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Hamiltonian flow over deformations of ordinary double points

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Abstract

We cannot see $M_t = \{(z_1, \ldots, z_n)\colon \sum_{i=1}^n |z_i|^2 = 1,\ z_1^2 + \cdots + z_n^2 = t\}$ from

$$M = \left\{(z_1, \ldots, z_n)\colon \sum_{i=1}^n |z_i|^2 = 1,\ z_1^2 + \cdots + z_n^2 = 0\right\},$$

from the point of view of the Hamiltonian mechanics, even though their CR structures are so different. Nevertheless, in this paper, by using a special kind of the Hamiltonian flow, we write down the Kodaira–Spencer class of this deformation, M_t is an element of primitive forms.

Keywords: Hamiltonian flow; CR structure; Ordinary double point; Isolated singularity; Deformation theory

The first version of this paper is written almost five years ago and distributed as a preprint. While, recently, from the point of view of symplectic geometry, CR structures and even more its deformations have been discussed. Hence we conclude that: as a concrete example of deformations CR structures, related to the Hamiltonian mechanics, this paper should be published.

Let (V, o) be an isolated singularity in a complex euclidean space (C^N, o). Let M be the intersection of this V and the hypersurface $S_\epsilon^{2N-1}(o)$, centered at o with the radius ϵ. Then, naturally, over this C^∞ manifold M, a CR structure $(M, {}^0T'')$ is induced from V. Linked with

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the deformation theory of isolated singularities, the deformation theory of CR structures has been successfully developed (see [1,2] for the case $\dim_R M = 2n-1 \geqslant 7$, and [6] for the case $\dim_R M = 5$). And as an analogy for Calabi–Yau manifolds in the complex manifolds case, a subspace of the infinitesimal deformation space, Z^1 is introduced. While in treating the deformation theory of CR structures in the case $\dim_R M = 5$ (the problem is reduced to solving a non-linear $\bar\partial_{T'}$-equation), we introduce a new differential complex in order to solve this non-linear equation. As for this complex, we find that the $\bar\partial_{T'}$ harmonic space over the T'-valued one forms is defined by the fourth order partial differential operators (which corresponds to the Zariski tangent space of infinitesimal deformations of CR structures) (see [6]). And also we find that: this set includes as a subset, $\{\phi\colon \phi \in \Gamma(M, E_1),\ \bar\partial_1\phi = 0,\ \delta''\phi = 0\}$, where δ'' is the formal adjoint operator of $\bar\partial_{T'}$, so $\bar\partial_1^*\bar\partial_1 + \bar\partial_{T'}\delta''$ is a second order partial differential operator. While, in studying an analogy for Calabi–Yau manifolds, we introduce Z^1 space (see [4,5]). By the simple computation with Kaehler identities, Z^1 corresponds to $\{\phi\colon \phi \in \Gamma(M, E_1),\ \bar\partial_1\phi = 0,\ \delta''\phi = 0\}$ (see Theorem 2.3 in this paper) if our canonical line bundle is trivial in the CR sense. We set a non-trivial smooth deformation of CR structures (Hamiltonian flow with respect to the ambient euclidean space) in the case rational double point, and from this non-trivial deformation, we construct a non-trivial Z^1-element. And as an application of this existence theorem, by using these elements, we give an affirmative answer to $d'd''$ lemma (see Lemma 5.1 in this paper), which is treated by [4], and also give a criterion (see Theorem 5.1) about the smoothness of versal family. Of course, the ratinal double point is a hypersurface singularity. While, a hypersurface singularity is unobstructed, because its canonical line bundle is trivial and $\mathbf{H}^{n-1,2}_{d''}$ (obstruction space of standard deformation theory) vanishes. However, in this paper, by the assumptions: $Z^1 = \mathbf{H}^{n-1,1}_{d''}$ (the tangent space of infinitesimal deformation and $Z^2 = \mathbf{H}^{n-2,2}_{d''}$ (we note that our assumptions are different from the vanishing of the obstruction cohomology group) (see Theorem 5.1), we see that the deformation of CR structures, in the case the rational double point, is unobstructed.

1. A CR structure and its versal family

Let $(M, {}^0T'')$ be a CR structure. This means that: M is a C^∞ differentiable manifold, and ${}^0T''$ is a complex subbundle of the complexfied tangent bundle $C \otimes TM$, satisfying

$$ {}^0T'' \cap {}^0T' = 0, \qquad \dim_C \frac{C \otimes TM}{{}^0T'' + {}^0T'} = 1, \tag{1.1} $$

$$ \left[\Gamma\left(M, {}^0T''\right), \Gamma\left(M, {}^0T''\right)\right] \subset \Gamma\left(M, {}^0T''\right), \tag{1.2} $$

where ${}^0T' = \overline{{}^0T''}$. We assume that there is a real vector field ξ on M satisfying for every point p of M,

$$ \xi_p \notin {}^0T''_p + {}^0T'_p. $$

We call this ξ a supplement vector field. By using this ξ, we set a C^∞ vector bundle decomposition

$$ C \otimes TM = {}^0T'' + {}^0T' + C \otimes \xi. \tag{1.3} $$

Henceforth we use the notation

$$ T' = {}^0T' + C \otimes \xi, $$

and we have the Levi form by: for $X, Y \in {}^0T''$,

$$g(X,Y) = -\sqrt{-1}[X,\overline{Y}]_{C\otimes\xi},$$

where $[X,\overline{Y}]_{C\otimes\xi}$ means the coefficient of $[X,\overline{Y}]$ with respect to ξ according to (1.3). If this Levi form is positive definite or negative definite, then our CR structure is called strongly pseudoconvex. Now we assume that $(M, {}^0T'')$ is strongly pseudoconvex. More we assume that M is compact and $\dim_R M = 2n-1 \geqslant 5$. Under these assumptions, we studied the deformation theory of CR structures of $(M, {}^0T'')$ in our series of papers [1,2,6], and we have a versal a family of deformations of CR structures. We, briefly recall the results. First, we recall $\bar\partial_{T'}$ operator, and the standard deformation complex. $\bar\partial_{T'}$ operator is defined as follows. For $u \in \Gamma(M,T')$, we set an element $\bar\partial_{T'}u$ of $\Gamma(M, T'\otimes({}^0T'')^*)$ by

$$\bar\partial_{T'}u(X) = [X,u]_{T'}, \quad \text{for } X \in \Gamma\big(M, {}^0T''\big).$$

This is a first order differential operator from $\Gamma(M,T')$ to $\Gamma(M, T'\otimes({}^0T'')^*)$. Like in the case for scalar valued differential forms, we have

$$\bar\partial_{T'}^{(p)} : \Gamma\left(M, T'\otimes\bigwedge^{p}\big({}^0T''\big)^*\right) \to \Gamma\left(M, T'\otimes\bigwedge^{p+1}\big({}^0T''\big)^*\right). \tag{1.4}$$

