Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Math. Anal. Appl. 323 (2006) 119–130
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A description of norm-convergent martingales on vector-valued L^p-spaces

Stuart F. Cullender *, Coenraad C.A. Labuschagne [1]

School of Mathematics, University of the Witwatersrand, Private Bag 3, PO WITS 2050, South Africa

Received 8 August 2005
Available online 16 November 2005
Submitted by J. Diestel

Abstract

Norm-convergent martingales on tensor products of Banach spaces are considered in a measure-free setting. As a consequence, we obtain the following characterization for convergent martingales on vector-valued L^p-spaces: Let (Ω, Σ, μ) be a probability space, X a Banach space and (Σ_n) an increasing sequence of sub σ-algebras of Σ. In order for $(f_n, \Sigma_n)_{n=1}^{\infty}$ to be a convergent martingale in $L^p(\mu, X)$ $(1 \leqslant p < \infty)$ it is necessary and sufficient that, for each $i \in \mathbb{N}$, there exists a convergent martingale $(x_i^{(n)}, \Sigma_n)_{n=1}^{\infty}$ in $L^p(\mu)$ and $y_i \in X$ such that, for each $n \in \mathbb{N}$, we have

$$f_n(s) = \sum_{i=1}^{\infty} x_i^{(n)}(s) y_i \quad \text{for all } s \in \Omega,$$

where $\| \sum_{i=1}^{\infty} |\lim_{n\to\infty} x_i^{(n)}| \|_{L^p(\mu)} < \infty$ and $\lim_{i\to\infty} \|y_i\| \to 0$.

Keywords: Filtration; Convergent martingale; Banach lattice; Banach space

1. Introduction

Classical martingale theory in vector-valued L^p-spaces has proved to be a useful tool in the study of the geometry of Banach spaces (see [4–6]). In this work, we are concerned with provid-

* Corresponding author.
E-mail address: scullender@maths.wits.ac.za (S.F. Cullender).
[1] This research was funded by the John Knopfmacher Centre for Applicable Analysis and Number Theory.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.10.032

ing a representation of convergent vector-valued martingales in terms of convergent scalar-valued martingales and constant sequences from the underlying Banach space.

Let (Ω, Σ, μ) denote a probability space. Then, for $1 \leqslant p < \infty$ and X a Banach space, let $L^p(\mu, X)$ denote the space of (classes of a.e. equal) Bochner p-integrable functions $f : \Omega \to X$ and denote the Bochner norm on $L^p(\mu, X)$ by Δ_p, i.e.,

$$\Delta_p(f) = \left(\int_\Omega \|f\|_X^p \, d\mu \right)^{1/p}.$$

If Σ_1 is a sub σ-algebra of Σ, the *conditional expectation* of $f \in L^p(\mu, X)$ relative to Σ_1, denoted by $\mathbb{E}(f \mid \Sigma_1)$, is a Σ_1-measurable element of $L^p(\mu, X)$ which is given by

$$\int_A \mathbb{E}(f \mid \Sigma_1) \, d\mu = \int_A f \, d\mu \quad \text{for all } A \in \Sigma_1. \tag{1.1}$$

A monotone increasing sequence (Σ_n) of sub σ-algebras of Σ is called a *filtration*. For a filtration (Σ_n) and $n \leqslant m$, it follows from (1.1) that

$$\mathbb{E}(\cdot \mid \Sigma_n) = \mathbb{E}\big(\mathbb{E}(\cdot \mid \Sigma_m) \mid \Sigma_n\big) = \mathbb{E}\big(\mathbb{E}(\cdot \mid \Sigma_n) \mid \Sigma_m\big),$$

which implies $\mathcal{R}(\mathbb{E}(\cdot \mid \Sigma_n)) \subseteq \mathcal{R}(\mathbb{E}(\cdot \mid \Sigma_m))$. Here, we use the notation $\mathcal{R}(T)$ to denote the range of a function T.

If (Σ_n) is a filtration, a sequence (f_n) in $L^p(\mu, X)$ is called a *martingale* relative to (Σ_n) if each f_n is Σ_n-measurable and

$$\mathbb{E}(f_m \mid \Sigma_n) = f_n \quad \text{for all } n \leqslant m.$$

A martingale (f_n) is *norm-convergent* if there exists $f \in L^p(\mu, X)$ such that $\Delta_p(f - f_n) \to 0$ as $n \to \infty$. From this point on we shall simply refer to a norm-convergent martingale as *convergent*.

Our aim is to prove the following result which describes convergent martingales on $L^p(\mu, X)$.

Theorem 1.1. *Let (Ω, Σ, μ) denote a probability space, $(\Sigma_n)_{n=1}^\infty$ a filtration, X a Banach space and $1 \leqslant p < \infty$. Then, in order for $(f_n, \Sigma_n)_{n=1}^\infty$ to be a convergent martingale in $L^p(\mu, X)$, it is necessary and sufficient that, for each $i \in \mathbb{N}$, there exists a convergent martingale $(x_i^{(n)}, \Sigma_n)_{n=1}^\infty$ in $L^p(\mu)$ and $y_i \in X$ such that, for each $n \in \mathbb{N}$, we have*

$$f_n(s) = \sum_{i=1}^\infty x_i^{(n)}(s) y_i \quad \textit{for all } s \in \Omega,$$

where

$$\left\| \sum_{i=1}^\infty \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\|_{L^p(\mu)} < \infty \quad \textit{and} \quad \lim_{i\to\infty} \|y_i\| \to 0.$$

Theorem 1.1 is a special case of Theorem 5.3 below, and its proof is given at the end of Section 5.

