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Solution of soliton equations in terms of neutral fermion particles

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ABSTRACT

In this paper we exploit the algebraic structure of the soliton equations and find solutions in terms of neutral free fermion particles. We show how pfaffians arise naturally in the fermionic approach to soliton equations. We write the τ-function for neutral free fermions in terms of pfaffians. Examples of how to get soliton, rational and dromion solutions from τ-functions for the various soliton equations are given.

1. Introduction

In soliton theory solutions to soliton equations have been given in many ways such as by means of Wronskians [1], Grammians [2], Pfaffians [3,13], etc. The diversity of expressing solutions reflects the richness of algebraic structures which the soliton equations possess in common. It is Sato [7] that unveiled the structures by means of the method of algebraic analysis in the study of the Kadomtsev–Petviashvili (KP) hierarchy.

The study of algebraic structure of soliton equations give rise to solutions in terms of fermion particles [9]. These particles can either be charged or neutral [10,15], and they can have one component structure or they can have more than one, depending on the structure of the equation. The algebraic structure of soliton equations for the charged fermion particles has been studied in [15]. In this paper we are interested in the solution of the soliton equations in terms of neutral free fermions. An example of their fermionic structure is shown, both in charged and neutral case, in Table 1 for KP, Davey–Stewartson (DS), BKP and Novikov–Veselov–Nizhnik (NVN) equations.

We write the τ-function [6] for neutral free fermions in terms of pfaffians, in the following form

$$\tau_\phi = \mathrm{Pf}(A)\,\mathrm{Pf}\big(A' + V\big),$$

where A, A' are constant triangular matrices, A' is the analogue of the inverse A and V is also a triangular matrix with the entries of neutral fermions. These are explained in more detail in the later sections. This τ_ϕ-function in pfaffian form is analogue to τ_ψ-function in determinant form for the charged fermions (see [15]).

This paper is organized as follows. In Section 2, we introduce some properties of pfaffians. In Section 3, we recall some results from [9] and apply *Wick's theorem* to compute the expectation values of fermions. In Section 4, we give time evolution to the fermions via Hamiltonian in order to use fermion particles with time variable. Next we introduce a polynomial $\tau(\underline{x}, g)$ function in (4), where $\underline{x}$ is time variable and g represents fermions. As a new result the $\tau(\underline{x}, g)$ function in terms of Schur's Q-functions give rational solutions of the BKP equation. In Section 5 we derive a new general formulae for the neutral free fermions from which the one-component or two-component fermion solutions can be obtained. We give general formulae for the soliton solutions of the 1-component and 2-component BKP hierarchies. Examples of how to get the soliton solution to BKP equation and dromion solution to NVN equations, from τ-functions are also given.

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Table 1

Fermions	1 component	2 component
charged (ψ_i)	KP	DS
neutral (ϕ_i)	BKP	NVN

2. Pfaffians

Let

$$A = (a_{ij})$$

be an $n \times n$ skew-symmetric matrix (i.e. $a_{ij} = -a_{ji}$ and consequently $a_{ii} = 0$ for $i, j = 1, 2, \dots, n$). It is known that if n is odd, then $\det(A)$ is zero, but if n is even $\det(A)$ is a perfect square of a polynomial in the entries a_{ij}, called the *pfaffian* of A and denoted by $\mathrm{Pf}(A)$. Roughly speaking, a pfaffian is the square root of the determinant of a skew-symmetric matrix. To be precise, for even n

$$\mathrm{Pf}(A) = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{23} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \end{vmatrix} = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),\sigma(2)} \cdots a_{\sigma(n-1),\sigma(n)},$$

where σ runs over the permutations of $\{1, \dots, n\}$ such that

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n), \qquad \sigma(1) < \sigma(3) < \cdots < \sigma(n-1),$$

and $\epsilon(\sigma)(= \pm 1)$ is the parity of this permutation. For example, we have $\mathrm{Pf}(A) = a_{12}$ for $n = 2$ and

$$\mathrm{Pf}(A) = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{a_{34}} \end{vmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

for $n = 4$. The a_{ij} is taken to be skew-symmetric, and therefore $a_{ij} = -a_{ji}$.

