J. Math. Anal. Appl. 394 (2012) 744–760

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Weighted norm inequalities for the maximal operator on variable Lebesgue spaces

D. Cruz-Uribe, SFO [a,*], A. Fiorenza [b,c], C.J. Neugebauer [d]

[a] Department of Mathematics, Trinity College, Hartford, CT 06106-3100, USA
[b] Dipartimento di Architettura, Università di Napoli, Via Monteoliveto, 3, I-80134 Napoli, Italy
[c] Istituto per le Applicazioni del Calcolo "Mauro Picone", sezione di Napoli, Consiglio Nazionale delle Ricerche, via Pietro Castellino, 111, I-80131 Napoli, Italy
[d] Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395, USA

ARTICLE INFO

Article history:
Received 1 February 2012
Available online 27 April 2012
Submitted by Maria J. Carro

Keywords:
Variable Lebesgue space
Maximal operators
Weights

ABSTRACT

We prove weighted strong and weak-type norm inequalities for the Hardy–Littlewood maximal operator on the variable Lebesgue space $L^{p(\cdot)}$. Our results generalize both the classical weighted norm inequalities on L^p and the more recent results on the boundedness of the maximal operator on variable Lebesgue spaces.

1. Introduction

Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, hereafter referred to simply as an exponent, we define the variable Lebesgue space $L^{p(\cdot)}$ to be the set of measurable functions on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f) = \rho(f/\lambda) = \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}\, dx < \infty.$$

This is a Banach space (see [1–4]) when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho(f/\lambda) \le 1\}.$$

The variable Lebesgue spaces are a special case of the Musielak–Orlicz spaces [5] and generalize the classical Lebesgue spaces: when $p(x) = p_0$ is constant, $L^{p(\cdot)} = L^{p_0}$.

Variable Lebesgue spaces were first studied by Orlicz [6] in 1931, but in the past two decades there has been a great deal of interest in them, particularly for their applications to partial differential equations with non-standard growth conditions and to modeling electrorheological fluids. See [7,8] for the history and references.

Much attention has been focused on finding conditions on the exponent function $p(\cdot)$ so that the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}$. Recall that given a locally integrable function f on $\mathbb{R}^n$, the maximal function Mf is defined on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy = \sup_{Q \ni x} \fint_Q |f(y)|\, dy,$$

* Corresponding author.
E-mail addresses: david.cruzuribe@trincoll.edu (D. Cruz-Uribe, SFO), fiorenza@unina.it (A. Fiorenza), neug@math.purdue.edu (C.J. Neugebauer).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.04.044

where the supremum is taken over all cubes Q containing x whose sides are parallel to the coordinate axes. The following result first proved by the authors and their collaborators (see [9–11]) is nearly optimal. Hereafter, given a set E, let

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \qquad p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x).$$

If $E = \mathbb{R}^n$, for brevity we will write simply p_- and p_+.

Theorem 1.1. *Given $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, suppose that $1 \le p_- \le p_+ < \infty$ and that $p(\cdot)$ satisfies the local log-Hölder continuity condition*

$$LH_0 : |p(x) - p(y)| \le \frac{C}{-\log|x-y|}, \quad x, y \in \mathbb{R}^n,\ |x-y| < 1/2, \tag{1.1}$$

and is log-Hölder continuous at infinity: there exists p_∞, $1 \le p_\infty < \infty$, such that

$$LH_\infty : |p(x) - p_\infty| \le \frac{C}{\log(e+|x|)}, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Then the Hardy–Littlewood maximal operator satisfies the weak-type inequality

$$\|t\chi_{\{x\in\mathbb{R}^n : Mf(x)>t\}}\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}, \quad t > 0.$$

If $p_- > 1$, then it is bounded on $L^{p(\cdot)}$:

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}.$$

Theorem 1.1 was first proved by Diening [12] assuming that $p(\cdot)$ is constant outside of a large ball. It was proved independently by Nekvinda [13] (see also [9]) with (1.2) replaced by a somewhat more general condition. The log-Hölder continuity conditions are not necessary; see Lerner [14]. However, if they are relaxed it is possible to construct counter-examples: see [11,15].

In this paper we generalize Theorem 1.1 to variable Lebesgue spaces with weights. For classical Lebesgue spaces the following result is due to Muckenhoupt [16] (see also [17,18]). Hereafter, we use the notation $\fint_Q f(x)\,dx = |Q|^{-1}\int_Q f(x)\,dx$.

Theorem 1.2. *Given p, $1 \le p < \infty$, and a locally integrable function w such that $0 < w(x) < \infty$ almost everywhere, the following are equivalent:*

(a) *$w \in A_p$: for every cube Q*

$$\fint_Q w(x)\,dx \left(\fint_Q w(x)^{1-p'}\,dx\right)^{p-1} \le K < \infty, \tag{1.3}$$

if $p > 1$, and

$$\fint_Q w(x)\,dx \le K w(y),$$

for almost every $y \in Q$ if $p = 1$.

(b) *The maximal operator is weak (p,p): for every $t > 0$,*

$$w(\{x\in\mathbb{R}^n : Mf(x) > t\}) \le \frac{C}{t^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{1.4}$$

(c) *If $p > 1$, the maximal operator is strong (p,p):*

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \le C\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{1.5}$$

Remark 1.3. In the definition of A_p, the smallest constant K is referred to as the A_p constant of w.

There are two possible ways to generalize Theorem 1.2 to variable Lebesgue spaces. One way is to treat $w\,dx$ as a measure, and define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ with respect to this measure. This approach is in one respect a natural generalization of Theorem 1.2 and was adopted by Diening and Hästo [19]. They proved that if $1 < p_- \le p_+ < \infty$ and $p(\cdot)$ is log-Hölder continuous locally and at infinity, then the maximal operator is bounded on $L^{p(\cdot)}(w)$ if and only if

$$|Q|^{-p_-(Q)} w(Q)\|w^{-1}\chi_Q\|_{p'(\cdot)/p(\cdot)} \le K < \infty.$$

Here, $p'(\cdot)$ is the conjugate exponent, defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

where we let $p'(x) = \infty$ if $p(x) = 1$. The second norm is defined as before even if $p'(\cdot)/p(\cdot)$ is less than 1.

We take a different approach: we recast the inequalities so that the weight w acts as a multiplier. More precisely, in inequalities (1.4) and (1.5) we replace the weight w by w^p and rewrite the integrals in terms of L^p norms to get

$$\|t\chi_{\{x\in\mathbb{R}^n:Mf(x)>t\}}w\|_p \le C\|fw\|_p,$$
$$\|(Mf)w\|_p \le C\|fw\|_p.$$

Similarly, with w replaced by w^p, the A_p condition can be rewritten as

$$\|w\chi_Q\|_p\|w^{-1}\chi_Q\|_{p'} \le K|Q|. \tag{1.6}$$

While this is not the standard way to write either the A_p condition or weighted norm inequalities for the maximal operator, it is the natural way to write the off-diagonal weighted inequalities for fractional integrals (see, for example, Muckenhoupt and Wheeden [20]) and has also been used in the study of two-weight norm inequalities (see [21]).

Definition 1.4. Given an exponent function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and a weight w–that is, a locally integrable function such that $0 < w(x) < \infty$ a.e.–we say that $w \in A_{p(\cdot)}$ if there exists a constant K such that for every cube Q,

$$\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} \le K|Q|.$$

Our main result is the following weighted norm inequality.

Theorem 1.5. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$. If $1 < p_- \le p_+ < \infty$ and $p(\cdot)$ satisfies the LH_0 and LH_∞ conditions (1.1) and (1.2), then given any $w \in A_{p(\cdot)}$,*

$$\|(Mf)w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}.$$

If $p_- \ge 1$, then

$$\|t\chi_{\{x\in\mathbb{R}^n:Mf(x)>t\}}w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}.$$

Conversely, given a weight w and an exponent function $p(\cdot)$ such that the maximal operator satisfies the strong or weak-type inequality, then $w \in A_{p(\cdot)}$.

Remark 1.6. While this paper was in preparation, the strong-type inequality Theorem 1.5 when $p_- > 1$ was proved using a very different approach by the first author, Diening, and Hästö [22]. That proof, while straightforward, depends heavily on the sophisticated machinery developed by Diening [23] (see also [2]) to give the necessary and sufficient condition for the boundedness of the maximal operator. Our approach here has two advantages: it is direct, and it makes clear the connection between the $A_{p(\cdot)}$ condition and the classical Muckenhoupt A_p condition.

Remark 1.7. Separately, Diening and Hästö [19] proved when $p_- > 1$ that the $A_{p(\cdot)}$ condition is necessary for the strong-type inequality.

Remark 1.8. Particular results of this type for power weights (and generalizations of power weights) have been proved by Kokilashvili et al. [24–28] and Khabazi [29]. Other authors have also considered weighted norm inequalities for the maximal operator on variable Lebesgue spaces. See, for instance, [30–33].

In Theorem 1.5 the necessity of the $A_{p(\cdot)}$ condition is proved without using any continuity assumptions on the exponent function $p(\cdot)$. Given that the classical A_p condition is necessary and sufficient for the maximal operator to be bounded on weighted Lebesgue spaces, it was extremely tempting to conjecture that the $A_{p(\cdot)}$ condition, without additional continuity assumptions on $p(\cdot)$, is sufficient for the maximal operator to be bounded on variable Lebesgue spaces. However, even when $w = 1$ this is false; a counter-example is due independently to Diening [34] and Kopaliani [35].

There is a connection between the variable $A_{p(\cdot)}$ condition and the classical A_p condition: if $w \in A_{p(\cdot)}$, then $w^{p(\cdot)} \in A_\infty$—see Section 3—and this plays a critical role in our proofs. Earlier, Lerner [36] showed that there is a connection between the classical A_p condition and the boundedness of the maximal operator on $L^{p(\cdot)}(\Omega)$ for bounded Ω. A deeper understanding of the inter-relationship between weights and variable Lebesgue spaces remains to be elucidated.