2. Preliminaries

It is well known that $L^p(\mu, X)$ is isometrically isomorphic to the norm completion $L^p(\mu) \mathbin{\widetilde{\otimes}_{\Delta_p}} X$ of $L^p(\mu) \otimes_{\Delta_p} X$. An isometric isomorphism can be established in the following way (see [1–4]): Define $\psi : L^p(\mu) \times X \to L^p(\mu, X)$ by $\psi(g, x)(s) = g(s)x$ for all $s \in \Omega$.

Then ψ is bilinear. Hence, there is a unique linear map $\Psi : L^p(\mu) \otimes X \to L^p(\mu, X)$ for which $\Psi \circ \otimes = \psi$. The image of the step functions

$$S(\mu) \otimes X = \left\{ \sum_{k=1}^{n} \chi_{A_k} \otimes x_k \colon n \in \mathbb{N},\ \chi_{A_k} \text{integrable},\ x_k \in X \right\}$$

under Ψ is the set of X valued step functions in $L^p(\mu, X)$, denoted by

$$S_p(X) := \left\{ \sum_{k=1}^{n} x_k \chi_{A_k} \colon n \in \mathbb{N},\ \chi_{A_k} \text{ integrable},\ x_k \in X \right\}.$$

Consequently, $L^p(\mu) \otimes X$ is dense in $L^p(\mu, X)$; i.e., the continuous extension $\widetilde{\Psi} : L^p(\mu) \widetilde{\otimes}_{\Delta_p} X \to L^p(\mu, X)$ of Ψ is a surjective isometry.

For the convenience of the reader, we recall the construction of a conditional expectation on $L^p(\mu, X)$, as can be found in [4], in terms of the above completed tensor product. Let Σ_1 be a sub σ-algebra of Σ. Define $\mathbb{E}(\cdot \mid \Sigma_1) : S_p(X) \to S_p(X)$ by

$$\mathbb{E}\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i \,\middle|\, \Sigma_1 \right) = \left(\mathbb{E}(\cdot \mid \Sigma_1) \otimes \mathrm{id}_X \right) \left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i \right),$$

where $\mathbb{E}(\chi_{A_i} \mid \Sigma_1)$ denotes the conditional expectation of $\chi_{A_i} \in L^p(\mu)$. By Jensen's inequality, it can be shown that

$$\Delta_p\left(\mathbb{E}\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i \,\middle|\, \Sigma_1 \right) \right) \leqslant \Delta_p\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i \right).$$

The *conditional expectation* $\mathbb{E}(\cdot \mid \Sigma_1) : L^p(\mu, X) \to L^p(\mu, X)$, of $f \in L^p(\mu, X)$ relative to Σ_1, is defined as the continuous extension of the operator $\mathbb{E}(\cdot \mid \Sigma_1) \otimes \mathrm{id}_X$ on $S_p(X)$ to $L^p(\mu, X)$; it satisfies (1.1) and is a contractive projection.

We assume that the reader is familiar with the basic concepts and notation of Banach spaces, Banach lattices, vector-valued L^p-spaces and Riesz spaces as can be found in [4,13–15,17].

3. Martingales on Banach spaces

A measure-free approach to martingales on Banach lattices is studied in [16], where it is observed that if F is a Banach lattice, then $L^p(\mu, F)$ is also a Banach lattice, which grants access to this approach. However, for a general Banach space X, $L^p(\mu, X)$ is not a Banach lattice, in which case, a large portion of [16] is not applicable. We introduce the notion of a martingale on a Banach space and present a characterization of norm-convergence of such martingales.

Motivated by the above discussion and the measure-free approaches to martingales on Banach lattices and Riesz spaces given in [10,16], we introduce the following.

Definition 3.1. Let X be a Banach space.

(a) If $T_i : X \to X$ is a contractive projection and $T_{i \wedge j} = T_i T_j$ for each $i, j \in \mathbb{N}$, then the sequence of projections (T_i) is called a *BS-filtration* on X.
(b) If (T_i) is a BS-filtration on X, then (f_i, T_i) is called a *martingale* on X if $T_i f_j = f_i$ for all $i \leqslant j$.

Notice in the above definition, for $i \leqslant j$, we have that $\mathcal{R}(T_i) \subseteq \mathcal{R}(T_j)$. If X is a Banach space and (T_i) a BS-filtration on X, let

$$M(X, T_i) = \big\{(f_i, T_i)\colon\ (f_i, T_i) \text{ is a martingale on } X\big\}.$$

Then $M(X, T_i)$ is a vector space if we define $(f_i, T_i) + (g_i, T_i) = (f_i + g_i, T_i)$ and $\lambda(f_i, T_i) = (\lambda f_i, T_i)$ for all $\lambda \in \mathbb{R}$. The map $\Theta : M(X, T_i) \to X^{\mathbb{N}}$, defined by $\Theta((f_i, T_i)) = (f_i)$, is a linear injection. It is well known that the space of all norm bounded sequences on X, denoted by $\ell_\infty(X) := \{(x_i) \in X^{\mathbb{N}}\colon \sup \|x_i\| < \infty\}$, is a Banach space with respect to the norm $\|(x_i)\| := \sup_i \|x_i\|$. Define $\|\cdot\|$ on $M(X, T_i)$ by $\|(f_i, T_i)\| = \sup_i \|f_i\|$ and let $\mathcal{M}(X, T_i)$ denote the space of *norm bounded martingales* on X; i.e.,

$$\mathcal{M}(X, T_i) = \big\{(f_i, T_i) \in M(X, T_i)\colon\ \big\|(f_i, T_i)\big\| < \infty\big\}.$$

Then $\mathcal{M}(X, T_i)$ is a normed space with respect to $\|\cdot\|$. It is readily verified that Θ is an isometry from $\mathcal{M}(X, T_i)$ into $\ell_\infty(X)$ and that $\mathcal{M}(X, T_i)$ is norm complete.

