



Characterization of the variable exponent Bessel potential spaces via the Poisson semigroup

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ABSTRACT

Under the standard assumptions on the variable exponent $p(x)$ (log- and decay conditions), we give a characterization of the variable exponent Bessel potential space $\mathfrak{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ in terms of the rate of convergence of the Poisson semigroup P_t . We show that the existence of the Riesz fractional derivative $\mathbb{D}^\alpha f$ in the space $L^{p(\cdot)}(\mathbb{R}^n)$ is equivalent to the existence of the limit $\frac{1}{t^\alpha}(I - P_t)^\alpha f$. In the pre-limiting case $\sup_x p(x) < \frac{n}{\alpha}$ we show that the Bessel potential space is characterized by the condition $\|(I - P_\varepsilon)^\alpha f\|_{p(\cdot)} \leq C\varepsilon^\alpha$.

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1. Introduction

The Bessel potential space $\mathfrak{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$, $\alpha > 0$, defined as the range of the Bessel potential operator over the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, was recently studied in [2], under assumptions on $p(x)$ typical for *variable exponent analysis*, where in particular it was shown that the space $\mathfrak{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ may be characterized as the Sobolev space

$$L^{\alpha,p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}; \mathbb{D}^\alpha f \in L^{p(\cdot)}\}, \tag{1}$$

with the Riesz fractional derivative $\mathbb{D}^\alpha f$ realized as a hypersingular integral, the justification of the coincidence $\mathfrak{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = L^{\alpha,p(\cdot)}(\mathbb{R}^n)$ being given in [2] in the “under-limiting” case $\sup_{x \in \mathbb{R}^n} p(x) < \frac{n}{\alpha}$. In [2], in the case of integer α , it was also verified that $\mathfrak{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ coincides with the standard Sobolev space, defined in terms of partial derivatives, the same having been also checked in [11].

In the case of constant p it was also known that the Bessel potential space $\mathfrak{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ may be characterized in terms of the rate of convergence of identity approximations. For instance, with the usage of the Poisson semigroup $P_t, t > 0$, the space $\mathfrak{B}^\alpha[L^p(\mathbb{R}^n)]$ is described as the subspace of $L^p(\mathbb{R}^n)$ of functions f for which there exists the limit $\lim_{t \rightarrow 0} \frac{1}{t^\alpha}(I - P_t)^\alpha f$, besides this

$$\lim_{\substack{t \rightarrow 0 \\ (L^p)}} \frac{1}{t^\alpha}(I - P_t)^\alpha f = \mathbb{D}^\alpha f \tag{2}$$

see for instance Theorem B in [23], where the simultaneous existence of the left- and right-hand sides in (2) and their coincidence was proved under the assumption that f and $\mathbb{D}^\alpha f$ may belong to $L^r(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ with different p and r .

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In the case $p = r$ this was proved in [20] where the case of the Weierstrass semigroup was also considered. Relations of type (2) go back to Westphal's formula [28]

$$(-A)^\alpha f = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} (I - T_h)^\alpha f$$

for fractional powers of the generator of a semigroup T_h in a Banach space. The latter in its turn has the origin in the Grünwald–Letnikov approach [10,17] to fractional derivatives of functions of one variable, under which the fractional derivative is defined as $\lim_{h \rightarrow 0^+} \frac{\Delta_h^\alpha f}{h^\alpha}$, where $\Delta_h^\alpha f$ is the difference of fractional order $\alpha > 0$.

What is now called *variable exponent analysis* (VEA) was intensively developed during the last two decades, variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ being the core of VEA. The progress in VEA was inspired both by difficult open problems in this theory, and possible applications shown in [21]. Not going here into historical details, we refer to original papers [27,15] and surveying papers [7,12,14,25]. As is known, extension of various facts valid for constant p to the case of variable $p(x)$ encountered enormous difficulties and required essential efforts from various groups of researchers. Among the reasons we could remind that variable exponent spaces are not invariant with respect to translations and dilations, Young theorem for convolution operators is no more valid, the Minkowsky integral inequality proves to be a very rough inequality, etc.

Although expected, the validity of (2) in the variable exponent setting was not easy to justify, in particular because the apparatus of the Wiener algebra of Fourier transforms of integrable function, based on the Young theorem, is not applicable. Another natural approach, based on Fourier multipliers, extended in [4] to the variable exponent setting, may be already applicable, which is used in this paper. However, because of the specific behaviour of the Bessel functions appearing under the usage of the Mihklin–Hörmander Theorem, this approach also encountered essential difficulties, see Sections 4 and 5.1.

The paper is arranged as follows. In Section 2 we provide some necessary preliminaries. Section 3 contains formulations of the main results of the paper, see Theorems 14, 16, 17 and Corollary 15. In Section 4 we prove some important technical lemmas and in Section 5 we give the proofs of the main results. In particular in Section 5.1 we show that some specific functions are Fourier $p(x)$ -multipliers, which required the most efforts. The result on these Fourier multipliers is then used in Sections 5.2–5.4 to obtain the characterization of Bessel potential type spaces in terms of the rate of convergence of the Poisson semigroup.

2. Preliminaries

We refer to papers [27,15,22] and surveys [7,12,25] for details on variable Lebesgue spaces, but give some necessary definitions. For a measurable function $p : \mathbb{R}^n \rightarrow [1, \infty)$ we put

$$p_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x).$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of functions for which

$$\varrho_p(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

In the sequel, we suppose that $p(x)$ satisfies one of the following standard conditions:

$$1 \leq p_- \leq p(x) \leq p_+ < \infty, \tag{3}$$

or

$$1 < p_- \leq p(x) \leq p_+ < \infty. \tag{4}$$

Equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_p \left(\frac{f}{\lambda} \right) \leq 1 \right\},$$

this is a Banach space. By $p'(x)$ we denote the conjugate exponent: $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$. We make use of the well-known log-condition

$$|p(x) - p(y)| \leq \frac{C}{-\ln(|x - y|)}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n, \tag{5}$$

and assume that there exists $p(\infty) = \lim_{x \rightarrow \infty} p(x)$ and there holds the decay condition

$$|p(x) - p(\infty)| \leq \frac{A}{\ln(2 + |x|)}, \quad x \in \mathbb{R}^n. \tag{6}$$

Definition 1. By $\mathcal{P}(\mathbb{R}^n)$ we denote the set of all bounded measurable functions $p : \mathbb{R}^n \rightarrow [1, \infty)$ which satisfy assumptions (4)–(6).

Definition 2. By $\mathfrak{M}(\mathbb{R}^n)$ we denote the set of exponents $p(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$ such that the Hardy–Littlewood maximal operator is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$. As is known [5], $\mathcal{P}(\mathbb{R}^n) \subset \mathfrak{M}(\mathbb{R}^n)$.

2.1. Identity approximations

Let $\phi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$. For each $t > 0$, we put $\phi_t := t^{-n}\phi(x t^{-1})$. Following [3], we say that $\{\phi_t\}$ is a potential-type approximate identity, if it has integrable radial majorant

$$\sup_{|y| \geq |x|} |\phi(y)| \in L^1(\mathbb{R}^n).$$

Convergence of potential-type approximate identities in the setting of variable exponent Lebesgue spaces $L^{p(\cdot)}$ was known from [6] under the assumption that the maximal operator is bounded. (An extension to some weighted spaces was given in [19].) The following Proposition 3 proved in [3, Theorem 2.3], does not use the information about the maximal operator and allows to include the cases where $p(x)$ may be equal to 1.

