



# An expectation formula with applications

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## ABSTRACT

In this paper, we derive an expectation formula of a random variable having distribution  $W(x; q)$ . As applications of the expectation formula, we give a transformation formula and an expansion of Sears'  ${}_3\phi_2$  transformation formula.

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## 1. Introduction and main result

The probabilistic method is a useful tool in the study of basic hypergeometric functions. There are some works available in the literature. For example, K.W.J. Kadell [11] gave a probabilistic proof of Ramanujan's  ${}_1\psi_1$  sum based on the order statistics. J. Fulman [7] presented a probabilistic proof of Rogers–Ramanujan identity using Markov chain on the non-negative integers. R. Chapman [6] extended J. Fulman's methods to prove the Andrews–Gordon identity. In particular, the present author [19] established the following new probability distribution  $W(x; q)$ :

$$P(\xi = x^n q^k) = \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty}, \quad (1.1)$$

where  $x < 0$ ;  $0 < q < 1$ ;  $n = 0, 1$ ;  $k = 0, 1, 2, \dots$ , and gave some applications of this distribution in  $q$ -series. In this paper, we derive an expectation formula of  $W(x; q)$  and provide some probabilistic derivations of  $q$ -identities which include an extension of Sears'  ${}_3\phi_2$  transformation formula.

We recall some definitions, notation and known results in [4,8] which will be used in this paper. Throughout the whole paper, it is supposed that  $0 < q < 1$ . The  $q$ -shifted factorials are defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \quad (1.2)$$

We also adopt the following compact notation for multiple  $q$ -shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n, \quad (1.3)$$

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where  $n$  is an integer or  $\infty$ . We may extend the definition (1.2) of  $(a; q)_n$  to

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty}, \quad (1.4)$$

for any complex number  $\alpha$ . In particular,

$$(a; q)_{-n} = \frac{(a; q)_\infty}{(aq^{-n}; q)_\infty} = \frac{1}{(aq^{-n}; q)_n} = \frac{(-q/a)^n}{(q/a; q)_n} q^{\binom{n}{2}}. \quad (1.5)$$

Heine introduced the  ${}_{r+1}\phi_r$  basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r \left( \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \dots, b_r; q)_n}. \quad (1.6)$$

The  $q$ -binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \quad (1.7)$$

The  $q$ -Gauss summation formula

$${}_2\phi_1 \left( \begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab} \right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad \left| \frac{c}{ab} \right| < 1. \quad (1.8)$$

Sears'  ${}_3\phi_2$  transformation formula

$${}_3\phi_2 \left( \begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q, \frac{b_1 b_2}{a_1 a_2 a_3} \right) = \frac{(b_2/a_3, b_1 b_2/a_1 a_2; q)_\infty}{(b_2, b_1 b_2/a_1 a_2 a_3; q)_\infty} {}_3\phi_2 \left( \begin{matrix} b_1/a_1, b_1/a_2, a_3 \\ b_1, b_1 b_2/a_1 a_2 \end{matrix}; q, \frac{b_2}{a_3} \right),$$

$$\left| \frac{b_1 b_2}{a_1 a_2 a_3} \right| < 1, \quad \left| \frac{b_2}{a_3} \right| < 1. \quad (1.9)$$

The bilateral basic hypergeometric series  ${}_r\psi_s$  is defined by

$${}_r\psi_s \left( \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} (-1)^{(s-r)n} q^{\binom{s-r}{2}n} z^n. \quad (1.10)$$

The following is the well-known Ramanujan's  ${}_1\psi_1$  summation formula

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1. \quad (1.11)$$

F.H. Jackson defined the  $q$ -integral by [10]

$$\int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n, \quad (1.12)$$

and

$$\int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t. \quad (1.13)$$

He also defined an integral on  $(0, \infty)$  by

$$\int_0^\infty f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \quad (1.14)$$

On the interval  $(-\infty, \infty)$  the bilateral  $q$ -integral is defined by

$$\int_{-\infty}^\infty f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)] q^n. \quad (1.15)$$

The  $q$ -integral is important in the theory and applications of basic hypergeometric series. For example, the present author gives some applications of the  $q$ -integral [14–18,20]. These papers can help the reader to understand the main idea of the present paper. The following is the Andrews–Askey integral [2], which can be derived from Ramanujan's  ${}_1\psi_1$  summation:

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}, \quad (1.16)$$

provided that no zero factors occur in the denominator of the integrals. Al-Salam and Verma gave an extension of the Andrews–Askey integral, which is called the Al-Salam and Verma [1]  $q$ -integral

$$\int_c^d \frac{(qt/c, qt/d, et; q)_\infty}{(at, bt, ft; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, e/a, e/b, e/f; q)_\infty}{(ac, ad, bc, bd, fc, fd; q)_\infty}, \quad (1.17)$$

provided that no zero factors occur in the denominator of the integrals, where  $e = abcd f$ .

