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A Turán-type inequality for the gamma function

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ABSTRACT

We prove that the following Turán-type inequality holds for Euler's gamma function. For all odd integers $n \geqslant 1$ and real numbers $x > 0$ we have

$$\alpha \leqslant \Gamma^{(n-1)}(x)\Gamma^{(n+1)}(x) - \Gamma^{(n)}(x)^2,$$

with the best possible constant

$$\alpha = \min_{1.5 \leqslant x \leqslant 2} \Gamma(x)^2 \psi'(x) = 0.6359\ldots.$$

1. Introduction

In 1950, Turán [22] proved the inequality

$$0 \leqslant P_{n-1}(x)P_{n+1}(x) - P_n(x)^2 \quad (-1 \leqslant x \leqslant 1,\ n = 1, 2, \ldots),$$

where P_n denotes the Legendre polynomial of degree n. This inequality has attracted much attention, so that numerous inequalities of the same type were published for other special functions. In 1986, Csordas, Norfolk and Varga [7] proved a Turán-type inequality, which is a necessary condition for the validity of the famous Riemann hypothesis. Inequalities of Turán-type are studied in various fields, like, for example, complex analysis, number theory, combinatorics, and theory of mean-values. Also, they have applications in statistics and control theory. We refer to [6,8,9,11,13–19,21] and the references given therein.

In this paper we are concerned with a Turán-type inequality for Euler's gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad (x > 0).$$

There exist many inequalities for this important function and its relatives (see [12,20]), but inequalities involving higher order derivatives of the Γ-function are difficult to find in the literature. It is known that the Cauchy–Schwarz inequality can be applied to obtain estimates for special functions; see, for example, [10] and [14]. We find for odd integers $n \geqslant 1$ and real numbers $x > 0$:

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$$\Gamma^{(n)}(x)^2 = \left(\int_0^\infty \left[e^{-t}t^{x-1}\log(t)^{n-1}\right]^{1/2}\left[e^{-t}t^{x-1}\log(t)^{n+1}\right]^{1/2}dt\right)^2$$
$$\leqslant \int_0^\infty e^{-t}t^{x-1}\log(t)^{n-1}\,dt\int_0^\infty e^{-t}t^{x-1}\log(t)^{n+1}\,dt = \Gamma^{(n-1)}(x)\Gamma^{(n+1)}(x).$$

(As usual, $\Gamma^{(0)} = \Gamma$.) Thus we have

$$0 \leqslant \Gamma^{(n-1)}(x)\Gamma^{(n+1)}(x) - \Gamma^{(n)}(x)^2 = \Delta_n(x).$$

(In what follows we maintain this notation.) Is it possible to replace the lower bound 0 by a positive constant? It is our aim to give an affirmative answer to this question. More precisely we determine the largest real number α, which is independent of n and x, such that we have

$$\alpha \leqslant \Delta_n(x) \quad (x > 0,\ n = 1, 3, 5, \ldots). \tag{1.1}$$

In the next section we collect some lemmas. They play an important role in the proof of our main result, given in Section 3. The numerical and algebraic computations have been carried out by 'MAPLE V, Release 5.1.'

2. Lemmas

The first five lemmas provide properties of the psi function, $\psi = \Gamma'/\Gamma$, and its derivatives. We denote by $x_0 = 1.461\ldots$ the only positive zero of ψ.

Lemma 1.

(i) *ψ is strictly increasing on* $(0, \infty)$.
(ii) *ψ' and ψ''' are positive and strictly decreasing on* $(0, \infty)$.
(iii) *ψ'' is negative and strictly increasing on* $(0, \infty)$.

Lemma 1 follows from the integral formula

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty e^{-xt}\frac{t^n}{1-e^{-t}}\,dt \quad (x > 0,\ n = 1, 2, \ldots), \tag{2.1}$$

see, for instance, [1, p. 260].

Lemma 2.