Because of (1.2)

$$\bar\partial_{T'}^{(p)}\bar\partial_{T'}^{(p-1)} = 0, \quad p = 1,2,\ldots,$$

so we have a differential complex

$$0 \to \Gamma(M,T') \xrightarrow{\bar\partial_{T'}} \Gamma\big(M, T'\otimes\big({}^0T''\big)^*\big) \xrightarrow{\bar\partial_{T'}^{(1)}} \Gamma\Big(M, T'\otimes\bigwedge^{2}\big({}^0T''\big)^*\Big) \to . \tag{1.5}$$

This is called the standard deformation complex (an analogy as in the case complex manifolds). In the complex manifold case, the $\bar\partial$ is elliptic. While in the CR case, the $\bar\partial_b$ operator is not elliptic (there is one missing direction), but a subelliticity holds at some degree. Therefore, in order to obtain the versal family of deformations of CR structures just by the method as in the complex manifolds, there is an essential difficulty (convergence). So in order to overcome this point, we introduce E_j bundles (see [1]). We, briefly, sketch this. First, we set a C^∞ vector bundle decomposition of $T'\otimes\bigwedge^j({}^0T'')^*$ by (1.3)

$$T'\otimes\bigwedge^{j}\big({}^0T''\big)^* = {}^0T'\otimes\bigwedge^{j}\big({}^0T''\big)^* + (C\otimes\xi)\otimes\bigwedge^{j}\big({}^0T''\big)^*, \tag{1.6}$$

and set

$$\Gamma_j = \left\{u\colon\ u \in \Gamma\left(M, {}^0T'\otimes\bigwedge^{j}\big({}^0T''\big)^*\right),\ \big(\bar\partial_{T'}^{(j)}u\big)_{(C\otimes\xi)\otimes\bigwedge^j({}^0T'')^*}\right\} \tag{1.7}$$

for Γ_j.

Proposition 1.1. *$\bar\partial_{T'}^{(j)}\Gamma_j \subset \Gamma_{j+1}$. So, $(\Gamma_j, \bar\partial^{(j)})$, $\bar\partial^{(j)} = \bar\partial_{T'}^{(j)}|_{\Gamma(M,E_j)}$ is a subcomplex of the standard deformation complex $(\Gamma(M, T'\otimes\bigwedge^i({}^0T'')^*), \bar\partial_{T'}^{(i)})$.*

Proposition 1.2. *There is a subvector bundle E_j of ${}^0T'\otimes\bigwedge^j({}^0T'')^*$, satisfying*

$$\Gamma(M,E_j) = \Gamma_j \quad \textit{and} \quad E_0 = 0$$

(*see Proposition* 2.1 *in* [1]).

Theorem 1.1. *(See Theorem 2.4 in [1].)*

$$\frac{\operatorname{Ker}\bar\partial^{(j)}}{\operatorname{Im}\bar\partial^{(j-1)}} \simeq \frac{\operatorname{Ker}\bar\partial^{(j)}_{T'}}{\operatorname{Im}\bar\partial^{(j-1)}_{T'}}, \quad 2 \leqslant j \leqslant n-2.$$

Theorem 1.2. *(See Theorem 4.1 in [1].) With the assumption* $\dim_R M = 2n-1 \geqslant 7$, *over* E_2, *a subelliptic estimate holds.*

In [6], we set

$$H_0 = \{u:\ u \in \Gamma(M, T'),\ (\bar\partial_{T'}u)_{(C\otimes\xi)\otimes({}^0T'')^*} = 0\},$$

and introduce a differential complex

$$0 \to H_0 \xrightarrow{\bar\partial_{T'}} \Gamma(M, E_1) \xrightarrow{\bar\partial_1} \Gamma(M, E_2). \tag{1.8}$$

We should mention about the adjoint operator of $\bar\partial_{T'}$. By using the Levi metric, we complete the prehilbert spaces H_0, $\Gamma(M, E_i)$. We use the same notation for these spaces H_0, $\Gamma(M, E_i)$. And $\bar\partial_0^*$ means the hilbert space adjoint operator of $\bar\partial_{T'}$.

Theorem 1.3. *Under the above assumptions,*

$$\bar\partial_0^* = \pi_{\widetilde{H}_0} \circ \delta'',$$

where δ'' *means the formal adjoint operator of* $\bar\partial_{T'}$ *and* $\pi_{\widetilde{H}_0}$ *means the projection map of* $\Gamma_2(M, T')$ *to* $\widetilde{H}_0$ *(see Lemma* 5.3 *in* [6]). *Here* $\Gamma_2(M, T')$ *means the completion of* $\Gamma(M, T')$ *and* $\widetilde{H}_0$ *means the completion of* H_0 *with respect to the Levi metric.*

And our versal family is defined by

$$\{\phi:\ \phi \in \Gamma(M, E_1),\ P(\phi) = \bar\partial^{(1)}\phi + R_2(\phi) = 0,\ \bar\partial_0^*\phi = 0\}$$

(see Section 7 in [6]). Here $\bar\partial_0^*$ is the composition of the standard adjoint operator δ'' of $\bar\partial_{T'}$ and the projection map, which includes the first order derivatives. So obviously, the Zariski tangent space of the versal family is

$$\{\psi:\ \psi \in \Gamma(M, E_1),\ \bar\partial\psi = 0,\ \bar\partial_0^*\psi = 0\}$$

and so it includes

$$\{\psi:\ \psi \in \Gamma(M, E_1), \bar\partial\psi = 0,\ \delta''\psi = 0\}.$$

2. Mixed Hodge structure

Though it is discussed in [4,5], we pick up some results (it is arranged in our setting). Let $(M, {}^0T'')$ be a CR structure, which is strongly pseudoconvex CR structure. Henceforth, we assume that there is a real global vector field ζ which satisfies

$$\zeta_p \notin {}^0T' + {}^0T'', \quad \text{for every point } p \text{ of } M, \tag{2.1}$$

$$[\zeta, \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''). \tag{2.2}$$

We adopt this ζ as a supplement vector field and also we set $T' = {}^0T' + C \otimes \zeta$. With this ζ, we set a real differential one form θ by

$$\theta|_{{}^0T' + {}^0T''} = 0, \tag{2.3}$$

$$\theta(\zeta) = 1. \tag{2.4}$$

We set a C^∞ vector bundle decomposition of C^∞ k-complex valued differential forms as follows

$$\begin{aligned} \bigwedge^k (C \otimes TM)^* = &\sum_{r>0,\, s>0,\, r+s=k} \bigwedge^r \left({}^0T'\right)^* \wedge \bigwedge^s \left({}^0T''\right)^* \\ &+ \sum_{r>0,\, s>0,\, r+s=k-1} \theta \wedge \bigwedge^r \left({}^0T'\right)^* \wedge \bigwedge^s \left({}^0T''\right)^*. \end{aligned} \tag{2.5}$$