The main result of this paper is the following theorem:

**Theorem 1.** Let  $\xi$  denote random variable having distribution  $W(x; q)$ ,  $-1 < x < 0$ , then we have

$$E \left\{ \frac{(cx\xi; q)_\infty}{(a\xi, b\xi, c\xi; q)_\infty} \right\} = \frac{(acx, bcx, x; q)_\infty}{(a, ax, b, bx, c, cx; q)_\infty} {}_3\phi_2 \left( \begin{matrix} ab, c, cx \\ acx, bcx \end{matrix}; q, x \right), \quad (1.18)$$

provided that  $\max\{|a|, |b|, |c|\} < 1$ , where  $E(X)$  denotes expected value of the random variable  $X$ .

## 2. The proof of Theorem 1

In this section, we use the Al-Salam and Verma [1]  $q$ -integral to give the proof of Theorem 1. We also need to use the following well-known theorems:

Analytic continuation theorem: If  $f$  and  $g$  are analytic at  $z_0$  and agree at infinitely many points which include  $z_0$  as an accumulation point, then  $f = g$ .

Lebesgue's dominated convergence theorem: Suppose that  $\{X_n, n \geq 1\}$  is a sequence of random variables, that  $X_n \rightarrow X$  pointwise almost everywhere as  $n \rightarrow \infty$ , and that  $|X_n| \leq Y$  for all  $n$ , where random variable  $Y$  is integrable. Then  $X$  is integrable, and

$$\lim_{n \rightarrow \infty} EX_n = EX. \quad (2.1)$$

The analytic continuation theorem is useful in the study of basic hypergeometric functions. Two well-known examples are, Ismail's proof of the Ramanujan's  ${}_1\psi_1$  sum [9] and the Askey–Ismail proof of Bailey's sum of the very well poised  ${}_6\psi_6$  [5].

Now we give the proof.

**Proof.** Considering the following sequence of random variables (on a probability space):

$$\eta_n = \frac{(abcx\xi; q)_\infty}{(a\xi, b\xi, c\xi; q)_\infty} \sum_{k=0}^n \frac{(ab, c\xi; q)_k x^k}{(q, abcx\xi; q)_k},$$

where  $n = 0, 1, 2, \dots$

Since,

$$\begin{aligned} |\eta_n| &= \left| \frac{(abcx\xi; q)_\infty}{(a\xi, b\xi, c\xi; q)_\infty} \sum_{k=0}^n \frac{(ab, c\xi; q)_k x^k}{(q, abcx\xi; q)_k} \right| \\ &= \left| \sum_{k=0}^n \frac{(abcxq^k\xi; q)_\infty}{(a\xi, b\xi, cq^k\xi; q)_\infty} \cdot \frac{(ab; q)_k x^k}{(q; q)_k} \right| \\ &\leq \frac{(-|abcx|; q)_\infty}{(|a|, |b|, |c|; q)_\infty} \sum_{k=0}^n \left| \frac{(ab; q)_k x^k}{(q; q)_k} \right| \end{aligned}$$

and the series

$$\sum_{k=0}^{\infty} \frac{(ab; q)_k x^k}{(q; q)_k}$$

converges absolutely. Using Lebesgue's dominated convergence theorem obtains:

$$\lim_{n \rightarrow \infty} E\eta_n = E\left(\lim_{n \rightarrow \infty} \eta_n\right). \quad (2.2)$$

Using the  $q$ -Gauss theorem (1.8), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \eta_n &= \frac{(abcx\xi; q)_{\infty}}{(a\xi, b\xi, c\xi; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(ab, c\xi; q)_k x^k}{(q, abcx\xi; q)_k} \\ &= \frac{(abcx\xi; q)_{\infty}}{(a\xi, b\xi, c\xi; q)_{\infty}} \cdot \frac{(cx\xi, abx; q)_{\infty}}{(abcx\xi, x; q)_{\infty}} \\ &= \frac{(abx; q)_{\infty}}{(x; q)_{\infty}} \cdot \frac{(cx\xi; q)_{\infty}}{(a\xi, b\xi, c\xi; q)_{\infty}}. \end{aligned} \quad (2.3)$$