A classical notation for the pfaffian of A [4] is

$$\mathrm{Pf}(A) = (1, 2, \dots, n),$$

where $(i, j) = a_{ij}$. One expansion rule for pfaffians is given by

$$(1, 2, \dots, n) = \sum_{i=2}^{n} (-1)^i (1, i)(2, 3, \dots, \hat{i}, \dots, n),$$

where $\hat{}$ indicates that the index underneath should be deleted. For example for $n = 4$ we can write the pfaffian representation as

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).$$

(See [14] for more information on pfaffians.)

3. Preliminaries

We recall some results from [9]. Let **A** be an associative algebra over $\mathbb{C}$ with generators ϕ_n ($n \in \mathbb{Z}$). In [15] we constructed charged free fermions ψ_n and ψ_n^* ($n \in \mathbb{Z}$) for the KP-hierarchy. Here we exploit neutral free fermions ϕ_n ($n \in \mathbb{Z}$) [8], for the BKP-hierarchy, satisfying the anti-commutator relation

$$[\phi_m, \phi_n]_+ = (-1)^m \delta_{m,-n} \tag{1}$$

where $[X, Y]_+ = XY + YX$. The generators ϕ_m, ϕ_n ($m, n \in \mathbb{Z}$) will be referred to as *free fermions*.

The charged free fermions introduced in [15] can be split into two sets of neutral free fermions. Namely, if we set

$$\phi_m = \frac{\psi_m + (-1)^m \psi_{-m}^*}{\sqrt{2}}, \qquad \hat{\phi}_m = \mathrm{i}\frac{\psi_m - (-1)^m \psi_{-m}^*}{\sqrt{2}} \quad (m \in \mathbb{Z}),$$

we have

$$[\phi_m, \phi_n]_+ = (-1)^m \delta_{m,-n}, \qquad [\hat{\phi}_m, \hat{\phi}_n]_+ = (-1)^m \delta_{m,-n}$$

and

$$[\phi_m, \hat{\phi}_n]_+ = 0.$$

The expectation values of neutral free fermions are defined by

$$\langle \phi_m \phi_n \rangle = \begin{cases} (-1)^m \delta_{m,-n}, & n > 0, \\ \frac{1}{2}\delta_{m,0}, & n = 0, \\ 0, & n < 0, \end{cases}$$

where $\langle\ \rangle$ denotes a linear form on **A**, called the (*vacuum*) *expectation value*.

For a general product $w_1 \cdots w_r$ of free fermions w_i, we apply *Wick's theorem* to compute the expectation values

$$\langle w_1 \cdots w_r \rangle = \begin{cases} 0 & (r \text{ odd}), \\ \sum_\sigma \operatorname{sgn}\sigma \langle w_{\sigma(1)} w_{\sigma(2)} \rangle \cdots \langle w_{\sigma(r-1)} w_{\sigma(r)} \rangle & (r \text{ even}), \end{cases}$$

where σ runs over the permutations such that $\sigma(1) < \sigma(2), \dots, \sigma(r-1) < \sigma(r)$ and $\sigma(1) < \sigma(3), \dots, \sigma(r-1)$. We see that this theorem gives the expectation value of the general product of free fermions $w_1 \cdots w_r$ in terms of a pfaffian [14]. Therefore, Wick's theorem can be expressed in terms of pfaffians in the following way

$$\langle w_1 \cdots w_r \rangle = \begin{cases} 0 & (r \text{ odd}), \\ \operatorname{Pf}(\langle w_i w_j \rangle) & (r \text{ even}). \end{cases}$$

4. Neutral free fermions

It is convenient to use the generating functions for neutral free fermions ϕ_m, ϕ_n, defined as follows

$$\phi(p) := \sum_{i \in \mathbb{Z}} \phi_i p^i, \qquad \phi(q) := \sum_{i \in \mathbb{Z}} \phi_i q^i. \tag{2}$$

Theorem 4.1. *The expectation values for the generating functions $\phi(p)$, $\phi(q)$ are given by*

$$\langle \phi(p_i) \phi(q_j) \rangle = \frac{1}{2} \frac{p_i - q_j}{p_i + q_j}, \qquad \langle \phi(p_i) \phi_0 \rangle = \frac{1}{2}.$$