Let $I_i = \mathrm{id}_X$ for all $i \in \mathbb{N}$, where id_X denotes the identity map on X. Then (I_i) is a BS-filtration on X and

$$(f_i, I_i) \in \mathcal{M}(X, I_i) \quad \Longleftrightarrow \quad (f_i) \text{ is a constant sequence in } X.$$

If we define $\Psi : X \to \mathcal{M}(X, I_i)$ by $\Psi(f) = (f_i, I_i)$, where $f = f_i$ for all $i \in \mathbb{N}$, then X is isometrically isomorphic to $\mathcal{M}(X, I_i)$.

Let $\mathcal{M}_{\mathrm{nc}}(X, T_i)$ denote the space of *norm convergent martingales* on X; i.e.,

$$\mathcal{M}_{\mathrm{nc}}(X, T_i) = \big\{(f_i, T_i) \in \mathcal{M}(X, T_i)\colon\ (f_i) \text{ is norm convergent in } X\big\}.$$

To describe $\mathcal{M}_{\mathrm{nc}}(X, T_i)$, we use the following analogue of a familiar result from the vector-valued L^p-setting (see [4, Chapter 5, §2, Corollary 2]).

Proposition 3.2. *Let X be a Banach space and let (T_i) be a BS-filtration on X. Then $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ if and only if $\|T_i f - f\| \to 0$, where $\overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ denotes the norm closure of $\bigcup_{i=1}^\infty \mathcal{R}(T_i)$ in X.*

Proof. Suppose that $\lim_{i\to\infty} T_i f = f$. It is evident that $T_i f \in \mathcal{R}(T_i)$ for each $i \in \mathbb{N}$ so that $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$. Conversely, suppose that $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$. Then there exists a sequence $(f_n) \subseteq \bigcup_{i=1}^\infty \mathcal{R}(T_i)$ such that $\lim_{n\to\infty} f_n = f$. Thus, for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ so that $\|f_n - f\| < \varepsilon/2$. Since (T_i) is a filtration on X, there exists an $I_n \in \mathbb{N}$ such that $i \geqslant I_n$ implies $f_n \in \mathcal{R}(T_i)$. Hence, $\|T_i f - f\| \leqslant \|T_i f - f_n\| + \|f_n - f\| = \|T_i(f - f_n)\| + \|f_n - f\| \leqslant \|f - f_n\| + \|f_n - f\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ completes the proof. □

Corollary 3.3. *Let X be a Banach space and let (f_i, T_i) be a martingale in X. Then (f_i, T_i) converges to f if and only if $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ and $f_i = T_i f$ for all $i \in \mathbb{N}$.*

Proof. Suppose (f_i, T_i) converges to f, then it is clear that $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$. Also, for $i \leqslant j$, we have $T_i f_j = f_i$ so that $\lim_{j\to\infty} T_i f_j = T_i f = f_i$. Conversely, by the above proposition, we have $\|T_i f - f\| = \|f_i - f\| \to 0$, which completes the proof. □

Proposition 3.4. *Let X be a Banach space, (T_i) be a BS-filtration on X. Then $L : \mathcal{M}_{\mathrm{nc}}(X, T_i) \to \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$, defined by $L((f_i, T_i)) = \lim_i f_i$, is a surjective isometry.*

Proof. It follows easily that L is well defined, linear and

$$\|L\| = \sup\{\|L(f_i, T_i)\|: \|(f_i, T_i)\| \leqslant 1\} \leqslant 1.$$

To see that L is a surjection, let $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Then $T_i f \to f$ in norm and $(T_i f, T_i)$ is a martingale on X such that $L((T_i f, T_i)) = f$. Also, $L((f_i, T_i)) = 0$ implies that $\lim_i f_i = 0$ and Corollary 3.3 assures us that $f_i = T_i 0 = 0$ for each $i \in \mathbb{N}$. Thus, it follows that L is injective. Furthermore,

$$\|L^{-1}\| = \sup\{\|L^{-1} f\|: \|f\| \leqslant 1\} = \sup\Big\{\sup_i \|T_i f\|: \|f\| \leqslant 1\Big\} \leqslant 1,$$

which completes the proof that L is a surjective isometry. □

Corollary 3.5. *Let (T_i) be a BS-filtration on a Banach space X. Then $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = X$ if and only if $\mathcal{M}_{\mathrm{nc}}(X, T_i)$ is isometrically isomorphic to $\mathcal{M}(X, I_i)$ where $I_i = \mathrm{id}_X$ for all $i \in \mathbb{N}$.*

Proof. This can easily been seen from the fact that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \mathcal{M}_{\mathrm{nc}}(X, T_i)$ and $X = \mathcal{M}(X, I_i)$. □

We define an ordering on the space of martingales defined on a Banach lattice.

Definition 3.6. Let E be a Banach lattice. If (T_i) is a BS-filtration on a Banach lattice E such that each $T_i \geqslant 0$ and (f_i, T_i) is a martingale, define

$$(f_i, T_i) \geqslant 0 \quad \Longleftrightarrow \quad f_i \geqslant 0 \quad \text{for all } i \in \mathbb{N}.$$

Proposition 3.7. *Let E be a Banach lattice and (T_i) a BS-filtration on E for which each $T_i \geqslant 0$ and $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is a closed Riesz subspace of E. If $L : \mathcal{M}_{\mathrm{nc}}(E, T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is defined by $L((f_i, T_i)) = \lim_i f_i$, then $\mathcal{M}_{\mathrm{nc}}(E, T_i)$ is a Banach lattice and $L : \mathcal{M}_{\mathrm{nc}}(E, T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is a surjective Riesz isometry.*

Proof. It was shown in Proposition 3.4 that L is a surjective isometry. To see that L is positive is trivial, because if $(f_i, T_i) \geqslant 0$, then $f_i \geqslant 0$ for each $i \in \mathbb{N}$ and $\lim_i f_i \geqslant 0$. Similarly, L^{-1} is positive, because if $0 \leqslant f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ then $T_i f \geqslant 0$ for each $i \in \mathbb{N}$; hence, $L^{-1}(f) = (T_i f, T_i) \geqslant 0$.

