d-dimensional Minkowski contents of S. Let S_0 be a closed (full) unit square, divided into nine squares with sides $\frac{1}{3}$. In the first step of the construction, we remove an open square with side $\frac{1}{3}$ from the centre of the unit square, and in that way 8 squares with the side $\frac{1}{3}$ are left behind (we denote their union by S_1). Generally, in the nth step we remove 8^{n-1} squares with side $\frac{1}{3^n}$, while 8^n squares are left behind (we denote their union by S_n). See Fig. 1. The Sierpinski carpet S is obtained by continuing this procedure ad infinitum, that is, $S = \bigcap_{n=0}^{\infty} S_n$.

Now, we would like to compute the Lebesgue measure of the r-neighbourhood of S. Let $r > 0$ be given. We look at the r-neighbourhood of S, denoted by $S(r)$, from two sides of

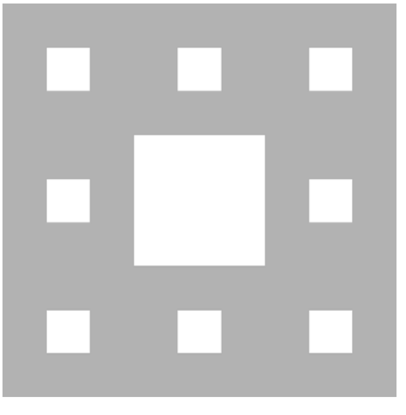

Fig. 1. First and second step in the construction of the Sierpinski carpet.

the unit square, and therefore we can divide it into two parts: $A^{(r)}$ is the r-neighbourhood of S inside the unit square S_0 (inner r-neighbourhood) and $B^{(r)}$ is the r-neighbourhood of S outside the unit square S_0 (outer r-neighbourhood), which are defined by

$$A^{(r)} := S(r) \cap S_0, \qquad B^{(r)} := S(r) \setminus S_0.$$

Note that $S(r) = A^{(r)} \cup B^{(r)}$. Furthermore, if we closely examine the inner r-neighbourhood $A^{(r)}$, we could note that there is a step in the construction of S after which the inner r-neighbourhood covers all the squares (the removed ones as well). Hence, there is the condition on n such that a full square in the nth generation is not contained in the r-neighbourhood of its boundary:

$$r < \frac{1}{2}3^{-n}.$$

It is easily shown that the largest such n is given by

$$n(r) = \left\lfloor \log_3\left(\frac{1}{2r}\right)\right\rfloor.$$

Therefore, the inner r-neighbourhood is changing in each step until the $n(r)$th step, so let $A_1^{(r)} := A_{n(r)}^{(r)} \setminus S_{n(r)}$, where $A_{n(r)}^{(r)}$ is the inner r-neighbourhood of S until the $n(r)$th step. After that step the inner r-neighbourhood is constant and we denote it by $A_2^{(r)}$. Actually, $A_2^{(r)} = S_{n(r)}$.

First, we look at the case $1 \leqslant k \leqslant n(r)$. We compute the area of the inner r-neighbourhood of any square in the kth generation to be $4r\frac{1}{3^k} - 4r^2$, so the measure of the union of all inner r-neighbourhoods of squares over generations k with $1 \leqslant k \leqslant n(r)$ is

$$\left|A_1^{(r)}\right| = \sum_{k=1}^{n(r)} 8^{k-1}\left(4r\frac{1}{3^k} - 4r^2\right) = \frac{4}{5}r\left(\frac{8}{3}\right)^{n(r)} - \frac{4}{7}r^2 8^{n(r)} - \frac{4}{5}r + \frac{4}{7}r^2.$$

After the $n(r)$th step the area of the inner r-neighbourhood $A_2^{(r)}$ is

$$\left|A_2^{(r)}\right| = 8^{n(r)}\left(\frac{1}{3^{n(r)}}\right)^2 = \left(\frac{8}{9}\right)^{n(r)}.$$

