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Rates of convergence in certain limit theorem for extreme values

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ABSTRACT

Let $X^*_{n1},\dots,X^*_{nn}$ be independent random variables with the common negative binomial distribution with parameters $r>0$ and $1/n$, where r is not necessarily an integer. We determine the limiting distribution of the random variable $M^*_n=\max\{X^*_{n1},\dots,X^*_{nn}\}$ as $n\to\infty$, corresponding normalizing constants and the rate of convergence. For an integer r the connection with certain waiting time problems is indicated.

1. Introduction

Negative binomial distribution appears naturally in connection with the number of independent trials necessary to obtain r occurrences of an event that has the same probability p of occurring in each trial. It is well known that negative binomial distribution does not belong to any of maximum domains of attraction of extreme value distributions. But if there is n possible outcomes of each trial and the probability p of occurring of each outcome in every trial is equal to $1/n$, then the number of trials needed to obtain all outcomes at least r times is the maximum of n dependent random variables with the same negative binomial distribution which limiting distribution is the Gumbel double exponential distribution. This problem and some of its variations have been considered by many authors and different methods were employed. See, for example, Erdös and Rényi [2], Baum and Bilingsley [1], Holst [6–8], Johnson and Kotz [10], Flato [4], Mladenović [13–15]. All employed methods are based significantly on the fact that the parameter r of the corresponding negative binomial distribution is an integer. Some approaches involved in consideration independent random variables with the same negative binomial distribution also.

It is well known that the negative binomial distribution can be defined for any positive r, not necessarily being an integer. For the genesis of the negative binomial distribution, historical remarks, some properties and applications see Johnson and Kotz [9] and Feller [3]. In this paper we shall determine the limiting distribution of the random variable $M^*_n=\max\{X^*_{n1},\dots,X^*_{nn}\}$ as $n\to\infty$, where $X^*_{n1},\dots,X^*_{nn}$ are independent random variables with the common negative binomial distribution with parameters $r>0$ and $p=1/n$. Corresponding normalizing constants for the maximum M_n and the rate of convergence are also determined. This will give a new insight to different rates of convergence in the cases $r=1$ and $r\neq 1$. Rates of convergence in limit theorems for maxima have been studied by many authors, see Leadbetter and Rootzén [12] and references therein.

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2. Some preliminaries and notation

Let $X_{n1}^*, X_{n2}^*, \dots, X_{nn}^*$ be independent random variables with the common negative binomial distribution with parameters $r > 0$ and $p \in (0,1)$, where r is not necessarily an integer, that is

$$P\{X_{nj}^* = k\} = \binom{r+k-1}{k}(1-p)^k p^r, \quad k = 0,1,2,\dots, \tag{2.1}$$

where

$$\binom{r+k-1}{k} = \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}, \tag{2.2}$$

and $\Gamma(r)$ is the Gamma function given by

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt, \quad r > 0. \tag{2.3}$$

We shall use Stirling's formula and its consequence, see Graham et al. [5]:

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \cdot \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + o\left(\frac{1}{x^2}\right)\right\}, \quad x \to \infty, \tag{2.4}$$

$$\frac{\Gamma(x+t)}{\Gamma(x)} = x^t \cdot \left\{1 + \frac{t^2-t}{2x} + o\left(\frac{1}{x}\right)\right\}, \quad x \to \infty \ (t > 0 \text{ fixed}). \tag{2.5}$$

The following property of the Gamma function will also be used:

$$\Gamma(x)\Gamma(a-x) \geqslant \Gamma(y)\Gamma(a-y), \quad \text{for } 1 < x \leqslant y \leqslant \frac{a}{2}, \tag{2.6}$$

where $a > 2$ is a constant. In order to clarify the property (2.6), define the function $b(x) = \frac{\Gamma(x)\Gamma(a-x)}{\Gamma(a)}$, $0 < x < a$. Then, $b(x) = B(x, a-x) = \int_0^1 t^{x-1}(1-t)^{a-x-1} dt$, where $B(\cdot,\cdot)$ is the usual notation for the Beta function. Obviously $b(x) = b(a-x)$ for $0 < x < a$. It is easy to check that $b''(x) > 0$ for $1 < x < a-1$. Hence, the function $b(x)$ is convex on the interval $(1, a-1)$, decreases on $(1, a/2]$, and the property (2.6) follows easily.

3. Results

Theorem 3.1. *Let $M_n^* = \max\{X_{n1}, \dots, X_{nn}^*\}$, where $X_{n1}^*, \dots, X_{nn}^*$ are independent random variables with the common negative binomial distribution with parameters $r > 0$ and $p = 1/n$, and let $u_n(x) = n(x + \ln n + (r-1)\ln\ln n - \ln\Gamma(r))$. For any $x \in \mathbb{R}$ the following equality then holds*

$$\lim_{n\to\infty} P\{M_n^* \leqslant u_n(x)\} = \exp(-e^{-x}). \tag{3.1}$$

Let us denote $\Delta_n(r,x) = P\{M_n^* \leqslant u_n(x)\} - \exp(-e^{-x})$. The rate of convergence of the maximum M_n^*, as $n \to \infty$, is given by:

$$\Delta_n(r,x) \sim \frac{e^{-x}\exp(-e^{-x})}{2}\frac{\ln n}{n}, \quad \text{if } r = 1, \tag{3.2}$$

$$\Delta_n(r,x) \sim -(r-1)^2 e^{-x}\exp(-e^{-x})\frac{\ln\ln n}{\ln n}, \quad \text{if } r \neq 1. \tag{3.3}$$