Since $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is a Riesz space, it follows that $\mathcal{M}_{\mathrm{nc}}(E, T_i)$ is also a Riesz space. Indeed, for $f, g \in \mathcal{M}_{\mathrm{nc}}(E, T_i)$, it is readily verified that $L^{-1}(L(f) \vee L(g))$ is the least upper bound in $\mathcal{M}_{\mathrm{nc}}(E, T_i)$ of $\{f, g\}$.

Thus, by the preceding part, L is a surjective Riesz isometry. Since $\|\cdot\|_E$ is a Riesz norm, the martingale norm is also a Riesz norm. Furthermore, since $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is a Banach lattice, $\mathcal{M}_{\mathrm{nc}}(E, T_i)$ is a Banach lattice. □

For further reading on the space of bounded martingales defined on a Banach lattice, see [16].

4. Filtrations on the l-tensor product

If X and Y are Banach spaces and α is a norm on $X \otimes Y$, we denote the normed space $(X \otimes Y, \alpha)$ by $X \otimes_\alpha Y$, its norm completion by $X \widetilde{\otimes}_\alpha Y$ and its continuous dual by $(X \otimes_\alpha Y)'$.

The norm of an element $u \in X \widetilde{\otimes}_\alpha Y$ will be denoted $\alpha_{X,Y}(u)$ when there is a need to distinguish the Banach spaces involved or simply $\alpha(u)$ if there is no risk of ambiguity. A norm α on $X \otimes Y$ is called a *reasonable cross norm* (cf. [2–4,9]) if α satisfies the conditions:

(a) For $x \in X$ and $y \in Y$, $\alpha(x \otimes y) \leqslant \|x\| \, \|y\|$.
(b) For $x' \in X'$ and $y' \in Y'$, $x' \otimes y' \in (X \otimes_\alpha Y)'$ and $\|x' \otimes y'\| \leqslant \|x'\| \, \|y'\|$.

It is well known that the inequalities in (a) and (b) may be replaced by equality.

Let X, X_0, Y and Y_0 be Banach spaces. If $S : X_0 \to X$ and $T : Y_0 \to Y$ are bounded linear maps, then a reasonable cross norm α is called a *uniform cross norm* if $S \otimes T : X_0 \otimes_\alpha Y_0 \to X \otimes_\alpha Y$ satisfies

$$\|S \otimes T\| \leqslant \|S\| \, \|T\|.$$

Since the inequality $\|S \otimes T\| \geqslant \|S\|\|T\|$ holds for all reasonable cross norms α, equality holds in the definition of uniform cross norms. In the case where X_0 is a closed subspace of X, Y_0 is a closed subspace of Y and α is a uniform cross norm, we have that $\alpha_{X_0,Y_0}(u) \leqslant \alpha_{X,Y}(u)$. This inequality can be strict and thus $E_0 \widetilde{\otimes}_\alpha Y_0$ need not be a subspace of $E \widetilde{\otimes}_\alpha Y$. A uniform cross norm for which $\alpha_{X_0,Y_0}(u) = \alpha_{X,Y}(u)$ holds for each closed subspace X_0 of X and Y_0 of Y is called *injective*.

Pisier noted that the Bochner norm Δ_p is not an injective uniform cross norm for $1 < p < \infty$ (see [2, p. 147]). However, for $1 \leqslant p < \infty$, it is known the Bochner norm Δ_p on $L^p(\mu, X)$ has the property that if $0 \leqslant S : L^p(\mu) \to L^p(\mu)$ (note that any positive operator between Banach lattices is bounded, thus S is also bounded) and $T : X \to X$ is a bounded map, then $S \otimes T : L^p(\mu, X) \to L^p(\mu, X)$ has the property that

$$\|S \otimes T\| = \|S\| \, \|T\| \tag{4.1}$$

(see [4,12]).

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice and a Banach space (see [1,15]). If E is a Banach lattice and Y is a Banach space, then the l-norm of $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$ is given by

$$\|u\|_l = \inf\left\{\left\|\sum_{i=1}^n \|y_i\| \, |x_i|\right\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

Furthermore, if $E = L^p(\mu)$ where (Ω, Σ, μ) is a σ-finite measure space, then we have that $E \widetilde{\otimes}_l Y$ is isometric to $L^p(\mu, Y)$.

If X is a Banach space and F is a Banach lattice, then the transpose of the l-norm is called the m-norm and is given by the formula

$$\|u\|_m = \inf\left\{\left\|\sum_{i=1}^n \|x_i\| \, |y_i|\right\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}$$

for all $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes F$.