Recently, there has been a great deal of interest in the sharp constant (in terms of the constant in the A_p condition) in weighted norm inequalities for various operators: see, for example, [37] and the references it contains. The sharp constant for the maximal operator in Theorem 1.2 is due to Buckley [38]; also see Lerner [39]. An open (and difficult) question is to determine the corresponding results in the variable Lebesgue spaces. For a discussion of a related problem, see [19].

Organization

The remainder of this paper is organized as follows. In Section 2 we gather together some basic results about variable Lebesgue spaces. In Section 3 we prove some basic properties of the $A_{p(\cdot)}$ condition. Finally, in Sections 4 and 5 we prove Theorem 1.5.

Throughout this paper all notation is standard or will be defined as needed. In order to emphasize that we are dealing with variable exponents, we will always write $p(\cdot)$ instead of p to denote an exponent function. Unless otherwise specified, C and c will denote positive constants which will depend only on the dimension n, any underlying sets (such as a fixed cube Q), and the exponent $p(\cdot)$. If we write $A \approx B$, then we mean that there exist positive constants c and C such that $cA \leq B \leq CA$. By cubes we will always mean cubes whose sides are parallel to the coordinate axes. Given a cube Q and $r > 0$, let rQ denote the cube with the same center as Q and such that $\ell(rQ) = r\ell(Q)$. A weight w will always be assumed to be locally integrable and positive almost everywhere.

2. Variable Lebesgue spaces

In this section we gather some basic results on variable Lebesgue spaces. Unless otherwise noted, proofs of these results can be found in [1–4].

Lemma 2.1. *Given $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $p_+ < \infty$, then $\|f\|_{p(\cdot)} \leq C_1$ if and only if*

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \leq C_2.$$

Moreover, if either constant equals 1 we can take the other equal to 1 as well.

Lemma 2.2. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be such that $p_+ < \infty$. Then given any set Ω,*

(a) *if $\|f\chi_\Omega\|_{p(\cdot)} \leq 1$, $\|f\chi_\Omega\|_{p(\cdot)}^{p_+(\Omega)} \leq \int_\Omega |f(x)|^{p(x)}\, dx \leq \|f\chi_\Omega\|_{p(\cdot)}^{p_-(\Omega)}$.*
(b) *if $\|f\chi_\Omega\|_{p(\cdot)} \geq 1$, $\|f\chi_\Omega\|_{p(\cdot)}^{p_-(\Omega)} \leq \int_\Omega |f(x)|^{p(x)}\, dx \leq \|f\chi_\Omega\|_{p(\cdot)}^{p_+(\Omega)}$.*

In particular, if $\|f\|_{p(\cdot)} \leq 1$,

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \leq \|f\|_{p(\cdot)}.$$

Lemma 2.3. *If $p_+ < \infty$ and $\|f\|_{p(\cdot)} = 1$, then*

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx = 1.$$

Lemma 2.4. *If $p_+ < \infty$, bounded functions of compact support are dense in $L^{p(\cdot)}$.*

When the exponent $p(\cdot)$ equals $+\infty$ on a set of positive measure, we modify the definition of the norm as follows. Let $\Omega_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$, and redefine the modular ρ by

$$\rho(f) = \int_{\mathbb{R}^n \setminus \Omega_\infty} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

With this definition $\|\cdot\|_{p(\cdot)}$ is still a norm and $L^{p(\cdot)}$ a Banach space. Moreover, we have the following generalization of Hölder's inequality and an equivalent expression for the norm.

Lemma 2.5 (*Hölder's Inequality*)**.** *Given an exponent $p(\cdot)$, there exists a constant C such that*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

Lemma 2.6. *Given an exponent $p(\cdot)$ and $f \in L^{p(\cdot)}$, there exists $g \in L^{p'(\cdot)}$, $\|g\|_{p'(\cdot)} \leq 1$, such that*

$$\|f\|_{p(\cdot)} \approx \int_{\mathbb{R}^n} f(x)g(x)\, dx.$$

The next two lemmas were proved in [9, Lemmas 2.7, 2.8] for Lebesgue measure. The exact same proofs work in general for non-negative measures μ. They are central to applying the LH_∞ condition.

Lemma 2.7. *Given a set G and two exponent $s(\cdot)$ and $r(\cdot)$ such that*

$$|s(y) - r(y)| \leq \frac{C_0}{\log(e + |y|)},$$

and given a non-negative measure μ, then for every $t \geq 1$ there exists a constant $C = C(t, C_0)$ such that for all functions f with $|f(y)| \leq 1$,

$$\int_G |f(y)|^{s(y)}\, d\mu(y) \leq C \int_G |f(y)|^{r(y)}\, d\mu(y) + \int_G \frac{1}{(e+|y|)^{tnr_-(G)}}\, d\mu(y). \tag{2.1}$$

Lemma 2.8. *Given a set G and two exponent $s(\cdot)$ and $r(\cdot)$ such that*

$$0 \leq r(y) - s(y) \leq \frac{C_0}{\log(e+|y|)},$$

and given a non-negative measure μ, then for every $t \geq 1$ there exists a constant $C = C(t, C_0)$ such that for all functions f, inequality (2.1) holds.

The final lemma is due to Diening [12] and is a crucial tool for applying the LH_0 condition. His result is for balls instead of cubes, but the proof is essentially the same, changing only the dependence of the constant on the dimension n.

Lemma 2.9. *Given an exponent $p(\cdot)$, the following are equivalent:*

(a) *$p(\cdot)$ satisfies the LH_0 condition (1.1);*
(b) *There exists a constant C such that for every cube Q,*

$$|Q|^{p_-(Q)-p_+(Q)} \leq C.$$

3. The variable A_p condition

In this section we prove some basic properties of the weights in $A_{p(\cdot)}$. We begin with some equivalent definitions of A_∞; for a proof of this well-known result, see [17,18].

Lemma 3.1. *Given a weight W, the following are equivalent:*

(a) *$W \in A_\infty = \bigcup_{p\geq 1} A_p$;*
(b) *There exist constants $0 < \alpha, \beta < 1$ such that given any cube Q and measurable set $E \subset Q$, if $|E| > \alpha|Q|$, then $W(E) > \beta W(Q)$;*
(c) *There exist constants $\delta > 0$ and $C_1 > 1$ such that given any cube Q and any measurable set $E \subset Q$,*

$$\frac{W(E)}{W(Q)} \leq C_1 \left(\frac{|E|}{|Q|}\right)^{\delta};$$

(d) *There exist constants $\epsilon > 0$ and $C_2 > 1$ such that given any cube Q and measurable set $E \subset Q$,*

$$\frac{|E|}{|Q|} \leq C_2 \left(\frac{W(E)}{W(Q)}\right)^{\epsilon}.$$

Weights in $A_{p(\cdot)}$ satisfy an A_∞-type condition in terms of the variable Lebesgue space norm.

Lemma 3.2. *Given an exponent $p(\cdot)$, if $w \in A_{p(\cdot)}$, then there exists a constant C depending on $p(\cdot)$ and w such that given any cube Q and measurable set $E \subset Q$,*

$$\frac{|E|}{|Q|} \leq C \frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_Q\|_{p(\cdot)}}.$$

Proof. Fix Q and $E \subset Q$; then by Hölder's inequality (Lemma 2.5) and the $A_{p(\cdot)}$ condition,

$$\begin{aligned}|E| &= \int_{\mathbb{R}^n} w(x)\chi_E w(x)^{-1}\chi_Q\, dx \\ &\leq C\|w\chi_E\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} \leq C\|w\chi_E\|_{p(\cdot)}\|w\chi_Q\|_{p(\cdot)}^{-1}|Q|. \quad \square\end{aligned}$$

The $A_{p(\cdot)}$ condition and log-Hölder continuity together imply that a weighted version of Lemma 2.9 holds.

Lemma 3.3. *Given an exponent $p(\cdot)$ such that (1.1) and (1.2) hold, if $w \in A_{p(\cdot)}$, then there exists a constant C_0 depending on $p(\cdot)$ and w such that for all cubes Q,*

$$\|w\chi_Q\|_{p(\cdot)}^{p_-(Q)-p_+(Q)} \leq C_0. \tag{3.1}$$

Proof. Fix Q. Clearly it suffices to assume that $\|w\chi_Q\|_{p(\cdot)} \leq 1$. Let $Q_0 = Q(0, 1)$. The proof of (3.1) depends on the relative size of Q and Q_0 and their distance from one another. We will consider the case $|Q| \leq |Q_0|$; the case when $|Q| > |Q_0|$ is proved in the same way, exchanging the roles of Q and Q_0.

Suppose $\mathrm{dist}(Q, Q_0) \le \ell(Q_0)$. Then $Q \subset 5Q_0$, and so by Hölder's inequality (Lemma 2.5) and the $A_{p(\cdot)}$ condition,

$$\begin{aligned}|Q| &= \int_Q w(x)w(x)^{-1}\,dx \le C\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} \\ &\le C5^n\|w\chi_Q\|_{p(\cdot)}|5Q_0|^{-1}\|w^{-1}\chi_{5Q_0}\|_{p'(\cdot)} \le C\|w\chi_Q\|_{p(\cdot)}\|w\chi_{5Q_0}\|_{p(\cdot)}^{-1}.\end{aligned}$$

If we rearrange terms and raise both sides to the power $p_+(Q) - p_-(Q)$, by Lemma 2.9 we have that

$$\|w\chi_Q\|_{p(\cdot)}^{p_-(Q)-p_+(Q)} \le C|Q|^{p_-(Q)-p_+(Q)}\|w\chi_{5Q_0}\|_{p(\cdot)}^{p_-(Q)-p_+(Q)} \le C(1 + \|w\chi_{5Q_0}\|_{p(\cdot)}^{-1})^{p_+-p_-}.$$

The right-hand side is a constant independent of Q and so we get (3.1).