By using this decomposition, we introduce our first order differential operators d', d'' on C^∞ differential forms. For $u \in \Gamma(M, \theta \wedge \bigwedge^r ({}^0T')^* \wedge \bigwedge^s ({}^0T'')^*)$, we set

$$d'u := (du)_{\theta \wedge \bigwedge^{r+1} ({}^0T')^* \wedge \bigwedge^s ({}^0T'')^*}, \tag{2.6}$$

and

$$d''u := (du)_{\theta \wedge \bigwedge^{r} ({}^0T')^* \wedge \bigwedge^{s+1} ({}^0T'')^*}. \tag{2.7}$$

Here $(du)_{\theta \wedge \bigwedge^r ({}^0T')^* \wedge \bigwedge^s ({}^0T'')^*}$ means the projection of du to $\theta \wedge \bigwedge^r ({}^0T')^* \wedge \bigwedge^s ({}^0T'')^*$ part according to (2.5). In [3], we proposed the notion of mixed Hodge structure. We recall this notion. For a pair of non-negative integers, (p, q), which satisfies $p + q \geqslant n - 1$, we set a subspace of differential forms of total degree $1 + p + q \geqslant n$ by

$$F^{1,p,q} = \left\{u\colon\ u \in \theta \wedge \bigwedge^p \left({}^0T'\right)^* \wedge \bigwedge^q \left({}^0T''\right)^*,\ Lu = 0\right\}. \tag{2.8}$$

Here Lu means $d\theta \wedge u$. For $F^{1,p,q}$, without proof, we mention several theorems.

Theorem 2.1. *(See Theorem 3.1 in [3].)*

$$d'F^{1,p,q} \subset F^{1,p+1,q}, \tag{2.9}$$

$$d''F^{1,p,q} \subset F^{1,p,q+1}, \tag{2.10}$$

and moreover,

$$d'd' = 0, \quad d''d'' = 0, \quad d'd'' + d''d' = 0 \quad \text{on } F^{1,p,q}. \tag{2.11}$$

So, we have a differential double complex $(F^{1,p,q}, d', d'')$. For this complex, we have

Theorem 2.2. *(See Theorem 3.2 in [3].) If $2n - 1 \geqslant p + q \geqslant n - 1$, then*

$$\frac{\operatorname{Ker} d'' \cap F^{1,p,q}}{d''F^{1,p,q-1}} \simeq H^q\left(M, \bigwedge^{1+p} (T')^*\right) \tag{2.12}$$

where $H^q(M, \bigwedge^{1+p} (T')^*)$ means the standard Kohn–Rossi cohomology.

By inspired by the Rumin work, this complex is developed as follows. For the total degree $n-1$ case, we set

$$H^{n-1-q,q} = \Big\{ v:\ v \in \Gamma\Big(M, \bigwedge^{n-1-q} (T')^* \wedge \bigwedge^{q} \big({}^0T''\big)^*\Big), \\ (dv)_{\bigwedge^{n-1-q}({}^0T')^* \wedge \bigwedge^{q+1}({}^0T'')^*} = 0 \Big\}. \tag{2.13}$$

(This is a prehilbert space and in this degree, $n-1$, the equation $(dv)_{\bigwedge^{n-1-q}({}^0T')^* \wedge \bigwedge^{q+1}({}^0T'')^*} = 0$ includes the first order derivatives of v.)

$$\begin{array}{ccccccc}
 & & & & H^{n-2-q,q+1} & \xrightarrow{d''} & F^{1,n-3-q,q+2} \\
 & & & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
 & & H^{n-1-q,q} & \xrightarrow{d''} & F^{1,n-2-q,q+1} & \xrightarrow{d''} & F^{1,n-2-q,q+2} \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
H^{n-q,q-1} & \xrightarrow{d''} & F^{1,n-1-q,q} & \xrightarrow{d''} & F^{1,n-1-q,q+1} & \xrightarrow{d''} & F^{1,n-1-q,q+2}
\end{array}$$

Especially, if $q=1$,

$$\begin{array}{ccccccc}
 & & & & H^{n-3,2} & \xrightarrow{d''} & F^{1,n-4,3} \\
 & & & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
 & & H^{n-2,1} & \xrightarrow{d''} & F^{1,n-3,2} & \xrightarrow{d''} & F^{1,n-3,3} \\
 & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
H^{n-1,0} & \xrightarrow{d''} & F^{1,n-2,1} & \xrightarrow{d''} & F^{1,n-2,2} & \xrightarrow{d''} & F^{1,n-2,3}
\end{array}$$

And if $n=3$ and $p=1$, even on $F^{1,1,1}$, the Kodaira–Hodge decomposition theorem holds (see [6]). By the assumption: that our ζ satisfies

$$\big[\zeta, \Gamma\big(M, {}^0T''\big)\big] \subset \Gamma\big(M, {}^0T''\big).$$

Then, $(\Gamma(M, \theta \wedge \bigwedge^p({}^0T')^* \wedge \bigwedge^q({}^0T'')^*), d'')$ becomes a differential complex. By Tanaka (see [7]), the standard exterior derivative d induces the differential complex

$$\Big(\Gamma\Big(M, \bigwedge^r (T')^* \wedge \bigwedge^q \big({}^0T''\big)^*\Big), \bar\partial_b\Big).$$

Here $\bar\partial_b$ is defined by: for $u \in \Gamma(M, \bigwedge^r (T')^* \wedge \bigwedge^q({}^0T'')^*)$,

$$\bar\partial_b u = (du)_{\bigwedge^r (T')^* \wedge \bigwedge^q({}^0T'')^*}$$

according to (2.5).

We recall the Kohn–Rossi cohomology. We set a C^∞ vector bundle decomposition of C^∞ k-complex valued differential forms

$$\bigwedge^k (C \otimes TM)^* = \sum_{r>0,\, s>0,\, r+s=k} \bigwedge^r (T')^* \wedge \bigwedge^s \big({}^0T''\big)^* \tag{2.14}$$

and for $u \in \Gamma(M, \bigwedge^r (T')^* \wedge \bigwedge^s ({}^0T'')^*)$,

$$\bar{\partial}_b u := (du)_{\bigwedge^r (T')^* \wedge \bigwedge^s ({}^0T'')^*}. \tag{2.15}$$

Here $(du)_{\bigwedge^r (T')^* \wedge \bigwedge^s ({}^0T'')^*}$ means the projection of du to $\bigwedge^r (T')^* \wedge \bigwedge^s ({}^0T'')^*$ part according to the above decomposition. Then, since our CR structure$(M, {}^0T'')$ is integrable (see (1.2) in this paper)

$$\bar{\partial}_b \bar{\partial}_b = 0 \tag{2.16}$$

(this always holds even without the assumption of normality).