Hence, we get the right-hand side of (2.2):

$$E\left(\lim_{n \rightarrow \infty} \eta_n\right) = \frac{(abx; q)_{\infty}}{(x; q)_{\infty}} E\left\{ \frac{(cx\xi; q)_{\infty}}{(a\xi, b\xi, c\xi; q)_{\infty}} \right\}. \quad (2.4)$$

Now we begin to calculate the left-hand side of (2.2). Observe that

$$E\eta_n = E\left\{ \frac{(abcx\xi; q)_{\infty}}{(a\xi, b\xi, c\xi; q)_{\infty}} \sum_{k=0}^n \frac{(ab, c\xi; q)_k x^k}{(q, abcx\xi; q)_k} \right\} = \sum_{k=0}^n \frac{(ab; q)_k x^k}{(q; q)_k} E\left\{ \frac{(abcxq^k\xi; q)_{\infty}}{(a\xi, b\xi, cq^k\xi; q)_{\infty}} \right\}. \quad (2.5)$$

Employing the Al-Salam and Verma  $q$ -integral (1.17) gives

$$\begin{aligned} E\left\{ \frac{(abcxq^k\xi; q)_{\infty}}{(a\xi, b\xi, cq^k\xi; q)_{\infty}} \right\} &= \sum_{n=0}^1 \sum_{m=0}^{\infty} \frac{(-x)^n (x^{n-1}q^{m+1}, x^nq^{m+1}, abcx^{n+1}q^{k+m}; q)_{\infty} q^m}{(q, q/x, x, ax^nq^m, bx^nq^m, cq^{k+m}x^n; q)_{\infty}} \\ &= \frac{1}{(1-q)(q, q/x, x; q)_{\infty}} \left[ (1-q) \sum_{m=0}^{\infty} \frac{(q^{m+1}/x, q^{m+1}, abcxq^{k+m}; q)_{\infty} q^m}{(aq^m, bq^m, cq^{k+m}; q)_{\infty}} \right. \\ &\quad \left. - x(1-q) \sum_{m=0}^{\infty} \frac{(q^{m+1}, xq^{m+1}, abcx^2q^{k+m}; q)_{\infty} q^m}{(axq^m, bxq^m, cq^{k+m}x; q)_{\infty}} \right] \\ &= \frac{1}{(1-q)(q, q/x, x; q)_{\infty}} \int_x^1 \frac{(qt/x, qt, abcxq^kt; q)_{\infty}}{(at, bt, cq^kt; q)_{\infty}} d_q t \\ &= \frac{(abx, acxq^k, bcxq^k; q)_{\infty}}{(a, ax, b, bx, cq^k, cxq^k; q)_{\infty}}. \end{aligned} \quad (2.6)$$

Substituting (2.6) into (2.5) yields

$$E\eta_n = \sum_{k=0}^n \frac{(ab; q)_k x^k}{(q; q)_k} E\left\{ \frac{(abcxq^k\xi; q)_{\infty}}{(a\xi, b\xi, cq^k\xi; q)_{\infty}} \right\} = \frac{(abx, acx, bcx; q)_{\infty}}{(a, ax, b, bx, c, cx; q)_{\infty}} \sum_{k=0}^n \frac{(ab, c, cx; q)_k x^k}{(q, acx, bcx; q)_k}. \quad (2.7)$$

Hence, we get the left-hand side of (2.2)

$$\lim_{n \rightarrow \infty} E\eta_n = \frac{(abx, acx, bcx; q)_{\infty}}{(a, ax, b, bx, c, cx; q)_{\infty}} {}_3\phi_2\left(\begin{matrix} ab, c, cx \\ acx, bcx \end{matrix}; q, x\right). \quad (2.8)$$

Substituting (2.4) and (2.8) into (2.2) yields (1.18).  $\square$

### 3. Some applications

The expectation formula (1.18) can be used to derive identities about the basic hypergeometric functions. In this section, we show some applications of Theorem 1. The following transformation formula can be derived directly from Theorem 1 and  $q$ -binomial theorem.

