



# Normal forms of necessary conditions for dynamic optimization problems with pathwise inequality constraints

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## ABSTRACT

There has been a longstanding interest in deriving conditions under which dynamic optimization problems are normal, that is, the necessary conditions of optimality (NCO) can be written with a nonzero multiplier associated with the objective function. This paper builds upon previous results on nondegenerate NCO for trajectory constrained optimal control problems to provide even stronger, normal forms of the conditions. The NCO developed may address problems with nonsmooth, less regular data. The particular case of calculus of variations problems is here explored to show a favorable comparison with existent results.

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## 1. Introduction

In this paper, we study Necessary Conditions of Optimality (NCO) for Dynamic Optimization Problems with pathwise inequality constraints. In particular, we are interested in normal forms of the NCO, i.e., forms in which the scalar multiplier associated with the objective function – here called  $\lambda$  – is nonzero. The normal forms of the NCO are guaranteed to supply non-trivial information, in the sense that they guarantee that the objective function is taken into account when selecting candidates to optimal processes.

Many important applications of NCO would benefit or even require normal forms. In engineering applications or in decision making contexts, the NCO are used to select a candidate (or a small number of candidates) for optimal solution. If we do not guarantee normality and allow  $\lambda = 0$ , then the NCO identify a set of candidates in which the objective function is not used in the selection, and such an identified set is typically too large. This is even more critical in applications where the NCO are used to find a solution without human intervention (e.g. synthesis of controls for autonomous vehicles), and thus we have to guarantee that the NCO remain informative.

Normal forms of NCO are also important in establishing results on the regularity properties of optimal solutions and to establish second-order conditions. In most results of such nature, the possibility of selecting  $\lambda \neq 0$  has to be assumed (e.g. [1–5]) or conditions are imposed so as to guarantee that the system of first-order conditions is normal (e.g. [6,7]).

The importance of studying normal forms of NCO is well illustrated in the history of Mathematical Programming [8,9]. The Kuhn–Tucker conditions [10], one of the most cited results in optimization, are a strengthened, nondegenerate version of some earlier conditions, now less known, of Fritz John [11].

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There has been a growing interest and literature on strengthened forms of NCO for Optimal Control Problems (OCP), reporting both nondegenerate and normal forms of the maximum principle (MP). (See e.g. [12] for what appears to be the first result on the subject, the recent works [13,14] and references therein, as well as [15] which provides references to an extensive Russian literature on the subject.) The normality results reported in literature require different degrees of regularity on the problem data [16–21]. Requiring very little regularity on the data, we can find strengthened NCO in [22] which, although not ensuring normality, are able to avoid certain sets of degenerate multipliers. Building upon the nondegeneracy results in [22], we develop here an even stronger form of NCO: a normal form. An advantage of our result comparing with similar results in literature is the fact that it addresses problems with less regular, nonsmooth data. However, the additional hypotheses under which our result is valid, known as constraint qualification (CQ), involve the optimal control which we do not know in advance, and consequently, in general, it is not so easy to verify whether the CQ is satisfied for the problem we have in hands. Nevertheless, in some cases, the conditions we propose compare favorably with existing results. One such case is the application of our result to calculus of variations problems (CVP). We study normality of NCO for CVP as a consequence of the results on normality of NCO for OCP here developed. The special structure of CVP permits the derivation of CQ that are much easier to verify than in the optimal control case. The conditions thereby obtained generalize a result in [16] to the nonsmooth case.

This paper is organized as follows. In a brief Preliminaries section, we provide some of the concepts and notation that are used throughout the paper. Section 3 describes the context of our results: optimal control problems with state constraints and the nonsmooth maximum principle that is to be strengthened in later sections. We also describe the case of CVP with inequality constraints and its necessary conditions of optimality. Section 4 provides a main result of this paper: a normal form of NCO valid under a suitable constraint qualification. In Section 5, we apply the previous result to a CVP and deduce CQs which are specific for this problem and have the advantage that they are easy to verify. In Section 6 we compare the results obtained in the previous section with other results when applied to CVPs. Finally, in Sections 7 and 8, we prove the main results and lemmas of this paper.

## 2. Preliminaries

Throughout,  $\mathbb{B}$  denotes the *closed unit ball*,  $\text{co } S$  denotes the *convex hull* of a set  $S$ ,  $\text{supp}\{\mu\}$  denotes the support of measure  $\mu$ , and  $\delta_{\{0\}}$  denotes the Dirac unit measure concentrated at 0. We also make reference to the space  $W^{1,1}$  of absolutely continuous functions,  $C^*$  the dual space of continuous functions, and  $C^{1,1}$  the space of functions which are continuously differentiable with locally Lipschitz continuous derivatives.