The area of outer r-neighbourhood $B^{(r)}$ is $|B^{(r)}| = 4r + r^2\pi$. Now we have

$$\begin{aligned}\left|S(r)\right| &= \left|A_1^{(r)}\right| + \left|A_2^{(r)}\right| + |B^{(r)}| \\ &= -\frac{4}{7}r^2 8^{n(r)} + \frac{4}{5}r\left(\frac{8}{3}\right)^{n(r)} + \left(\frac{8}{9}\right)^{n(r)} + \left(\frac{4}{7} + \pi\right)r^2 + \frac{16}{5}r.\end{aligned}$$

Hence,

$$\frac{|S(r)|}{r^{2-d}} = -\frac{4}{7}r^d 8^{n(r)} + \frac{4}{5}r^{d-1}\left(\frac{8}{3}\right)^{n(r)} + \left(\frac{8}{9}\right)^{n(r)} r^{d-2} + \left(\frac{4}{7} + \pi\right)r^d + \frac{16}{5}r^{d-1}.$$

We see that

$$\frac{|S(r)|}{r^{2-d}} = f(r) + O(r^{d-1}),$$

where

$$f(r) = -\frac{4}{7}r^d 8^{n(r)} + \frac{4}{5}r^{d-1}\left(\frac{8}{3}\right)^{n(r)} + \left(\frac{8}{9}\right)^{n(r)} r^{d-2}.$$

Since $d = \log_3 8 > 1$, we have that $O(r^{d-1}) = (\frac{4}{7} + \pi)r^d + \frac{16}{5}r^{d-1} \to 0$ as $r \to 0$.

Now let us fix any natural number n and choose r that satisfies

$$n < \log_3 \frac{1}{2r} \leqslant n + 1,$$

that is,

$$r \in I_n := [r_{n+1}, r_n), \quad r_n := \frac{1}{2}3^{-n}.$$

For such r note that $n(r) = n$.

We would like to find the points of minimum and maximum in I_n for the function $f_n : I_n \to \mathbb{R}$, defined by $f_n(r) = f(r)$. It is easy to show that minimum r_- and maximum r_+ are given by

$$r_\pm = \frac{1}{3^n}d_\pm,$$

where

$$d_\pm = \frac{7}{2d}\left(\frac{d-1}{5} \pm \sqrt{\frac{(d-1)^2}{25} + \frac{d(d-2)}{7}}\right).$$

An easy computation shows that the values of $f_n(r_-)$ and $f_n(r_+)$ do not depend on n. Therefore,

$$\begin{aligned}\mathcal{M}^d_*(S) &= \liminf_{r\to 0} \frac{|S(r)|}{r^{2-d}} = \liminf_{r\to 0} f(r) = f_n(r_-) \\ &= \frac{(d_-)^{d-2}}{d}\left(\frac{4d_-}{5} + 2\right) \approx 1.350670209701 \qquad (17)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^{*d}(S) &= \limsup_{r\to 0} \frac{|S(r)|}{r^{2-d}} = \limsup_{r\to 0} f(r) = f_n(r_+) \\ &= \frac{(d_+)^{d-2}}{d}\left(\frac{4d_+}{5} + 2\right) \approx 1.355617082261. \qquad (18)\end{aligned}$$

Oscillating nature of the function $f(r)$ near $r = 0$ can nicely be seen by plotting its graph, see Fig. 2.

(b) The first claim follows from Theorem 3, using nondegeneracy of Minkowski contents in (a). The asymptotic behaviour of I_r follows immediately from [11, estimate (3.4), using (3.5)].