**Theorem 2.** Let  $|a| < 1$ ,  $|b| < 1$ ,  $|x| < 1$  and  $|z| < 1$ , then

$$\sum_{n=0}^{\infty} \frac{(a, ax; q)_n z^n}{(q, abx; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2 z q^n, b, bx \\ abx q^n, abz \end{matrix}; q, x \right) = \frac{(az, azx; q)_{\infty}}{(z, abz; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} b, bx \\ abx \end{matrix}; q, x \right). \quad (3.1)$$

**Proof.** Rewrite  $q$ -binomial theorem (1.7) as

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} \cdot \frac{1}{(aq^n, az; q)_{\infty}} = \frac{1}{(z; q)_{\infty}} \cdot \frac{1}{(a; q)_{\infty}}. \quad (3.2)$$

Letting  $a = a\xi$  in (3.2) gives

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} \cdot \frac{1}{(aq^n \xi, az\xi; q)_{\infty}} = \frac{1}{(z; q)_{\infty}} \cdot \frac{1}{(a\xi; q)_{\infty}}, \quad (3.3)$$

where  $\xi$  denotes random variable having distribution  $W(x; q)$  and  $-1 < x < 0$ . Multiplying both sides of (3.3) by

$$\frac{(bx\xi; q)_{\infty}}{(b\xi; q)_{\infty}}$$

and then applying the expectation operator  $E$  on both sides of (3.3), it follows that

$$E \left\{ \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} \cdot \frac{(bx\xi; q)_{\infty}}{(aq^n \xi, az\xi, b\xi; q)_{\infty}} \right\} = \frac{1}{(z; q)_{\infty}} E \left\{ \frac{(bx\xi; q)_{\infty}}{(a\xi, b\xi; q)_{\infty}} \right\}. \quad (3.4)$$

Since,

$$\left| \frac{z^n}{(q; q)_n} \cdot \frac{(bx\xi; q)_{\infty}}{(aq^n \xi, az\xi, b\xi; q)_{\infty}} \right| \leq \frac{(-|b|; q)_{\infty}}{(|a|, |az|, |b|; q)_{\infty}} \cdot \frac{|z|^n}{(q; q)_n}$$

and the series

$$\sum_{n=0}^{\infty} \frac{|z|^n}{(q; q)_n}$$

converges. Using Lebesgue's dominated convergence theorem yields

$$E \left\{ \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} \cdot \frac{(bx\xi; q)_{\infty}}{(aq^n \xi, az\xi, b\xi; q)_{\infty}} \right\} = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} E \left\{ \frac{(bx\xi; q)_{\infty}}{(aq^n \xi, az\xi, b\xi; q)_{\infty}} \right\}. \quad (3.5)$$

Hence,

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} E \left\{ \frac{(bx\xi; q)_{\infty}}{(aq^n \xi, az\xi, b\xi; q)_{\infty}} \right\} = \frac{1}{(z; q)_{\infty}} E \left\{ \frac{(bx\xi; q)_{\infty}}{(a\xi, b\xi; q)_{\infty}} \right\}. \quad (3.6)$$

Employing (1.18) gives

$$E \left\{ \frac{(bx\xi; q)_{\infty}}{(aq^n \xi, az\xi, b\xi; q)_{\infty}} \right\} = \frac{(abxq^n, abzx, x; q)_{\infty}}{(aq^n, axq^n, az, azx, b, bx; q)_{\infty}} {}_3\phi_2 \left( \begin{matrix} a^2 z q^n, b, bx \\ abx q^n, abz \end{matrix}; q, x \right) \quad (3.7)$$

and

$$E \left\{ \frac{(bx\xi; q)_{\infty}}{(a\xi, b\xi; q)_{\infty}} \right\} = \frac{(abx, x; q)_{\infty}}{(a, ax, b, bx; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} b, bx \\ abx \end{matrix}; q, x \right). \quad (3.8)$$

Substituting (3.7) and (3.8) into (3.6) yields

$$\sum_{n=0}^{\infty} \frac{(a, ax; q)_n z^n}{(q, abx; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2 z q^n, b, bx \\ abx q^n, abz \end{matrix}; q, x \right) = \frac{(az, azx; q)_{\infty}}{(z, abz; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} b, bx \\ abx \end{matrix}; q, x \right), \quad (3.9)$$

where  $-1 < x < 0$ . By analytic continuation, we may replace the assumption  $-1 < x < 0$  by  $|x| < 1$ . Thus, we get (3.1).  $\square$

From the theorem, we can easily get the summation formula involving  ${}_3\phi_2$ :