By the definition, for $u \in \Gamma(M, \theta \wedge \bigwedge^p ({}^0T')^* \wedge \bigwedge^q ({}^0T'')^*)$,

$$d'' u = (\bar{\partial}_b u)_{\theta \wedge \bigwedge^p ({}^0T')^* \wedge \bigwedge^{q+1} ({}^0T'')^*}. \tag{2.17}$$

Since our M is normal,

$$\bar{\partial}_b \Gamma\left(M, \bigwedge^{p+1} \left({}^0T'\right)^* \wedge \bigwedge^{q} \left({}^0T''\right)^*\right) \subset \Gamma\left(M, \bigwedge^{p+1} \left({}^0T'\right)^* \wedge \bigwedge^{q+1} \left({}^0T''\right)^*\right)$$

so induced d' satisfies also

$$d'' d'' = 0 \quad \text{on } \Gamma\left(M, \theta \wedge \bigwedge^{p} \left({}^0T'\right)^* \wedge \bigwedge^{q} \left({}^0T''\right)^*\right).$$

This is shown as follows. For $u \in \Gamma(M, \theta \wedge \bigwedge^p ({}^0T')^* \wedge \bigwedge^q ({}^0T'')^*)$,

$$\begin{aligned} d'' d'' u &= \left(\bar{\partial}_b (\bar{\partial}_b u)_{\theta \wedge \bigwedge^p ({}^0T')^* \wedge \bigwedge^{q+1} ({}^0T'')^*}\right)_{\theta \wedge \bigwedge^p ({}^0T')^* \wedge \bigwedge^{q+2} ({}^0T'')^*} \\ &= (\bar{\partial}_b \bar{\partial}_b u)_{\theta \wedge \bigwedge^p ({}^0T')^* \wedge \bigwedge^{q+2} ({}^0T'')^*} \\ &= 0. \end{aligned} \tag{2.18}$$

In this case, over M, several Kaehler identities hold (see [7]).

Theorem 2.3.

$$\left\{u:\ u \in \Gamma\left(M, F^{1,n-p,p-1}\right),\ d'' u = 0,\ \delta'' u = 0\right\} \tag{2.19}$$

is equal to

$$\left\{u:\ u \in \Gamma\left(M, F^{1,n-p,p-1}\right),\ d'' u = 0,\ d' u = 0\right\}. \tag{2.20}$$

Proof. First, we see that for $u \in \Gamma(M, F^{1,n-p,p-1})$, satisfying: $d''u = 0$, $d'u = 0$, $\delta''u = 0$ holds. From the definition of $F^{1,r,s}$ and the Kaehler identity

$$\delta'' L - L\delta'' = \sqrt{-1} d' \tag{2.21}$$

we have $-L\delta'' u = 0$. We recall

$$\Lambda L - L\Lambda = (n - 1 - r - s) \quad \text{on } F^{1,r,s}. \tag{2.22}$$

As $\delta'' u$ is of type $(1, n - p, p - 2)$,

$$\Lambda L\delta'' u - L\Lambda\delta'' u = \left(n - 1 - (n - p) - (p - 2)\right)\delta'' u = \delta'' u.$$

Hence

$$-(L\Lambda\delta''u,\delta''u)=(\delta''u,\delta''u),$$
$$-(\Lambda\delta''u,\Lambda\delta''u)=(\delta''u,\delta''u).$$

So $\delta''u=0$ must hold. We show the converse. For $u\in\Gamma(M,F^{1,n-p,p-1})$, satisfying that; $d''u=0$, $\delta''u=0$, by (2.17), we have

$$\sqrt{-1}d'u=0. \tag{2.23}$$

Hence our theorem follows. □

Corollary 2.1. $\{u\colon u\in\Gamma(M,F^{1,n-p,p-1}), d''u=0, d'u=0\}$ *is a subset of* $\mathbf{H}_{d''}^{n-p+1,p-1}$. *Here*

$$\mathbf{H}_{d''}^{n-p+1,p-1}=\{u\colon u\in\Gamma(M,F^{1,n-p,p-1}), d''u=0, d''^{*}u=0\}.$$

We prove that only in the case $\dim_R M=2n-1=5$ and $p=2$, we show that $\mathbf{H}^{2,1}$ is finite-dimensional (see [6]), but this seems to hold for general n, p.

3. Hamiltonian flow

Let (V,o) be an isolated singularity in a complex euclidean space C^N. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be the defining equations of V in C^N, where Λ is a finite set. We construct a new isolated singularity (V',o) in a complex euclidean space C^{N+m} by

$$V'=\{(z_1,z_2,\dots,z_N,\dots,z_{N+m})\colon f_\lambda(z)+z_{N+1}^2+\cdots+z_{N+m}^2=0, \lambda\in\Lambda\}. \tag{3.1}$$

By this procedure, from two-dimensional rational double point, we have enough higher-dimensional singularities. For example, we take the ordinary double point. Then, we have a higher-dimensional ordinary double point. This is explicitly defined as follows. Let $\{z_1,z_2,\dots,z_{n+1}\}$ be the complex coordinates of C^{n+1}. We consider a holomorphic function on C^{n+1} by

$$f=z_1^2+z_2^2+\cdots+z_{n+1}^2. \tag{3.2}$$

And consider the ordinary double point

$$V_0=\{(z_1,z_2,\dots,z_{n+1})\colon z_1^2+z_2^2+\cdots+z_{n+1}^2=0\}. \tag{3.3}$$

Obviously our ordinary double point has an singularity only at the origin o. Now we study the boundary of this singularity. Let

$$M=V_0\cap\left\{(z_1,z_2,\dots,z_{n+1})\colon \sum_{i=1}^{n+1}|z_i|^2=1\right\}, \tag{3.4}$$

and consider the CR structure over this manifold. We show that the canonical line bundle of this M is trivial in the CR sense. In fact, first, on V_0-o, the canonical line bundle is trivial. This is shown as follows. On C^{n+1}, we have a canonical holomophoic $n+1$-form $dz_1\wedge dz_2\wedge\cdots\wedge dz_{n+1}$. This $n+1$-form induces a canonical n form, ω' on V_0-o. Ordinarily, this is done by the residue map by

$$\omega'=\mathrm{Res}_{V_0}\left[\frac{\omega}{f}\right].$$

Here we adopt another approach. We set

$$L := \sqrt{-1}\sum_{i=1}^{n+1} dz_i \wedge d\bar{z}_i. \tag{3.5}$$

By using the metric associated to (3.5), we set the Hamiltonian vector field of type $(1,0)$, Z_f, defined on a neighborhood of M, by: for $(1,0)$ vector field X,