(i) *The function $x \mapsto x\psi(x)$ is strictly decreasing on $(0, r_0]$ and strictly increasing on $[r_0, \infty)$, where $r_0 = 0.216\ldots$.*
(ii) *The functions $x \mapsto x^2\psi'(x)$ and $x \mapsto -x^3\psi''(x)$ are strictly increasing on $(0, \infty)$.*

A proof for part (i) can be found in [3]. Part (ii) is a special case of a more general monotonicity theorem, which is proved in [4].

Lemma 3. *For all $x > 0$ we have*

$$0 < \psi'(x)\psi'''(x) - \psi''(x)^2. \tag{2.2}$$

An application of (2.1) and of the Cauchy–Schwarz inequality leads to (2.2). See [5] and [14] for corresponding inequalities involving higher derivatives.

Lemma 4. *For all integers $n \geqslant 0$ and $x > 0$ we have*

$$\psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n\frac{n!}{x^{n+1}}.$$

This recurrence formula is given in [1, p. 260].

Lemma 5. *For all $x > 0$ we have*

$$\log(x) - \frac{1}{x} < \psi(x) < \log(x), \tag{2.3}$$

$$\frac{1}{x} < \psi'(x) \quad \textit{and} \quad -\frac{1}{x^2} - \frac{2}{x^3} < \psi''(x). \tag{2.4}$$

Proof. A proof for (2.3) can be found in [2]. Using the integral representations (2.1) and

$$\frac{(n-1)!}{x^n} = \int_0^\infty e^{-xt} t^{n-1}\, dt \quad (x > 0,\ n = 1, 2, \ldots)$$

as well as

$$e^z - 1 - z > 0 \quad (z \neq 0)$$

we obtain for $x > 0$:

$$\psi'(x) - \frac{1}{x} = \int_0^\infty e^{-xt} \frac{e^{-t} - 1 + t}{1 - e^{-t}}\, dt > 0$$

and

$$\psi''(x) + \frac{1}{x^2} + \frac{2}{x^3} = \int_0^\infty e^{-(x+1)t} \frac{t(e^t - 1 - t)}{1 - e^{-t}}\, dt > 0.$$

This proves (2.4). $\square$

The next three lemmas present properties of Δ_n.

Lemma 6. *Let $n \geqslant 1$ be an odd integer. Then, Δ_n is convex on $(0, \infty)$.*

Proof. We set $n = 2k - 1$ with $k \geqslant 1$. Differentiation gives for $x > 0$:

$$\Delta''_{2k-1}(x) = \Gamma^{(2k-2)}(x)\Gamma^{(2k+2)}(x) - \Gamma^{(2k)}(x)^2.$$

Applying the Cauchy–Schwarz inequality yields $\Delta''_{2k-1}(x) \geqslant 0$. $\square$

Remark. Let $n \geqslant 1$ be an odd integer. Since Δ_n is nonnegative and convex on $(0, \infty)$, we conclude that the following Schur-type functional inequality holds for all real numbers $x, y, z > 0$:

$$0 \leqslant (x - y)(x - z)\Delta_n(x) + (y - x)(y - z)\Delta_n(y) + (z - x)(z - y)\Delta_n(z),$$

see [23].

Lemma 7. *For all $x > 0$ we have*

$$\Delta_1(x) < \Delta_3(x).$$

Proof. We have

$$\Delta_3(x) - \Delta_1(x) = H(x)\Gamma(x)^2,$$

where

$$\begin{aligned} H(x) = {} & \psi(x)^2\psi'''(x) + 2\psi(x)^3\psi''(x) + 3\psi'(x)^3 + \psi'(x)\psi'''(x) + \psi'(x)\psi(x)^4 \\ & - 2\psi(x)\psi'(x)\psi''(x) - \psi'(x) - \psi''(x)^2. \end{aligned}$$

To prove that H is positive on $(0, \infty)$ we distinguish four cases.

Case 1. $0 < x \leqslant 1/4$.