Proof. From the definition (2)

$$\langle \phi(p_i) \phi(q_j) \rangle = \sum_{m,n \in \mathbb{Z}} \langle \phi_m \phi_n \rangle p_i^m q_j^n = \frac{1}{2} + \sum_{n>0} (-1)^m \delta_{m,-n} p_i^m q_j^n = \frac{1}{2} + \sum_{n>0}^{\infty} (-1)^n \left(\frac{q_j}{p_i} \right)^n = \frac{1}{2} \frac{p_i - q_j}{p_i + q_j}$$

and similarly

$$\langle \phi(p_i) \phi_0 \rangle = \sum_{m,n \in \mathbb{Z}} \langle \phi_m \phi_0 \rangle p_i^m = \sum_{m \geqslant 0} \frac{1}{2} \delta_{m,0} p_i^m = \frac{1}{2}. \qquad \Box$$

Next we wish to express the time evolution for the neutral free fermions by the following Hamiltonian

$$H(\underline{x}) = \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ i = 1, 3, \dots}} (-1)^{n+1} x_i \phi_n \phi_{-n-l}$$

where $\underline{x}$ is the time variable.

Note first of all that

$$\begin{aligned} [H(\underline{x}), \phi(p)] &= \frac{1}{2} \sum_{n \geqslant 1, k \in \mathbb{Z}} x_n \left[\sum_{i \in \mathbb{Z}} \phi_i \phi_{i+n}, \phi_k \right] p^k = \sum_{n \geqslant 1, k \in \mathbb{Z}} x_n p^k \left(\sum_{i \in \mathbb{Z}} \delta_{k,i+n} \phi_i \right) = \sum_{n \geqslant 1, k \in \mathbb{Z}} x_n p^k \phi_{k-n} \\ &= \sum_{n \geqslant 1} x_n p^n \phi(p) = \xi(\underline{x}, p) \phi(p), \end{aligned}$$

where $\xi(\underline{x}, p) := \sum_{n \geqslant 1} x_n p^n$.

Then

$$\phi(\underline{x}, p_i) = e^{H(\underline{x})}\phi(p_i)e^{-H(\underline{x})} = e^{(\operatorname{ad} H(\underline{x}))}\phi(p_i) = \left(1 + \operatorname{ad} H(\underline{x}) + \frac{1}{2}\left(\operatorname{ad} H(\underline{x})\right)^2 + \cdots\right)\phi(p_i),$$

where $(\operatorname{ad} H(\underline{x}))X = [H(\underline{x}), X]$ and so $(\operatorname{ad} H(\underline{x}))\phi(p_i) = (\xi(\underline{x}, p_i))\phi(p_i)$, hence

$$\phi(\underline{x}, p_i) = e^{\xi(\underline{x}, p_i)}\phi(p_i). \tag{3}$$

Later we will use $e^{\xi(\underline{x}, p_i)}$ to generate polynomials.

Next we call a polynomial $\tau(x)$ a τ-*function* if it is representable in the following form for some g:

$$\tau(\underline{x}, g) = \langle g(\underline{x})\rangle, \tag{4}$$

where we choose

$$g = \phi(p_1)\cdots\phi(p_{2r}), \tag{5}$$

for some r.

From Wick's theorem we write the following expectation value

$$\tau_\phi(\underline{x}, g) = e^{\sum_{i=1}^{2r}\xi(\underline{x}, p_i)}\langle g\rangle = e^{\sum_{i=1}^{2r}\xi(\underline{x}, p_i)}\begin{vmatrix} \langle\phi(p_1)\phi(p_2)\rangle & \langle\phi(p_1)\phi(p_3)\rangle & \cdots & \langle\phi(p_1)\phi(p_{2r})\rangle \\ & \langle\phi(p_2)\phi(p_3)\rangle & \cdots & \langle\phi(p_2)\phi(p_{2r})\rangle \\ & & \ddots & \vdots \\ & & & \langle\phi(p_{2r-1})\phi(p_{2r})\rangle \end{vmatrix}$$