Now suppose that $\mathrm{dist}(Q, Q_0) \ge \ell(Q_0)$. Then there exists a cube $\tilde{Q}$ such that $Q,\ Q_0 \subset \tilde{Q}$ and $\ell(\tilde{Q}) \approx \mathrm{dist}(Q, Q_0) \approx \mathrm{dist}(Q, 0) = d_Q$. If argue as we did above, with $5Q_0$ replaced by $\tilde{Q}$, we get that

$$|Q| \le C|\tilde{Q}|\|w\chi_Q\|_{p(\cdot)}\|w\chi_{\tilde{Q}}\|_{p(\cdot)}^{-1} \le C|\tilde{Q}|\|w\chi_Q\|_{p(\cdot)}\|w\chi_{\tilde{Q}}\|_{p(\cdot)}^{-1}.$$

We could continue the above argument and get (3.1) provided that

$$|\tilde{Q}|^{p_+(Q)-p_-(Q)} \le C. \tag{3.2}$$

To see that this is the case: by the LH_0 condition, $p(\cdot)$ is continuous, so $p_-(Q) = p(x_1)$ and $p_+(Q) = p(x_2)$ where $x_1,\ x_2 \in Q$. (More precisely, they may be in the closure of Q, but this makes no difference.) Since $|x_1|,\ |x_2| \approx d_Q$, by the LH_∞ condition,

$$p_+(Q) - p_-(Q) \le |p(x_2) - p_\infty| + |p(x_1) - p_\infty| \le \frac{C}{\log(e + d_Q)}.$$

If we combine this with the fact that $|\tilde{Q}| \le C(e + d_Q)^n$, we get (3.2). □

Lemma 3.4. *Given an exponent $p(\cdot)$ such that (1.1) and (1.2) hold, if $w \in A_{p(\cdot)}$, then $W(\cdot) = w(\cdot)^{p(\cdot)} \in A_\infty$.*

Proof. Fix a cube Q and let $E \subset Q$ be a measurable set. We will show that there exists a constant C (independent of E and Q) such that

$$\frac{|E|}{|Q|} \le C\left(\frac{W(E)}{W(Q)}\right)^{1/p_+};$$

it will then follow at once from Lemma 3.1 that $W \in A_\infty$.

We consider three cases. If $\|w\chi_Q\|_{p(\cdot)} \le 1$, then by Lemmas 3.2 and 2.2 (applied twice) and Lemma 3.3,

$$\begin{aligned}\frac{|E|}{|Q|} &\le C\frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_Q\|_{p(\cdot)}^{p_-(Q)/p_+(Q)}\|w\chi_Q\|_{p(\cdot)}^{1-p_-(Q)/p_+(Q)}} \\ &\le C\left(\frac{W(E)}{W(Q)}\right)^{1/p_+(Q)}\|w\chi_Q\|_{p(\cdot)}^{p_-(Q)/p_+(Q)-1} \le C\left(\frac{W(E)}{W(Q)}\right)^{1/p_+}.\end{aligned}$$

If $\|w\chi_E\|_{p(\cdot)} \le 1 \le \|w\chi_Q\|_{p(\cdot)}$, then again by Lemmas 3.2 and 2.2 we get that

$$\frac{|E|}{|Q|} \le C\left(\frac{W(E)}{W(Q)}\right)^{1/p_+(Q)} \le C\left(\frac{W(E)}{W(Q)}\right)^{1/p_+}.$$

Finally, suppose $\|w\chi_Q\|_{p(\cdot)} \ge \|w\chi_E\|_{p(\cdot)} \ge 1$. Let $\lambda = \|w\chi_Q\|_{p(\cdot)}$. Since $p(\cdot)$ satisfies the LH_∞ condition, by Lemma 2.7 with $d\mu = w(\cdot)^{p(\cdot)}\,dx$, for all $t > 1$,

$$\int_Q \lambda^{-p_\infty}w(x)^{p(x)}\,dx \le C_t\int_Q\left(\frac{w(x)}{\lambda}\right)^{p(x)}dx + \int_Q\frac{w(x)^{p(x)}}{(e + |x|)^{tnp_-}}\,dx.$$

By Lemma 2.3, the first integral on the right-hand side equals 1. We claim that there exists $t > 1$ independent of Q such that the second integral is less than 1. To see this, let $Q_k = Q(0, 2^k)$. Then by Lemma 2.2,

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{w(x)^{p(x)}}{(e + |x|)^{tnp_-}}\,dx &\le e^{-ntp_-}W(Q_0) + \sum_{k=1}^\infty\int_{Q_k\setminus Q_{k-1}}\frac{w(x)^{p(x)}}{(e + |x|)^{tnp_-}}\,dx \\ &\le e^{-ntp_-}W(Q_0) + C\sum_{k=1}^\infty 2^{-ktnp_-}W(Q_k) \\ &\le e^{-ntp_-}W(Q_0) + C\sum_{k=1}^\infty 2^{-ktnp_-}\max\big(\|w\chi_{Q_k}\|_{p(\cdot)}^{p_-}, \|w\chi_{Q_k}\|_{p(\cdot)}^{p_+}\big).\end{aligned}$$

By Lemma 3.2,

$$\|w\chi_{Q_k}\|_{p(\cdot)} \leq C\frac{|Q_k|}{|Q_0|}\|w\chi_{Q_0}\|_{p(\cdot)} \leq C2^{nk}.$$

Combining these two estimates we have that

$$\int_{\mathbb{R}^n} \frac{w(x)^{p(x)}}{(e+|x|)^{tnp_-}}\,dx \leq e^{-ntp_-}W(Q_0) + C\sum_{k=1}^{\infty} 2^{nkp_+-ktnp_-}. \tag{3.3}$$

For $t > p_+/p_-$ the sum converges, and by choosing t sufficiently large (depending only on w and $p(\cdot)$) we can make the right-hand side less than 1. This gives us the desired bound. It follows from this that

$$W(Q)^{1/p_\infty} \leq (C_t+1)^{1/p_\infty}\|w\chi_Q\|_{p(\cdot)}.$$

We now repeat the above argument, replacing Q with E and exchanging the roles of p_∞ and $p(\cdot)$. By Lemma 2.3,

$$1 = \int_E \lambda^{-p(x)}w(x)^{p(x)}\,dx \leq C_t\int_E \lambda^{-p_\infty}w(x)^{p(x)}\,dx + \int_E \frac{w(x)^{p(x)}}{(e+|x|)^{tnp_-}}\,dx.$$

Arguing as before we can make the second term on the right less than $1/2$; if we rearrange terms, we get that

$$\|w\chi_E\|_{p(\cdot)} \leq CW(E)^{1/p_\infty}.$$

Therefore, by Lemma 3.2

$$\frac{|E|}{|Q|} \leq C\frac{\|w\chi_E\|_{p(\cdot)}}{\|w\chi_Q\|_{p(\cdot)}} \leq C\left(\frac{W(E)}{W(Q)}\right)^{1/p_\infty} \leq C\left(\frac{W(E)}{W(Q)}\right)^{1/p_+}$$

This completes the proof. □

As corollaries to the proof of Lemma 3.4 we get two lemmas that we will use repeatedly below.

Lemma 3.5. *Given an exponent $p(\cdot)$ such that (1.1) and (1.2) hold, if $w \in A_{p(\cdot)}$, Q is a cube and $E \subset Q$ is such that $\|w\chi_E\|_{p(\cdot)} \geq 1$, then*

$$\frac{|E|}{|Q|} \leq C\left(\frac{W(E)}{W(Q)}\right)^{1/p_\infty}.$$

Lemma 3.6. *Given an exponent $p(\cdot)$ such that (1.1) and (1.2) hold, if $w \in A_{p(\cdot)}$ and Q is a cube such that $\|w\chi_Q\|_{p(\cdot)} \geq 1$, then $\|w\chi_Q\|_{p(\cdot)} \approx W(Q)^{1/p_\infty}$.*

The $A_{p(\cdot)}$ condition can be characterized in terms of averaging operators. Given any cube Q, define the operator A_Q by

$$A_Qf(x) = \fint_Q |f(y)|\,dy\,\chi_Q(x).$$

Proposition 3.7. *Given an exponent $p(\cdot)$ and a weight w, $w \in A_{p(\cdot)}$ if and only if*

$$\sup_Q \|(A_Qf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

Proposition 3.7 is implicit in [2]. When $p(\cdot)$ is constant this is a lesser known property of the Muckenhoupt A_p weights; it was first given by Jawerth [40]. For completeness we sketch the short proof.

Proof. First suppose that $w \in A_{p(\cdot)}$. Then given any cube Q, by Hölder's inequality (Lemma 2.5),

$$\begin{aligned}\|(A_Qf)w\|_{p(\cdot)} &= \fint_Q |f(x)|\,dx\|w\chi_Q\|_{p(\cdot)} \\ &\leq C|Q|^{-1}\|fw\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)}\|w\chi_Q\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.\end{aligned}$$

Since the constant C is independent of Q, we have that the A_Q are uniformly bounded.

Now assume that the A_Q are uniformly bounded. Fix a cube Q; then by Lemma 2.6 (exchanging the roles of $p(\cdot)$ and $p'(\cdot)$) there exists a function g with $\|gw\|_{p(\cdot)} \leq 1$ such that

$$\begin{aligned}\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} &\leq C\|w\chi_Q\|_{p(\cdot)}\int_{\mathbb{R}^n}\chi_Q g(x)\,dx \\ &= C|Q|\|(A_Qg)w\|_{p(\cdot)} \leq C|Q|\|gw\|_{p(\cdot)} \leq C|Q|.\end{aligned}$$

It follows that $w \in A_{p(\cdot)}$. □

Finally, we note that when $w \equiv 1$, the log-Hölder continuity conditions LH_0 and LH_∞ imply the $A_{p(\cdot)}$ condition. This follows from the necessity in Theorem 1.5 but it is also possible to give a direct proof. This is done, for instance, in [2]; for the convenience of the reader we sketch the details.