The *limiting normal cone* of a closed set  $C \subset \mathbb{R}^n$  at  $\bar{x} \in C$  is defined to be

$$N_C^L(\bar{x}) := \{\eta \in \mathbb{R}^n : \exists \text{ sequences } \{M_i\} \in \mathbb{R}^+, x_i \rightarrow \bar{x}, \eta_i \rightarrow \eta \text{ such that} \\ x_i \in C \text{ and } \eta_i \cdot (y - x_i) \leq M_i \|y - x_i\|^2 \text{ for all } y \in \mathbb{R}^n, i = 1, 2, \dots\}.$$

Given a lower semicontinuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  the *limiting subdifferential* of  $f$  at a point  $\bar{x} \in \mathbb{R}^n$  such that  $f(\bar{x}) < +\infty$  is the set

$$\partial^L f(\bar{x}) := \{\eta \in \mathbb{R}^n : (\eta, -1) \in N_{\text{epi } f}^L(\bar{x}, f(\bar{x}))\};$$

where  $\text{epi } f := \{(x, \alpha) : \alpha \geq f(x)\}$ . We also make use of the *hybrid partial subdifferential* of  $h$  in the  $x$ -variable defined as

$$\partial_x^> h(t, x) := \text{co}\{\xi : \text{there exist } (t_i, x_i) \rightarrow (t, x) \text{ s.t.} \\ h(t_i, x_i) > 0, h(t_i, x_i) \rightarrow h(t, x), \text{ and } h_x(t_i, x_i) \rightarrow \xi\}.$$

We refer to [23–25] for further concepts of nonsmooth analysis and optimal control. See also [26] for a review using a notation similar to the one used here.

## 3. Context

Consider the fixed left-endpoint Optimal Control Problem (OCP) with inequality state constraints:

$$(\text{OCP}_1) \begin{cases} \text{Minimize} & g(x(1)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\ & x(0) = x_0 \\ & u(t) \in \Omega(t) \quad \text{a.e. } t \in [0, 1] \\ & h(t, x(t)) \leq 0 \quad \text{for all } t \in [0, 1]. \end{cases}$$

The data for this problem comprise functions  $g : \mathbb{R}^n \mapsto \mathbb{R}$ ,  $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ ,  $h : [0, 1] \times \mathbb{R}^n \mapsto \mathbb{R}$ , an initial state  $x_0 \in \mathbb{R}^n$ , and a multifunction  $\Omega : [0, 1] \rightrightarrows \mathbb{R}^m$ .

The set of *control functions* for  $(\text{OCP}_1)$ , denoted  $\mathcal{U}$ , is the set of measurable functions  $u : [0, 1] \rightarrow \mathbb{R}^m$  such that  $u(t) \in \Omega(t)$  a.e.  $t \in [0, 1]$ . A *state trajectory* is an absolutely continuous function which satisfies the differential equation in the constraints for some control function  $u$ . The domain of the above optimization problem is the set of *admissible processes*,

namely pairs  $(x, u)$  comprising a control function  $u$  and a corresponding state trajectory  $x$  which satisfy the constraints of  $(\text{OCP}_1)$ . We say that an admissible process  $(\bar{x}, \bar{u})$  is a *local minimizer* if there exists  $\delta > 0$  such that

$$g(\bar{x}(1)) \leq g(x(1)),$$

for all admissible processes  $(x, u)$  satisfying

$$\|x(t) - \bar{x}(t)\|_{L^\infty} \leq \delta.$$

We develop here refinements of the nonsmooth maximum principle below that is valid under the following hypotheses, some of which refer to a minimizer  $(\bar{x}, \bar{u})$  or a  $\delta'$  neighborhood of it.

H1 The function  $(t, u) \rightarrow f(t, x, u)$  is  $\mathcal{L} \times \mathcal{B}^m$  measurable for each  $x$ . ( $\mathcal{L} \times \mathcal{B}^m$  denotes the product  $\sigma$ -algebra generated by the Lebesgue subsets  $\mathcal{L}$  of  $[0, 1]$  and the Borel subsets of  $\mathbb{R}^m$ .)

H2 There exists a  $\mathcal{L} \times \mathcal{B}^m$  measurable function  $k(t, u)$  such that  $t \mapsto k(t, \bar{u}(t))$  is integrable and

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|$$

for  $x, x' \in \bar{x}(t) + \delta'\mathbb{B}$ ,  $u \in \Omega(t)$  a.e.  $t \in [0, 1]$ .

H3 The function  $g$  is Lipschitz continuous on  $\bar{x}(1) + \delta'\mathbb{B}$ .

H4 The graph of  $\Omega$  is  $\mathcal{L} \times \mathcal{B}^m$  measurable.

H5 The function  $h$  is upper semicontinuous in  $t$  and there exists a scalar  $K_h > 0$  such that the function  $x \rightarrow h(t, x)$  is Lipschitz of rank  $K_h$  for all  $t \in [0, 1]$ .