Remark 3.2. For positive integers r relations (3.2) and (3.3) were proved by Mladenović [13,14] in i.i.d. settings as well as for dependent random variables that appeared in connection with the coupon collector's problem. The techniques employed there used significantly the fact that r is a positive integer and combinatorial interpretation of the corresponding negative binomial distribution, and it cannot be applied if r is not an integer.

4. Proof of Theorem 3.1

Let $F_{n,r}(x)$ be the distribution function of a random variable X with the negative binomial distribution with parameters $r > 0$ and $p = 1/n$. For any positive integer m the tail $P\{X > m\}$ is then given by

$$1 - F_{n,r}(m) = \sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1 - \frac{1}{n}\right)^k \left(\frac{1}{n}\right)^r. \tag{4.1}$$

Lemma 4.1. *Let x be a real number and $r > 0$. The asymptotic equality*

$$m = n\big(\ln n + (r-1)\ln\ln n + x - \ln\Gamma(r) + o(1)\big), \quad n \to \infty, \tag{4.2}$$

is the necessary and sufficient condition for the asymptotic relation

$$\sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1-\frac{1}{n}\right)^k\left(\frac{1}{n}\right)^r \sim \frac{e^{-x}}{n}, \quad \text{as } n \to \infty. \tag{4.3}$$

Proof. (a) Let $l = n[\ln\ln n]$. We shall first prove the following statement: if (4.3) holds, then $m \geqslant l$.

(a1) Case $r \geqslant 1$. Suppose that $m < l$. Note that the inequalities

$$\frac{\Gamma(r+k)}{\Gamma(k+1)} \geqslant \frac{\Gamma(r+l)}{\Gamma(1+l)} \quad \text{for } k \in \{l, l+1, \ldots, 2l\},$$

hold as consequences of (2.6) and the facts $1+l = \min\{r+k, r+l, k+1, 1+l\}$ and $(r+k)+(1+l) = (r+l)+(k+1)$. Now, it follows that

$$\begin{aligned}\sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1-\frac{1}{n}\right)^k\frac{1}{n^r} &\geqslant \sum_{k=l}^{2l} \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1-\frac{1}{n}\right)^k\frac{1}{n^r}\\
&\geqslant \frac{1}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{2l}\frac{1}{n^r}\sum_{k=l}^{2l}\frac{\Gamma(r+k)}{\Gamma(k+1)} \geqslant \frac{1}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{2l}\frac{l+1}{n^r}\cdot\frac{\Gamma(r+l)}{\Gamma(1+l)}\\
&\geqslant \frac{1}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{2l}\frac{n+1}{n^r}\cdot\frac{\Gamma(r+l)}{\Gamma(1+l)} \sim \frac{1}{\Gamma(r)}\cdot\frac{1}{\ln^2 n}\cdot\frac{1}{n^{r-1}}\cdot l^{r-1} \geqslant \frac{1}{\Gamma(r)\ln^2 n},\end{aligned}$$

and consequently (4.3) does not hold.

(a2) Case $0 < r < 1$. Suppose that $m < l$. In this case we shall use the inequalities

$$\frac{\Gamma(r+k)}{\Gamma(k+1)} \geqslant \frac{\Gamma(r+l+n)}{\Gamma(1+l+n)} \quad \text{for } k \in \{l, l+1, \ldots, l+n\},$$

that are consequences of (2.6) and the facts $r+k = \min\{r+k, r+l+n, k+1, 1+l+n\}$ and $(r+k)+(1+l+n) = (r+l+n)+(k+1)$. It follows that

$$\begin{aligned}\sum_{k=l}^{l+n} \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1-\frac{1}{n}\right)^k\frac{1}{n^r} &\geqslant \frac{1}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{l+n}\frac{1}{n^r}\sum_{k=l}^{l+n}\frac{\Gamma(r+k)}{\Gamma(k+1)}\\
&\geqslant \frac{1}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{l+n}\frac{n+1}{n^r}\cdot\frac{\Gamma(r+l+n)}{\Gamma(1+l+n)} \sim \frac{1}{\Gamma(r)}\cdot\frac{1}{e\ln n}\cdot\frac{1}{n^{r-1}}\cdot(l+n)^{r-1}\\
&\geqslant \frac{1}{e\Gamma(r)\ln n}\cdot\frac{l^{r-1}}{n^{r-1}} \sim \frac{1}{e\Gamma(r)\ln n}\cdot\frac{1}{(\ln\ln n)^{r-1}},\end{aligned}$$

and again (4.3) does not hold.

(b) Next, we shall prove the following statement: if (4.3) holds, then $m \leqslant 2n\ln n$.