Proposition 3.8. *Suppose the exponent $p(\cdot)$ satisfies (1.1) and (1.2) and $p_- > 1$. Then $1 \in A_{p(\cdot)}$: there exists K such that*

$$\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \le K|Q|.$$

Proof. Fix a cube Q. If $|Q| \le 1$, then by Lemma 2.2,

$$\|\chi_Q\|_{p(\cdot)} \le C|Q|^{1/p_+(Q)}, \qquad \|\chi_Q\|_{p'(\cdot)} \le C|Q|^{1/p'_+(Q)} = C|Q|^{1-1/p_-(Q)}.$$

Therefore,

$$\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \le C|Q|\,|Q|^{1/p_+(Q)-1/p_-(Q)} \le K|Q|,$$

where the last inequality follows from Lemma 2.9.

Now suppose that $|Q| > 1$. Then by an argument that is essentially the same as that in the proof of Lemma 3.4, by Lemma 2.7 we have that

$$\|\chi_Q\|_{p(\cdot)} \le K|Q|^{1/p_\infty}, \qquad \|\chi_Q\|_{p'(\cdot)} \le K|Q|^{1/p'_\infty}.$$

Multiplying the two inequalities we get the desired result. □

4. The necessity of the $A_{p(\cdot)}$ condition

In this section we show that the $A_{p(\cdot)}$ condition is necessary in Theorem 1.5.

Proposition 4.1. *Given an exponent $p(\cdot)$ and a weight w, if the maximal operator satisfies the strong-type inequality $\|Mf\,w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}$ or the weak-type inequality $\|t\chi_{\{x:Mf(x)>t\}}w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}$, $t > 0$, then $w \in A_{p(\cdot)}$.*

Proof. Since the strong-type inequality implies the weak-type inequality, it will suffice to show that the latter implies the $A_{p(\cdot)}$ condition. Our proof is modeled on the proof of necessity in Theorem 1.2.

Fix a cube Q; since the weak-type inequality and the $A_{p(\cdot)}$ condition are both homogeneous, we may assume without loss of generality that $\|w^{-1}\chi_Q\|_{p'(\cdot)} = 1$. Define the sets

$$F_0 = \{x \in Q : p'(x) < \infty\}, \qquad F_\infty = \{x \in Q : p'(x) = \infty\},$$

and fix λ, $1/2 < \lambda < 1$. Then by the definition of the norm,

$$1 < \rho_{p'(\cdot)}\left(\frac{w^{-1}\chi_Q}{\lambda}\right) = \int_{F_0}\left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} dx + \lambda^{-1}\|w^{-1}\chi_{F_\infty}\|_\infty.$$

We consider two cases. First suppose that

$$\lambda^{-1}\|w^{-1}\chi_{F_\infty}\|_\infty > \frac{1}{2}.$$

Fix s, $s > \|w^{-1}\chi_{F_\infty}\|_\infty^{-1} = \operatorname{ess\,inf}_{x\in F_\infty} w(x)$ (where we take $1/\infty = 0$). Then there exists a set $E \subset F_\infty$, $|E| > 0$, such that for all $x \in E$, $w(x) \le s$. Define $f = \chi_E$. Since $p(x) = 1$ on F_∞,

$$\|fw\|_{p(\cdot)} = \|w\chi_E\|_{p(\cdot)} = w(E).$$

Further, for all $x \in Q$,

$$Mf(x) \ge \frac{|E|}{|Q|}.$$

Fix $t < |E|/|Q|$. Then by the weak-type inequality,

$$Cw(E) = C\|fw\|_{p(\cdot)} \ge t\|w\chi_{\{x:Mf(x)>t\}}\|_{p(\cdot)} \ge t\|w\chi_Q\|_{p(\cdot)}.$$

Taking the supremum over all such t and re-arranging terms, we get that

$$|Q|^{-1}\|w\chi_Q\|_{p(\cdot)} \le C\frac{w(E)}{|E|} \le Cs.$$

If we now take the infimum over all such s we get

$$|Q|^{-1}\|w\chi_Q\|_{p(\cdot)} \le C\|w^{-1}\chi_{F_\infty}\|_\infty^{-1} \le \frac{2C}{\lambda} \le 4C.$$

Since $\|w^{-1}\chi_Q\|_{p'(\cdot)} = 1$, we get that the $A_{p(\cdot)}$ condition holds on Q.

Now suppose

$$\int_{F_0} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} dx > 1/2.$$

If $p'(\cdot)$ is unbounded, this integral could be infinite. To avoid this complication, define

$$F_R = \{x \in F_0 : p'(x) < R\}, \quad R > 1.$$

Then there exists R such that

$$1/2 < \int_{F_R} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} dx < \infty.$$

(The lower bound is gotten by taking R sufficiently large. The upper bound comes from the fact that by Lemma 2.1, $w^{-p'(\cdot)}\chi_{F_R}$ is integrable.) We claim that there exists $E \subset F_R$ such that

$$\int_{E} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} dx \approx 1/2,$$

and indeed this integral can be made as close to $1/2$ as desired. To see this, fix $\epsilon > 0$; then by the continuity of the integral (cf. Rudin [41, Theorem 6.11]) we can decompose F_R into a finite number of disjoint sets such that the integral of $(w(x)^{-1}/\lambda)^{p'(x)}$ on each set is less than ϵ. We can take E to be the union of some collection of these sets.

Now define the function

$$f(x) = \frac{w(x)^{-p'(x)}}{\lambda^{p'(x)-1}}\chi_E.$$

Then

$$\rho_{p(\cdot)}(fw) = \int_{E} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} dx \approx 1/2 < 1.$$

Hence, $\|fw\|_{p(\cdot)} \le 1$. On the other hand, for all $x \in Q$,

$$Mf(x) \ge \fint_Q f(x)\,dx = \frac{\lambda}{|Q|}\int_{E} \left(\frac{w(x)^{-1}}{\lambda}\right)^{p'(x)} dx \approx \frac{\lambda}{2|Q|}.$$

Fix $t < \frac{\lambda}{2|Q|}$. Then we can argue as before using the weak-type inequality to get

$$C \ge C\|fw\|_{p(\cdot)} \ge t\|w\chi_Q\|_{p(\cdot)};$$

if we take the supremum over all such t, we get the $A_{p(\cdot)}$ condition as desired. □

5. The sufficiency of the $A_{p(\cdot)}$ condition

In this section we prove that the $A_{p(\cdot)}$ condition is sufficient in Theorem 1.5. We will consider the proof of the strong-type inequality in detail; the proof of the weak-type inequality is very similar but simpler at key steps, and we will sketch the changes at the end of the section.

We begin with several lemmas. The first two give the essential properties of the Calderón–Zygmund decomposition. For a proof see [42, Appendix A].

Lemma 5.1. *Given a function f such that $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$, then for each $\lambda > 0$ there exists a set of pairwise disjoint dyadic cubes $\{Q_j\}$ (called the Calderón–Zygmund (CZ) cubes of f at height λ) such that*

$$\Omega_\lambda = \{x \in \mathbb{R}^n : Mf(x) > 4^n\lambda\} \subset \bigcup_j 3Q_j.$$

Further, these cubes have the property that for all j,

$$\lambda < \fint_{Q_j} |f(x)|\,dx \le 3^n \fint_{3Q_j} |f(x)|\,dx.$$

Lemma 5.2. *Given any $a > 2^n$, let $\{Q_j^k\}$, $k \in \mathbb{Z}$, be the CZ cubes of a function f at height a^k. Then for each j and k there exists a set $\widetilde{Q}_j^k \subset Q_j^k$ such that the sets $\widetilde{Q}_j^k$ are pairwise disjoint (for all j and k), and there exists α, $0 < \alpha < 1$, such that $|\widetilde{Q}_j^k| \ge \alpha|Q_j^k|$.*

The final lemma generalizes the classical inequalities for the Hardy–Littlewood maximal operator. For a proof, see [18, p. 146].

Definition 5.3. Given a weight σ and a locally integrable function f, define the maximal operator M_σ by

$$M_\sigma f(x) = \sup_{Q \ni x} \frac{1}{\sigma(Q)} \int_Q |f(y)|\sigma(y)\,dy,$$

where the supremum is taken over all cubes containing x.

Lemma 5.4. *Given a weight $\sigma \in A_\infty$, for $1 < p < \infty$ and $f \in L^p(\sigma)$,*

$$\int_{\mathbb{R}^n} M_\sigma f(x)^p\,\sigma(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p\,\sigma(x)\,dx.$$

The constant depends on p, n and σ.

Our proof of sufficiency is loosely adapted from the proof of Theorem 1.2 due to Christ and Fefferman [43]. Fix f; without loss of generality we may assume that f is non-negative and $\|fw\|_{p(\cdot)} = 1$. We want to form the Calderón–Zygmund cubes of f; in order to do so, we need to show that $\fint_Q f(y)\,dy \to 0$ as $|Q| \to \infty$. By Lemma 3.4, $W(x) = w(x)^{p(x)}$ is in A_∞. Therefore, given cubes $Q_1 \subset Q_2$, $|Q_2| = r|Q_1|$, by Lemma 3.1 we have that for some $\delta > 0$,

$$W(Q_2) \ge cr^\delta W(Q_1).$$

Therefore, it follows that $W(Q) \to \infty$ as $|Q| \to \infty$. Hence, by Hölder's inequality (Lemma 2.5) the $A_{p(\cdot)}$ condition and Lemma 2.2, for all cubes sufficiently large

$$\fint_Q f(y)\,dy \le C\|fw\|_{p(\cdot)}|Q|^{-1}\|w^{-1}\chi_Q\|_{p'(\cdot)} \le C\|w\chi_Q\|_{p(\cdot)}^{-1} \le CW(Q)^{-1/p_+}.$$

This gives us the desired limit.