Proposition 3. Let a function $p : \mathbb{R}^n \rightarrow [1, \infty)$ satisfy conditions (3), (5) and (6). If $\{\phi_t\}$ is a potential-type approximate identity then:

- (i) $\|\phi_t * f\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$, for all $t > 0$, with $C > 0$ not depending on t and f , and
- (ii) $\lim_{t \rightarrow 0} \|\phi_t * f - f\|_{p(\cdot)} = 0$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

2.2. Fourier $p(x)$ -multipliers

Let $m \in L^\infty(\mathbb{R}^n)$. We define the operator T_m by

$$\widehat{T_m f}(\xi) = m(\xi) \widehat{f}(\xi) \tag{7}$$

where $\widehat{f}(\xi) = Ff(\xi)$ is the Fourier transform given by

$$Ff(\xi) = \int_{\mathbb{R}^n} e^{ix \cdot y} f(x) dx.$$

When T_m generates a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$, we say that m is a *Fourier $p(\cdot)$ -multiplier*. The following Mihlin-type multiplier theorem for variable Lebesgue spaces is known, see [13, Theorem 4.5] where it was proved in a weighted setting; note that a similar theorem in the form of Hörmander criterion (for variable exponents proved in [4, Section 2.5]) requires to check the behaviour of less number of derivatives (up to order $[\frac{n}{2}] + 1$), but leads to stronger restrictions on $p(x)$.

Theorem 4. Let a function $m(x)$ be continuous everywhere in \mathbb{R}^n , except for probably the origin, have the mixed distributional derivative $\frac{\partial^{\alpha} m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$ and the derivatives $D^{\alpha} m = \frac{\partial^{|\alpha|} m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ of orders $|\alpha| = \alpha_1 + \dots + \alpha_n \leq n - 1$ continuous beyond the origin and

$$|x|^{|\alpha|} |D^{\alpha} m(x)| \leq C, \quad |\alpha| \leq n, \tag{8}$$

where the constant $C > 0$ does not depend on x . If p satisfies condition (4) and $p \in \mathfrak{M}(\mathbb{R}^n)$, then m is a Fourier $p(\cdot)$ -multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$.

It is easily seen that Mihlin condition (8) for radial functions $\mathcal{M}(r) = m(|x|)$ is reduced to

$$\left| r^k \frac{d^k}{dr^k} \mathcal{M}(r) \right| \leq C < \infty, \quad k = 0, 1, \dots, n. \tag{9}$$

Note that (9) is equivalent to

$$\left| \left(r \frac{d}{dr} \right)^k \mathcal{M}(r) \right| \leq C < \infty, \quad k = 0, 1, \dots, n, \tag{10}$$

since $(r \frac{d}{dr})^k = \sum_{j=1}^k C_{k,j} r^j \frac{d^j}{dr^j}$ with constant $C_{k,j}$, where $C_{k,1} = C_{k,k} = 1$.

Lemma 5. Let a function m satisfy Mihlin’s condition (8). Then the function $m_\varepsilon(x) := m(\varepsilon x)$ satisfies (8) uniformly in ε , with the same constant C .

The proof is obvious since $D^\alpha m(\varepsilon \cdot)(\xi) = \varepsilon^{|\alpha|} (D^\alpha m)(\varepsilon \xi)$.

We need the following lemma on the identity approximations. Note that in Lemma 6 no information on the kernel is required: the only requirement is that its Fourier transform satisfies the Mihlin multiplier condition.

Lemma 6. Suppose that $m(x)$ satisfies Mihlin’s condition (8). If

$$\lim_{\varepsilon \rightarrow 0} m(\varepsilon x) = 1$$

for almost all $x \in \mathbb{R}^n$ and $p \in \mathcal{P}(\mathbb{R}^n)$, then

$$\lim_{\varepsilon \rightarrow 0} \|T_\varepsilon f - f\|_{p(\cdot)} = 0, \tag{11}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where T_ε is the operator generated by the multiplier $m(\varepsilon x)$.

Proof. The statement of the lemma is well known in the case of constant $p \in (1, \infty)$, see [23, Lemma 12], being valid in this case for an arbitrary Fourier p -multiplier m . By Lemma 5 and Theorem 4, the family of operators $\{T_\varepsilon\}$ is uniformly bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, it suffices to check (11) on a dense set in $L^{p(\cdot)}(\mathbb{R}^n)$, for instance for $f \in L^{p^-}(\mathbb{R}^n) \cap L^{p^+}(\mathbb{R}^n)$. Since $\|f\|_{p(\cdot)} \leq \|f\|_{p^-} + \|f\|_{p^+}$, for such functions f we have

$$\|T_\varepsilon f - f\|_{p(\cdot)} \leq \|T_\varepsilon f - f\|_{p^-} + \|T_\varepsilon f - f\|_{p^+}$$

from which the conclusion follows, in view of the validity of the theorem in case of constant p . \square

Since we will also deal with Fourier $p(\cdot)$ -multipliers which do not satisfy Mihlin condition (8), we will need the following lemma.

Lemma 7. Let a function $m(x)$, $x \in \mathbb{R}^n$ be given as $m(x) = M(x) + \phi(x)$, where $M(x)$ satisfies Mihlin’s condition (8) and $F^{-1}\phi(x) := \Phi(x)$ has its radial non-increasing majorant $\tilde{\Phi}(x) := \sup_{|y| \geq |x|} |\Phi(y)|$ integrable. Then $m(x)$ is a Fourier $p(\cdot)$ -multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot) \in \mathfrak{M}(\mathbb{R}^n)$.

Proof. This follows from Theorem 4, Proposition 3 and the fact that

$$T_m f(x) = F^{-1}[M(\xi)\widehat{f}(\xi)](x) + \Phi * f(x). \quad \square$$

2.3. On finite differences

By a finite difference of integer order ℓ and step $h \in \mathbb{R}$, in this paper we always mean a non-centered difference

$$\Delta_h^\ell f(x) = (I - \tau_h)^\ell f(x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(x - jh) \tag{12}$$

where I is the identity operator and $\tau_h f(x) = f(x - h)$ is the translation operator. We refer to [24, Chapter 3] and [26, Sections 25–26] for more information on centered or non-centered finite differences and their role in fractional calculus and the theory of hypersingular integrals.

The difference of fractional order α is defined as

$$\Delta_h^\alpha f(x) = (I - \tau_h)^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh), \quad \alpha > 0, \tag{13}$$

where the series converges absolutely and uniformly for each $\alpha > 0$ and for every bounded function f , which follows from the fact that $\sum_{j=1}^{\infty} |\binom{\alpha}{j}| < \infty$, see for instance [26, Section 20.1], for properties of fractional order differences.