$$df(X) = L(X, \overline{Z_f}) \quad \text{on a neighborhood of } M. \tag{3.6}$$

In our case, this L is the same as in Section 2, if we choose

$$\zeta = \frac{\sqrt{-1}}{2}\left(\sum_{i=1}^{n+1} z_i \frac{\partial}{\partial z_i} - \sum_{i=1}^{n+1} \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\right)$$

as a supplement vector field. And set a real one form θ on M, by

$$\theta = \sqrt{-1}\sum_{i=1}^{n+1} z_i d\bar{z}_i \quad \text{on } M, \tag{3.7}$$

and

$$L = d\theta.$$

By the complex coordinates, we can write down Z_f as follows

$$Z_f = \sqrt{-1}\sum_{i=1}^{n+1} \overline{\left(\frac{\partial f}{\partial z_i}\right)} \frac{\partial}{\partial z_i}. \tag{3.8}$$

In our case,

$$Z_f = 2\sqrt{-1}\sum_{i=1}^{n+1} \overline{z_i} \frac{\partial}{\partial z_i} \tag{3.9}$$

so

$$Z_f\left(\sum_{i=1}^{n+1} |z_i|^2 - 1\right) = 2\sqrt{-1}\sum_{i=1}^{n+1} \bar{z}_i \cdot \bar{z}_i = 0 \quad \text{on } V_0 - o. \tag{3.10}$$

Now we introduce a differential n-form ω', which is defined on a neighborhood of M in $C^{n+1} - o$ by

$$\omega'(X_1, \ldots, X_n) := \omega(Z_f, X_1, \ldots, X_n). \tag{3.11}$$

We note that by the residue operator, it is defined on only $V_0 - o$. We study this ω' precisely. By the definition of Z_f, the vector X on $C^{n+1} - o$ is perpendicular to Z_f with respect to the euclidean metric, if and only if

$$Xf = 0. \tag{3.12}$$

Proposition 3.1. *Let* $\{X_i\}_{1\leqslant i\leqslant n+1}$ *be type* $(1,0)$ *vectors or* $(0,1)$ *vectors on* M. *If*

$$X_i\left(\sum_{i=1}^{n+1}|z_i|^2-1\right)=0 \quad \text{on } M, \qquad \text{then } \omega'(X_1,\dots,X_n)=0 \quad \text{on } M.$$

Proof. By the type of differential form ω, it is enough to see the case, X_i being $(1,0)$-type. However, let

$$^0T'_{C^{n+1}}=\left\{X\colon\ X\in T'C^{n+1},\ X\left(\sum_{i=1}^{n+1}|z_i|^2-1\right)=0\right\},$$

then $\dim_C {}^0T'_{C^{n+1}}=n$ and, as is proved, Z_f is also a section of $^0T'_{C^{n+1}}$ on M (see (3.10)). Hence, $\omega(Z_f,X_1,\dots,X_n)=0$. So we have our proposition. □

Lemma 3.1.

$$\omega=df\wedge\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega' \quad \text{on a neighborhood of } M \text{ in } C^{n+1}-o.$$

Proof. We set

$$\omega''=\omega-df\wedge\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega'.$$

By the definition of ω', this is of type $(n,0)$. So, for the proof, it is enough to see

$$\omega''(Z_f,X_1,\dots,X_n)=0 \quad \text{on a neighborhood of } M \text{ in } C^{n+1}-o, \tag{3.13}$$

$$\omega''(X_1,\dots,X_{n+1})=0 \quad \text{on a neighborhood of } M \text{ in } C^{n+1}-o, \tag{3.14}$$

for type $(1,0)$ vectors X_i, $1\leqslant i\leqslant n+1$, which satisfies $X_if=0$ on M. While

$$(df)(Z_f)=\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2 \quad \text{on a neighborhood of } M \text{ in } C^{n+1}-o.$$

So, the first one is valid, because of the definition of ω' with Proposition 3.1. The second one is obvious because of $\omega''(X_1,\dots,X_{n+1})=0$. So we have our lemma. □

By restricting this ω' to V_0-o, we have the corresponding canonical form. This approach has one advantage. Because, ω' is defined on a neighborhood of M in $C^{n+1}-o$, So, the differentiation of ω' makes sense in $C^{n+1}-o$. By using this fact, we have the following lemma for $\omega'|_M$, the restriction of ω' to M (for brevity, we write ω').

Proposition 3.2.

$$d\omega'=0 \quad \text{on } M.$$

Proof. By Lemma 3.1,

$$\omega=df\wedge\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega' \quad \text{on a neighborhood of } M \text{ in } C^{n+1}-o.$$

Hence,

$$\begin{aligned} d\omega &= ddf \wedge \left(\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega'\right) + (-1)\, df \wedge d\left(\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega'\right) \\ &= -df \wedge d\left(\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega'\right) \quad \text{on a neighborhood of } M \text{ in } C^{n+1} - o. \quad (3.15) \end{aligned}$$

While,

$$d\omega = 0 \text{ on } C^{n+1}.$$

Hence on a neighborhood of M in $C^{n+1} - o$, we have $df \wedge d((\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2})\omega') = 0$. Hence restricting $(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2})\omega'$ to $V_0 - o$,

$$d\left(\left(\frac{1}{\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2}\right)\omega'\right) = 0.$$

Hence on M, we have our lemma. □

Our V_0 has a natural family of deformations (versal family), $\{V_t\}$, defined by

$$V_t = \{(z_1, \ldots, z_{n+1}):\ z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = t\}. \quad (3.16)$$

For this V_t, we set a real hypersurface M_t by

$$M_t = \left\{(z_1, \ldots, z_{n+1}):\ z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = t,\ \sum_{i=1}^{n+1}|z_i|^2 = 1\right\}. \quad (3.17)$$

This real hypersurface has a natural contact structure, $\theta_t = \sqrt{-1}\sum_{i=1}^{n+1} z_i\, d\bar{z}_i$ on M_t. For these real hypersurfaces, we can make the same procedure. Take the canonical $n+1$ form ω on C^{n+1}. We introduce a n form ω'_t on V_t, which should be related with the residue $\mathrm{Res}_{V_t}[\frac{\omega}{f-t}]$, by

$$\omega'_t(X_1, \ldots, X_n) := \omega\left(Z_f - 2\sqrt{-1}\bar{t}\sum_{i=1}^{n+1} z_i \frac{\partial}{\partial z_i}, X_1, \ldots, X_n\right)$$
$$\text{on a neighborhood of } M_t \text{ in } C^{n+1} - o. \quad (3.18)$$

Then we have the same lemma as Lemma 3.1.