Define $\sigma(x) = w(x)^{-p'(x)}$. If we write $f = f_1 + f_2$, where $f_1 = f\chi_{\{f\sigma^{-1}>1\}}$ and $f_2 = f\chi_{\{f\sigma^{-1}\le 1\}}$, then $Mf \le Mf_1 + Mf_2$ and for $i = 1, 2$, by Lemma 2.2,

$$\int_{\mathbb{R}^n} |f_i(x)|^{p(x)} w(x)^{p(x)}\,dx \le \|f_i w\|_{p(\cdot)} \le \|fw\|_{p(\cdot)} = 1. \tag{5.1}$$

Hence, by Lemma 2.1, to prove the desired inequality it will suffice to show that there exists a constant C depending on $p(\cdot)$ and w such that

$$\int_{\mathbb{R}^n} Mf_i(x)^{p(x)} w(x)^{p(x)}\,dx \le C, \quad i = 1, 2. \tag{5.2}$$

Estimate (5.2) for $\mathbf{f_1}$. Let $a = 4^n$ and for each $k \in \mathbb{Z}$ let

$$\Omega_k = \{x \in \mathbb{R}^n : Mf_1(x) > a^{k+1}\}.$$

Since $f \in L^1_{\text{loc}}$ and $\fint_Q f(y)\,dy \to 0$ as $|Q| \to \infty$, $0 < Mf_1(x) < \infty$ a.e., and $\mathbb{R}^n = \bigcup_k \Omega_k \setminus \Omega_{k+1}$ (up to a set of measure 0).

By Lemma 5.1, let $\{Q_j^k\}$ be the CZ cubes of f_1 at height a^k. Then for all k,

$$\Omega_k \subset \bigcup_j 3Q_j^k.$$

For each k define the sets E_j^k inductively: $E_1^k = \Omega_k \setminus \Omega_{k+1} \cap 3Q_1^k$, $E_2^k = (\Omega_k \setminus \Omega_{k+1} \cap 3Q_2^k) \setminus E_1^k$, $E_3^k = (\Omega_k \setminus \Omega_{k+1} \cap 3Q_3^k) \setminus (E_1^k \cup E_2^k)$, etc. The sets E_j^k are pairwise disjoint for all j and k and $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E_j^k$. We now estimate as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf_1(x)^{p(x)} w(x)^{p(x)}\,dx &= \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} Mf_1(x)^{p(x)} w(x)^{p(x)}\,dx \\
&\le a^{2p_+} \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} a^{kp(x)} w(x)^{p(x)}\,dx \\
&\le C \sum_{k,j} \int_{E_j^k} \left(\frac{1}{|3Q_j^k|} \int_{3Q_j^k} f_1(y)\,dy \right)^{p(x)} w(x)^{p(x)}\,dx \\
&= C \sum_{k,j} \int_{E_j^k} \left(\int_{3Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)\,dy \right)^{p(x)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx.
\end{aligned}$$

Since $f_1\sigma^{-1} \geq 1$ or $f_1\sigma^{-1} = 0$, by (5.1),

$$\int_{3Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)/p_-(3Q_j^k)}\sigma(y)\,dy \leq \int_{3Q_j^k} (f_1(y)\sigma(y)^{-1})^{p(y)}\sigma(y)\,dy = \int_{3Q_j^k} f_1(y)^{p(y)}w(y)^{p(y)}\,dy \leq 1.$$

Therefore, we can first raise $f_1\sigma^{-1}$ to the power $p(y)/p_-(3Q_j^k)$ and then decrease the exponent of the whole integral to $p_-(3Q_j^k)$ to get

$$\sum_{k,j}\int_{E_j^k}\left(\int_{3Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)\,dy\right)^{p(x)}|3Q_j^k|^{-p(x)}w(x)^{p(x)}\,dx$$

$$\leq \sum_{k,j}\left(\int_{3Q_j^k}(f_1(y)\sigma(y)^{-1})^{p(y)/p_-(3Q_j^k)}\sigma(y)\,dy\right)^{p_-(3Q_j^k)}\int_{E_j^k}|Q_j^k|^{-p(x)}w(x)^{p(x)}\,dx. \tag{5.3}$$

If we multiply and divide by $\sigma(3Q_j^k)$ and then apply Hölder's inequality, we get

$$\leq \sum_{k,j}\left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k}(f_1(y)\sigma(y)^{-1})^{p(y)/p_-(3Q_j^k)}\sigma(y)\,dy\right)^{p_-(3Q_j^k)}\int_{E_j^k}\sigma(3Q_j^k)^{p_-(3Q_j^k)}|Q_j^k|^{-p(x)}w(x)^{p(x)}\,dx$$

$$\leq \sum_{k,j}\left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k}(f_1(y)\sigma(y)^{-1})^{p(y)/p_-}\sigma(y)\,dy\right)^{p_-}\int_{E_j^k}\sigma(3Q_j^k)^{p_-(3Q_j^k)}|Q_j^k|^{-p(x)}w(x)^{p(x)}\,dx. \tag{5.4}$$

Assume for the moment that

$$\int_{E_j^k}\sigma(3Q_j^k)^{p_-(3Q_j^k)}|Q_j^k|^{-p(x)}w(x)^{p(x)}\,dx \leq C\sigma(3Q_j^k). \tag{5.5}$$

By Lemma 3.4 applied to $w^{-1} \in A_{p'(\cdot)}$, $\sigma \in A_\infty$, so by Lemmas 3.1 and 5.2,

$$\sigma(3Q_j^k) \leq C\sigma(Q_j^k) \leq C\sigma(\tilde{Q}_j^k). \tag{5.6}$$

Therefore, (5.4) is bounded by

$$\begin{aligned} C\sum_{k,j}\left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k}(f_1(y)\sigma(y)^{-1})^{p(y)/p_-}\sigma(y)\,dy\right)^{p_-}\sigma(\tilde{Q}_j^k) &\leq C\sum_{k,j}\int_{\tilde{Q}_j^k} M_\sigma((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)\,dx \\ &\leq C\int_{\mathbb{R}^n} M_\sigma((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)\,dx; \end{aligned}$$

by Lemma 5.4 and (5.1),

$$\begin{aligned} &\leq C\int_{\mathbb{R}^n} f_1(x)^{p(x)}\sigma(x)^{-p(x)}\sigma(x)\,dx \\ &= C\int_{\mathbb{R}^n} f_1(x)^{p(x)}w(x)^{p(x)}\,dx \\ &\leq C. \end{aligned}$$

We will now show that (5.5) holds for all j and k. If $p(\cdot) = p$ is constant, then this reduces to the A_p condition (1.6). We will show that given our hypotheses it is implied by the $A_{p(\cdot)}$ condition. For brevity, let $Q = 3Q_j^k$. Since $w \in A_{p(\cdot)}$, we have that

$$\|(C|Q|)^{-1}\|w^{-1}\chi_Q\|_{p'(\cdot)}\,w\chi_Q\|_{p(\cdot)} \leq 1,$$

so

$$\int_Q \|w^{-1}\chi_Q\|_{p'(\cdot)}^{p(x)}|Q|^{-p(x)}w(x)^{p(x)}\,dx \leq C. \tag{5.7}$$

The left-hand side of (5.5) is dominated by

$$\left(\frac{\sigma(Q)}{\|w^{-1}\chi_Q\|_{p'(\cdot)}}\right)^{p_-(Q)}\int_Q \|w^{-1}\chi_Q\|_{p'(\cdot)}^{p_-(Q)-p(x)}\|w^{-1}\chi_Q\|_{p'(\cdot)}^{p(x)}|Q|^{-p(x)}w(x)^{p(x)}\,dx,$$

so by (5.7) it will suffice to show that

$$\left(\frac{\sigma(Q)}{\|w^{-1}\chi_Q\|_{p'(\cdot)}}\right)^{p_-(Q)} \le C\sigma(Q) \tag{5.8}$$

and

$$\|w^{-1}\chi_Q\|_{p'(\cdot)}^{p_-(Q)-p(x)} \le C. \tag{5.9}$$

We first show (5.9); we may assume that $\|w^{-1}\chi_Q\|_{p'(\cdot)} \le 1$ since otherwise the inequality is trivial with $C = 1$. Let $(p')_+$ be the supremum of $p'(\cdot)$ and $(p')_-$ be the infimum. By the definition of $p'(\cdot)$,

$$\begin{aligned} p(x) - p_-(Q) &= \frac{p'(x)}{p'(x)-1} - \frac{(p')_+(Q)}{(p')_+(Q)-1} \\ &= \frac{(p')_+(Q) - p'(x)}{[p'(x)-1][(p')_+(Q)-1]} \le \frac{(p')_+(Q)-(p')_-(Q)}{[(p')_- - 1]^2}. \end{aligned} \tag{5.10}$$

Therefore, by Lemma 3.3 (applied to $w^{-1} \in A_{p'(\cdot)}$) we have that (5.9) holds.