In a similar way there is introduced a generalized difference of fractional order α , if one replaces the translation operator τ_h by any semigroup of operators. In this paper we make use of the Poisson semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} P(x - y, t) f(y) dy \tag{14}$$

where

$$P(x, t) = \frac{c_n t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \quad c_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{\frac{n+1}{2}},$$

so that

$$(I - P_t)^\alpha f(x) = \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} P_{kt} f(x). \tag{15}$$

The Poisson kernel $P(x, t)$ is a potential-type approximate identity in accordance with the definition of Section 2.1. Therefore, by Proposition 3 the operators $P_t f$ are uniformly bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$ under the assumptions of that Proposition on $p(\cdot)$. Then

$$\|(I - P_t)^\alpha f\|_{p(\cdot)} \leq C c(\alpha) \|f\|_{p(\cdot)}, \quad c(\alpha) = \sum_{k=0}^\infty \left| \binom{\alpha}{k} \right| < \infty, \tag{16}$$

where C is the constant from the uniform estimate $\|P_t f\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$, when $p : \mathbb{R}^n \rightarrow [1, \infty]$ satisfies conditions (3), (5) and (6).

2.4. Riesz potential operator and Riesz fractional derivative

Recall that the Riesz potential operator, also known as fractional integral operator, is given by

$$I^\alpha g(x) := \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{g(y)}{|x - y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \tag{17}$$

with the normalizing constant $\gamma_n(\alpha) = 2^\alpha \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$. The hypersingular integral

$$\mathbb{D}^\alpha f(x) := \frac{1}{d_{n,\ell}(\alpha)} \int_{\mathbb{R}^n} \frac{\Delta_y^\ell f(x)}{|y|^{n+\alpha}} dy, \tag{18}$$

where $\Delta_y^\ell f(x)$ is a finite difference of order $\ell > 2[\frac{\alpha}{2}]$, is known as the Riesz fractional derivative, see [24, Chapter 3] or [26, Sections 26] for the value of the normalizing constant $d_{n,\ell}(\alpha)$. The condition $\ell > 2[\frac{\alpha}{2}]$ is known in the theory of hypersingular integrals, see [24], and is a weaker assumption than just $\ell > \alpha$, although applicable only in the case of non-centered finite differences. It allows to make use of differences of lower order. For instance, in the case $0 < \alpha < 2$ one may take $\ell = 1$, not necessarily $\ell = 2$.

It is known [24,26] that operator (18) is left inverse to the operator I^α within the frameworks of L^p -spaces, which was extended to variable exponent spaces $L^{p(\cdot)}(\mathbb{R}^n)$ in [1].

Everywhere in the sequel, $\ell > \alpha$ and is even.

When considered on functions in the range $I^\alpha(X)$ of the operator I^α over this or that space X , the integral in (18) is always interpreted as the limit $\mathbb{D}^\alpha f := \lim_{\varepsilon \rightarrow 0} \mathbb{D}_\varepsilon^\alpha f$ in the norm of the space X , of the truncated operators

$$\mathbb{D}_\varepsilon^\alpha f(x) = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{\Delta_y^\ell f(x)}{|y|^{n+\alpha}} dy. \tag{19}$$

The following proposition was proved in [1, Theorem 5.5].

Proposition 8. Let $p \in \mathfrak{M}(\mathbb{R}^n)$ and $1 < p_-(\mathbb{R}^n) \leq p_+(\mathbb{R}^n) < \frac{n}{\alpha}$. Then

$$\mathbb{D}^\alpha I^\alpha \varphi = \varphi, \quad \varphi \in L^{p(\cdot)}(\mathbb{R}^n),$$

where the hypersingular operator \mathbb{D}^α is understood as convergent in $L^{p(\cdot)}$ -norm.

We will also use the following result for variable exponent spaces, proved in [2, Theorems 3.2–3.3]. (Note that in [2] this statement was formulated for p satisfying the log- and decay condition, but the analysis of the proof shows that it uses only the fact that the maximal operator is bounded.)

Proposition 9. Let $p \in \mathfrak{M}(\mathbb{R}^n)$, $1 < p_-(\mathbb{R}^n) \leq p_+(\mathbb{R}^n) < \frac{n}{\alpha}$, and let f be a locally integrable function. Then $f \in I^\alpha[L^{p(\cdot)}]$, if and only if $f \in L^{q(\cdot)}$, with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and

$$\|\mathbb{D}_\varepsilon^\alpha f\|_{p(\cdot)} \leq C \tag{20}$$

where C does not depend on $\varepsilon > 0$.

It is known (see [24, p. 70]) that

$$F(\mathbb{D}_\varepsilon^\alpha f)(x) = \widehat{K_{\ell,\alpha}(\varepsilon x)} |x|^\alpha \widehat{f}(x), \quad f \in C_0^\infty(\mathbb{R}^n), \tag{21}$$

where $\widehat{K_{\ell,\alpha}}(x)$ is the Fourier transform of the function $K_{\ell,\alpha}(x)$ with the property

$$K_{\ell,\alpha}(x) \in L^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} K_{\ell,\alpha}(x) dx = 1. \tag{22}$$

The function $\widehat{K_{\ell,\alpha}}(x)$ is given explicitly by

$$\widehat{K_{\ell,\alpha}}(x) = \frac{(2i)^\ell}{d_{n,\ell}(\alpha)} \int_{|y|>|x|} \frac{\sin^\ell(y_1)}{|y|^{n+\alpha}} dy. \tag{23}$$

For brevity of notation, we will denote $\widehat{K_{\ell,\alpha}}(x)$ simply as $w(|x|)$, therefore

$$w(|x|) = c \int_{|y|>|x|} \frac{\sin^\ell(y_1)}{|y|^{n+\alpha}} dy = c \int_{|x|}^\infty \frac{V(\rho)}{\rho^{1+\alpha}} d\rho \tag{24}$$

where

$$V(\rho) = \int_{S^{n-1}} \sin^\ell(\rho\sigma_1) d\sigma \quad \text{and} \quad c = \frac{(2i)^\ell}{d_{n,\ell}(\alpha)}. \tag{25}$$

Lemma 10. The following formula is valid

$$V(\rho) = \lambda + \sum_{i=0}^{\frac{\ell}{2}-1} C_i \frac{J_{\nu-1}(\ell_i\rho)}{(\ell_i\rho)^{\nu-1}}, \tag{26}$$

where $\ell = 2, 4, 6, \dots$, $J_{\nu-1}(r)$ is the Bessel function of the first kind, $\nu = \frac{n}{2}$, $\ell_i = \ell - 2i$ and λ and C_i are constants:

$$\lambda = \frac{4\pi^{\frac{n}{2}} \Gamma(\frac{\ell+1}{2})}{\ell \Gamma(\frac{\ell}{2}) \Gamma(\frac{n}{2})}, \quad C_i = (-1)^{\frac{\ell}{2}-i} (2\pi)^{\frac{n}{2}} 2^{1-\ell} \binom{\ell}{i}. \tag{27}$$

Proof. Formula (26) is a consequence of the Catalan formula

$$\int_{S^{n-1}} \sin^\ell(\rho\sigma_1) d\sigma = |S^{n-2}| \int_{-1}^1 \sin^\ell(\rho t) (1-t^2)^{\frac{n-3}{2}} dt \tag{28}$$