Property (4.1) extends to the l-tensor and the m-tensor products as stated below; proofs of which may be found in [12]:

Let E_1 and E_2 be Banach lattices and Y_1 and Y_2 Banach spaces. If $S : E_1 \to E_2$ is a positive linear operator and $T : Y_1 \to Y_2$ a bounded linear operator, then

$$\|(S \otimes T)u\|_l \leqslant \|S\| \, \|T\| \, \|u\|_l \quad \text{for all } u \in E_1 \otimes Y_1.$$

Let X_1 and X_2 be Banach spaces and F_1 and F_2 Banach lattices. If $S : X_1 \to X_2$ is a bounded linear operator and $T : F_1 \to F_2$ a positive linear operator, then

$$\left\|(S \otimes T)u\right\|_m \leqslant \|S\|\,\|T\|\,\|u\|_m \quad \text{for all } u \in X_1 \otimes F_1.$$

In [12] it is shown that if E and E_0 are Banach lattices, Y and Y_0 are Banach spaces, $S : E_0 \to E$ is a Riesz isometry and $T : Y_0 \to Y$ is an isometry, then both $S \otimes_l \mathrm{id}_Y : E_0 \widetilde{\otimes}_l Y \to E \widetilde{\otimes}_l Y$ and $\mathrm{id}_E \otimes_l T : E \widetilde{\otimes}_l Y_0 \to E \widetilde{\otimes}_l Y$ are isometries. It now follows from the fact that $S \otimes_l \mathrm{id}_{Y_0} : E_0 \widetilde{\otimes}_l Y_0 \to E \widetilde{\otimes}_l Y_0$ is an isometry that the composition

$$S \otimes_l T = (\mathrm{id}_E \otimes_l T)(S \otimes_l \mathrm{id}_{Y_0}) : E_0 \widetilde{\otimes}_l Y_0 \to E \widetilde{\otimes}_l Y$$

is also an isometry. Thus the l-norm exhibits a weaker form of injectivity: if E_0 is a closed Riesz subspace of E and Y_0 is a closed subspace of Y, then $E_0 \widetilde{\otimes}_l Y_0$ is a closed subspace of $E \widetilde{\otimes}_l Y$.

A symmetrical statement also holds for the m-norm. These properties motivate the following definition.

Definition 4.1. If E and E_0 are Banach lattices, Y and Y_0 are Banach spaces, $0 \leqslant S : E_0 \to E$ and $T : Y_0 \to Y$ are bounded linear maps, then a reasonable cross norm α is called

(a) *left order uniform* (or in short, *left uniform*) if $\|S \otimes T\| \leqslant \|S\|\|T\|$;
(b) *left order injective* (or in short, *left injective*) if $S \otimes T : E_0 \widetilde{\otimes}_\alpha Y_0 \to E \widetilde{\otimes}_\alpha Y$ is an isometry, provided that S is a Riesz isometry and T is an isometry.

The notions of a *right order uniform cross norm* and a *right order injective cross norm* are defined in a symmetrical manner.

Lemma 4.2.

(a) *Let E be a Banach lattice and Y a Banach space. If α is a left uniform, left injective cross norm on $E \otimes Y$, $0 \leqslant S : E \to E$ and $T : Y \to Y$ are bounded projections respectively and $\mathcal{R}(S)$ is a (closed) Riesz subspace of E, then $S \otimes_\alpha T : E \widetilde{\otimes}_\alpha Y \to E \widetilde{\otimes}_\alpha Y$ is a bounded projection with range $S(E) \widetilde{\otimes}_\alpha T(Y)$, which is a closed subspace of $E \widetilde{\otimes}_\alpha Y$.*
(b) *A symmetrical result holds if α is a right uniform, right injective cross norm on $E \otimes Y$.*

Proof. Since α is a left order uniform cross norm, it follows that $\|S \otimes T\| = \|S\|\,\|T\|$; consequently, the continuous extension $S \otimes_\alpha T : E \widetilde{\otimes}_\alpha Y \to E \widetilde{\otimes}_\alpha Y$ is bounded. To see that $S \otimes_\alpha T$ is a projection, let $u \in E \widetilde{\otimes}_\alpha Y$. Then there exists a sequence $(u_j) \subseteq E \otimes Y$ such that $u_j \to u$ in norm. Representing each u_j as $\sum_{i=1}^{n_j} x_i^{(j)} \otimes y_i^{(j)}$, we conclude that

$$(S \otimes_\alpha T)^2(u_j) = \sum_{i=1}^{n_j} S^2\big(x_i^{(j)}\big) \otimes T^2\big(y_i^{(j)}\big) = \sum_{i=1}^{n_j} S\big(x_i^{(j)}\big) \otimes T\big(y_i^{(j)}\big) = (S \otimes_\alpha T)(u_j).$$

By the continuity of $S \otimes_\alpha T$, it follows that $(S \otimes_\alpha T)^2(u) = (S \otimes_\alpha T)(u)$. As $S(E)$ is a closed Riesz subspace of E and $T(Y)$ a closed subspace of Y, the left order injectivity of the α-norm gives

$$(S \otimes_\alpha T)(E \otimes Y) = S(E) \otimes T(Y) \subseteq S(E) \widetilde{\otimes}_\alpha T(Y) \hookrightarrow E \widetilde{\otimes}_\alpha Y \quad \text{(isometrically)}.$$

Thus, $S(E) \otimes T(Y) \subseteq (S \otimes_\alpha T)(E \widetilde{\otimes}_\alpha Y) \subseteq S(E) \widetilde{\otimes}_\alpha T(Y)$. As $S \otimes_\alpha T$ is a bounded projection and thus has closed range, it follows that $(S \otimes_\alpha T)(E \widetilde{\otimes}_\alpha Y) = S(E) \widetilde{\otimes}_\alpha T(Y)$. $\square$

The preceding lemma and the fact that the range of a classical conditional expectation on $L^p(\mu)$ is a closed Riesz subspace motivates the following definition.

Definition 4.3. Let E be a Banach lattice. A BS-filtration (T_i) on E for which each $T_i \geqslant 0$ and $\mathcal{R}(T_i)$ is a closed Riesz subspace of E is called a *BL-filtration* on E.