(b1) Case $r \geqslant 1$. Suppose that $m > 2n\ln n$. Since

$$\frac{1}{\Gamma(r)} < 2, \qquad \left(1-\frac{1}{n}\right)^n < \frac{1}{e}, \qquad \frac{\Gamma(r+k)}{\Gamma(k+1)} < 2k^{r-1} \quad \text{for } k \geqslant k_0,$$

we obtain that

$$\begin{aligned}\sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1-\frac{1}{n}\right)^k &\leqslant 4\sum_{k>2n\ln n} k^{r-1}\left(1-\frac{1}{n}\right)^k\\
&= 4\Bigg\{\sum_{2n\ln n<k\leqslant 2n\ln(2n)} k^{r-1}\left(1-\frac{1}{n}\right)^k + \sum_{2n\ln(2n)<k\leqslant 2n\ln(4n)} k^{r-1}\left(1-\frac{1}{n}\right)^k\\
&\quad + \sum_{2n\ln(4n)<k\leqslant 2n\ln(8n)} k^{r-1}\left(1-\frac{1}{n}\right)^k + \cdots\Bigg\}\end{aligned}$$

$$\begin{aligned}
&\leqslant 4\left\{\left(2n\ln(2n)\right)^{r-1}\left(1-\frac{1}{n}\right)^{2n\ln n}(2n\ln 2+1)\right.\\
&\quad\left.+\left(2n\ln(4n)\right)^{r-1}\left(1-\frac{1}{n}\right)^{2n\ln(2n)}(2n\ln 2+1)+\cdots\right\}\\
&\leqslant 4(2n)^{r-1}(2n\ln 2+1)\left\{\frac{(\ln(2n))^{r-1}}{e^{2\ln n}}+\frac{(\ln(4n))^{r-1}}{e^{2\ln(2n)}}+\cdots\right\}\\
&=\frac{4(2n)^{r-1}(2n\ln 2+1)}{n^{3/2}}\left\{\frac{(\ln(2n))^{r-1}}{n^{1/2}}+\frac{(\ln(4n))^{r-1}}{(2n)^{1/2}\cdot 2^{3/2}}+\frac{(\ln(8n))^{r-1}}{(4n)^{1/2}\cdot 4^{3/2}}+\cdots\right\}\\
&\leqslant\frac{4K_1(2n)^{r-1}(2n\ln 2+1)}{n^{3/2}}\left(1+\frac{1}{2^{3/2}}+\frac{1}{4^{3/2}}+\cdots\right)\leqslant\frac{K_2n^r}{n^{3/2}},
\end{aligned}$$

for sufficiently large n and some constants $K_1>0$ and $K_2>0$. Consequently (4.3) does not hold for $m>2n\ln n$.

(b2) Case $0<r<1$. Suppose that $m>2n\ln n$. Using similar arguments as in the case (b1) we obtain that

$$\begin{aligned}
\sum_{k=m+1}^{\infty}\frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}\left(1-\frac{1}{n}\right)^k &\leqslant 4\left\{(2n\ln n)^{r-1}\left(1-\frac{1}{n}\right)^{2n\ln n}(2n\ln 2+1)\right.\\
&\quad\left.+\left(2n\ln(2n)\right)^{r-1}\left(1-\frac{1}{n}\right)^{2n\ln(2n)}(2n\ln 2+1)+\cdots\right\}\\
&\leqslant\frac{4(2n)^{r-1}(2n\ln 2+1)}{n^{3/2}}\left\{\frac{(\ln n)^{r-1}}{n^{1/2}}+\frac{(\ln(2n))^{r-1}}{(2n)^{1/2}\cdot 2^{3/2}}+\frac{(\ln(4n))^{r-1}}{(4n)^{1/2}\cdot 4^{3/2}}+\cdots\right\}\\
&\leqslant\frac{K_3n^r}{n^{3/2}},
\end{aligned}$$

for sufficiently large n and some constant $K_3>0$. Consequently (4.3) does not hold for $m>2n\ln n$.

(c) We shall use the following notation: $1-\frac{1}{n}=q$, $r-1=s$, $m+1=p$, $k-1=i$, $k+1=j$, $s-[s]=r-[r]=\vartheta$, $l=n[\ln\ln n]$ and

$$\Psi(p,s)=\sum_{k=p}^{\infty}\frac{\Gamma(s+k+1)q^k}{\Gamma(s+1)\Gamma(k+1)}.$$

Relation (4.3) can be equivalently rewritten as follows:

$$\Psi(p,s)\sim n^se^{-x},\quad \text{as } n\to\infty. \tag{4.4}$$

Since $1-q=\frac{1}{n}$, we obtain for $r\geqslant 1$,

$$\begin{aligned}
\frac{\Psi(p,s)}{n}&=\Psi(p,s)-q\Psi(p,s)\\
&=\frac{\Gamma(s+p+1)q^p}{\Gamma(s+1)\Gamma(p+1)}+\left\{\sum_{k=p+1}^{\infty}\frac{\Gamma(s+k+1)q^k}{\Gamma(s+1)\Gamma(k+1)}-\sum_{k=p}^{\infty}\frac{\Gamma(s+k+1)q^{k+1}}{\Gamma(s+1)\Gamma(k+1)}\right\}\\
&=\frac{\Gamma(s+p+1)q^p}{\Gamma(s+1)\Gamma(p+1)}+\left\{\sum_{i=p}^{\infty}\frac{\Gamma(s+i+2)q^{i+1}}{\Gamma(s+1)\Gamma(i+2)}-\sum_{k=p}^{\infty}\frac{\Gamma(s+k+1)q^{k+1}}{\Gamma(s+1)\Gamma(k+1)}\right\}\\
&=\frac{\Gamma(s+p+1)q^p}{\Gamma(s+1)\Gamma(p+1)}+\sum_{k=p}^{\infty}\frac{\Gamma(s+k+1)q^{k+1}}{\Gamma(s+1)\Gamma(k+1)}\left(\frac{s+k+1}{k+1}-1\right)\\
&=\frac{\Gamma(s+p+1)q^p}{\Gamma(s+1)\Gamma(p+1)}+\sum_{j=p+1}^{\infty}\frac{\Gamma(s+j)q^j}{\Gamma(s)\Gamma(j+1)},
\end{aligned}$$

and consequently

$$\Psi(p,s)=\frac{nq^p\Gamma(s+p+1)}{\Gamma(s+1)\Gamma(p+1)}+n\Psi(p+1,s-1),$$
$$\Psi(p+1,s-1)=\frac{nq^{p+1}\Gamma(s+p+1)}{\Gamma(s)\Gamma(p+2)}+n\Psi(p+2,s-2),$$
$$\vdots$$
$$\Psi\big(p+[s]-1,1+\vartheta\big)=\frac{nq^{p+[s]-1}\Gamma(s+p+1)}{\Gamma(2+\vartheta)\Gamma(p+[s])}+n\Psi\big(p+[s],\vartheta\big).$$

Multiplying previous equations by 1, $n,\dots,n^{[s]-1}$ respectively, and summing them we obtain the following statement: for $r\geqslant 1$ the relation (4.3) is equivalent to the following two relations:

$$\Psi(p,s)=\frac{(n-1)^{m+1}\Gamma(r+m+1)}{n^m}\left\{\frac{1}{\Gamma(r)\Gamma(m+2)}+\frac{n-1}{\Gamma(r-1)\Gamma(m+3)}+\cdots+\frac{(n-1)^{[r]-2}}{\Gamma(2+\vartheta)\Gamma(m+[r])}\right\}$$
$$+n^{[r]-1}\Psi\big(m+[r],\vartheta\big)\sim n^{r-1}e^{-x},\quad n\to\infty, \tag{4.5}$$

$$\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)}+\frac{(n-1)\Gamma(r+m+1)}{\Gamma(r-1)\Gamma(m+3)}+\cdots+\frac{(n-1)^{[r]-2}\Gamma(r+m+1)}{\Gamma(2+\vartheta)\Gamma(m+[r])}$$
$$+\frac{n^{m+[r]-1}\Psi(m+[r],\vartheta)}{(n-1)^{m+1}}\sim\frac{n^{m+r-1}e^{-x}}{(n-1)^{m+1}},\quad n\to\infty. \tag{4.6}$$

Note also the following: if $1\leqslant r<2$, then the left-hand side of Eq. (4.6) reduces to the last addend only.

(d) Next, we shall prove that the left-hand side of Eq. (4.6) is asymptotically equivalent to the addend $\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)}$. The proof will be given separately in the cases $r\geqslant 2$ and $r\in[1,2)$. We suppose that $n[\ln\ln n]\leqslant m\leqslant 2n\ln n$.

(d1) Case $r\geqslant 2$. Using (2.5) we obtain that $\frac{\Gamma(n+\vartheta)}{\Gamma(n)}\leqslant 2n^{\vartheta}$ for sufficiently large n, say $n\geqslant n_0$. Now, we obtain

$$\begin{aligned}\Psi\big(m+[r],\vartheta\big)&=\sum_{k=m+[r]}^{\infty}\frac{\Gamma(\vartheta+k+1)q^k}{\Gamma(\vartheta+1)\Gamma(k+1)}\leqslant\frac{2}{\Gamma(\vartheta+1)}\sum_{k=m+[r]}^{\infty}(k+1)^{\vartheta}q^k\\&=\frac{2}{\Gamma(\vartheta+1)}\sum_{k=m+[r]}^{\infty}\frac{(k+1)q^k}{(k+1)^{1-\vartheta}}\leqslant 4\sum_{k=m+[r]}^{\infty}\frac{(k+1)q^k}{(m+[r])^{1-\vartheta}}\\&=\frac{(1-\frac{1}{n})^{m+[r]}(m+[r]+n)}{(m+[r])^{1-\vartheta}\cdot n}\leqslant\frac{C(1-\frac{1}{n})^m m^{\vartheta}}{n},\end{aligned}$$

for some $C>0$ and $n\geqslant n_0$. For the last addend on the left-hand side of (4.6) we obtain the following bound:

$$\frac{n^{m+[r]-1}\Psi(m+[r],\vartheta)}{(n-1)^{m+1}}\leqslant\frac{n^{m+[r]-1}C(n-1)^m m^{\vartheta}}{(n-1)^{m+1}n^{m+1}}=\frac{Cn^{[r]-2}m^{\vartheta}}{n-1}. \tag{4.7}$$