(see, for instance, [24, p. 13]), the Fourier expansion

$$\sin^\ell(t) = \frac{1}{2^{\ell-1}} \sum_{i=0}^{\frac{\ell}{2}-1} (-1)^{\frac{\ell}{2}-i} \binom{\ell}{i} \cos((\ell - 2i)t) + \frac{1}{2^\ell} \binom{\ell}{\ell/2} \tag{29}$$

of the function $\sin^\ell(t)$ with even ℓ (see, e.g., [18, Appendix I.1.9]), and the Poisson formula

$$J_\nu(\rho) = \frac{(\rho/2)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})} \int_{-1}^1 \cos(\rho t) (1-t^2)^{\nu-\frac{1}{2}} dt \tag{30}$$

with $\Re \nu > -\frac{1}{2}$ for the Bessel function (see, e.g., [16]). The values in (27) are obtained by direct calculations via properties of Gamma-function. \square

Following [23] (see also [24, p. 214]), we make use of the functions

$$A(x) = \frac{(1 - e^{-|x|})^\alpha}{|x|^\alpha w(|x|)} \quad \text{and} \quad B(x) = \frac{1}{A(x)}, \quad x \in \mathbb{R}^n, \tag{31}$$

which will play a central role in this paper.

Since the functions $A(x)$ and $B(x)$ are radial, we find it convenient to also use the notation

$$\mathcal{A}(r) = \frac{(1 - e^{-r})^\alpha}{r^\alpha w(r)} \quad \text{and} \quad \mathcal{B}(r) = \frac{r^\alpha w(r)}{(1 - e^{-r})^\alpha}. \tag{32}$$

2.5. Bessel potential operator

The Bessel potential of order $\alpha > 0$ of the density φ is defined by

$$\mathfrak{B}^\alpha \varphi(x) = \int_{\mathbb{R}^n} G_\alpha(x - y) \varphi(y) dy \tag{33}$$

where the Fourier transform of the Bessel kernel G_α is given by

$$\widehat{G}_\alpha(x) = (1 + |x|^2)^{-\alpha/2}, \quad x \in \mathbb{R}^n, \quad \alpha > 0.$$

Definition 11. We define the variable exponent Bessel potential space, sometimes also called Liouville space of fractional smoothness, as the range of the Bessel potential operator

$$\mathfrak{B}^\alpha [L^{p(\cdot)}(\mathbb{R}^n)] = \{f: f = \mathfrak{B}^\alpha \varphi, \varphi \in L^{p(\cdot)}(\mathbb{R}^n)\}, \quad \alpha > 0.$$

The following characterization of the variable exponent Bessel potential space via hypersingular integrals was given in [2].

Proposition 12. Let $0 < \alpha < n$. If $1 < p_- \leq p_+ < n/\alpha$ and $p(\cdot)$ satisfies conditions (5) and (6), then $\mathfrak{B}^\alpha(L^{p(\cdot)}) = L^{p(\cdot)} \cap I^\alpha(L^{p(\cdot)})$ with equivalent norms.

3. Main results

We first prove that the functions $A(x)$, $B(x)$ defined in (31) are Fourier $p(\cdot)$ -multipliers in $L^{p(\cdot)}(\mathbb{R}^n)$ under suitable exponents $p(\cdot)$, see Theorem 13, which proved to be the principal difficulty in extending the result in (2) to variable exponents.

Theorem 13. The functions $A(x)$ and $B(x)$ are Fourier $p(\cdot)$ -multipliers when $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

Then by means of Theorem 13 we prove the following statements.

Theorem 14. Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $\mathbb{D}_\varepsilon^\alpha f$ be the truncated hypersingular integral (19). The limits

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \mathbb{D}_\varepsilon^\alpha f \tag{34}$$

exist in $L^{p(\cdot)}(\mathbb{R}^n)$ simultaneously and coincide with each other.

Corollary 15. Let $\alpha > 0$ and $p \in \mathcal{P}(\mathbb{R}^n)$. The equivalent characterization of the space $L^{\alpha, p(\cdot)}(\mathbb{R}^n)$ defined in (1), is given by

$$L^{\alpha, p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}(\mathbb{R}^n): \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n) \right\}.$$

Theorem 16. Let $0 < \alpha < n$, $1 < p_- \leq p_+ < n/\alpha$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ belongs to $L^{\alpha, p(\cdot)}(\mathbb{R}^n)$ if and only if

$$\|(I - P_\varepsilon)^\alpha f\|_{p(\cdot)} \leq C \varepsilon^\alpha, \tag{35}$$

where C does not depend on ε ; condition (35) being fulfilled, it involves that $\mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)$ and (35) is also valid in the form $\|(I - P_\varepsilon)^\alpha f\|_{p(\cdot)} \leq C \|\mathbb{D}^\alpha f\|_{p(\cdot)} \varepsilon^\alpha$ where C does not depend on f and ε .

Theorem 17. Let $0 < \alpha < n$, $1 < p_- \leq p_+ < n/\alpha$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The variable exponent Bessel potential space $\mathfrak{B}^\alpha(L^{p(\cdot)})$ is the subspace in $L^{p(\cdot)}(\mathbb{R}^n)$ of functions f for which the limit $\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f$ exists.

4. Crucial lemmata

Lemma 18. *The function $\mathcal{B}(r)$ defined in (32) is non-vanishing: $\inf_{r \in \mathbb{R}_+^1} |\mathcal{B}(r)| > 0$.*

Proof. The function $\mathcal{B}(r)$ is continuous in $(0, \infty)$ and $|\mathcal{B}(r)| > 0$ for all $r \in (0, \infty)$. Therefore, it suffices to check that $\mathcal{B}(0) \neq 0$ and $\mathcal{B}(\infty) \neq 0$. From (22) it follows that $\mathcal{B}(0) = 1$, while $\mathcal{B}(\infty) = \frac{\lambda}{\alpha} \neq 0$ is seen from the asymptotics (40) proved in Lemma 20. \square

Lemma 19. *The following formula is valid*

$$\int_0^\infty f(t)t^\nu J_{\nu-1}(rt) dt = \frac{(-1)^m}{r^m} \sum_{k=1}^m c_{k,m} \int_0^\infty f^{(k)}(t)t^{\nu+k-m} J_{\nu+m-1}(rt) dt, \quad m \geq 1, \tag{36}$$

if

$$f(t)t^\nu J_\nu(t)|_0^\infty = 0$$

and

$$f^{(k)}(t)t^{\nu+k-j} J_{\nu+j}(t)|_0^\infty = 0, \quad k = 1, 2, \dots, j, \quad j = 1, 2, \dots, m - 1,$$

the latter appearing in the case $m \geq 2$.