and using Theorem 4.1, we get

$$\tau_\phi = \langle\phi(x, p_1)\cdots\phi(x, p_{2r})\rangle = \frac{1}{2^r}\prod_{i<j}\frac{p_i - p_j}{p_i + p_j}e^{\sum_{i=1}^{2r}\xi(x, p_i)} \tag{6}$$

and expand the entries of the determinant in (6), in the following way:

$$\frac{1}{2}\frac{p_i - p_j}{p_i + p_j} = \frac{1}{2} + \sum_{n=0}^{\infty}\left(-\frac{p_j}{p_i}\right)^{n+1}, \qquad e^{\xi(\underline{x}, p_i)} = \sum_{k=0}^{\infty} q_k(x)p_i^k, \tag{7}$$

where $q_k(x)$ are the complete symmetric functions.

Now we wish to express the τ-function τ_ϕ in terms of another class of symmetric functions, Schur's Q-functions [5,10,12]. In general, for a partition $\lambda = (\lambda_1, \dots, \lambda_m)$, a Q-function is defined by

$$Q_\lambda(x; t) = \prod_{1\leqslant i<j\leqslant m}\left(\partial^{(i)} - \partial^{(j)}\right)\left(\partial^{(i)} - t\partial^{(j)}\right)^{-1}\prod_{i=1}^{m} q_{\lambda_i}\left(x^{(i)}; t\right)\big|_{x^{(i)}} = x$$

which reduces to the Schur function $S_\lambda(x)$ when $t = 0$, so that $Q_\lambda(x; 0) = S_\lambda(x)$. In particular we have the Q-function for the partition $\lambda = (ij)$, namely

$$Q_{(ij)} = q_i q_j + 2\sum_{k=0}^{j-1}(-1)^{k+1}q_{i+k+1}q_{j-k-1}. \tag{8}$$

For a general λ, we define a triangular matrix $A = (Q_{(\lambda_i\lambda_j)})$. Then, if λ has even number of parts $Q_\lambda = \operatorname{Pf}(A)$, and if λ has odd number of parts then

$$Q_\lambda = \begin{vmatrix} & q_{\lambda_1} \\ A & q_{\lambda_2} \\ & \vdots \\ & q_{\lambda_m} \end{vmatrix}.$$

The element g is given by (5) can be written as

$$g = \sum_{i_1,\dots,i_{2r}\in\mathbb{Z}} p_1^{i_1}\cdots p_{2r}^{i_{2r}}g',$$

where

$$g' = \phi_{i_1}\cdots\phi_{i_{2r}}.$$

Thus (6) can be used as a generating function to determine $\tau_\phi(\underline{x}, g')$ by looking at the coefficients of $p_1^{i_1} \cdots p_{2r}^{i_{2r}}$ where $i_1 > i_2 > \cdots > i_{2r} \in \mathbb{Z}$, and the (i, j)th entry in (6) can be written as

$$\begin{aligned}
\big\langle \phi(\underline{x}, p_i)\phi(\underline{x}, p_j)\big\rangle &= \frac{1}{2}\frac{p_i - p_j}{p_i + p_j} e^{\xi(\underline{x}, p_i) + \xi(\underline{x}, p_j)} \\
&= \left(\frac{1}{2} + \sum_{n=0}^{\infty}\left(-\frac{p_j}{p_i}\right)^{n+1}\right)\sum_{k=0}^{\infty} q_k(x) p_i^k \sum_{l=0}^{\infty} q_l(x) p_j^l \\
&= \frac{1}{2}\sum_{\substack{k=0\\ l=0}}^{\infty} q_k q_l p_i^k p_j^l + \sum_{\substack{k=0\\ l=0}}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+1} q_k q_l p_i^{k-n-1} p_j^{l+n+1} \\
&= \frac{1}{2}\sum_{\substack{k=-1\\ l=0}}^{\infty} q_k q_l p_i^k p_j^l + \sum_{\substack{k=-1\\ l=0}}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+1} q_{k+n+1} q_{l-n-1} p_i^k p_j^l \\
&= \frac{1}{2}\sum_{\substack{k=-1\\ l=0}}^{\infty}\left(q_k q_l + 2\sum_{n=0}^{\infty}(-1)^{n+1} q_{k+n+1} q_{l-n-1}\right) p_i^k p_j^l,
\end{aligned} \tag{9}$$

where $q_i(x) = 0$ for $i < 0$ and $q_0(x) = 1$. The coefficients of $p_i^k p_j^l$ in (9) are the Q-functions defined in (8). Therefore, we can write the (i, j)th entry of neutral free fermions in terms of Q-functions as follows:

$$\big\langle \phi(\underline{x}, p_i)\phi(\underline{x}, p_j)\big\rangle = \frac{1}{2}\sum_{\substack{k=-1\\ l=0}}^{\infty} Q_{(kl)} p_i^k p_j^l.$$