Applying Lemmas 1 and 3 gives

$$H(x) > I(x) + J(x),$$

where

$$I(x) = 2\psi(x)^3\psi''(x) - 2\psi(x)\psi'(x)\psi''(x) \quad \text{and} \quad J(x) = 3\psi'(x)^3 - \psi'(x).$$

We have

$$I(x) = 2\psi(x)\psi''(x)K(x),$$

where

$$K(x) = \psi(x)^2 - \psi'(x).$$

Using Lemmas 1 and 4 leads to

$$-xK(x) = -x\psi(x+1)^2 + 2\psi(x+1) + x\psi'(x+1) < 2\psi(x+1) + x\psi'(x+1) \leqslant 2\psi(5/4) + \frac{1}{4}\psi'(1) = -0.04\ldots.$$

This proves $I(x) > 0$.

Furthermore, Lemma 1 yields

$$\frac{J(x)}{\psi'(x)} = 3\psi'(x)^2 - 1 \geqslant 3\psi'(1/4)^2 - 1 = 886.24\ldots.$$

Hence, $J(x) > 0$.

Case 2. $1/4 \leqslant x \leqslant x_0$.

Lemmas 1 and 3 give

$$H(x) \geqslant 3\psi'(x)^3 + \psi'(x)\psi(x)^4 - 2\psi(x)\psi'(x)\psi''(x) - \psi'(x) = \frac{\psi'(x)M(x)}{x^4},$$

where

$$M(x) = 3\bigl[x^2\psi'(x)\bigr]^2 + \bigl[x\psi(x)\bigr]^4 - 2\bigl[-x\psi(x)\bigr]\bigl[-x^3\psi''(x)\bigr] - x^4.$$

Let $1/4 \leqslant r \leqslant x \leqslant s \leqslant x_0$. Applying Lemma 2 we obtain

$$M(x) \geqslant 3\bigl[r^2\psi'(r)\bigr]^2 + \bigl[s\psi(s)\bigr]^4 - 2\bigl[-r\psi(r)\bigr]\bigl[-s^3\psi''(s)\bigr] - s^4 = N(r,s), \quad \text{say}.$$

Since the numbers

$$N(0.25, 0.46), \quad N(0.46, 0.75), \quad N(0.75, 1.18), \quad N(1.18, x_0)$$

are positive, we conclude that $M(x) > 0$ for $x \in [1/4, x_0]$. This implies that H is positive on $[1/4, x_0]$.

Case 3. $x_0 \leqslant x \leqslant 5$.

Using (2.2) leads to

$$H(x) > \psi(x)^2\psi'''(x) + 2\psi(x)^3\psi''(x) + 3\psi'(x)^3 + \psi'(x)\psi(x)^4 - 2\psi(x)\psi'(x)\psi''(x) - \psi'(x).$$

Let $x_0 \leqslant r \leqslant x \leqslant s \leqslant 5$. Then we get from Lemma 1:

$$H(x) > \psi(r)^2\psi'''(s) + 2\psi(s)^3\psi''(r) + 3\psi'(s)^3 + \psi'(s)\psi(r)^4 - 2\psi(r)\psi'(s)\psi''(s) - \psi'(r) = P(r,s), \quad \text{say}.$$

We have

$$P(x_0, 1.8) > 0 \quad \text{and} \quad P\bigl(1.8 + k/100, 1.8 + (k+1)/100\bigr) > 0 \quad \text{for } k = 0, 1, \ldots, 319.$$

This implies that $H(x) > 0$ for $x \in [x_0, 5]$.

Case 4. $x \geqslant 5$.

Applying Lemmas 1, 3, 5 and $\log(5) - 1/5 > 1$ gives

$$H(x) > \psi'(x)\big(\psi(x)^4 - 1\big) + 2\psi(x)^3\psi''(x)$$
$$> \frac{1}{x}\left[\left(\log(x) - \frac{1}{x}\right)^4 - 1\right] + 2\log(x)^3\left(-\frac{1}{x^2} - \frac{2}{x^3}\right) = Q(x), \quad \text{say}. \tag{2.5}$$