Theorem 4.4.

(a) *Let E be a Banach lattice and Y a Banach space. If α is a left uniform, left injective cross norm on $E \otimes Y$, (S_i) is a BL-filtration on E and (T_i) is a BS-filtration on Y, then $(S_i \otimes_\alpha T_i)$ is a BS-filtration on $E \widetilde{\otimes}_\alpha Y$.*
(b) *A symmetrical result holds if α is a right uniform, right injective cross norm on $E \otimes Y$.*

Proof. The proof is a simple application of Lemma 4.2. □

Our development of filtrations on tensor products of Banach lattices uses properties of Fremlin's tensor product of Archimedean Riesz spaces. Thus, we give a brief account of it, as required in the sequel.

Let E and F be Archimedean Riesz spaces. We denote the *projective cone* of $E \otimes F$ by

$$E_+ \otimes F_+ := \left\{ \sum_{i=1}^{n} x_i \otimes y_i \colon (x_i, y_i) \in E_+ \times F_+,\ n \in \mathbb{N} \right\}.$$

D.H. Fremlin [7] constructed an Archimedean Riesz space $E \overline{\otimes} F$ with the following properties:

(RBi) If $(x, y) \in E \times F$, then $|x| \otimes |y| = |x \otimes y|$ in $E \overline{\otimes} F$.
(F) If G is any Archimedean Riesz space such that $E \otimes F$ is a vector subspace of G and $|x| \otimes |y| = |x \otimes y|$ in G for all $(x, y) \in E \times F$, then $E \overline{\otimes} F$ is the Riesz subspace of G generated by $E \otimes F$.
(SS) If E_0 and F_0 are Riesz subspaces of E and F respectively, then $E_0 \overline{\otimes} F_0$ is a Riesz subspace of $E \overline{\otimes} F$.
(ru-D)$_+$ If $z \in (E \overline{\otimes} F)_+$, then there exists $(x, y) \in E_+ \times F_+$ with the property that for each $\varepsilon > 0$ there exists $v_\varepsilon \in E_+ \otimes F_+$ such that $|z - v_\varepsilon| \leqslant \varepsilon x \otimes y$; moreover, $v_\varepsilon \in E_+ \otimes F_+$ may be chosen such that $v_\varepsilon \leqslant z$ (see [8]).

Let E and F be Banach lattices. We are interested in those reasonable cross norms α on $E \otimes F$ which have extensions to $E \overline{\otimes} F$ in such a way that (the extension of) α is a Riesz norm on $E \overline{\otimes} F$. Such reasonable cross norms are called *order reasonable cross norms*. It is known that if α is an order reasonable cross norm on $E \otimes F$, then $E \overline{\otimes} F$ is a dense Riesz subspace of the Banach lattice $E \widetilde{\otimes}_\alpha F$ with positive cone $(E \widetilde{\otimes}_\alpha F)_+$, which is the α-closure of $E_+ \otimes F_+$ (see [11]).

A left (right) uniform cross norm that is also an order reasonable cross norm will be referred to as a *left* (*right*) *uniform Riesz cross norm*. A left (right) injective cross norm that is also an order reasonable cross norm will be referred to as a *left* (*right*) *injective Riesz cross norm*.

The l-norm (m-norm) is an example of a left (right) uniform, left (right) injective Riesz cross norm (see [12]).

Lemma 4.5.

(a) *Let E and F be Banach lattices. If α is a left uniform, left injective Riesz cross norm on $E \otimes F$, $S : E \to E$ and $T : F \to F$ are positive contractive projections with ranges (closed) Riesz subspaces of E and F respectively, then $(S \otimes_\alpha T) : E \widetilde{\otimes}_\alpha F \to E \widetilde{\otimes}_\alpha F$ is a positive contractive projection with range $S(E) \widetilde{\otimes}_\alpha T(F)$, which is a closed Riesz subspace of $E \widetilde{\otimes}_\alpha F$.*

(b) *A symmetrical result holds if α is a right uniform, right injective Riesz cross norm on $E \otimes F$.*

Proof. By Lemma 4.2, it suffices to show that $S \otimes_\alpha T \geqslant 0$ with range $S(E) \widetilde{\otimes}_\alpha T(F)$ a Riesz subspace of $E \widetilde{\otimes}_\alpha F$.

Since α is an order reasonable cross norm, it follows that $E \widetilde{\otimes}_\alpha F$ is a Banach lattice with $E_+ \otimes F_+$ α-dense in $(E \widetilde{\otimes}_\alpha F)_+$. Since $(S \otimes T)(E_+ \otimes F_+) \subseteq E_+ \otimes F_+$ and $S \otimes T : E \otimes_\alpha F \to E \otimes_\alpha F$ is continuous, in fact $\|S \otimes T\| = \|S\| \, \|T\|$, we get that $0 \leqslant S \otimes_\alpha T : E \widetilde{\otimes}_\alpha F \to E \widetilde{\otimes}_\alpha F$.

By the left order injectivity of α, we have that $S(E) \widetilde{\otimes}_\alpha T(F)$ is a closed subspace of $E \widetilde{\otimes}_\alpha F$. Also, by property (SS), we get that $S(E) \overline{\otimes} T(F)$ is a Riesz subspace of $E \overline{\otimes} F$ and is thus also a Riesz subspace of $E \widetilde{\otimes}_\alpha F$. Since $S(E) \overline{\otimes} T(F)$ is dense in $S(E) \widetilde{\otimes}_\alpha T(F)$, it follows that $S(E) \widetilde{\otimes}_\alpha T(F)$ is a closed Riesz subspace of $E \widetilde{\otimes}_\alpha F$. □

As an easy consequence of the above lemma, we obtain the following proposition.