Hence the τ-function in (6) can be written in the following form:

$$\tau_\phi(\underline{x}, g) = 2^{-r}\,\mathrm{Pf}\left(\sum_{\substack{k_i=-1\\ l_j=0}}^{\infty} Q_{(k_i l_j)} p_i^{k_i} p_j^{l_j}\right) = 2^{-r}\sum_{\substack{k_1,\ldots,k_r=-1\\ l_1,\ldots,l_r=0}}^{\infty}\mathrm{Pf}(Q_{(k_i l_j)}) p_1^{k_1}\cdots p_r^{k_r} p_1^{l_1}\cdots p_r^{l_r}$$

and

$$\tau_\phi\big(\underline{x}, g'\big) = 2^{-r}\,\mathrm{Pf}(Q_{(ij)}).$$

Hence by (8) each τ_ϕ is a Q-function. These give rational solutions [8] of the BKP equation, where $u = 2\partial_x(\log\tau_\phi)$.

5. Neutral free fermions in general

We wish to express g more generally for the neutral free fermions in the following form:

$$g = e^{\sum_{i<j=1}^{2n} a_{ij}\phi^i\phi^j},$$

where ϕ^i, ϕ^j ($i < j = 1, \ldots, 2N$) can be either one-component or two-component fermions. For example we will take $\phi^i = \phi(p_i)$ for the one-component case and $\phi^i = \phi^{(1)}(p^{(1)})$ or $\phi^i = \phi^{(2)}(p^{(2)})$ for the two-component case. Then the τ-function is

$$\begin{aligned}
\tau_\phi = \big\langle g(\underline{x})\big\rangle &= 1 + \sum_{i_1<j_1=1}^{2N} a_{i_1 j_1}\big\langle\phi^{i_1}\phi^{j_1}\big\rangle + \sum_{i_1<j_1, i_2<j_2=1}^{2N} a_{i_1 j_1} a_{i_2 j_2}\big\langle\phi^{i_1}\phi^{j_1}\phi^{i_2}\phi^{j_2}\big\rangle + \cdots \\
&\quad + \sum_{i_1<j_1,\ldots,i_N<j_N=1}^{2N} a_{i_1 j_1}\cdots a_{i_N j_N}\big\langle\phi^{i_1}\phi^{j_1}\cdots\phi^{i_N}\phi^{j_N}\big\rangle.
\end{aligned} \tag{10}$$

Next we define the following expectation value

$$\big\langle\phi^i, \phi^i\big\rangle := 0. \tag{11}$$

Using the definition of the expectation value in (11) and Wick's theorem, the τ_ϕ function in (10) can be written in the following pfaffian form

$$\tau_\phi = \mathrm{Pf}(A)\,\mathrm{Pf}(S), \tag{12}$$

where A is constant tri-angular matrix with the entries $A_{i<j} = [a_{ij}]$ and S is the tri-angular matrix with the entries $S_{i<j} = [a'_{ij} + \langle \phi^i, \phi^j \rangle]$, and $A'_{i<j} = [a'_{ij}]$ is the analogue constant tri-angular matrix of the inverse A. Namely A and A' have the relation $(\mathrm{Pf}(A))^{-1} = \mathrm{Pf}(A')$.