Moreover, we have

$$\frac{x}{\log(x)^4} Q(x) = \left(1 - \frac{1}{x\log(x)}\right)^4 - \frac{1}{\log(x)^4} - \frac{2}{\log(x)}\left(\frac{1}{x} + \frac{2}{x^2}\right)$$
$$\geqslant \left(1 - \frac{1}{5\log(5)}\right)^4 - \frac{1}{\log(5)^4} - \frac{2}{\log(5)}\left(\frac{1}{5} + \frac{2}{25}\right) = 0.091\ldots. \tag{2.6}$$

From (2.5) and (2.6) we obtain $H(x) > 0$ for $x \geqslant 5$. $\square$

Lemma 8. *For all $x > 0$ we have*

$$\Delta_1(x) \geqslant \min_{1.5\leqslant t\leqslant 2} \Delta_1(t) = 0.6359\ldots. \tag{2.7}$$

Proof. Let $\tilde{x} = 1.8746$. Then, $\Delta_1'(\tilde{x}) > 0$. Applying Lemma 6 gives for $x \geqslant 1.5$:

$$\Delta_1(x) \geqslant \Delta_1(\tilde{x}) + (x - \tilde{x})\Delta_1'(\tilde{x}) \geqslant \Delta_1(\tilde{x}) + (1.5 - \tilde{x})\Delta_1'(\tilde{x}) = 0.63596\ldots.$$

This yields

$$\min_{1.5\leqslant x\leqslant 2} \Delta_1(x) \geqslant 0.63596.$$

Furthermore, we have

$$\min_{1.5\leqslant x\leqslant 2} \Delta_1(x) \leqslant \Delta_1(\tilde{x}) = 0.635994\ldots \leqslant 0.635995.$$

Thus, $\min_{1.5\leqslant x\leqslant 2} \Delta_1(x) = 0.6359\ldots$.

Next, we show that $\Delta_1(x) \geqslant 0.639$ for $x \in (0, 1.5] \cup [2, \infty)$. If $x \in (0, 1.5]$, then

$$\Delta_1(x) = \Gamma(x)^2\psi'(x) \geqslant \Gamma(x_0)^2\psi'(1.5) = 0.73\ldots.$$

Let $x \geqslant 2$. We define

$$\tau(x) = \Gamma(x) - 0.498x \quad \text{and} \quad \bar{x} = 2.09.$$

Since τ is convex on $[2, \infty)$ and $\tau'(\bar{x}) > 0$, we get

$$\tau(x) \geqslant \tau(\bar{x}) + (x - \bar{x})\tau'(\bar{x}) \geqslant \tau(\bar{x}) + (2 - \bar{x})\tau'(\bar{x}) = 0.0005\ldots.$$

Hence,

$$\frac{\Gamma(x)}{x} > 0.498.$$

Using this estimate and Lemma 2(ii) we obtain

$$\Delta_1(x) = \left(\frac{\Gamma(x)}{x}\right)^2\big[x^2\psi'(x)\big] > (0.498)^2 \cdot 4\psi'(2) = 0.639\ldots.$$

The proof of Lemma 8 is complete. $\square$

3. Main result

With the help of Lemmas 7 and 8 we are now in a position to determine the best possible constant lower bound α in inequality (1.1).

Theorem. *For all odd integers $n \geqslant 1$ and real numbers $x > 0$ we have*

$$\alpha \leqslant \Gamma^{(n-1)}(x)\Gamma^{(n+1)}(x) - \Gamma^{(n)}(x)^2, \tag{3.1}$$

with the best possible constant

$$\alpha = \min_{1.5 \leqslant x \leqslant 2} \Gamma(x)^2 \psi'(x) = 0.6359\ldots. \tag{3.2}$$