Lemma 3.2.

$$\omega = df \wedge \left(\frac{1}{4\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2 - 2\sqrt{-1}|t|^2}\right)\omega'_t \quad \textit{on a neighborhood of } M_t \textit{ in } C^{n+1} - o.$$

Proof. We set

$$\omega'' = \omega - df \wedge \left(\frac{1}{4\sqrt{-1}\sum_{i=1}^{n+1}|z_i|^2 - 2\sqrt{-1}|t|^2}\right)\omega'_t.$$

For the proof, it is enough to see

$$\omega''\left(Z_f - 2\sqrt{-1}\bar{t}\sum_{i=1}^{n+1} z_i \frac{\partial}{\partial z_i}, X_1, \ldots, X_n\right) = 0 \tag{3.19}$$

on a neighborhood of M in $C^{n+1} - o$,

$$\omega''(X_1, \ldots, X_{n+1}) = 0 \quad \text{on a neighborhood of } M \text{ in } C^{n+1} - o, \tag{3.20}$$

for type $(1, 0)$ vectors X_i, $1 \leqslant i \leqslant n+1$, which satisfies $X_i f = 0$ on M. While

$$(df)\left(Z_f - 2\sqrt{-1}\,\bar{t}\sum_{i=1}^{n+1} z_i \frac{\partial}{\partial z_i}\right) = 4\sqrt{-1}\sum_{i=1}^{n+1} |z_i|^2 - 4\sqrt{-1}|t|^2$$

on a neighborhood of M in $C^{n+1} - o$. So, the first one is OK, because of the definition of ω' and by Proposition 3.1. The second one is obvious because of $\omega''(X_1, \ldots, X_{n+1}) = 0$. So we have our lemma. □

For this ω'_t, we have the same result.

Lemma 3.3.

$$d\left(\frac{1}{4\sqrt{-1} - 2\sqrt{-1}|t|^2}\omega'_t\right) = 0 \quad \textit{on } M_t.$$

So, if t is sufficiently close to the origin o, we have

Proposition 3.3.

$$d\omega'_t = 0 \quad \textit{on } M_t.$$

And also with

$$\left(Z_f - 2\sqrt{-1}\bar{t}\sum_{i=1}^{n+1} z_i \frac{\partial}{\partial z_i}\right)\left(\sum_{i=1}^{n+1} |z_i|^2 - 1\right) = 0 \quad \text{on } M_t, \tag{3.21}$$

we have

Theorem 3.1. *Let $\{X_i\}_{1 \leqslant i \leqslant n}$ be type $(1, 0)$ vectors on M_t. If*

$$X_i\left(\sum_{i=1}^{n+1} |z_i|^2 - 1\right) = 0 \quad \textit{on } M_t,$$

then

$$\omega'_t(X_1, \ldots, X_n) = 0 \quad \textit{on } M_t \ (\textit{compare Proposition 3.1}). \tag{3.22}$$

Now from the point of view of deformation there, we consider a family of deformations of CR structures of M, $\{M_t\}$. Our family has a natural C^∞ trivialization i_t from M to M_t, defined by

$$i_t : (z_1, \ldots, z_i, \ldots, z_{n+1}) \to \left(\frac{z_1 + (\frac{1}{2})t\bar{z}_1}{\sqrt{1 + (\frac{1}{4})|t|^2}}, \ldots, \frac{z_i + (\frac{1}{2})t\bar{z}_i}{\sqrt{1 + (\frac{1}{4})|t|^2}}, \ldots, \frac{z_{n+1} + (\frac{1}{2})t\bar{z}_{n+1}}{\sqrt{1 + (\frac{1}{4})|t|^2}}\right).$$

This C^∞ isomorphism map is a contact transformation,

$$i_t^*\theta_t = \sqrt{-1}\sum_{i=1}^{n+1}\left(\frac{z_i + (\frac{1}{2})t\bar{z}_i}{\sqrt{1+(\frac{1}{4})|t|^2}}\right)d\left(\frac{\bar{z}_i + (\frac{1}{2})tz_i}{\sqrt{1+(\frac{1}{4})|t|^2}}\right) = \theta \quad \text{on } M. \tag{3.23}$$

Now we introduce types $(n-1,1)$, $(n-2,2)$ forms ω_1, ω_2 on V by

$$(i_t)^*\omega_t' = \omega' + \omega_1 t + \omega_2 t^2 + O(\bar{t}, t^3), \tag{3.24}$$

where $O(\bar{t}, t^3)$ means a linear combination of $a(t,\bar{t})\bar{t} + b(t,\bar{t})t^3$, where $a(t,\bar{t}), b(t,\bar{t})$ are real analytic functions of $t, \bar{t}$. And henceforth, we use this notation for such a term.

We see that ω_i' is of type $(n-i,i)$.

Proposition 3.4. *ω_i' is of type $(n-i,i)$.*

Proof.

$$i_t^*\omega_t' = i_t^*\omega' + i_t^*(\omega_t' - \omega'). \tag{3.25}$$

We note that the second term of the right-hand side of this equality includes $\bar{t}$. So in order to see ω_i', by the definition, it is enough to check $i_t^*\omega'$. This is done as follows. As $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1}$ and $Z_f = 2\sqrt{-1}\sum_{i=1}^{n+1}\bar{z}_i\frac{\partial}{\partial z_i}$,

$$\omega'(X_1,\ldots,X_n) = \omega(Z_f, X_1,\ldots,X_n).$$

So, ω' can be written explicitly as follows

$$\omega' = 2\sqrt{-1}\sum_{i=1}^{n+1}(-1)^{i+1}\bar{z}_i\, dz_1 \wedge \cdots \wedge d\check{z}_i \wedge \cdots \wedge dz_{n+1}.$$

Hence

$$\begin{aligned}
i_t^*\omega' = \omega' &+ t\sqrt{-1}\big\{\bar{z}_1(d\bar{z}_2 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_2 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots \\
&+ dz_2 \wedge \cdots \wedge d\bar{z}_{n+1}) \\
&- \bar{z}_2(d\bar{z}_1 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_1 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots \\
&+ dz_1 \wedge \cdots \wedge d\bar{z}_{n+1}) + \cdots \\
&+ (-1)^{n+2}\bar{z}_{n+1}(d\bar{z}_1 \wedge dz_2 \wedge \cdots \wedge dz_n + dz_1 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n + \cdots \\
&+ dz_1 \wedge \cdots \wedge d\bar{z}_n)\big\} \\
&+ \frac{1}{2}t^2\sqrt{-1}\left\{\bar{z}_1\left(\sum_{i,j} dz_2 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots d\bar{z}_j \wedge \cdots \wedge dz_{n+1}\right)\right. \\
&- \bar{z}_2\left(\sum_{i,j} dz_1 \wedge d\check{z}_2 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_{n+1}\right) + \cdots \\
&\left.+ (-1)^{n+2}\bar{z}_{n+1}\left(\sum_{i,j} dz_1 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_n\right)\right\} \\
&+ O(\bar{t}, t^3).
\end{aligned} \tag{3.26}$$