Since

$$\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)}\sim\frac{m^{r-1}}{\Gamma(r)}\quad\text{and}\quad\frac{n^{[r]-2}m^{\vartheta}}{n}=o\big(m^{r-1}\big),\quad\text{as } n\to\infty,$$

it follows that

$$\frac{n^{m+[r]-1}\Psi(m+[r],\vartheta)}{(n-1)^{m+1}}=o\left(\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)}\right),\quad\text{as } n\to\infty. \tag{4.8}$$

For $\nu\in\{1,2,\dots,[r]-2\}$ the following relations hold:

$$\frac{(n-1)^{\nu}\Gamma(r+m+1)}{\Gamma(r-\nu)\Gamma(m+2+\nu)}\cdot\left\{\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)}\right\}^{-1}$$
$$=\frac{\Gamma(r)}{\Gamma(r-\nu)}\cdot\frac{(n-1)^{\nu}\Gamma(m+2)}{\Gamma(m+2+\nu)}\sim\frac{\Gamma(r)}{\Gamma(r-\nu)}\cdot\frac{n^{\nu}}{m^{\nu}}=o(1),\quad\text{as } n\to\infty. \tag{4.9}$$

It follows from (4.8) and (4.9) that the left-hand side of (4.6) is asymptotically equivalent to $\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)}$ for $r\geqslant 2$.

(d2) Case $r\in[1,2)$. Eq. (4.6) can be rewritten as follows

$$\frac{n^m\Psi(m+1,r-1)}{(n-1)^{m+1}}\sim\frac{n^{m+r-1}e^{-x}}{(n-1)^{m+1}},\quad n\to\infty.$$

The next goal is to prove the following relation:

$$\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)} \sim \frac{n^m \Psi(m+1, r-1)}{(n-1)^{m+1}}, \quad \text{as } n \to \infty,$$

or, equivalently,

$$\frac{\Gamma(r+m+1)}{\Gamma(m+2)} \sim \frac{n^m}{(n-1)^{m+1}} \sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)q^k}{\Gamma(k+1)}, \quad \text{as } n \to \infty. \tag{4.10}$$

Using (2.5) we obtain that

$$\frac{\Gamma(r+k)}{\Gamma(k+1)} = (k+1)^{r-1}\left(1 + \frac{\psi}{k+1}\right), \quad \text{where } |\psi| \leqslant Q,$$

for some constant Q and sufficiently large k, say $k > k_0$. For $m > k_0$ we obtain

$$\begin{aligned}
\sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)q^k}{\Gamma(k+1)} &= \sum_{k=m+1}^{\infty} (k+1)^{r-1} q^k + \sum_{k=m+1}^{\infty} (k+1)^{r-2} \psi q^k \\
&\sim \sum_{k=m+1}^{\infty} (k+1)^{r-1} q^k = \sum_{k=m+1}^{\infty} \frac{(k+1)q^k}{(k+1)^{2-r}} \leqslant \frac{1}{(m+1)^{2-r}} \sum_{k=m+1}^{\infty} kq^k \\
&= \frac{1}{(m+1)^{2-r}} \left\{ n(m+1)q^{m+1} + n^2 q^{m+2} \right\} \sim \frac{nmq^{m+1}}{(m+1)^{2-r}} \sim nm^{r-1} q^{m+1}.
\end{aligned}$$

On the other hand we obtain

$$\sum_{k=m+1}^{\infty} (k+1)^{r-1} q^k \geqslant (m+2)^{r-1} \sum_{k=m+1}^{\infty} q^k = (m+2)^{r-1} \frac{q^{m+1}}{1-q} \sim nm^{r-1} q^{m+1}.$$

All asymptotic relations hold as $n \to \infty$. It follows that

$$\sum_{k=m+1}^{\infty} (k+1)^{r-1} q^k \sim nm^{r-1} q^{m+1} = nm^{r-1} \frac{(n-1)^{m+1}}{n^{m+1}}.$$

Consequently

$$\frac{n^m}{(n-1)^{m+1}} \sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)q^k}{\Gamma(k+1)} \sim \frac{n^m}{(n-1)^{m+1}} \cdot \frac{nm^{r-1}(n-1)^{m+1}}{n^{m+1}} = m^{r-1} \sim \frac{\Gamma(r+m+1)}{\Gamma(m+2)}, \quad n \to \infty.$$

Hence, for $r \geqslant 1$ relation (4.3) is equivalent to the following one

$$\frac{\Gamma(r+m+1)}{\Gamma(r)\Gamma(m+2)} \sim \frac{n^{m+r-1} e^{-x}}{(n-1)^{m+1}}, \quad \text{as } n \to \infty. \tag{4.11}$$

(e) Let $0 < r < 1$. Relation (4.3) can be rewritten equivalently as follows

$$\sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)q^k}{\Gamma(k+1)} \sim e^{-x} n^{r-1} \Gamma(r), \quad \text{as } n \to \infty.$$