Proof. We have

$$\begin{aligned}
\Delta_n(x) &= \int_0^\infty e^{-t}t^{x-1}\log(t)^{n-1}\,dt \int_0^\infty e^{-t}t^{x-1}\log(t)^{n+1}\,dt - \left(\int_0^\infty e^{-t}t^{x-1}\log(t)^n\,dt\right)^2 \\
&= \int_0^\infty\int_0^\infty e^{-s-t}(st)^{x-1}\left[\log(s)^{n-1}\log(t)^{n+1} - \log(s)^n\log(t)^n\right] ds\,dt \\
&= \int_0^\infty\int_0^\infty e^{-s-t}(st)^{x-1}\left[\log(s)\log(t)\right]^{n-1}\left[\log(t)^2 - \log(s)\log(t)\right] ds\,dt \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty e^{-s-t}(st)^{x-1}\left[\log(s)\log(t)\right]^{n-1}\left[\log(s) - \log(t)\right]^2 ds\,dt.
\end{aligned}$$

Let $k \geqslant 1$ be an integer. We get the integral representation

$$\Delta_{2k+1}(x) - \Delta_{2k-1}(x) = \frac{1}{2}\int_0^\infty\int_0^\infty e^{-s-t}(st)^{x-1}\left[\log(s)^2\log(t)^2\right]^{k-1}\left[\log(s)^2\log(t)^2 - 1\right]\left[\log(s) - \log(t)\right]^2 ds\,dt.$$

Using the elementary inequality

$$z^{k-1}(z-1) \geqslant z-1 \quad (z \geqslant 0,\ k = 1, 2, \ldots)$$

with $z = \log(s)^2\log(t)^2$ gives

$$\Delta_{2k+1}(x) - \Delta_{2k-1}(x) \geqslant \Delta_3(x) - \Delta_1(x). \tag{3.3}$$

From (3.3) and Lemma 7 we obtain

$$\Delta_{2k-1}(x) < \Delta_{2k+1}(x) \quad (x > 0,\ k = 1, 2, \ldots). \tag{3.4}$$

Combining (3.4) with (2.7) yields (3.1). Moreover, we conclude that the lower bound given in (3.2) is best possible. □

Remarks. (1) In view of (3.1) it is natural to look for an upper bound for $\Delta_n(x)$, which is valid for all odd $n \geqslant 1$ and real $x > 0$. We show that such a bound does not exist. Using the Leibniz rule for differentiation gives for $k \geqslant 0$:

$$\Gamma^{(k)}(x) = \left(x^{-1}\Gamma(x+1)\right)^{(k)} = \sum_{\nu=0}^{k}\binom{k}{\nu}\left(x^{-1}\right)^{(\nu)}\Gamma^{(k-\nu)}(x+1).$$

We have

$$\left(x^{-1}\right)^{(\nu)} = (-1)^\nu \nu! x^{-\nu-1},$$

so that we obtain

$$x^{k+1}\Gamma^{(k)}(x) = \sum_{\nu=0}^{k-1}\binom{k}{\nu}(-1)^\nu \nu! x^{k-\nu}\Gamma^{(k-\nu)}(x+1) + (-1)^k k!\Gamma(x+1).$$

This yields

$$\lim_{x\to 0} x^{k+1}\Gamma^{(k)}(x) = (-1)^k k!.$$

Since

$$x^{2n+2}\Delta_n(x) = x^n\Gamma^{(n-1)}(x)\cdot x^{n+2}\Gamma^{(n+1)}(x) - \left[x^{n+1}\Gamma^{(n)}(x)\right]^2,$$

we get for $n \geqslant 1$:

$$\lim_{x\to 0} x^{2n+2}\Delta_n(x) = (n-1)!n!.$$

This leads to

$$\lim_{x\to 0}\Delta_n(x) = \infty. \tag{3.5}$$

(2) From (3.5) we also conclude that there is no constant upper bound for $\Delta_n(x)$, which holds for all even integers $n \geqslant 2$ and positive real numbers x. Does there exist a lower bound, which is independent of n and x? We have

$$\Delta_2(1.13) \approx -0.9, \qquad \Delta_4(1.42) \approx -6, \qquad \Delta_6(1.69) \approx -97, \qquad \Delta_8(1.94) \approx -2493, \qquad \Delta_{10}(2.18) \approx -90\,701.$$

It is tempting to *conjecture* that there is no real number c such that we have $\Delta_n(x) \geqslant c$ for all even $n \geqslant 2$ and $x > 0$.

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