So,

$$\begin{aligned}\omega_1' = {} & \bar{z}_1(d\bar{z}_2 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_2 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots + dz_2 \wedge \cdots \wedge d\bar{z}_{n+1}) \\ & - \bar{z}_2(d\bar{z}_1 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_1 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots \\ & + dz_1 \wedge \cdots \wedge d\bar{z}_{n+1}) + \cdots \\ & + (-1)^{n+2}\bar{z}_{n+1}\left(\sum_{i=1}^{n} dz_1 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge dz_n\right) \end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\omega_2' = {} & \bar{z}_1\left(\sum_{i,j} d\check{z}_1 \wedge dz_2 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_{n+1}\right) \\ & - \bar{z}_2\left(\sum_{i,j} dz_1 \wedge d\check{z}_2 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_{n+1}\right) + \cdots \\ & + (-1)^{n+2}\bar{z}_{n+1}\left(\sum_{i,j} dz_1 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_n\right). \end{aligned} \tag{3.28}$$

Therefore our proposition is now clear. □

And for these forms, we have

Theorem 3.2. $d\omega_1 = 0$, $d\omega_2 = 0$ *on* M.

Proof. Because of Proposition 3.3,

$$\begin{aligned} d\big((i_t)^*\omega_t'\big) &= (i_t)^*\, d\omega_t' \quad \text{on } M \\ &= 0 \quad \text{on } M, \end{aligned} \tag{3.29}$$

so we have

$$d\big(\omega' + \omega_1 t + \omega_2 t^2 + O\big(\bar{t}, t^3\big)\big) = 0 \quad \text{on } M. \tag{3.30}$$

Hence we have our theorem. □

4. The spaces Z^1, Z^2

We set

$$Z^1 = \big\{u\colon\ u \in \Gamma\big(M, F^{1,n-2,1}\big),\ d'u = 0,\ d''u = 0\big\}, \tag{4.1}$$

$$Z^2 = \big\{u\colon\ u \in \Gamma\big(M, F^{1,n-3,2}\big),\ d'u = 0,\ d''u = 0\big\}. \tag{4.2}$$

Then our Z^1 (respectively Z^2) is a subspace of $\mathbf{H}_{d''}^{n-1,1} = \{u\colon\ u \in F^{1,n-2,1},\ d''^*u = 0,\ d''u = 0\}$ (respectively $\mathbf{H}_{d''}^{n-2,2} = \{u\colon\ u \in F^{1,n-3,2}, d''^*u = 0, d''u = 0\}$).

Theorem 4.1. *For the CR structure, which is constructed in Section* 3, *from the ordinary double point, we have*

$$Z^1 = \mathbf{H}_{d''}^{n-1,1}, \tag{4.3}$$
$$Z^2 = \mathbf{H}_{d''}^{n-2,2}. \tag{4.4}$$

In Section 3, we construct an element ω_1' (respectively ω_2') of $\Gamma(M, \bigwedge^{n-1}(T')^* \wedge \bigwedge({}^0T'')^*)$ (respectively $\Gamma(M, \bigwedge^{n-2}(T')^* \wedge \bigwedge^2({}^0T'')^*)$. By S.S. Yau's work (see [8]), we have that

$$\dim_C \mathbf{H}_{d''}^{n-1,1} = \dim_C \mathbf{H}_{d''}^{n-2,2} = 1.$$

Therefore if we see that ω_1' (respectively ω_2') are really of Z_1' (respectively Z_2'), defined as above and also non-vanishing, then we have our theorem, for this, we have to show, over M,

$$\begin{cases} \omega_i' \wedge \theta = 0, \\ L\omega_i' = 0, \\ d'\omega' = 0, \\ d''\omega_i' = 0, \quad i = 1, 2. \end{cases}$$

We see these.

4.1. *The proof of $\omega_i' \wedge \theta = 0$ on M*

For this, it is enough to show that

$$\big(\theta \wedge \omega_i'\big)(\zeta, Z_1, \ldots, Z_n) = 0, \tag{4.5}$$
$$\big(\theta \wedge \omega_i'\big)(Y_1, \ldots, Y_{n+1}) = 0 \quad \text{on } M,\ i = 1, 2. \tag{4.6}$$

Here, $Z_i, Y_j \in {}^0T' + {}^0T''$. For (4.5), we have

$$\big(\theta \wedge \omega_i'\big)(\zeta, Z_1, \ldots, Z_n) = \theta(\zeta)\omega_i'(Z_1, \ldots, Z_n) = \omega_i(Z_1, \ldots, Z_n). \tag{4.7}$$

While

$$\begin{aligned}(i_t)^*\omega_t'(Z_1, \ldots, Z_n) &= \omega_t'\big((i_t)_*Z_1, \ldots, (i_t)_*Z_n\big) \quad \text{for } Z_i \in {}^0T' + {}^0T'' \text{ on } M \\ &= 0 \quad \text{(by Proposition 3.1)}.\end{aligned} \tag{4.8}$$

So (4.5) holds. By the definition of θ, (4.6) is obvious. So we have 4.1.

4.2. *The proof of $L\omega_i' = 0$*

By 4.1, we have

$$\begin{aligned} d\big(\theta \wedge \omega_i'\big) &= d\theta \wedge \omega_i' - \theta \wedge d\omega_i', \\ 0 &= d\theta \wedge \omega_i' - \theta \wedge d\omega_i'. \end{aligned} \tag{4.9}$$

While

$$d\omega_i' = 0 \quad \text{(Theorem 3.2)}, \tag{4.10}$$

we have 4.2.

Now the others are obvious, because of the definitions of d' and d''.