Proof. A relation of type (36) is known in the form

$$\int_0^\infty f(t)t^\nu J_{\nu-1}(t|x|) dt = \frac{(-1)^m}{|x|^m} \int_0^\infty f^{(m)}(t)t^{\nu+m} J_{\nu+m-1}(t|x|) dt, \tag{37}$$

under the conditions

$$f^{(j)}(t)t^{\nu+j} J_{\nu+j}(t)|_0^\infty = 0, \quad j = 0, 1, 2, \dots, m - 1, \tag{38}$$

see formula (8.133) in [23], where it is denoted $f^{(m)}(t) = (\frac{1}{t} \frac{d}{dt})^m f(t)$. Then (36) follows from (37) if one observes that

$$\left(\frac{1}{t} \frac{d}{dt}\right)^m f(t) = \sum_{k=1}^m c_{k,m} \frac{f^{(k)}(t)}{t^{2m-k}}, \tag{39}$$

where $c_{k,m}$ are constants (here and in the sequel, by c, c_k, c_{km}, c_{js} , etc., we denote constants the exact values of which are not important for us). \square

The following lemmas are crucial for our purposes.

Lemma 20. *The function $\mathcal{B}(r)$ has the following structure at infinity:*

$$\mathcal{B}(r) = \frac{\lambda}{\alpha} + \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{3}{2}}}\right), \quad r \rightarrow \infty, \tag{40}$$

where λ and C_i are constants from (27).

Proof. By (24) and (26), we obtain

$$r^\alpha w(r) = \frac{\lambda}{\alpha} + r^\alpha \sum_{i=0}^{\frac{\ell}{2}-1} C_i \int_r^\infty \frac{J_{\nu-1}(\ell_i t)}{t^{1+\alpha}(\ell_i t)^{\nu-1}} dt. \tag{41}$$

By the well-known differentiation formula $\frac{J_\nu(t)}{t^{\nu-1}} = -\frac{d}{dt}[\frac{J_{\nu-1}(t)}{t^{\nu-1}}]$ for the Bessel functions, via integration by parts we arrive at the relation

$$\int_r^\infty \frac{J_\nu(t)}{t^\beta} dt = \frac{J_{\nu-1}(r)}{r^\beta} + (\nu - \beta - 1) \int_r^\infty \frac{J_{\nu-1}(t)}{t^{\beta+1}} dt, \tag{42}$$

for $r > 0$ and $\beta > -\frac{1}{2}$. Applying repeatedly this formula two times, we transform (41) to

$$r^\alpha w(r) = \frac{\lambda}{\alpha} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} \left[\frac{J_{\nu-2}(\ell_i r)}{r^\nu} - \frac{2 + \alpha}{\ell_i} \frac{J_{\nu-3}(\ell_i r)}{r^{\nu+1}} \right] + (\alpha + 2)(\alpha + 4) \sum_{i=0}^{\frac{\ell}{2}-1} \frac{r^\alpha}{\ell_i} \int_r^\infty \frac{J_{\nu-3}(\ell_i t)}{t^{\alpha+\nu+2}} dt, \quad 0 < r < \infty,$$

whence (40) follows, since $\mathcal{B}(r) = r^\alpha w(r) + O(e^{-r})$ as $r \rightarrow \infty$. \square

Lemma 21. *The function $\mathcal{A}(r)$ has the following structure at infinity*

$$\mathcal{A}(r) = \frac{\alpha}{\lambda} + \frac{C}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) + h(r) + m(r) =: \mathcal{A}_3(r) + m(r), \tag{43}$$

where $m(r)$ satisfies Mikhlin's condition and $h(r) = O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right)$.

Proof. We have

$$\frac{1}{r^\alpha w(r)} = \frac{\alpha}{\lambda} + \frac{1 - \frac{\alpha}{\lambda} r^\alpha w(r)}{r^\alpha w(r)}.$$

Making use of the asymptotics obtained in Lemma 20 and the fact that $r^\alpha w(r) > C \neq 0$ for sufficiently large r , we obtain

$$\frac{1}{r^\alpha w(r)} = \frac{\alpha}{\lambda} - \left(\frac{\alpha}{\lambda}\right)^2 \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) + \frac{s(r)}{r^\alpha w(r)} - \frac{\alpha}{\lambda} \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) \left[\frac{1}{r^\alpha w(r)} - \frac{\alpha}{\lambda} \right]$$

where

$$s(r) = -\frac{\alpha(\alpha + 2)}{\lambda} \left((\alpha + 4) \sum_{i=0}^{\frac{\ell}{2}-1} \frac{r^\alpha}{\ell_i} \int_r^\infty \frac{J_{\nu-3}(\ell_i t)}{t^{\alpha+\nu+2}} dt - \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} \frac{J_{\nu-3}(\ell_i r)}{r^{\nu+1}} \right). \tag{44}$$

Applying the same process to the factor $\left[\frac{1}{r^\alpha w(r)} - \frac{\alpha}{\lambda}\right]$ we obtain

$$\frac{1}{r^\alpha w(r)} = \frac{\alpha}{\lambda} - \left(\frac{\alpha}{\lambda}\right)^2 \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) + h(r) - \underbrace{\left(\frac{\alpha}{\lambda}\right)^2 \frac{1}{r^{\alpha+\nu} w(r)} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r)}_{:=m(r)} \tag{45}$$

where

$$h(r) = s(r) \left(\frac{1}{r^\alpha w(r)} - \frac{\alpha}{\lambda r^{\alpha+\nu} w(r)} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) \right). \tag{46}$$

To see that $m(r)$ satisfies Mikhlin's condition is a matter of direct calculations. To obtain (43), we just need to take into account (44)–(46) and the fact that $\mathcal{A}(r) = \frac{1}{r^\alpha w(r)} + O(e^{-r})$. \square

Lemma 22. *The derivatives $\mathcal{B}^{(k)}(r)$ have the following structure at infinity:*

$$\mathcal{B}^{(k)}(r) = \frac{c}{r^k} + \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} C_i J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), \quad r \rightarrow \infty, \tag{47}$$

where c and c_i are constants.

Proof. By Leibniz' formula it suffices to show that the derivatives $[r^\alpha w(r)]^{(k)}$ have the same asymptotics at infinity as in (47). We have

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum_{j=1}^k c_{k,j} r^{\alpha+j-k} w^{(j)}(r). \tag{48}$$

From (24) we have $w'(r) = \frac{V(r)}{r^{1+\alpha}}$, so that

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum_{j=0}^{k-1} c_{k,j+1} r^{\alpha+j+1-k} \frac{d^j}{dr^j} \left(\frac{V(r)}{r^{1+\alpha}} \right). \tag{49}$$

Hence

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum_{i=0}^{k-1} c_{k,i} r^{i-k} V^{(i)}(r). \tag{50}$$

We make use of the relation

$$\left(\frac{d}{dr} \right)^i = \sum_{s=0}^{[\frac{i}{2}]} c_{i,s} r^{i-2s} \mathfrak{D}^{i-s}, \quad \mathfrak{D} = \frac{d}{r dr}$$

and transform (50) to

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum_{s=0}^{k-1} c_{k,s} r^{2s-k} \mathfrak{D}^s V(r),$$

keeping in mind formula (26). Then by (26) and the following formula $\mathfrak{D}^s \left[\frac{J_\nu(r)}{r^\nu} \right] = (-1)^s \frac{J_{\nu+s}(r)}{r^{\nu+s}}$, after easy transformations we arrive at the equality

$$[r^\alpha w(r)]^{(k)} = c r^{\alpha-k} w(r) + c_1 r^{-k} + \sum_{s=0}^{k-1} r^{-\nu-s} \sum_{i=0}^{\frac{\ell}{2}-1} c_{s,i} J_{\nu+k-s-2}(\ell_i r)$$

for $0 < r < \infty$. In view of (40), we arrive then at (47). \square

Lemma 23. *The derivatives $\mathcal{A}_3^{(k)}(r)$ have the following structure at infinity*

$$\mathcal{A}_3^{(k)}(r) = \frac{c}{r^k} + \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} c_i J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), \quad r \rightarrow \infty, \tag{51}$$

where c, c_i are constants and $\mathcal{A}_3(r)$ is given in (43).