We now prove (5.8). If $\|w^{-1}\chi_Q\|_{p'(\cdot)} > 1$, then by Lemma 2.2,

$$\left(\frac{\sigma(Q)}{\|w^{-1}\chi_Q\|_{p'(\cdot)}}\right)^{p_-(Q)} \le \left(\sigma(Q)^{1-1/(p')_+(Q)}\right)^{p_-(Q)} = \sigma(Q).$$

If $\|w^{-1}\chi_Q\|_{p'(\cdot)} \le 1$, then by Lemma 2.2 (applied twice) and Lemma 3.3,

$$\begin{aligned} \left(\frac{\sigma(Q)}{\|w^{-1}\chi_Q\|_{p'(\cdot)}}\right)^{p_-(Q)} &\le \left(\|w^{-1}\chi_Q\|_{p'(\cdot)}^{(p')_-(Q)-1}\right)^{p_-(Q)} \\ &\le \left(C\|w^{-1}\chi_Q\|_{p'(\cdot)}^{(p')_-(Q)-1+(p')_+(Q)-(p')_-(Q)}\right)^{p_-(Q)} \\ &\le C\left(\|w^{-1}\chi_Q\|_{p'(\cdot)}^{(p')_+(Q)-1}\right)^{p_-(Q)} \\ &\le C\left(\sigma(Q)^{\frac{(p')_+(Q)-1}{(p')_+(Q)}}\right)^{p_-(Q)} \\ &= C\left(\sigma(Q)^{\frac{p_-(Q)'-1}{p_-(Q)'}}\right)^{p_-(Q)} \\ &= C\sigma(Q). \end{aligned}$$

This completes the proof of (5.5) and so the proof of (5.2) for f_1.

Estimate (5.2) *for* $\mathbf{f_2}$. This argument is considerably more technical. We begin with a geometric observation. Since by Lemma 3.4, σ and $W = w(\cdot)^{p(\cdot)}$ are in A_∞, by the properties of A_∞ weights given in Lemma 3.1, we can find a cube

$$P = \bigcup_{i=1}^{2^n} P_i$$

that is the union of 2^n dyadic cubes adjacent to the origin and such that for each i, $|P_i| \ge C$, $W(P_i) \ge C$, and $\sigma(P_i) \ge C$, where $C > 1$ is chosen so that if Q is any cube adjacent to one of the P_i and the same size, then $W(Q),\ \sigma(Q) \ge 1$. Below we will make repeated use of the fact that $W,\ \sigma \in A_\infty$ without further reference.

We now decompose the integral of Mf_2 as we did above for Mf_1 to get, with the same notation as before,

$$\begin{aligned} \int_{\mathbb{R}^n} Mf_2(x)^{p(x)} w(x)^{p(x)}\,dx &\le C\sum_{k,j}\int_{E_j^k}\left(\frac{1}{|3Q_j^k|}\int_{3Q_j^k} f_2(y)\,dy\right)^{p(x)} w(x)^{p(x)}\,dx \\ &= C\left(\sum_{(k,j)\in\mathcal{F}} + \sum_{(k,j)\in\mathcal{G}} + \sum_{(k,j)\in\mathcal{H}}\right) \\ &= C(I_1 + I_2 + I_3), \end{aligned}$$

where

$$\mathcal{F} = \{(k,j) : Q_j^k \subset P\},$$

$$\mathcal{G} = \{(k,j) : Q_j^k \not\subset P,\ \mathrm{dist}(0, 3Q_j^k) = 0\},$$

$$\mathcal{H} = \{(k,j) : Q_j^k \not\subset P,\ \mathrm{dist}(0, 3Q_j^k) > 0\}.$$

We estimate each sum in turn, which is also in order of increasing difficulty.

Estimate of I_1. As before, let $\tilde{Q}_j^k$ be the pairwise disjoint sets associated with the cubes Q_j^k (by Lemma 5.2). If $(k,j) \in \mathcal{F}$, then $3Q_j^k \subset 3P$. Therefore, since $f_2\sigma^{-1} \le 1$, by inequalities (5.5) and (5.6),

$$\begin{aligned}
I_1 &\le \sum_{(k,j)\in\mathcal{F}} \int_{E_j^k} \left(\fint_{3Q_j^k} \sigma(y)\,dy \right)^{p(x)} w(x)^{p(x)}\,dx \\
&\le \sum_{(k,j)\in\mathcal{F}} \int_{E_j^k} \sigma(3Q_j^k)^{p(x)-p_-(3Q_j^k)} \sigma(3Q_j^k)^{p_-(3Q_j^k)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx \\
&\le \sum_{(k,j)\in\mathcal{F}} (1+\sigma(3Q_j^k))^{p_+(3Q_j^k)-p_-(3Q_j^k)} \int_{E_j^k} \sigma(3Q_j^k)^{p_-(3Q_j^k)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx \\
&\le C(1+\sigma(3P))^{p_+-p_-} \sum_{(k,j)\in\mathcal{F}} \sigma(3Q_j^k) \\
&\le C(1+\sigma(3P))^{p_+-p_-} \sum_{(k,j)\in\mathcal{F}} \sigma(\tilde{Q}_j^k) \\
&\le C(1+\sigma(3P))^{p_+-p_-} \sigma(3P).
\end{aligned}$$

The last term is a constant depending only on w and $p(\cdot)$ as required.

Estimate of I_2. Since the cubes Q_j^k are dyadic, either $\operatorname{dist}(Q_j^k, 0) = 0$ or $\operatorname{dist}(Q_j^k, 0) \ge \ell(Q_j^k)$. Hence, if $(k,j) \in \mathcal{G}$, then we must have that for some i, $P_i \subset 3Q_j^k$. In particular, we have that $W(3Q_j^k),\ \sigma(3Q_j^k) \ge 1$. Hence, by Lemmas 3.5 and 3.6 (applied to $w^{-1} \in A_{p'(\cdot)}$)

$$|3Q_j^k|^{-1} \le C|P_i|^{-1}\sigma(P_i)^{1/p'_\infty}\sigma(3Q_j^k)^{-1/p'_\infty} \le C\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{-1}. \tag{5.11}$$

The final constant C depends on σ and P_i, and so on w and $p(\cdot)$. Hence, by Hölder's inequality (Lemma 2.5),

$$\begin{aligned}
\fint_{3Q_j^k} f_2(y)\,dy &\le C\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{-1} \int_{3Q_j^k} f_2(y)\,dy \\
&\le C\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{-1} \|f_2 w\|_{p(\cdot)} \|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)} \le C.
\end{aligned}$$

Therefore, by Lemma 2.7 we can estimate as follows:

$$\begin{aligned}
I_2 &\le C \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \left(C^{-1} \fint_{3Q_j^k} f_2(y)\,dy \right)^{p(x)} w(x)^{p(x)}\,dx \\
&\le C_t \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \left(\fint_{3Q_j^k} f_2(y)\,dy \right)^{p_\infty} w(x)^{p(x)}\,dx + \sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \frac{w(x)^{p(x)}}{(e+|x|)^{ntp_-}}\,dx.
\end{aligned}$$

Arguing exactly as in the proof of Lemma 3.4, inequality (3.3), since the sets E_j^k are disjoint we can choose $t > 1$ (depending only on $p(\cdot)$ and w) so that

$$\sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \frac{w(x)^{p(x)}}{(e+|x|)^{ntp_-}}\,dx \le \int_{\mathbb{R}^n} \frac{w(x)^{p(x)}}{(e+|x|)^{ntp_-}}\,dx \le 1. \tag{5.12}$$

Therefore, to complete the estimate of I_2 we only have to show that the first sum is bounded by a constant. But we have that

$$\begin{aligned}
&\sum_{(k,j)\in\mathcal{G}} \int_{E_j^k} \left(\fint_{3Q_j^k} f_2(y)\,dy \right)^{p_\infty} w(x)^{p(x)}\,dx \\
&= \sum_{(k,j)\in\mathcal{G}} \left(\frac{1}{\sigma(3Q_j^k)} \int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy \right)^{p_\infty} \left(\frac{\sigma(3Q_j^k)}{|3Q_j^k|} \right)^{p_\infty} W(E_j^k).
\end{aligned}$$

Again by Lemma 3.6 and by the $A_{p(\cdot)}$ condition,

$$\sigma(3Q_j^k)^{p_\infty-1} = \sigma(3Q_j^k)^{p_\infty/p'_\infty} \le C\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{p_\infty} \le C\left(\frac{|3Q_j^k|}{\|w\chi_{3Q_j^k}\|_{p(\cdot)}} \right)^{p_\infty} \le C\frac{|3Q_j^k|^{p_\infty}}{W(3Q_j^k)}.$$

Therefore, if we apply this estimate, by Lemmas 5.4 and 2.7 and inequality (5.6),

$$\begin{aligned}
&\sum_{(k,j)\in\mathcal{G}} \left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy\right)^{p_\infty}\left(\frac{\sigma(3Q_j^k)}{|3Q_j^k|}\right)^{p_\infty} W(E_j^k)\\
&\quad\le C\sum_{(k,j)\in\mathcal{G}} \left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy\right)^{p_\infty}\sigma(3Q_j^k)W(3Q_j^k)^{-1}W(E_j^k)\\
&\quad\le C\sum_{(k,j)\in\mathcal{G}} \left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy\right)^{p_\infty}\sigma(\tilde{Q}_j^k)\\
&\quad\le C\sum_{(k,j)\in\mathcal{G}} \int_{\tilde{Q}_j^k} M_\sigma(f_2\sigma^{-1})(x)^{p_\infty}\sigma(x)\,dx\\
&\quad\le C\int_{\mathbb{R}^n} M_\sigma(f_2\sigma^{-1})(x)^{p_\infty}\sigma(x)\,dx\\
&\quad\le C\int_{\mathbb{R}^n} (f_2(x)\sigma^{-1}(x))^{p_\infty}\sigma(x)\,dx\\
&\quad\le C_t\int_{\mathbb{R}^n} (f_2(x)\sigma^{-1}(x))^{p(x)}\sigma(x)\,dx+\int_{\mathbb{R}^n}\frac{\sigma(x)}{(e+|x|)^{tnp_-}}\,dx\\
&\quad\le C_t\int_{\mathbb{R}^n} f_2(x)w(x)^{p(x)}\,dx+\int_{\mathbb{R}^n}\frac{\sigma(x)}{(e+|x|)^{tnp_-}}\,dx.
\end{aligned}\tag{5.13}$$

By inequality (5.1), the first term is bounded by a constant; using an argument identical to that used to prove (5.12), replacing $w^{p(\cdot)}$ by σ, we can find t so that the second term is bounded by 1. This completes the estimate of I_2.