Proposition 4.6.

(a) *Let E and F be Banach lattices. If α is a left uniform, left injective Riesz cross norm on $E \otimes F$, (S_i) and (T_i) BL-filtrations on E and F respectively, then $(S_i \otimes_\alpha T_i)$ is a BL-filtration on $E \widetilde{\otimes}_\alpha F$.*

(b) *A symmetrical result holds if α is a right uniform, right injective Riesz cross norm on $E \otimes F$.*

5. Convergent martingales on the l-tensor product

To prove Theorem 1.1, we first derive:

Lemma 5.1.

(a) *If (S_i) is a BL-filtration on the Banach lattice E and (T_i) is a BS-filtration on the Banach space Y, then $\overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i)} \widetilde{\otimes}_l \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)} = \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i \otimes_l T_i)}$.*

(b) *If (J_i) is a BS-filtration on the Banach space X and (K_i) is a BL-filtration on the Banach lattice F, then $\overline{\bigcup_{i=1}^\infty \mathcal{R}(J_i)} \widetilde{\otimes}_m \overline{\bigcup_{i=1}^\infty \mathcal{R}(K_i)} = \overline{\bigcup_{i=1}^\infty \mathcal{R}(J_i \otimes_m K_i)}$.*

Proof. We will prove the first equality, the second is derived similarly.

($\supseteq$): Let $y \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i \otimes_l T_i)}$ and $\varepsilon > 0$ be given. Select $y_0 \in \mathcal{R}(S_i \otimes_l T_i)$ for some $i \in \mathbb{N}$ such that $\|y - y_0\|_l < \varepsilon$. Since $\mathcal{R}(S_i \otimes_l T_i) = S_i(E) \widetilde{\otimes}_l T_i(Y)$ and $S_i(E) \widetilde{\otimes}_l T_i(Y) \subseteq \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i)} \widetilde{\otimes}_l \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ by the left order injectivity of the l-norm, it follows that $y \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i)} \widetilde{\otimes}_l \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$.

($\subseteq$): Let $y \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \widetilde{\otimes}_l \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and $\varepsilon > 0$ be given. Select $y_0 \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \otimes \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ such that $\|y - y_0\|_l < \varepsilon/2$. Let $y_0 = \sum_{i=1}^{n_0} a_i \otimes y_i$, where $a_i \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}$ and $y_i \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Select $v_i \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ such that

$$\|y_i - v_i\|_Y < \frac{\varepsilon}{4\sum_{i=1}^{n_0} \|a_i\|}$$

and select $b_i \in \bigcup_{i=1}^{\infty} \mathcal{R}(S_i)$ such that

$$\|a_i - b_i\|_E < \frac{\varepsilon}{4\sum_{i=1}^{n_0} \|v_i\|}.$$

Let $z_1 = \sum_{i=1}^{n_0} b_i \otimes v_i$. Then $z_1 \in \bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)$,

$$y_0 - z_1 = \sum_{i=1}^{n_0} \bigl(a_i \otimes (y_i - v_i) + (a_i - b_i) \otimes v_i\bigr),$$

$$\|y_0 - z_1\|_l \leqslant \left\| \sum_{i=1}^{n_0} \bigl(\|y_i - v_i\| |a_i| + \|v_i\| |a_i - b_i|\bigr) \right\|_E < \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

and

$$\|y - z_1\|_l \leqslant \|y - y_0\|_l + \|y_0 - z_1\|_l \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $y \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)}$. $\square$

Thus, one has the following distributive property:

Corollary 5.2.

(a) *If (S_i) is a BL-filtration on the Banach lattice E and (T_i) is a BS-filtration on the Banach space Y, then $\mathcal{M}_{\mathrm{nc}}(E \widetilde{\otimes}_l Y, S_i \otimes_l T_i) = \mathcal{M}_{\mathrm{nc}}(E, S_i) \widetilde{\otimes}_l \mathcal{M}_{\mathrm{nc}}(Y, T_i)$.*
(b) *If (J_i) is a BS-filtration on the Banach space X and (K_i) is a BL-filtration on the Banach lattice F, then $\mathcal{M}_{\mathrm{nc}}(X \widetilde{\otimes}_m F, J_i \otimes_m K_i) = \mathcal{M}_{\mathrm{nc}}(X, J_i) \widetilde{\otimes}_m \mathcal{M}_{\mathrm{nc}}(F, K_i)$.*

Proof. We only prove (a), since the proof for (b) is similar.

By Propositions 3.4, 3.7 and Theorem 4.4, we have that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}$ is Riesz isometric to $\mathcal{M}_{\mathrm{nc}}(E, S_i)$, $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is isometric to $\mathcal{M}_{\mathrm{nc}}(Y, T_i)$ and $\mathcal{M}_{\mathrm{nc}}(E \widetilde{\otimes}_l Y, S_i \otimes_l T_i)$ is isometric to

$\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)}$. By Lemma 5.1, we have

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)} = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)},$$

from which the above assertion is now clear. □

It is shown in [12] that, if E is a Banach lattice and Y a Banach space, then $u \in E \,\widetilde{\otimes}_l\, Y$ if and only if $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, where

$$\left\| \sum_{i=1}^{\infty} |x_i| \right\|_E < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\|_Y = 0. \tag{5.1}$$

As an easy consequence of this result and Lemma 5.1, we obtain the following main result of this section.