Note that

$$\frac{\Gamma(r+k)}{\Gamma(k+1)} = \frac{\Gamma(r+k)}{k\Gamma(k)} = k^{r-1}\left\{1 + \frac{r^2 - r}{2k} + o\left(\frac{1}{k}\right)\right\}, \quad k \to \infty.$$

The asymptotic behavior of the sum $\sum_{k=m+1}^{\infty} \frac{\Gamma(r+k)q^k}{\Gamma(k+1)} \sim \sum_{k=m+1}^{\infty} k^{r-1} q^k$ can be determined as follows:

$$\begin{aligned}
\sum_{k=m+1}^{\infty} k^{r-1} q^k &= \sum_{k=m+1}^{\infty} \frac{kq^k}{k^{2-r}} \leqslant \frac{1}{(m+1)^{2-r}} \sum_{k=m+1}^{\infty} kq^k \sim \frac{mnq^{m+1}}{(m+1)^{2-r}}, \\
\sum_{k=m+1}^{\infty} k^{r-1} q^k &= \sum_{k=m+1}^{\infty} \frac{k^r q^k}{k} \geqslant (m+1)^r \sum_{k=m+1}^{\infty} \frac{q^k}{k}.
\end{aligned}$$

Let $U=\sum_{k=m+1}^{\infty}\frac{q^k}{k}$. Then the following relations hold as $n\to\infty$:

$$U-qU=\frac{q^{m+1}}{m+1}+\left\{\frac{q^{m+2}}{(m+1)(m+2)}+\frac{q^{m+3}}{(m+2)(m+3)}+\cdots\right\}\leqslant\frac{q^{m+1}}{m+1}+\frac{1}{(m+1)^2}\big(q^{m+2}+q^{m+3}+\cdots\big)$$
$$=\frac{q^{m+1}}{m+1}+\frac{q^{m+2}n}{(m+1)^2}=\frac{q^{m+1}}{m+1}\big(1+o(1)\big).$$

It follows that

$$U-qU\sim\frac{q^{m+1}}{m+1},\qquad U\sim\frac{nq^{m+1}}{m+1},\qquad\sum_{k=m+1}^{\infty}k^{r-1}q^k\sim\frac{mnq^{m+1}}{(m+1)^{2-r}}.$$

Hence, for $0<r<1$ relation (4.3) is equivalent to the relation

$$\frac{mnq^{m+1}}{(m+1)^{2-r}}\sim e^{-x}n^{r-1}\Gamma(r),\quad\text{as } n\to\infty. \tag{4.12}$$

It is easy to check that (4.12) is equivalent to (4.11). Hence, relation (4.3) is equivalent to (4.12) for all $r>0$.

(f) Relation (4.12) can be rewritten equivalently as follows

$$\left(\frac{m}{n}\right)^{r-1}n\left(\frac{n-1}{n}\right)^m\to e^{-x}\Gamma(r),\quad\text{as } n\to\infty, \tag{4.13}$$

$$(r-1)\ln\frac{m}{n}+\ln n-m\ln\left(1+\frac{1}{n-1}\right)\to\ln\Gamma(r)-x,\quad\text{as } n\to\infty. \tag{4.14}$$

Using conclusions from points (a) and (b) we get that $\frac{\ln m}{\ln n}\to 1$ as $n\to\infty$. It follows from (4.14) that

$$(r-1)\frac{\ln m-\ln n}{\ln n}+\frac{\ln n}{\ln n}-\frac{m}{\ln n}\ln\left(1+\frac{1}{n-1}\right)\to 0,\quad\text{as } n\to\infty,$$
$$1-\frac{m}{n\ln n}\to 0,\quad\text{and}\quad m\sim n\ln n,\quad\text{as } n\to\infty.$$

Consequently (4.14) can be rewritten equivalently as follows

$$(r-1)\left(\ln\frac{m}{n\ln n}+\ln\ln n\right)+\ln n-\frac{m}{n}+m\left\{\frac{1}{n}-\ln\left(1+\frac{1}{n-1}\right)\right\}\to\ln\Gamma(r)-x,\quad\text{as } n\to\infty,$$
$$(r-1)\ln\ln n+\ln n-\frac{m}{n}=\ln\Gamma(r)-x+o(1),\quad\text{as } n\to\infty,$$
$$m=n\big(\ln n+(r-1)\ln\ln n+x-\ln\Gamma(r)+o(1)\big),\quad\text{as } n\to\infty,$$

and the proof of Lemma 4.1 is completed. □

Lemma 4.2. *Let $u_n(x)=n(\ln n+(r-1)\ln\ln n+x-\ln\Gamma(r))$, $m=[u_n(x)]$, $r_n=u_n(x)-m\in[0,1)$ and $\tau_{n,r}=n\{1-F_{n,r}(u_n(x))\}=n\{1-F_{n,r}(m)\}$.*

If $r\in(0,\infty)\setminus\{1\}$, then the following asymptotic relation holds:

$$\tau_{n,r}=e^{-x}\left\{1+\frac{(r-1)^2\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\},\quad n\to\infty. \tag{4.15}$$