Estimate of I_3. If $(k,j)\in\mathcal{H}$, because the cubes Q_j^k are dyadic, since $3Q_j^k$ is not adjacent to the origin, $\operatorname{dist}(3Q_j^k,0)\ge \ell(Q_j^k)$. Therefore, $|x|$ is essentially constant on $3Q_j^k$: more precisely, there exists a constant $R>1$ independent of (k,j) such that

$$\sup_{x\in 3Q_j^k}|x|\le R\inf_{x\in 3Q_j^k}|x|.\tag{5.14}$$

To estimate I_3 we actually need to divide $\mathcal{H}$ into two subsets:

$$\mathcal{H}_1=\{(k,j)\in\mathcal{H}:\sigma(3Q_j^k)\le 1\},\qquad \mathcal{H}_2=\{(k,j)\in\mathcal{H}:\sigma(3Q_j^k)>1\}.$$

We will estimate the sum in I_3 by first summing over $\mathcal{H}_1$ and then over $\mathcal{H}_2$.

For the first estimate, we want to apply Lemma 2.8 to replace $p(\cdot)$ by $p_+(3Q_j^k)$. Since $p(\cdot)$ is continuous (as a consequence of the LH_0 condition) there exists $x_+\in 3Q_j^k$ such that $p(x_+)=p_+(3Q_j^k)$. (More precisely, x_+ is in the closure of $3Q_j^k$, but this has no effect since we may replace all the cubes by their closures.) Hence, by the LH_∞ condition and (5.14), for all $x\in 3Q_j^k$,

$$|p_+(3Q_j^k)-p(x)|\le|p(x_+)-p_\infty|+|p(x)-p_\infty|\le\frac{C_\infty}{\log(e+|x_+|)}+\frac{C_\infty}{\log(e+|x|)}\le\frac{C}{\log(e+|x|)}.$$

Therefore, we can estimate as follows: by Lemma 2.8 and (5.12),

$$\begin{aligned}
&\sum_{(k,j)\in\mathcal{H}_1}\int_{E_j^k}\left(\fint_{3Q_j^k} f_2(y)\,dy\right)^{p(x)} w(x)^{p(x)}\,dx\\
&\quad\le C_t\sum_{(k,j)\in\mathcal{H}_1}\int_{E_j^k}\left(\fint_{3Q_j^k} f_2(y)\,dy\right)^{p_+(3Q_j^k)} w(x)^{p(x)}\,dx+\sum_{(k,j)\in\mathcal{H}_1}\int_{E_j^k}\frac{w(x)^{p(x)}}{(e+|x|)^{tnp_-}}\,dx\\
&\quad\le C_t\sum_{(k,j)\in\mathcal{H}_1}\int_{E_j^k}\left(\fint_{3Q_j^k} f_2(y)\,dy\right)^{p_+(3Q_j^k)} w(x)^{p(x)}\,dx+1.
\end{aligned}$$

To estimate the sum, by Lemma 2.9 we have that

$$\begin{aligned}
\sum_{(k,j)\in\mathcal{H}_1}\int_{E_j^k}\left(\fint_{3Q_j^k} f_2(y)\,dy\right)^{p_+(3Q_j^k)} w(x)^{p(x)}\,dx &\le C\sum_{(k,j)\in\mathcal{H}_1}\int_{E_j^k}\left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy\right)^{p_+(3Q_j^k)}\\
&\qquad\times\sigma(3Q_j^k)^{p_+(3Q_j^k)}|3Q_j^k|^{-p(x)}w(x)^{p(x)}\,dx;
\end{aligned}$$

since $f_2\sigma^{-1} \le 1$, by Lemma 2.7 we have that

$$\begin{aligned}
&\le C \sum_{(k,j)\in\mathcal{H}_1} \int_{E_j^k} \left(\frac{1}{\sigma(3Q_j^k)} \int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy \right)^{p_\infty} \sigma(3Q_j^k)^{p_+(3Q_j^k)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx \\
&\quad + \sum_{(k,j)\in\mathcal{H}_1} \int_{E_j^k} \sigma(3Q_j^k)^{p_+(3Q_j^k)} |3Q_j^k|^{-p(x)} \frac{w(x)^{p(x)}}{(e+|x|)^{tnp_-}}\,dx \\
&= J_1 + J_2.
\end{aligned}$$

To estimate J_2 we use (5.14) (since $E_j^k \subset 3Q_j^k$), the fact that $\sigma(3Q_j^k) \le 1$, (5.5) and (5.6) to get

$$\begin{aligned}
J_2 &\le \sum_{(k,j)\in\mathcal{H}_1} \sup_{x\in E_j^k} (e+|x|)^{-tnp_-} \int_{E_j^k} \sigma(3Q_j^k)^{p_-(3Q_j^k)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx \\
&\le C \sum_{(k,j)\in\mathcal{H}_1} \sup_{x\in E_j^k} (e+|x|)^{-tnp_-} \sigma(3Q_j^k) \\
&\le C \sum_{(k,j)\in\mathcal{H}_1} \int_{\tilde{Q}_j^k} \frac{\sigma(x)}{(e+|x|)^{tnp_-}}\,dx \\
&\le C;
\end{aligned}$$

the last inequality follows as it did above at the end of the estimate for I_2.

To estimate J_1 we again use the fact that $\sigma(3Q_j^k) \le 1$ and (5.5) to get

$$\begin{aligned}
J_1 &\le \sum_{(k,j)\in\mathcal{H}_1} \left(\frac{1}{\sigma(3Q_j^k)} \int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy \right)^{p_\infty} \int_{E_j^k} \sigma(3Q_j^k)^{p_-(3Q_j^k)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx \\
&\le C \sum_{(k,j)\in\mathcal{H}_1} \left(\frac{1}{\sigma(3Q_j^k)} \int_{3Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)\,dy \right)^{p_\infty} \sigma(3Q_j^k). \qquad (5.15)
\end{aligned}$$

The final sum is bounded by a constant. We can show this arguing exactly as we did in the estimate for I_2: use (5.6) to replace $\sigma(3Q_j^k)$ with $\sigma(\tilde{Q}_j^k)$ and then argue as we did from (5.13). This completes the bound for the sum over $\mathcal{H}_1$.

Finally, we estimate the sum over $\mathcal{H}_2$. By Hölder's inequality (Lemma 2.5),

$$\int_{3Q_j^k} f_2(y)\,dy \le c\|f_2 w\|_{p(\cdot)} \|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)} \le c\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}.$$

Therefore, by Lemma 2.7,

$$\begin{aligned}
\sum_{(k,j)\in\mathcal{H}_2} \int_{E_j^k} \left(\fint_{3Q_j^k} f_2(y)\,dy \right)^{p(x)} w(x)^{p(x)}\,dx &\le C \sum_{(k,j)\in\mathcal{H}_2} \int_{E_j^k} \left(c\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{-1} \int_{3Q_j^k} f_2(y)\,dy \right)^{p(x)} \\
&\qquad \times \left(\frac{\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}}{|3Q_j^k|} \right)^{p(x)} w(x)^{p(x)}\,dx \\
&\le C \sum_{(k,j)\in\mathcal{H}_2} \int_{E_j^k} \left(\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{-1} \int_{3Q_j^k} f_2(y)\,dy \right)^{p_\infty} \\
&\qquad \times \left(\frac{\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}}{|3Q_j^k|} \right)^{p(x)} w(x)^{p(x)}\,dx \\
&\quad + \sum_{(k,j)\in\mathcal{H}_2} \int_{E_j^k} \left(\frac{\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}}{|3Q_j^k|} \right)^{p(x)} \frac{w(x)^{p(x)}}{(e+|x|)^{ntp_-}}\,dx \\
&= K_1 + K_2.
\end{aligned}$$

To estimate K_2, note that since $\sigma(3Q_j^k) \ge 1$, by (5.6), $\sigma(\tilde{Q}_j^k) > \epsilon > 0$, where ϵ does not depend on (k,j). Therefore, by (5.7) and (5.14) we have that

$$K_2 \le \sum_{(k,j)\in\mathcal{H}_2} \sup_{x\in Q_j^k} (e+|x|)^{-ntp_-} \int_{3Q_j^k} \left(\frac{\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}}{|3Q_j^k|} \right)^{p(x)} w(x)^{p(x)}\,dx$$

$$\begin{aligned}
&\leq C \sum_{(k,j)\in\mathcal{H}_2} \sup_{x\in Q_j^k} (e+|x|)^{-ntp_-}\sigma(\tilde{Q}_j^k) \\
&\leq C \int_{\mathbb{R}^n} \frac{\sigma(x)}{(e+|x|)^{ntp_-}}\,dx \\
&\leq C;
\end{aligned}$$

the final bound is gotten as in the estimate of J_2 above.

To estimate K_1, we use Lemma 3.6 to get

$$\|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{-p_\infty}\sigma(3Q_j^k)^{p_\infty} \leq C\sigma(3Q_j^k)^{-p_\infty/p'_\infty+p_\infty} = C\sigma(3Q_j^k).$$

Therefore, by (5.7) and (5.6) we have that

$$\begin{aligned}
K_1 &= \sum_{(k,j)\in\mathcal{H}_2} \int_{E_j^k} \left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\,dy\right)^{p_\infty} \|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{p(x)-p_\infty} \frac{\sigma(3Q_j^k)^{p_\infty}}{|3Q_j^k|^{p(x)}} w(x)^{p(x)}\,dx \\
&\leq C \sum_{(k,j)\in\mathcal{H}_2} \left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\,dy\right)^{p_\infty} \sigma(3Q_j^k) \int_{3Q_j^k} \|w^{-1}\chi_{3Q_j^k}\|_{p'(\cdot)}^{p(x)} |3Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx \\
&\leq C \sum_{(k,j)\in\mathcal{H}_2} \left(\frac{1}{\sigma(3Q_j^k)}\int_{3Q_j^k} f_2(y)\,dy\right)^{p_\infty} \sigma(\tilde{Q}_j^k).
\end{aligned}$$

We can estimate the final term as we did in the estimate for J_1, inequality (5.15). This completes the bound for the sum over $\mathcal{H}_2$, and so gives us the desired estimate for I_3. This completes the estimate for f_2 and so the proof of the sufficiency of the $A_{p(\cdot)}$ condition for the strong-type inequality.