**Corollary 1.** Let  $|a| < |b| < 1$  and  $|z| < 1$ , then

$$\sum_{n=0}^{\infty} \frac{(a, a^2/b; q)_n z^n}{(q, a^2; q)_n} {}_3\phi_2 \left( \begin{matrix} a, b, a^2 z q^n \\ abz, a^2 q^n \end{matrix}; q, \frac{a}{b} \right) = \frac{(az, a^2 z/b, a, a^2/b; q)_{\infty}}{(z, a^2 z, a^2, a/b; q)_{\infty}}. \quad (3.10)$$

**Proof.** Letting  $x = a/b$  in (3.1) gives

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a, a^2/b; q)_n z^n}{(q, a^2; q)_n} {}_3\phi_2 \left( \begin{matrix} a, b, a^2 z q^n \\ abz, a^2 q^n \end{matrix}; q, \frac{a}{b} \right) &= \frac{(az, a^2 z/b; q)_{\infty}}{(z, a^2 z; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} a, b \\ a^2 \end{matrix}; q, \frac{a}{b} \right) \\ &= \frac{(az, a^2 z/b, a, a^2/b; q)_{\infty}}{(z, a^2 z, a^2, a/b; q)_{\infty}}, \end{aligned} \quad (3.11)$$

as desired.  $\square$

Sears'  ${}_3\phi_2$  transformation formula has been used by Andrews [3,12] in proving many of Ramanujan's identities for partial theta functions. Using the expectation formula (1.18), we can easily derive the following expansion of Sears'  ${}_3\phi_2$  transformation formula.

**Theorem 3.** Let  $|a| < |c| < |b| < 1$ ,  $|d| < 1$ ,  $|x| < 1$  and  $|z| < 1$ , then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(b, a, ax; q)_n z^n}{(q, c, adx; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2 bz q^n/c, d, dx \\ adx q^n, abd zx/c \end{matrix}; q, x \right) \\ = \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b, abz/c, abzx/c; q)_n (c/b)^n}{(q, bz, abd zx/c; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2 bz q^n, d, dx \\ abd zx q^n/c, adx \end{matrix}; q, x \right). \end{aligned} \quad (3.12)$$

**Proof.** By the following Rogers' iteration of Heine's transformation formula [4,8]

$${}_2\phi_1 \left( \begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} b, abz/c \\ bz \end{matrix}; q, \frac{c}{b} \right), \quad (3.13)$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} \cdot \frac{1}{(aq^n, abz/c; q)_{\infty}} = \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b; q)_n (c/b)^n}{(q, bz; q)_n} \cdot \frac{1}{(a, abz q^n/c; q)_{\infty}}. \quad (3.14)$$

Letting  $a = a\xi$  in (3.14) and multiplying both sides of (3.14) by

$$\frac{(dx\xi; q)_{\infty}}{(d\xi; q)_{\infty}}$$

yields,

$$\sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} = \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b; q)_n (c/b)^n}{(q, bz; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(a\xi, abz q^n \xi/c, d\xi; q)_{\infty}}, \quad (3.15)$$

where  $\xi$  denotes random variable having distribution  $W(x; q)$  and  $-1 < x < 0$ . Applying the expectation operator  $E$  on both sides of (3.15), it follows that

$$\begin{aligned}
& \mathbb{E} \left\{ \sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} \right\} \\
&= \mathbb{E} \left\{ \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b; q)_n (c/b)^n}{(q, bz; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(a\xi, abzq^n \xi/c, d\xi; q)_{\infty}} \right\}.
\end{aligned} \quad (3.16)$$

Since,

$$\left| \frac{(b; q)_n z^n}{(q, c; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} \right| \leq \frac{(-|d|; q)_{\infty}}{(|a|, |abz/c|, |d|; q)_{\infty}} \left| \frac{(|a|, b; q)_n z^n}{(q, c; q)_n} \right|$$

and the series

$$\sum_{m=0}^{\infty} \left| \frac{(|a|, b; q)_m z^m}{(q, c; q)_m} \right|$$

is convergent. Using Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E} \left\{ \sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} \right\} = \sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} \mathbb{E} \left\{ \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} \right\}. \quad (3.17)$$

Employing (1.18) gives

$$\mathbb{E} \left\{ \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} \right\} = \frac{(adxq^n, abdzx/c, x; q)_{\infty}}{(aq^n, axq^n, abz/c, abzx/c, d, dx; q)_{\infty}} {}_3\phi_2 \left( \begin{matrix} a^2bzq^n/c, d, dx \\ adxq^n, abdzx/c \end{matrix}; q, x \right). \quad (3.18)$$