Proof. Case $r\geqslant 2$. Note that the following relations hold:

$$\left(1-\frac{1}{n}\right)^{m+1}=e^{\{n(\ln n+(r-1)\ln\ln n+x-\ln\Gamma(r))-r_n+1\}\ln(1-\frac{1}{n})}=e^{\{n(\ln n+(r-1)\ln\ln n+x-\ln\Gamma(r))-r_n+1\}\{-\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2})\}}$$
$$=e^{-\ln n-(r-1)\ln\ln n-x+\ln\Gamma(r)-\frac{\ln n}{2n}+o(\frac{\ln n}{n})}=\frac{\Gamma(r)e^{-x}}{n(\ln n)^{r-1}}\left\{1-\frac{\ln n}{2n}+o\left(\frac{\ln n}{n}\right)\right\}, \tag{4.16}$$

$$\left(\frac{m+k}{n}\right)^{r-1}=\big(\ln n+(r-1)\ln\ln n+o(\ln\ln n)\big)^{r-1}$$
$$=(\ln n)^{r-1}\left\{1+\frac{(r-1)^2\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\},\quad\text{where } k=\text{const.}, \tag{4.17}$$

$$\frac{\Psi(m+[r],\vartheta)}{n^{\vartheta}} \leqslant \frac{C}{n}\left(\frac{m}{n}\right)^{\vartheta}\left(1-\frac{1}{n}\right)^{m} = \frac{C}{n}\left(\frac{m}{n}\right)^{\vartheta}\frac{\Gamma(r)e^{-x}}{n(\ln n)^{r-1}}\left\{1-\frac{\ln n}{2n}+o\left(\frac{\ln n}{n}\right)\right\}$$
$$\sim \frac{C\Gamma(r)e^{-x}}{n^{2}(\ln n)^{r-\vartheta-1}}, \quad n\to\infty. \tag{4.18}$$

Using (4.16)–(4.18) and expression for $\Psi(p,s)$ from (4.5), we obtain

$$\begin{aligned}
\tau_{n,r} &= \frac{\Psi(p,s)}{n^{r-1}} = \frac{(n-1)^{m+1}\Gamma(r+m+1)}{n^{m+r-1}\Gamma(r)\Gamma(m+2)}\left\{1+\frac{(n-1)\Gamma(r)\Gamma(m+2)}{\Gamma(r-1)\Gamma(m+3)}\right.\\
&\quad \left.+\frac{(n-1)^{2}\Gamma(r)\Gamma(m+2)}{\Gamma(r-2)\Gamma(m+4)}+\cdots+\frac{(n-1)^{[r]-2}\Gamma(r)\Gamma(m+2)}{\Gamma(2+\vartheta)\Gamma(m+[r])}\right\}+\frac{\Psi(m+[r],\vartheta)}{n^{\vartheta}}\\
&= \left(1-\frac{1}{n}\right)^{m+1}\left(\frac{m+2}{n}\right)^{r-1}\frac{n}{\Gamma(r)}\left\{1+\frac{(r-1)^{2}-(r-1)}{2(m+2)}+o\left(\frac{1}{m}\right)\right\}\\
&\quad\times\left\{1+\frac{\Gamma(r)}{\Gamma(r-1)\ln n}+o\left(\frac{1}{\ln n}\right)\right\}+\frac{\Psi(m+[r],\vartheta)}{n^{\vartheta}}\\
&= e^{-x}\left\{1+\frac{(r-1)^{2}\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\}, \quad n\to\infty.
\end{aligned}$$

Case $1<r<2$. Using some calculations from Lemma 4.1, relations (4.16)–(4.17) and equalities

$$1+\frac{n}{m} = 1+\frac{1}{\ln n+o(\ln n)} = 1+\frac{1}{\ln n}+o\left(\frac{1}{\ln n}\right), \quad n\to\infty,$$
$$\tau_{n,r} = \frac{\Psi(p,s)}{n^{r-1}} = \frac{1}{n^{r-1}\Gamma(r)}\sum_{k=m+1}^{\infty}\frac{\Gamma(r+k)q^{k}}{\Gamma(k+1)},$$

we obtain the following relations:

$$\begin{aligned}
\tau_{n,r} &\leqslant \frac{1}{n^{r-1}\Gamma(r)}\sum_{k=m+1}^{\infty}(k+1)^{r-1}q^{k}\left(1+\frac{Q}{m}\right) \leqslant \frac{1}{n^{r-1}\Gamma(r)}\left(1-\frac{1}{n}\right)^{m+1}\frac{n(m+1)+n^{2}-n}{(m+1)^{2-r}}\left(1+\frac{Q}{m}\right)\\
&\leqslant \frac{n}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{m+1}\left(\frac{m+1}{n}\right)^{r-1}\left(1+\frac{n}{m}\right)\left(1+\frac{Q}{m}\right) = e^{-x}\left\{1+\frac{(r-1)^{2}\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\}, \quad n\to\infty,\\
\tau_{n,r} &\geqslant \frac{1}{n^{r-1}\Gamma(r)}\sum_{k=m+1}^{\infty}(k+1)^{r-1}q^{k}\left(1-\frac{Q}{m}\right) \geqslant \left(\frac{m+2}{n}\right)^{r-1}\frac{n}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{m+1}\left(1+\frac{Q}{m}\right)\\
&= e^{-x}\left\{1+\frac{(r-1)^{2}\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\}, \quad n\to\infty,
\end{aligned}$$

and consequently relation (4.15) follows.