Proof. By the fact that $\frac{d^k}{dr^k} [\mathcal{A}_3(r) - h(r)] = O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right)$, we just need to take care of the asymptotic of $h(r)$.

By (46), we have

$$h(r) = s(r) v(r) \frac{1}{r^\alpha w(r)}$$

where

$$v(r) = 1 - \frac{\alpha}{\lambda r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i} J_{\nu-2}(\ell_i r).$$

Taking into account the fact that

$$s^{(k)}(r) = O\left(\frac{1}{r^{\nu+\frac{3}{2}}}\right), \quad v^{(k)}(r) = \begin{cases} O(1), & k = 0, \\ O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), & k = 1, \dots, n, \end{cases}$$

and

$$\left(\frac{1}{r^\alpha w(r)} \right)^{(k)} = \frac{c}{r^k} + \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} c_i J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right) \tag{52}$$

we arrive at (51). To obtain (52), we just use formula (72) and asymptotic (47). \square

5. Proofs of the main result

5.1. Proof of Theorem 13

When we verify that $\mathcal{A}(r)$ and $\mathcal{B}(r)$ are $p(\cdot)$ -Fourier multipliers, we split the functions into portions covered by different means, some by the Mikhlin condition, others by establishing properties of the corresponding kernels. Under this approach, the result for $\frac{1}{\mathcal{B}(r)}$ does not follow automatically from that for $\mathcal{B}(r)$ and we have to treat both. Since both $\mathcal{B}(r)$ and $\mathcal{A}(r)$ have similar behaviour at the origin and infinity, we give all the details of the proof for $\mathcal{B}(r)$ and mention principal points for the proof in the case of $\mathcal{A}(r)$, the details of the proof being the same.

We need to deal with the behaviour of $\mathcal{B}(r)$ in different ways near the origin and infinity. To this end, we make use of a unity partition $1 \equiv \mu_1(r) + \mu_2(r) + \mu_3(r)$, $\mu_i \in C^\infty$, $i = 1, 2, 3$, where

$$\mu_1(r) = \begin{cases} 1 & \text{if } 0 \leq x < \varepsilon, \\ 0 & \text{if } x \geq \varepsilon + \delta, \end{cases} \quad \mu_3(r) = \begin{cases} 0 & \text{if } 0 \leq x < N - \delta, \\ 1 & \text{if } x \geq N, \end{cases} \tag{53}$$

with $\text{supp } \mu_1 = [0, \varepsilon + \delta]$, $\text{supp } \mu_3 = [N - \delta, \infty)$, and represent $\mathcal{B}(r)$ as

$$\mathcal{B}(r) = \left(\frac{1 - e^{-r}}{r}\right)^{-\alpha} w(r)\mu_1(r) + \mathcal{B}(r)\mu_2(r) + (1 - e^{-r})^{-\alpha} r^\alpha w(r)\mu_3(r) =: \mathcal{B}_1(r) + \mathcal{B}_2(r) + \mathcal{B}_3(r). \tag{54}$$

The function $\mathcal{B}_2(r)$ vanishing in the neighbourhoods of the origin and infinity, is infinitely differentiable, so that it is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, we only have to take care of the multipliers $\mathcal{B}_1(r)$ and $\mathcal{B}_3(r)$ supported in neighbourhoods of origin and infinity, respectively. They will be treated in a different way. For $\mathcal{B}_1(r)$ we will apply the Mikhlin criterion for the spaces $L^{p(\cdot)}(\mathbb{R}^n)$, while the case of the multiplier $\mathcal{B}_3(r)$ proved to be more difficult. In the case $n = 1$ it is easily covered by means of the Mikhlin criterion, while for $n \geq 2$ we use another approach. Namely, we show that the kernel $b_3(|x|)$, corresponding to the multiplier

$$\mathcal{B}_3(r) - \mathcal{B}_3(\infty) = \mu_3(r)\mathcal{B}(r) - \mathcal{B}(\infty), \tag{55}$$

has an integrable radial non-increasing majorant, which will mean that $\mathcal{B}_3(r)$ is certainly a multiplier. However, this will require the usage of special facts on behaviour of the Bessel functions at infinity and an information on some of integrals of Bessel functions.

The proof of Theorem 13 follows from the study of the multipliers $\mathcal{B}_1(r)$ and $\mathcal{B}_3(r)$ made in Sections 5.1.1 and 5.1.2 combined with Lemma 7.

5.1.1. Proof for the case of the multiplier $\mathcal{B}_1(r)$

Lemma 24. *The function $\mathcal{B}_1(r)$ satisfies Mikhlin condition (9).*

Proof. We have to check condition (9) only near the origin.

The function $g(r) := \frac{r}{1 - e^{-r}}$ with $g(0) \neq 0$ is non-vanishing in any neighbourhood of the origin. Therefore, the function $(\frac{r}{1 - e^{-r}})^\alpha = [g(r)]^\alpha$ is infinitely differentiable on any finite interval $[0, N]$ and thereby satisfies conditions (9) on every neighbourhood of the origin.

Thus, to estimate $r^k \frac{d^k}{dr^k} \mathcal{B}_1(r)$, we only need to show the boundedness of $|r^k w^{(k)}(r)|$ as $r \rightarrow 0$. By the equivalence (9) \iff (10), we may estimate $(r \frac{d}{dr})^j w(r)$. Since $w'(r) = -cr^{-1-\alpha} V(r)$ by (24), we only have to prove the estimate

$$\left| \left(r \frac{d}{dr} \right)^j G(r) \right| \leq C < \infty, \quad j = 1, 2, \dots, n - 1, \text{ for } 0 < r < \varepsilon, \tag{56}$$

where $G(r) = r^{-\alpha} \int_{S^{n-1}} \sin^\ell(r\sigma_1) d\sigma$. We represent $G(r)$ as

$$G(r) = r^{\ell-\alpha} F(r), \quad F(r) := \int_{S^{n-1}} s(r\sigma_1) \sigma_1^\ell d\sigma$$

where $s(t) = (\frac{\sin t}{t})^\ell$ is an analytic function and therefore $F(r)$ is an analytic function in r . Then estimate (56) becomes obvious since $\ell - \alpha > 0$. \square

5.1.2. Proof for the case of the multiplier $\mathcal{B}_3(r)$

As mentioned above, we treat separately the cases $n = 1$ and $n \geq 2$.