Substituting (3.18) into (3.17), then we get the left-hand side of (3.16)

$$\begin{aligned}
& \mathbb{E} \left\{ \sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(aq^n \xi, abz\xi/c, d\xi; q)_{\infty}} \right\} \\
&= \frac{(adx, abdzx/c, x; q)_{\infty}}{(a, ax, abz/c, abzx/c, d, dx; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b, a, ax; q)_n z^n}{(q, c, adx; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2bzq^n/c, d, dx \\ adxq^n, abdzx/c \end{matrix}; q, x \right).
\end{aligned} \quad (3.19)$$

Similarly, we can obtain the right-hand side of (3.16)

$$\begin{aligned}
& \mathbb{E} \left\{ \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b; q)_n (c/b)^n}{(q, bz; q)_n} \cdot \frac{(dx\xi; q)_{\infty}}{(a\xi, abzq^n \xi/c, d\xi; q)_{\infty}} \right\} \\
&= \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b; q)_n (c/b)^n}{(q, bz; q)_n} \mathbb{E} \left\{ \frac{(dx\xi; q)_{\infty}}{(a\xi, abzq^n \xi/c, d\xi; q)_{\infty}} \right\} \\
&= \frac{(adx, abdzx/c, x; q)_{\infty}}{(a, ax, abz/c, abzx/c, d, dx; q)_{\infty}} \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b, abz/c, abzx/c; q)_n (c/b)^n}{(q, bz, abdzx/c; q)_n} \\
&\quad \times {}_3\phi_2 \left( \begin{matrix} a^2bzq^n, d, dx \\ abdzxq^n/c, adx \end{matrix}; q, x \right).
\end{aligned} \quad (3.20)$$

Substituting (3.19) and (3.20) into (3.17), we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(b, a, ax; q)_n z^n}{(q, c, adx; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2bzq^n/c, d, dx \\ adxq^n, abdzx/c \end{matrix}; q, x \right) \\
&= \frac{(c/b, bz; q)_{\infty}}{(z, c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b, abz/c, abzx/c; q)_n (c/b)^n}{(q, bz, abdzx/c; q)_n} {}_3\phi_2 \left( \begin{matrix} a^2bzq^n, d, dx \\ abdzxq^n/c, adx \end{matrix}; q, x \right),
\end{aligned} \quad (3.21)$$

where  $-1 < x < 0$ . By analytic continuation, we may replace the assumption  $-1 < x < 0$  by  $|x| < 1$ . Thus, we get (3.12).  $\square$

We point out that the two terms  ${}_3\phi_2$  transformation formula of Sears is a special case of (3.12). The two terms  ${}_3\phi_2$  transformation formula of Sears [13] can be stated as

**Corollary 2.** *We have*

$${}_3\phi_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q, \frac{b_1 b_2}{a_1 a_2 a_3}\right) = \frac{(b_2/a_3, b_1 b_2/a_1 a_2; q)_\infty}{(b_2, b_1 b_2/a_1 a_2 a_3; q)_\infty} {}_3\phi_2\left(\begin{matrix} b_1/a_1, b_1/a_2, a_3 \\ b_1, b_1 b_2/a_1 a_2 \end{matrix}; q, \frac{b_2}{a_3}\right). \quad (3.22)$$

**Proof.** Letting  $d = 0$  in (3.12) and using  $q$ -binomial theorem (1.7) gives

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(b, a, ax; q)_n z^n}{(q, c; q)_n} \cdot \frac{(a^2 b z x q^n / c; q)_\infty}{(x; q)_\infty} \\ &= \frac{(c/b, bz; q)_\infty}{(z, c; q)_\infty} \sum_{n=0}^{\infty} \frac{(b, abz/c, abzx/c; q)_n (c/b)^n}{(q, bz; q)_n} \cdot \frac{(a^2 b z x q^n; q)_\infty}{(x; q)_\infty}, \end{aligned} \quad (3.23)$$

which can be rewritten as

$${}_3\phi_2\left(\begin{matrix} b, a, ax \\ c, a^2 b z x / c \end{matrix}; q, z\right) = \frac{(c/b, bz; q)_\infty}{(z, c; q)_\infty} {}_3\phi_2\left(\begin{matrix} b, abz/c, abzx/c \\ bz, a^2 b z x / c \end{matrix}; q, \frac{c}{b}\right). \quad (3.24)$$

After replacing  $(a, b, ax, c, z)$  by  $(a_1, a_3, a_2, b_2, b_1 b_2 / a_1 a_2 a_3)$  in (3.23), we obtain the two terms  ${}_3\phi_2$  transformation formula of Sears (3.22).  $\square$

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