Case $0<r<1$. Now the following relations hold:

$$\begin{aligned}
\tau_{n,r} &\leqslant \frac{1}{n^{r-1}\Gamma(r)}\sum_{k=m+1}^{\infty}k^{r-1}q^{k}\left(1+\frac{Q}{m}\right) \leqslant \frac{1}{n^{r-1}\Gamma(r)(m+1)^{2-r}}\sum_{k=m+1}^{\infty}kq^{k}\left(1+\frac{Q}{m}\right)\\
&= \frac{n(m+1)q^{m+1}+n^{2}q^{m+2}}{n^{r-1}\Gamma(r)(m+1)^{2-r}}\left(1+\frac{Q}{m}\right) = \frac{n}{\Gamma(r)}\left(\frac{m+1}{n}\right)^{r-1}\left(1-\frac{1}{n}\right)^{m+1}\left(1+\frac{n-1}{m+1}\right)\left(1+\frac{Q}{m}\right)\\
&= e^{-x}\left\{1+\frac{(r-1)^{2}\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\}, \quad n\to\infty,\\
\tau_{n,r} &\geqslant \frac{1}{n^{r-1}\Gamma(r)}\sum_{k=m+1}^{\infty}k^{r-1}q^{k}\left(1-\frac{Q}{m}\right) \geqslant \frac{(m+1)^{r}}{n^{r-1}\Gamma(r)}\sum_{k=m+1}^{\infty}\frac{q^{k}}{k}\left(1-\frac{Q}{m}\right)\\
&\geqslant \frac{(m+1)^{r}}{n^{r-1}\Gamma(r)}\frac{nq^{m+1}}{m+1}\left(1-\frac{Q}{m}\right) = \frac{n}{\Gamma(r)}\left(1-\frac{1}{n}\right)^{m+1}\left(\frac{m+1}{n}\right)^{r-1}\left(1-\frac{Q}{m}\right)\\
&= e^{-x}\left\{1+\frac{(r-1)^{2}\ln\ln n}{\ln n}+o\left(\frac{\ln\ln n}{\ln n}\right)\right\}, \quad n\to\infty,
\end{aligned}$$

and again relation (4.15) follows immediately. □

Proof of Theorem 3.1. Let $u_n(x) = n(x + \ln n + (r-1)\ln\ln n - \ln\Gamma(r))$. Using Lemma 4.1 we obtain that $n\{1 - F_{n,r}(u_n(x))\} \to e^{-x}$ as $n \to \infty$. Equality (3.1) follows then from Leadbetter et al. [11], Theorem 1.5.1.

According to Remark 3.2 it remains to prove relation (3.3) for non-integer $r > 0$. For such values of r let $\tau = e^{-x}$ and $\tau_{n,r} = n\{1 - F_{n,r}(u_n(x))\}$. Note that

$$\Delta_n(r,x) = P\left\{M_n^* \leqslant u_n(x)\right\} - e^{-\tau_{n,r}} + e^{-\tau_{n,r}} - e^{-\tau}. \tag{4.19}$$

Using Leadbetter et al. [11], Theorem 2.4.2, we obtain that

$$P\left\{M_n^* \leqslant u_n(x)\right\} - e^{-\tau_{n,r}} \sim \frac{e^{-2x}\exp(-e^{-x})}{2n}, \quad n \to \infty. \tag{4.20}$$

It follows from (4.15) that

$$e^{-\tau_{n,r}} - e^{-\tau} = -(r-1)^2 e^{-x}\exp\left(e^{-x}\right)\frac{\ln\ln n}{\ln n} + o\left(\frac{\ln\ln n}{\ln n}\right), \quad n \to \infty. \tag{4.21}$$

Finally, relation (3.3) is a consequence of relations (4.19)–(4.21). □

Remark 4.3. For $\widetilde{u}_n(x) = n(\ln n + (r-1)\ln\ln n + x - \ln\Gamma(r) + \frac{(r-1)^2\ln\ln n}{\ln n})$, and $\widetilde{m} = [\widetilde{u}_{n(x)}]$, $\widetilde{\tau}_{n,r} = n\{1 - F_{n,r}(\widetilde{u}_n(x))\}$ one can obtain the following asymptotic relations as $n \to \infty$:

$$\left(1 - \frac{1}{n}\right)^{\widetilde{m}+1} = \frac{\Gamma(r)e^{-x}}{n(\ln n)^{r-1}}\left\{1 - \frac{(r-1)^2\ln\ln n}{\ln n} + o\left(\frac{\ln\ln n}{\ln n}\right)\right\},$$

$$\widetilde{\tau}_{n,r}\left(u_n(x)\right) = e^{-x}\left\{1 + O\left(\frac{1}{\ln n}\right)\right\},$$

and consequently $P\{M_n^* \leqslant \widetilde{u}_n(x)\} - e^{-\tau} \sim O(\frac{1}{\ln n})$ as $n \to \infty$.

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