Consider a triangular array $A = [a_{ij}]_{i<j=1,\dots,n}$ of size n, i.e.

$$A = \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{23} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \end{bmatrix}.$$

Then one may define an adjoint array $A^\dagger = [(-1)^{i+j+1} a^\dagger_{ij}]_{i<j=1,\dots,n}$ whose entries are pfamans of subarrays of A and have this relation $A = \mathrm{Pf}(A)(A')^\dagger$. To be precise $a^\dagger_{kl}$ is the pfaffian of the array obtained by deleting the kth and lth lines in A. Now we define

$$A' = \frac{1}{\mathrm{Pf}(A)} A^\dagger.$$

Note that this array is the analogue of the inverse of a matrix; for a matrix M,

$$M^{-1} = \frac{1}{\det(M)} M^\dagger.$$

Indeed if W and W' are the skew-symmetric matrices whose upper triangles are A and A' respectively, then $W' = W^{-1}$.

For example, for $N = 2$ from (10) we have the following solution:

$$\begin{aligned}
\tau_{\phi_2} &= 1 + \sum_{i_1<j_1=1}^{4} a_{i_1 j_1} \langle \phi^{i_1} \phi^{j_1} \rangle + \sum_{i_1<j_1, i_2<j_2=1}^{4} a_{i_1 j_1} a_{i_2 j_2} \langle \phi^{i_1} \phi^{j_1} \phi^{i_2} \phi^{j_2} \rangle \\
&= 1 + a_{12}\langle \phi^1 \phi^2 \rangle + a_{13}\langle \phi^1 \phi^3 \rangle + a_{14}\langle \phi^1 \phi^4 \rangle + a_{23}\langle \phi^2 \phi^3 \rangle + a_{24}\langle \phi^2 \phi^4 \rangle + a_{34}\langle \phi^3 \phi^4 \rangle \\
&\quad + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\big(\langle \phi^1 \phi^2 \rangle\langle \phi^3 \phi^4 \rangle - \langle \phi^1 \phi^3 \rangle\langle \phi^2 \phi^4 \rangle + \langle \phi^1 \phi^4 \rangle\langle \phi^2 \phi^3 \rangle\big) \\
&= \mathrm{Pf}(A_2)\bigg(\frac{1}{\mathrm{Pf}(A_2)} + \frac{a_{12}}{\mathrm{Pf}(A_2)}\langle \phi^1 \phi^2 \rangle + \frac{a_{13}}{\mathrm{Pf}(A_2)}\langle \phi^1 \phi^3 \rangle + \frac{a_{14}}{\mathrm{Pf}(A_2)}\langle \phi^1 \phi^4 \rangle + \frac{a_{23}}{\mathrm{Pf}(A_2)}\langle \phi^2 \phi^3 \rangle \\
&\quad + \frac{a_{24}}{\mathrm{Pf}(A_2)}\langle \phi^2 \phi^4 \rangle + \frac{a_{34}}{\mathrm{Pf}(A_2)}\langle \phi^3 \phi^4 \rangle + \langle \phi^1 \phi^2 \rangle\langle \phi^3 \phi^4 \rangle - \langle \phi^1 \phi^3 \rangle\langle \phi^2 \phi^4 \rangle + \langle \phi^1 \phi^4 \rangle\langle \phi^2 \phi^3 \rangle\bigg) \\
&= \mathrm{Pf}(A_2)\big(\mathrm{Pf}(A'_2) + a'_{34}\langle \phi^1 \phi^2 \rangle - a'_{24}\langle \phi^1 \phi^3 \rangle + a'_{23}\langle \phi^1 \phi^4 \rangle + a'_{14}\langle \phi^2 \phi^3 \rangle - a'_{13}\langle \phi^2 \phi^4 \rangle \\
&\quad + a'_{12}\langle \phi^3 \phi^4 \rangle + \langle \phi^1 \phi^2 \rangle\langle \phi^3 \phi^4 \rangle - \langle \phi^1 \phi^3 \rangle\langle \phi^2 \phi^4 \rangle + \langle \phi^1 \phi^4 \rangle\langle \phi^2 \phi^3 \rangle\big),
\end{aligned} \tag{13}$$

where

$$A_2 = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{bmatrix} = \mathrm{Pf}(A_2) \begin{bmatrix} a'_{34} & -a'_{24} & a'_{23} \\ & a'_{14} & -a'_{13} \\ & & a'_{12} \end{bmatrix}$$

and

$$A'_2 = \begin{bmatrix} a'_{12} & a'_{13} & a'_{14} \\ & a'_{23} & a'_{24} \\ & & a'_{34} \end{bmatrix}.$$