In the case $n = 1$ we just have to show that $\mathcal{B}(r)$ and $r\mathcal{B}'(r)$ are bounded on $[0, \infty]$. The boundedness of $\mathcal{B}_3(r)$ is evident on any subinterval (N, N_1) , $N_1 > N$ and it suffices to note that there exist the finite value $\mathcal{B}(\infty)$, see the proof of Lemma 18. To show that $r\mathcal{B}'_3(r)$ is bounded, it suffices to check that $r[r^\alpha w(r)]'$ is bounded for large r . From (24) we have

$$r[r^\alpha w(r)]' = r^\alpha w(r) - c \sin^\ell r,$$

which is bounded.

We pass now to the case $n \geq 2$.

Lemma 25. Let $n \geq 2$. The kernel $b_3(r)$ is vanishing at infinity faster than any power and admits the estimate:

$$|b_3(r)| \leq \frac{C}{r^{\frac{n-1}{2}}(1+r)^m}, \quad 0 < r < \infty, \tag{57}$$

where $m = 1, 2, 3, \dots$ is arbitrarily large, and $C = C(m)$ does not depend on r .

Proof. (1) Estimation as $r \rightarrow 0$. By the Fourier inversion formula for radial functions we have

$$b_3(r) = \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_0^\infty t^\nu J_{\nu-1}(rt) [\mathcal{B}_3(t) - \mathcal{B}_3(\infty)] dt, \quad \nu = \frac{n}{2}. \tag{58}$$

From (58) we have

$$|b_3(r)| \leq \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_{N-\delta}^N t^\nu |J_{\nu-1}(rt) [\mathcal{B}_3(t) - \mathcal{B}(\infty)]| dt + \left| \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_N^\infty t^\nu J_{\nu-1}(rt) [\mathcal{B}(t) - \mathcal{B}(\infty)] dt \right|. \tag{59}$$

We make use of the asymptotics obtained in (40) and get

$$|b_3(r)| \leq \frac{c}{r^{\nu-1/2}} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{c_i}{r^{\nu-1}} \left| \int_N^\infty J_{\nu-1}(rt) J_{\nu-2}(\ell_i t) dt \right| + \frac{c}{r^{\nu-1}} \int_N^\infty t^\nu |J_{\nu-1}(rt)| \frac{dt}{t^{\nu+\frac{3}{2}}}, \tag{60}$$

where c_i are constants. Since $|J_{\nu-1}(t)| \leq \frac{ct^{\nu-1}}{(t+1)^{\nu-1+\frac{1}{2}}}$, the last term is easily estimated:

$$\frac{c}{r^{\nu-1}} \int_N^\infty t^\nu |J_{\nu-1}(rt)| \frac{dt}{t^{\nu+\frac{3}{2}}} \leq \frac{c}{r^{\nu-\frac{1}{2}}} \int_N^\infty \frac{t^{\nu-1}}{(t+\frac{1}{r})^{\nu-\frac{1}{2}}} \frac{dt}{t^{\frac{3}{2}}} \leq \frac{c}{r^{\nu-\frac{1}{2}}}.$$

Thus

$$|b_3(r)| \leq \frac{c}{r^{\nu-\frac{1}{2}}} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{c_i}{r^{\nu-1}} \left| \int_N^\infty J_{\nu-1}(rt) J_{\nu-2}(\ell_i t) dt \right|$$

as $r \rightarrow 0$. It is known that the integral $\int_0^\infty J_{\nu-1}(rt) J_{\nu-2}(\ell_i t) dt$ converges when $n \geq 2$; it is equal to zero, if $n > 2$ and $-\frac{1}{\ell_i}$, if $n = 2$, see [8, formula 6.512.3] (use also the fact that $J_{\nu-2}(r) = J_{-1}(r) = -J_1(r)$ if $n = 2$). Then

$$|b_3(r)| \leq \frac{c}{r^{\nu-\frac{1}{2}}} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{c_i}{r^{\nu-1}} \left| \int_0^N J_{\nu-1}(rt) J_{\nu-2}(\ell_i t) dt \right| \leq \frac{c}{r^{\nu-\frac{1}{2}}}$$

which proves (57) as $r \rightarrow 0$.

(2) Estimation as $r \rightarrow \infty$. Since the integral in (58) is not absolutely convergent for large t , it is not easy to treat the case $r \rightarrow \infty$ starting from the representation (58). So we transform this representation. We interpret the integral in (58) in the sense of regularization:

$$b_3(r) = \lim_{\varepsilon \rightarrow 0} \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_0^\infty e^{-\varepsilon t} t^\nu J_{\nu-1}(rt) [\mathcal{B}_3(t) - \mathcal{B}_3(\infty)] dt \tag{61}$$

and before to pass to the limit in (61), apply formula (36) with $f(t) = e^{-\varepsilon t}[\mathcal{B}_3(t) - \mathcal{B}_3(\infty)]$. Then conditions (38) are satisfied so that formula (36) is applicable and after easy passage to the limit we obtain

$$b_3(r) = \frac{(-1)^m (2\pi)^{-\nu}}{r^{\nu+m-1}} \sum_{k=1}^m c_{m,k} \int_0^\infty t^{\nu+k-m} J_{\nu+m-1}(rt) \mathcal{B}_3^{(k)}(t) dt \tag{62}$$

for every $m \geq 1$. The last representation already allows to obtain the estimation as $r \rightarrow \infty$. From (62) we get

$$\begin{aligned} |b_3(r)| \leq & \frac{c}{r^{\nu+m-1}} \int_{N-\delta}^N t^\nu |J_{\nu+m-1}(rt) \mathcal{B}^{(m)}(t)| dt + \frac{c}{r^{\nu+m-1}} \left| \int_N^\infty t^\nu J_{\nu+m-1}(rt) \mathcal{B}^{(m)}(t) dt \right| \\ & + \sum_{k=1}^{m-1} \frac{c_k}{r^{\nu+m-1}} \int_N^\infty t^{\nu+k-m} |J_{\nu+m-1}(rt) \mathcal{B}^{(k)}(t)| dt. \end{aligned} \tag{63}$$

The function $\mathcal{B}^{(m)}(t)$ is bounded on $[N - \delta, N]$ so that the estimation of the first term is obvious. Since $|J_{\nu+m-1}(rt)| \leq \frac{c}{\sqrt{rt}}$ and $|\mathcal{B}^{(k)}(t)| \leq ct^{-\xi}$, $\xi = \min\{k, \nu + 1/2\}$, see (47), the last sum in (63) is estimated by $\frac{c}{r^{\nu+m-\frac{3}{2}}}$.

It remains to estimate the second term. We make use of the asymptotics in (47) again and obtain

$$|b_3(r)| \leq \frac{1}{r^{\nu+m-\frac{3}{2}}} \left(c + \sum_{i=1}^{\ell-1} c_i \left| \int_N^\infty J_{\nu+m-1}(rt) J_{\nu-2}(\ell_i t) dt \right| \right). \tag{64}$$

It is known that the last integral converges when $\nu + \frac{m}{2} > 1$ and

$$\int_0^\infty J_{\nu+m-1}(rt) J_{\nu-2}(\ell_i t) dt = \frac{\gamma}{r^{\nu-1}}, \quad r > \ell_i,$$

where γ is a constant ($\gamma = \ell_i^{\nu-2} \frac{\Gamma(\nu-1+\frac{m}{2})}{\Gamma(\nu-1)\Gamma(1+\frac{m}{2})}$), see [8, formula 6.512.1]. Then from (64) we get (57).