4.3. Non-vanishing of ω_i'

4.3.1. Non-vanishing of ω_1'

Let $p = (\frac{1}{\sqrt{2}}, \sqrt{\frac{-1}{2}}, 0, \ldots, 0)$. Then our p is an element of M. We consider ω_1' at p. For this, we find out a base of complex-valued differential forms, $(C \otimes TM_p)^*$. As p is an element of V_0, so

$$\frac{1}{\sqrt{2}} dz_1 + \sqrt{\frac{-1}{2}} dz_2 = 0.$$

As p is an element of the hypersphere, so

$$\frac{1}{\sqrt{2}} dz_1 + \frac{1}{\sqrt{2}} d\bar{z}_1 - \sqrt{\frac{-1}{2}} dz_2 + \sqrt{\frac{-1}{2}} d\bar{z}_2 = 0.$$

From these, we have

$$dz_1 = -\sqrt{-1}\, dz_2, \qquad d\bar{z}_2 = dz_2.$$

Hence as a base of $(C \otimes TM_p)^*$, we can adopt dz_2 and $\{dz_3, \ldots, dz_{n+1},\ d\bar{z}_3, \ldots, d\bar{z}_{n+1}\}$. We can rewrite ω_1' (see (3.27)) by this. Namely,

$$\begin{aligned}
\omega_1' = {} & \frac{1}{\sqrt{2}}\{d\bar{z}_2 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_2 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots \\
& + dz_2 \wedge \cdots \wedge d\bar{z}_{n+1}\} \\
& + \sqrt{\frac{-1}{2}}\{d\bar{z}_1 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_1 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots \\
& + dz_1 \wedge \cdots \wedge d\bar{z}_{n+1}\} \\
= {} & \frac{1}{\sqrt{2}}\{d\bar{z}_2 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} + dz_2 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots + dz_2 \wedge \cdots \wedge d\bar{z}_{n+1}\} \\
& + \sqrt{\frac{-1}{2}}\{\sqrt{-1}\, d\bar{z}_2 \wedge dz_3 \wedge \cdots \wedge dz_{n+1} - \sqrt{-1}\, dz_2 \wedge d\bar{z}_3 \wedge \cdots \wedge dz_{n+1} + \cdots \\
& - \sqrt{-1}\, dz_2 \wedge \cdots \wedge d\bar{z}_{n+1}\},
\end{aligned}$$

so the coefficient of ω' with respect to $dz_2 \wedge d\bar{z}_3 \wedge dz_4 \wedge \cdots$ is $\sqrt{2}$. Hence we have that ω_1' is not a zero element.

4.3.2. Non-vanishing of ω_2'

By the same way, at the same p as in (4.3.1), we see ω_2'.

$$\begin{aligned}
\omega_2' = {} & \frac{1}{\sqrt{2}}\left\{\sum_{i,j} d\check{z}_1 \wedge dz_2 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_{n+1}\right\} \\
& - \left(-\sqrt{\frac{-1}{2}}\right)\left\{\sum_{i,j} dz_1 \wedge d\check{z}_2 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_{n+1}\right\}.
\end{aligned} \tag{4.11}$$

If we see the coefficient with respect to $dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4 \wedge \cdots \wedge dz_{n+1}$, then by the same computation, we have the non-vanishing of ω_2'.

5. $d'd''$-lemma

Theorem 5.1. *Let* $(M, {}^0T'')$ *be a compact strongly pseudoconvex CR manifold with* $\dim_R M = 2n-1 \geqslant 5$. *We assume that our canonical line bundle is trivial in the CR sense. Then if*

$$Z^1 = \mathbf{H}_{d''}^{n-1,1},$$
$$Z^2 = \mathbf{H}_{d''}^{n-2,2},$$

any deformation is unobstructed.

Proof. We, essentially, use the new deformation complex, introduced by [4]. For the proof of our theorem, it is enough to see

$$J^2 = \frac{\operatorname{Ker} d'' \cap \operatorname{Im} d' \cap F^{1,n-2,2}}{\operatorname{Im} d'' \cap \operatorname{Im} d' \cap F^{1,n-2,2}} \quad \text{vanishes.} \tag{5.1}$$

For this, for $u \in \operatorname{Ker} d'' \cap \operatorname{Im} d' \cap F^{1,n-2,2}$, there is a $v \in F^{1,n-3,2}$ satisfying:

$$u = d'v.$$

For this v, by a Hodge–Kodaira decomposition of the mixed Hodge complex, we have

$$v = H_{d''}v + d''\alpha + d''^*d''\beta, \tag{5.2}$$

where $\alpha \in H^{n-2,1}$ and $\beta \in F^{1,n-3,2}$. So,

$$d'v = d'H_{d''}v + d'd''\alpha + d'd''^*d''\beta. \tag{5.3}$$

By the assumption, the first term vanishes. For the third term on $F^{1,n-3,2}$,

$$d'd''^*d'' = \left(\frac{1}{2}\right)d''^*d''d'. \tag{5.4}$$

In fact, on $\Gamma(M, F^{1,p,q})$, $p+q \geqslant n$, d''^* is written down as follows

$$d''^*w = \delta''w - \frac{1}{p+q-(n-1)}\Lambda L\delta''w, \quad \text{for } w \in \Gamma\big(M, F^{1,p,q}\big). \tag{5.5}$$

Because $L\delta''w$ is of type $(1, n-2, 3)$, so

$$(L\Lambda - \Lambda L)L\delta''w = 2L\delta''w.$$

Hence

$$\begin{aligned}\frac{1}{2}L\Lambda L\delta''w &= L\delta''w + \Lambda LL\delta''w \\ &= L\delta''w + \Lambda L(\delta''L - \sqrt{-1}d')w \quad (\text{by } L\delta'' = \delta''L - \sqrt{-1}d') \\ &= L\delta''w \end{aligned}\tag{5.6}$$

and

$$\delta''w - \frac{1}{2}\Lambda L\delta''w \in \Gamma\big(M, F^{1,n-3,2}\big).$$

Furthermore, for $w' \in \Gamma(M, F^{1,n-3,2})$,

$$\left(\delta'' w - \frac{1}{2} \Lambda L \delta'' w, w'\right) = (\delta'' w, w') - \frac{1}{2}(L\delta'' w, Lw') = (\delta'' w, w') = (w, d'' w'), \tag{5.7}$$

we have (5.5). Now we see

$$\begin{aligned} d' d''^*(d''\beta) &= d'\delta'' d''\beta - \frac{1}{2} d' \Lambda L \delta'' d''\beta \\ &= -\delta'' d' d''\beta - \frac{1}{2}(\Lambda d' - \sqrt{-1}\delta'') L\delta'' d''\beta. \end{aligned} \tag{5.8}$$

While

$$\Lambda d' L\delta'' d''\beta = \Lambda d'(\delta'' L - \sqrt{-1} d') d''\beta = 0, \tag{5.9}$$

the above becomes

$$\begin{aligned} -\frac{1}{2}\delta'' d' d''\beta &= \frac{1}{2}\delta'' d'' d'\beta. = \frac{1}{2}\left(\delta'' - \frac{1}{3}(\delta'' L \delta'')\right) d'' d'\beta \\ &= \frac{1}{2} d''^* d'' d'\beta. \end{aligned} \tag{5.10}$$

So, if $d'' d' d''^* d''\beta = 0$, we have $d''(\frac{1}{2}) d''^* d'' d'\beta = 0$. This implies $d'' d'\beta = 0$. Hence we have

$$u = d' d''\alpha = d''(-d'\alpha). \tag{5.11}$$

So we have $J^2 = 0$. □

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