Now we can write the expression τ_{ϕ_2} in (13) in the form of (12)

$$\tau_{\phi_2} = \mathrm{Pf}(A_2)\,\mathrm{Pf}(S_2), \tag{14}$$

where

$$S_2 = \begin{bmatrix} a'_{12} + \langle \phi^1 \phi^2 \rangle & a'_{13} + \langle \phi^1 \phi^3 \rangle & a'_{14} + \langle \phi^1 \phi^4 \rangle \\ & a'_{23} + \langle \phi^2 \phi^3 \rangle & a'_{24} + \langle \phi^2 \phi^4 \rangle \\ & & a'_{34} + \langle \phi^3 \phi^4 \rangle \end{bmatrix}.$$

Example 5.1. For the soliton solution for the one-component case, we put $\phi^1 = \phi(p_1)$, $\phi^2 = \phi(q_1)$, $\phi^3 = \phi(p_2)$, $\phi^4 = \phi(q_2)$ and choose $a_{12} = 2\frac{p_1+q_1}{p_1-q_1}$, $a_{34} = 2\frac{p_2+q_2}{p_2-q_2}$, $a_{13} = a_{14} = a_{23} = a_{24} = 0$. The τ-function from (14) is

$$\begin{aligned}\tau &= \begin{vmatrix} 1 + a_{12}\langle\phi(p_1)\phi(q_1)\rangle & a_{12}\langle\phi(p_1)\phi(p_2)\rangle & a_{12}\langle\phi(p_1)\phi(q_2)\rangle \\ & a_{34}\langle\phi(q_1)\phi(p_2)\rangle & a_{34}\langle\phi(q_1)\phi(q_2)\rangle \\ & & 1 + a_{34}\langle\phi(p_2)\phi(q_2)\rangle \end{vmatrix} \\ &= 1 + a_{12}\langle\phi(p_1)\phi(q_1)\rangle + a_{34}\langle\phi(p_2)\phi(q_2)\rangle + a_{12}a_{34}\big(\langle\phi(p_1)\phi(q_1)\rangle\langle\phi(p_2)\phi(q_2)\rangle \\ &\quad - \langle\phi(p_1)\phi(p_2)\rangle\langle\phi(q_1)\phi(q_2)\rangle + \langle\phi(p_1)\phi(q_2)\rangle\langle\phi(q_1)\phi(p_2)\rangle\big) \\ &= 1 + e^{\eta_1} + e^{\eta_2} + B_{12}e^{\eta_1+\eta_2},\end{aligned}$$

where $\eta_i = \xi(x, p_i) + \xi(x, q_i)$ $(i = 1, 2)$ and

$$B_{12} = \frac{(p_1 - p_2)(q_1 - q_2)(p_1 - q_2)(q_1 - p_2)}{(p_1 + p_2)(q_1 + q_2)(p_1 + q_2)(q_1 + p_2)}.$$

Hence $u = 2\partial_x(\log\tau)$ gives the 2-soliton solution [11,12] for the BKP equation

$$\big(u_t + 15uu_{3x} + 15u_x^3 - 15u_xu_y + u_{5x}\big)_x + 5u_{3x,y} - 5u_{yy} = 0.$$

Example 5.2. For the two-component case, we put $\phi^1 = \phi^{(1)}(p^{(1)})$, $\phi^2 = \phi^{(1)}(q^{(1)})$, $\phi^3 = \phi^{(2)}(p^{(2)})$, $\phi^4 = \phi^{(2)}(q^{(2)})$ in (14). Then the τ-function from (14) is

$$\begin{aligned}\tau &= \begin{vmatrix} 1 + a_{12}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle & a_{13}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle & a_{14}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle \\ & a_{23}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle & a_{24}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \\ & & 1 + a_{34}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \end{vmatrix} \\ &= 1 + a_{12}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle + a_{34}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \\ &\quad + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \\ &= 1 + a_{12}\frac{1}{2}\frac{p^{(1)} - q^{(1)}}{p^{(1)} + q^{(1)}}e^{\eta^{(1)}} + a_{34}\frac{1}{2}\frac{p^{(2)} - q^{(2)}}{p^{(2)} + q^{(2)}}e^{\eta^{(2)}} + B_{12}e^{\eta^{(1)}+\eta^{(2)}},\end{aligned}$$

where $\eta^{(i)} = \xi(x^{(i)}, p^{(i)}) + \xi(x^{(i)}, q^{(i)})$ $(i = 1, 2)$ and

$$B_{12} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\frac{1}{4}\frac{p^{(1)} - q^{(1)}p^{(2)} - q^{(2)}}{p^{(1)} + q^{(1)}p^{(2)} + q^{(2)}}.$$