The weak-type inequality

The proof of the weak-type inequality is very similar to the proof of the strong-type inequality. In the proof of the latter we use that $p_- > 1$ only to apply the strong-type norm inequality for M_σ. We can readily modify the proof to avoid this.

Define f_1 and f_2 as before. Then for all t,

$$\{x\in\mathbb{R}^n : Mf(x) > t\} \subset \{x\in\mathbb{R}^n : Mf_1(x) > t/2\} \cup \{x\in\mathbb{R}^n : Mf_2(x) > t/2\}.$$

Therefore, it will suffice to prove that f_1 and f_2 each satisfy the weak-type inequality.

We first consider f_1. Fix $t > 0$ and form the CZ cubes $\{Q_j\}$ of f at height $t/4^n$. Then

$$\{x\in\mathbb{R}^n : Mf_1(x) > t\} \subset \bigcup_j 3Q_j,$$

and by Lemma 5.1 we have that

$$\int_{\mathbb{R}^n} t^{p(x)}\chi_{\{x\in Q_0:Mf_1(x)>t\}}(x)w(x)^{p(x)}\,dx \leq C\sum_j \int_{3Q_j}\left(\fint_{Q_j} f_1(y)\,dy\right)^{p(x)} w(x)^{p(x)}\,dx.$$

We can then modify the integral of f_1 as in (5.3) (here using $p_- = 1$) and then use inequality (5.5) (again with $p_- = 1$), the fact that $\sigma \in A_\infty$, and the fact that the cubes Q_j are disjoint to get that

$$\begin{aligned}
\sum_j \int_{3Q_j}\left(\fint_{Q_j} f_1(y)\,dy\right)^{p(x)} w(x)^{p(x)}\,dx &\leq C\sum_j \int_{3Q_j}\left(\frac{1}{\sigma(Q_j)}\int_{Q_j}(f_1(y)\sigma(y)^{-1})^{p(y)}\sigma(y)\,dy\right) \\
&\qquad \times \sigma(3Q_j)|3Q_j|^{-p(x)}w(x)^{p(x)}\,dx \\
&\leq \sum_j \left(\frac{1}{\sigma(Q_j)}\int_{Q_j}(f_1(y)\sigma(y)^{-1})^{p(y)}\sigma(y)\,dy\right)\sigma(3Q_j) \\
&\leq C\int_{\mathbb{R}^n} f_1(y)^{p(y)}w(y)^{p(y)}\,dy \\
&\leq C.
\end{aligned}$$

The estimates for f_2, though longer, can be adapted in exactly the same way to complete the proof of the weak-type inequality.

Acknowledgments

The authors would like to thank Lars Diening and Peter Hästö for sharing with us many preprints of their work.

References

[1] D. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Birkhäuser, Basel, 2009.
[2] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, in: Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011.
[3] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2) (2001) 424–446.
[4] O. Kováčik, J. Rákosník, A on spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (116) (1991) 592–618. (4).
[5] J. Musielak, Orlicz Spaces and Modular Spaces, in: Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983.
[6] W. Orlicz, Über konjugierte Exponentenfolgen., Studia Math. 3 (1931) 200–211.
[7] L. Diening, P. Hästö, A. Nekvinda, Open problems in variable exponent lebesgue and sobolev spaces. in: FSDONA04 Proceedings Drabek and Rakosnik (Ed.), Milovy, Czech Republic, Academy of Sciences of the Czech Republic, Prague, 2005, pp. 38–58.
[8] S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators, Integral Transforms Spec. Funct. 16 (5–6) (2005) 461–482.
[9] C. Capone, D. Cruz-Uribe, A. Fiorenza, The fractional maximal operator and fractional integrals on variable L^p spaces, Rev. Mat. Iberoam. 23 (3) (2007) 743–770.
[10] D. Cruz-Uribe, L. Diening, A. Fiorenza, A new proof of the boundedness of maximal operators on variable Lebesgue spaces, Boll. Unione Mat. Ital. (9) 2 (1) (2009) 151–173.
[11] D. Cruz-Uribe, A. Fiorenza, C.J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. 28 (1) (2003) 223–238. See also errata [44].
[12] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, Math. Inequal. Appl. 7 (2) (2004) 245–253.
[13] A. Nekvinda, Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$, Math. Inequal. Appl. 7 (2) (2004) 255–265.
[14] A.K. Lerner, Some remarks on the Hardy–Littlewood maximal function on variable L^p spaces, Math. Z. 251 (3) (2005) 509–521.
[15] L. Pick, M. Růžička, An example of a space $L^{p(x)}$ on which the Hardy–Littlewood maximal operator is not bounded, Expo. Math. 19 (4) (2001) 369–371.
[16] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972) 207–226.
[17] J. Duoandikoetxea, Fourier analysis, in: Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001.
[18] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, in: North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985.
[19] L. Diening, P. Hästö, Muckenhoupt weights in variable exponent spaces, 2010, Preprint.
[20] B. Muckenhoupt, R.L. Wheeden, Weighted norm inequalities for fractional integrals, Trans. Amer. Math. Soc. 192 (1974) 261–274.
[21] D. Cruz-Uribe, J.M. Martell, C. Pérez, Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture, Adv. Math. 216 (2) (2007) 647–676.
[22] D. Cruz-Uribe, L. Diening, P. Hästö, The maximal operator on weighted variable Lebesgue spaces, Fract. Calc. Appl. Anal. 14 (3) (2011) 361–374.
[23] L. Diening, Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces, Bull. Sci. Math. 129 (8) (2005) 657–700.
[24] V. Kokilashvili, N. Samko, S. Samko, The maximal operator in variable spaces $L^{p(\cdot)}(\Omega, \rho)$ with oscillating weights, Georgian Math. J. 13 (1) (2006) 109–125.
[25] V. Kokilashvili, N. Samko, S. Samko, The maximal operator in weighted variable spaces $L^{p(\cdot)}$, J. Funct. Spaces Appl. 5 (3) (2007) 299–317.
[26] V. Kokilashvili, S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces, Proc. A. Razmadze Math. Inst. 129 (2002) 145–149.
[27] V. Kokilashvili, S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces, Rev. Mat. Iberoam. 20 (2) (2004) 493–515.
[28] V. Kokilashvili, S. Samko, Operators of harmonic analysis in weighted spaces with non-standard growth, J. Math. Anal. Appl. 352 (1) (2009) 15–34.
[29] M. Khabazi, Maximal operators in weighted $L^{p(x)}$ spaces, Proc. A. Razmadze Math. Inst. 135 (2004) 143–144.
[30] M. Asif, V. Kokilashvili, A. Meskhi, Boundedness criteria for maximal functions and potentials on the half-space in weighted Lebesgue spaces with variable exponents, Integral Transforms Spec. Funct. 20 (11–12) (2009) 805–819.
[31] V. Kokilashvili, A. Meskhi, Two-weighted inequalities for Hardy–Littlewood functions and singular integrals in $L^{p(\cdot)}$ spaces, J. Math. Sci. 173 (6) (2011) 656–673.
[32] F. Mamedov, Y. Zeren, On a two-weighted estimation of the maximal operator in the Lebesgue space with variable exponent, Ann. Mat. Pura Appl. 190 (2) (2011) 263–275.
[33] F. Mamedov, Y. Zeren, Two-weight inequalities for the maximal operator in a Lebesgue space with variable exponent, J. Math. Sci. 173 (6) (2011) 701–716.
[34] L. Diening, Lebesgue and Sobolev spaces with variable exponent. Habilitation, Universität Freiburg, 2007.
[35] T. Kopaliani, On the Muckenchaupt condition in variable Lebesgue spaces, Proc. A. Razmadze Math. Inst. 148 (2008) 29–33.
[36] A.K. Lerner, On some questions related to the maximal operator on variable L^p spaces, Trans. Amer. Math. Soc. 362 (8) (2010) 4229–4242.
[37] D. Cruz-Uribe, J.M. Martell, C. Pérez, Sharp weighted estimates for classical operators, Adv. Math. 229 (2011) 408–441.
[38] S.M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, Trans. Amer. Math. Soc. 340 (1) (1993) 253–272.
[39] A.K. Lerner, An elementary approach to several results on the Hardy–Littlewood maximal operator, Proc. Amer. Math. Soc. 136 (8) (2008) 2829–2833.
[40] B. Jawerth, Weighted inequalities for maximal operators: linearization, localization and factorization, Amer. J. Math. 108 (2) (1986) 361–414.
[41] W. Rudin, Real and Complex Analysis, third ed., McGraw-Hill Book Co., New York, 1987.
[42] D. Cruz-Uribe, J.M. Martell, C. Pérez, Weights, Extrapolation and the Theory of Rubio de Francia, in: Operator Theory: Advances and Applications, vol. 215, Birkhäuser, Springer, Basel AG, Basel, 2011.
[43] M. Christ, R. Fefferman, A note on weighted norm inequalities for the Hardy–Littlewood maximal operator, Proc. Amer. Math. Soc. 87 (3) (1983) 447–448.
[44] D. Cruz-Uribe, A. Fiorenza, C.J. Neugebauer, Corrections to: The maximal function on variable L^p spaces [Ann. Acad. Sci. Fenn. Math., 28, 2003, 1, 223–238], Ann. Acad. Sci. Fenn. Math. 29 (1) (2004) 247–249.