(c) This follows from the proof of [11, Theorem 2]. $\square$

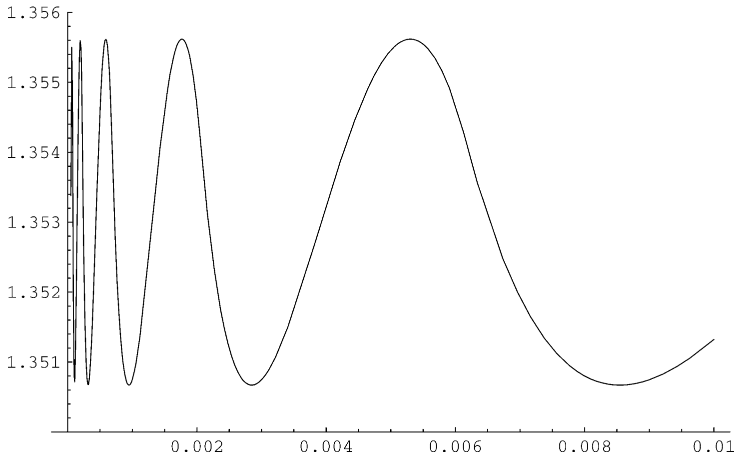

Fig. 2. Graph of the function $f(r)$ near $r = 0$.

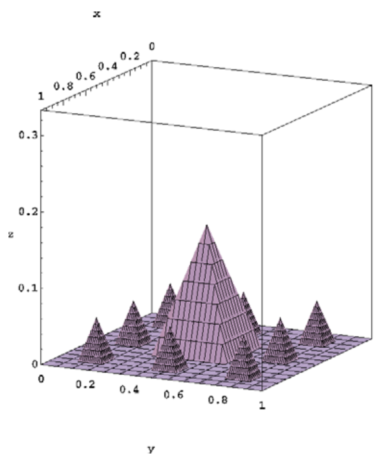

Fig. 3. Distance function from the Sierpinski carpet S (two iterations are shown).

Remark 6. From (16) we can see that the Sierpinski carpet is not Minkowski measurable, that is, $\mathcal{M}_*^d(S) < \mathcal{M}^{*d}(S)$. This fact is known, see Lapidus and van Frankenhuysen [7, p. 141].

It is interesting to visualize the distance function $x \mapsto d(x, A)$ from the Sierpinski carpet A, see Fig. 3.

The graph of the corresponding singular function $d(x, S)^{-\gamma}$ defined on the unit square in $\mathbb{R}^2$ is exhibited in Fig. 4. Visualization of the analogous graph of the function $d(x, A)^{-\gamma}$, $x \in (0, 1)$, where A is Cantor's set, can be seen in [13, p. 1286].

Example 2. In the construction of Sobolev function $G_\alpha * f \in L^{\alpha,p}(\mathbb{R}^N)$, $p > 1$, $\alpha p < N$, generated by (9), where the functions f_k are described in Example 1, instead of Cantor

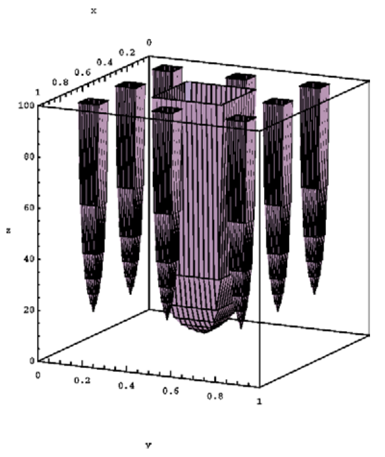

Fig. 4. The function $x \mapsto d(x, S)^{-\gamma}$ (only two iterations are shown) is Lebesgue integrable on the square $(0, 1) \times (0, 1)$ if and only if $\gamma < 2 - \log 8/\log 3$.

grills $A_k := C^{(a_k)} \times [0, 1]^{m_k}$ generating functions f_k we can consider for instance sets of the form $A_k := S \times C^{(a_k)} \times [0, 1]^{m_k}$. Here S is the Sierpinski carpet, and $a_k \in (0, 1/2)$, $m_k \in \mathbb{N}$ are suitably chosen constants, so that $f \in L^p(\mathbb{R}^N)$ and $\dim_H(\mathrm{Sing}(G_\alpha * f)) = N - \alpha p$.

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