Hence $u = \partial_{xy}(\log\tau)$ gives the 1-dromion solution [3] for the NVN equations

$$u_t = u_{xxx} + u_{yyy} + 3(\Phi_{xx}u) + 3(\Phi_{yy}u)_y, \qquad u = \Phi_{xy}.$$

We note that the τ-function in (12) for neutral fermions is the pfaffian analogue of the τ-function in [15] for charged fermions in determinantal form, namely from (12) τ can be written in the following form

$$\tau = \mathrm{Pf}(A)\,\mathrm{Pf}(S) = \mathrm{Pf}(A)\,\mathrm{Pf}\big(A' + V\big),$$

where $V_{i<j} = [\langle\phi^i\phi^j\rangle]$ and the corresponding determinantal form for the charged fermions in [15] can be written in the following form

$$\tau = \det(I + AV) = \det\big(A\big(A^{-1} + V\big)\big) = \det(A)\det\big(A^{-1} + V\big).$$

6. Conclusion

In this paper, we have elucidated the role of pfamans in determining new solutions by using neutral fermion particles. We have presented rational and soliton solutions to the BKP equation and dromion solution to the NVN equations by using the τ-function in terms of pfaffians. We have showed that each τ_ϕ function is a Schur's Q-function. We have derived new general formulae (12) for neutral fermions, from which higher order τ-functions and hence soliton solutions and dromion solutions for the BKP-hierarchy can be obtained.

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References

[1] N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg–de Vries and the Kadomtsev–Petviashvili equations: the Wronskian technique, Proc. R. Soc. Lond. A 389 (1983) 319–329.
[2] A. Nakamura, A bilinear N-soliton formula for the KP equation, J. Phys. Soc. Japan 58 (1989) 412–422.
[3] Y. Ohta, Pfaffian solutions for the Veselov–Novikov equation, J. Phys. Soc. Japan 61 (1992) 3928–3933.
[4] E.R. Caianiello, Combinatorics and Renormalization in Quantum Field Theory, Benjamin, Reading, MA, 1973.
[5] I.D. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, Oxford University Press, Oxford, 1995.
[6] E. Date, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, II. Vertex operators and tau functions, Proc. Japan Acad. A 57 (1981) 387–392.
[7] M. Sato, An introduction to Sato theory, RIMS Kyoto Univ. 439 (1981) 30.
[8] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, IV. A new hierarchy of soliton equations of KP-type, Phys. D 4 (1982) 343–365.
[9] M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras, RIMS Kyoto Univ. 19 (1983) 943–1001.
[10] Y. You, Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups, in: Infinite-Dimensional Lie Algebras and Groups, in: Adv. Ser. Math. Phys., vol. 7, World Science Publishing, Teaneck, NJ, 1990, pp. 449–464.
[11] R. Hirota, Soliton solutions to the BKP equations, I. The Pfaffian technique, J. Phys. Soc. Japan 58 (1989) 2285–2296.
[12] J.J.C. Nimmo, Hall–Littlewood symmetric functions and the BKP equation, J. Phys. A 23 (1990) 751–760.
[13] Guo-Fu Yu, Hon-Wah Tarn, On the $(2+l)$-dimensional Gardner equation: Determinant solutions and pfaffianization, J. Math. Anal. Appl. 330 (2007) 989–1001.
[14] M. Ünal, Application of the Pfaffian technique to the KR and mNVN equations, J. Math. Anal. Appl. 362 (2010) 224–230.
[15] M. Ünal, Fermionic approach to soliton equations, J. Math. Anal. Appl. 380 (2011) 782–793.