Theorem 5.3. *Let (S_n) be a BL-filtration on a Banach lattice E and (T_n) a BS-filtration on a Banach space Y. Then, in order for $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ to be a convergent martingale in $E \,\widetilde{\otimes}_l\, Y$, it is necessary and sufficient that, for each $i \in \mathbb{N}$, there exist convergent martingales $(x_i^{(n)}, S_n)_{n=1}^{\infty}$ and $(y_i^{(n)}, T_n)_{n=1}^{\infty}$ in E and Y respectively such that, for each $n \in \mathbb{N}$, we have*

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)},$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\| < \infty \quad \text{and} \quad \lim_{i\to\infty} \left\| \lim_{n\to\infty} y_i^{(n)} \right\| \to 0.$$

Proof. Let $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ be a convergent martingale in $E \,\widetilde{\otimes}_l\, Y$. Then by Lemma 5.1, M corresponds to an element

$$f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)} = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$$

and thus, by the remark preceding this theorem, we have $f = \sum_{i=1}^{\infty} x_i \otimes y_i$ where (5.1) holds. Then for each $n \in \mathbb{N}$, we have $f_n = (S_n \otimes_l T_n)(\sum_{i=1}^{\infty} x_i \otimes y_i)$. Now let $x_i^{(n)} := S_n(x_i)$ and $y_i^{(n)} := T_n(y_i)$ for each $i \in \mathbb{N}$. Then

$$f_n = \sum_{i=1}^{\infty} S_n(x_i) \otimes T_n(y_i) = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)},$$

where $(x_i^{(n)}, S_n)_{n=1}^{\infty}$ and $(y_i^{(n)}, T_n)_{n=1}^{\infty}$ are convergent martingales in E and Y with limits x_i and y_i respectively, so that $\| \sum_{i=1}^{\infty} |\lim_{n\to\infty} x_i^{(n)}|\| < \infty$ and $\lim_{i\to\infty} \|\lim_{n\to\infty} y_i^{(n)}\| \to 0$ hold.

Conversely, for each $i \in \mathbb{N}$, let $x_i = \lim_{n\to\infty} x_i^{(n)}$ and $y_i = \lim_{n\to\infty} y_i^{(n)}$. Then the sequences (x_i) and (y_i) satisfy (5.1) so that Lemma 5.1 implies

$$f := \sum_{i=1}^{\infty} x_i \otimes y_i \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)}.$$

Then, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)} = \sum_{i=1}^{\infty} S_n(x_i) \otimes T_n(y_i) = (S_n \otimes_l T_n) f.$$

It now follows that $M := (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ is a convergent martingale. □

Note that a symmetrical result holds for the m-norm.

Proof of Theorem 1.1. In the case where $E = L^p(\mu)$ $(1 \leqslant p < \infty)$, $S_n = \mathbb{E}(\cdot \mid \Sigma_n)$ (where (Σ_n) is a filtration in the classical sense) and $T_n = \mathrm{id}_X$ for each $n \in \mathbb{N}$, the proof of Theorem 1.1 now follows as a simple consequence of Theorem 5.3. □

References

[1] J. Chaney, Banach lattices of compact maps, Math. Z. 129 (1972) 1–19.
[2] A. Defant, K. Floret, Tensor Norms and Operator Ideals, North-Holland Math. Stud., vol. 176, North-Holland, Amsterdam, 1993.
[3] J. Diestel, J.H. Fourie, J. Swart, The metric theory of tensor products, part 1, Quaest. Math. 25 (1) (2002) 37–72.
[4] J. Diestel, J.J. Uhl, Vector Measures, Amer. Math. Soc. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[5] G.A. Edgar, L. Sucheston, Stopping Times and Directed Processes, Encyclopedia Math. Appl., Cambridge Univ. Press, 1992.
[6] L. Egghe, Stopping Time Techniques for Analysts and Probabilists, Cambridge Univ. Press, 1984.
[7] D.H. Fremlin, Tensor products of Archimedean vector lattices, Amer. J. Math. 94 (1972) 778–798.
[8] J.J. Grobler, C.C.A. Labuschagne, An f-algebra approach to the Riesz tensor product of Archimedean Riesz spaces, Quaest. Math. 12 (1989) 425–438.
[9] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Mat. São Paulo 8 (1956) 1–79.
[10] W.-C. Kuo, C.C.A. Labuschagne, B.A. Watson, Discrete-time stochastic processes on Riesz spaces, Indag. Math. (N.S.) 15 (2004) 435–451.
[11] C.C.A. Labuschagne, Riesz reasonable cross norms on tensor products of Banach lattices, Quaest. Math. 27 (2004) 243–266.
[12] C.C.A. Labuschagne, Characterizing the one-sided tensor norms Δ_p and ${}^t\Delta_p$, Quaest. Math. 27 (2004) 339–363.
[13] Y. Lindenstrauss, L. Tzafriri, Classical Banach Spaces, I and II, Springer-Verlag, 1979.
[14] P. Meyer-Nieberg, Banach Lattices, Springer-Verlag, 1991.
[15] H.H. Schaefer, Banach Lattices and Positive Operators, Springer-Verlag, 1974.
[16] V. Troitsky, Martingales in Banach lattices, Positivity, in press.
[17] A.C. Zaanen, Introduction to Operator Theory in Riesz Space, Springer-Verlag, 1997.

Further reading

[18] C. Aliprantis, O. Burkinshaw, Locally Solid Riesz Spaces, Academic Press, New York, 1978.
[19] D.H. Fremlin, On the completion of locally solid vector lattices, Pacific J. Math. 43 (1972) 341–347.
[20] N. Popa, Die Permanenzeigenschaften der Tensorprodukte von Banachverbänden, in: Romanian–Finnish Seminar on Complex Analysis, Proc., Bucharest, 1976, in: Lecture Notes in Math., vol. 743, Berlin, 1979, pp. 627–647.