Finally, for the function $\mathcal{A}(r)$, we only note that the splitting is similar:

$$\mathcal{A}(r) = \mathcal{A}_1(r) + \mathcal{A}_2(r) + \mathfrak{m}(r) + \mathcal{A}_3(r),$$

where $\text{supp } \mathcal{A}_i(r) = \text{supp } \mu_i$ and $\text{supp } \mathfrak{m}(r) = \text{supp } \mu_3$.

It is easy to see that the functions $\mathcal{A}_i(r)$, $i = 1, 2$ satisfy Mihlin's condition. The proof for $\mathcal{A}_3(r)$, prepared by Lemmas 21 and 23, is similar to that for \mathcal{B}_3 . \square

5.2. Proof of Theorem 14

Proof. Assume that the limit $\lim_{\varepsilon \rightarrow 0^+} \mathbb{D}_\varepsilon^\alpha f$ in (34) exists. We express $\frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f$ via $\varphi_\varepsilon(x) := \mathbb{D}_\varepsilon^\alpha f(x)$ in “averaging” terms:

$$\frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f(x) = c\varphi_\varepsilon(x) + \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} a\left(\frac{x-y}{\varepsilon}\right) \varphi_\varepsilon(y) dy \tag{65}$$

where $a(x) \in L^1(\mathbb{R}^n)$ and $a(x)$ is the inverse Fourier transform of the function $A(x) - A(\infty)$, $c = A(\infty)$ and

$$c + \int_{\mathbb{R}^n} a(y) dy = 1. \tag{66}$$

Representation (65)–(66), verified via Fourier transforms:

$$F\left(\frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f\right)(x) = A(\varepsilon x) F(\mathbb{D}_\varepsilon^\alpha f)(x) \tag{67}$$

was given in [23] for the case of constant p and thus valid for $f \in C_0^\infty(\mathbb{R}^n)$. Then (65) holds for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ by the continuity of the operators on the left-hand and right-hand sides in $L^{p(\cdot)}(\mathbb{R}^n)$; for the left-hand side see (16), while the boundedness of the convolution operator on the right-hand side follows from the fact that the Fourier transform of its kernel is a Fourier $p(\cdot)$ -multiplier by Theorem 13.

With $\varphi = \mathbb{D}^\alpha f = \lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon$, from (67) we have

$$\left\| \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f - \varphi \right\|_{p(\cdot)} \leq C \|\varphi_\varepsilon - \varphi\|_{p(\cdot)} + \|T_\varepsilon \varphi - \varphi\|_{p(\cdot)} \tag{68}$$

where T_ε is the operator generated by the multiplier $A(\varepsilon \cdot)$. The first term in the right-hand side of (68) tends to zero by the definition of φ and Proposition 8. Regarding the second term, we have

$$\|T_\varepsilon \varphi - \varphi\|_{p(\cdot)} \leq \|T_M \varphi - \varphi\|_{p(\cdot)} + \|T_a \varphi\|_{p(\cdot)} = I_1 + I_2$$

where T_M is the operator given by the multiplier $M(\varepsilon \cdot) := \mathcal{A}_1(\varepsilon \cdot) + \mathcal{A}_2(\varepsilon \cdot) + m(\varepsilon \cdot)$. Since M satisfies Mikhlin’s condition and $M(\varepsilon x) \rightarrow 1$ as $\varepsilon \rightarrow 0$ for almost all $x \in \mathbb{R}^n$, we have $I_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$ by Lemma 6.

For I_2 we observe that

$$T_a \varphi(x) = -\mathcal{A}_3(\infty) [(a_3)_\varepsilon * \varphi(x) - \varphi(x)] \tag{69}$$

where $(a_3)_\varepsilon$ is the dilatation of the kernel $a_3(x) = F^{-1}[\mathcal{A}_3(\cdot) - \mathcal{A}(\infty)](x)$ and then $I_2 \rightarrow 0$ as $\varepsilon \rightarrow 0$ by Proposition 3 and Lemma 25.

Suppose now that $\lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f$ exists in $L^{p(\cdot)}(\mathbb{R}^n)$. By (67) we have

$$F(\mathbb{D}_\varepsilon^\alpha f)(x) = B(\varepsilon x) F\left(\frac{(I - P_\varepsilon)^\alpha f}{\varepsilon^\alpha}\right)(x) \tag{70}$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where $B(x) = 1/A(x)$. Since $B(x)$ is also a Fourier multiplier by Theorem 13, the arguments are the same as in the above passage from $\lim_{\varepsilon \rightarrow 0} \mathbb{D}_\varepsilon^\alpha f$ to $\lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f$. \square

5.3. Proof of Theorem 16

The “only if” part of Theorem 16 is a consequence of Theorem 14.

To prove the “if” part, suppose that (35) holds. From (70) we obtain that

$$\|\mathbb{D}_\varepsilon^\alpha f\|_{p(\cdot)} \leq C \left\| \frac{1}{\varepsilon^\alpha} (I - P_\varepsilon)^\alpha f \right\|_{p(\cdot)} \leq C,$$

since $A(\varepsilon x)$ is a uniform Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 17 and Lemma 5. To finish the proof, it remains to refer to Theorems 9 and 8.

5.4. Proof of Theorem 17

Theorem 17 is an immediate consequence of Theorem 14 and Propositions 12 and 8.

Appendix A

The following recurrence formula for the k -th derivative of the quotient is valid (see, e.g., [29,9])

$$\left(\frac{u}{v}\right)^{(k)} = \frac{1}{v} \left(u^{(k)} - k! \sum_{j=1}^k \frac{v^{(k+1-j)}}{(k+1-j)!} \left(\frac{u}{v}\right)^{(j-1)} \right). \tag{71}$$

By means of this formula, by induction it is not hard to check the validity of the following formula for the k -th derivative of the fraction $\frac{1}{v(x)}$, $x \in \mathbb{R}^1$:

$$\frac{d^k}{dx^k} \left(\frac{1}{v}\right) = \frac{v^{(k)}}{v^2} + \sum_{j=1}^{k-1} \frac{A_{j,k}(D)v}{v^{j+1}} + (-1)^k k! \frac{(v')^k}{v^{k+1}} \tag{72}$$

where the differential operators $A_{j,k}(D)$ of order k have the form

$$A_{j,k}(D)v = a_j \left[\frac{d^{m_j} v}{dx^{m_j}} \right]^{\alpha_j} \left[\frac{d^{n_j} v}{dx^{n_j}} \right]^{\beta_j} + b_j \left[\frac{d^{p_j} v}{dx^{p_j}} \right]^{\gamma_j} \left[\frac{d^{q_j} v}{dx^{q_j}} \right]^{\delta_j}$$

where a_j and b_j are constants, $m_j, n_j, p_j, q_j, \alpha_j, \beta_j, \gamma_j, \delta_j$ are integers in $[1, k - 1]$ such that

$$m_j \alpha_j + n_j \beta_j = p_j \gamma_j + q_j \delta_j = k. \tag{73}$$

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