**Theorem 3.1** (Theorem 9.3.1 [26]). *Let  $(\bar{x}, \bar{u})$  be a local minimizer for  $(\text{OCP}_1)$ . Assume hypotheses H1–H5. Then, there exist  $p \in W^{1,1}([0, 1] : \mathbb{R}^n)$ , a measurable function  $\gamma$ , a non-negative measure  $\mu$  in  $C^*([0, 1] : \mathbb{R})$  and a scalar  $\lambda \in \{0, 1\}$  such that*

$$\mu\{[0, 1]\} + \|p\|_{L^\infty} + \lambda > 0, \quad (1)$$

$$-\dot{p}(t) \in \text{co } \partial_x^L(q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))) \quad \text{a.e. } t \in [0, 1], \quad (2)$$

$$-q(1) \in \lambda \partial^L g(\bar{x}(1)),$$

$$\gamma(t) \in \partial_x^> h(t, \bar{x}(t)) \quad \mu\text{-a.e.},$$

$$\text{supp}\{\mu\} \subset \{t \in [0, 1] : h(t, \bar{x}(t)) = 0\},$$

and, for almost every  $t \in [0, 1]$ ,  $\bar{u}(t)$  maximizes over  $\Omega(t)$

$$u \mapsto q(t) \cdot f(t, \bar{x}(t), u),$$

where

$$q(t) = \begin{cases} p(t) + \int_{[0,t)} \gamma(s)\mu(ds) & t \in [0, 1) \\ p(t) + \int_{[0,1]} \gamma(s)\mu(ds) & t = 1. \end{cases}$$

When the pathwise constraint is active at the initial instant of time, i.e. when

$$h(0, x_0) = 0,$$

the set of multipliers (degenerate multipliers)

$$\lambda = 0, \quad \mu = \beta \delta_{t=0}, \quad p = -\beta \gamma \quad \text{with } \gamma \in \partial_x^>(0, x_0) \text{ for some } \beta > 0 \quad (3)$$

satisfy the maximum principle (MP) for any admissible process  $(x, u)$ . This can be easily seen by noting that the quantity

$$p(t) + \int_{[0,t)} \gamma(s)\mu(ds)$$

vanishes almost everywhere and all conditions of the MP, (Theorem 3.1), are satisfied independently of the value of  $\bar{x}$  or  $\bar{u}$ . In this case, the NCO are said to degenerate.

In the literature, there exist strengthened forms of the MP to avoid this kind of degenerate multipliers, see for example [15,22,27,28]. Here, we are interested in developments of the strengthened form introduced by Ferreira, Fontes and Vinter in [22], which will be extended to guarantee normality.

The strengthened MP in [22], ensures that the nontriviality condition of the MP can be written as

$$\mu\{(0, 1]\} + \|q\|_{L^\infty} + \lambda > 0,$$

in place of (1)  $\mu\{[0, 1]\} + \|p\|_{L^\infty} + \lambda > 0$ , eliminating in this way the degenerate multipliers (3). This nondegenerate result holds when the data of the problem satisfies, besides the basic hypotheses, a slightly stronger form of H2, as follows:

H2' there exist scalars  $K_f > 0$  and  $\epsilon' > 0$  such that

$$\|f(t, x, u) - f(t, x', u)\| \leq K_f \|x - x'\|,$$

for  $x, x' \in \bar{x}(0) + \delta'\mathbb{B}$ ,  $u \in \Omega(t)$  a.e.  $t \in [0, \epsilon']$

and the constraint qualification:

CQ if  $h(0, x_0) = 0$ , then there exist positive constants  $K_u, \epsilon, \epsilon_1, \delta$ , and a control  $\hat{u} \in \mathcal{U}$  such that for a.e.  $t \in [0, \epsilon)$

$$\|f(t, x_0, \bar{u}(t))\| \leq K_u, \quad (4)$$

$$\|f(t, x_0, \hat{u}(t))\| \leq K_u, \quad (5)$$

and

$$\zeta \cdot [f(t, x_0, \hat{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta, \quad (6)$$

for all  $\zeta \in \partial_x^> h(s, x)$ ,  $s \in [0, \epsilon)$ ,  $x \in \{x_0\} + \epsilon_1\mathbb{B}$ .

**Theorem 3.2** ([22]). Assume hypotheses H1–H5, H2' and the constraint qualification CQ. Then, in Theorem 3.1, the nontriviality condition (1) can be replaced by

$$\mu\{[0, 1]\} + \|q\|_{L^\infty} + \lambda > 0.$$

In a recent publication [13], we discuss this and other forms of constraint qualification that guarantee nondegeneracy. In the next section, we provide an even stronger form of these necessary conditions: a form guaranteeing normality, i.e.  $\lambda > 0$ .

#### 4. Normality in optimal control problems

The main result here is that for all problems satisfying the constraint qualification  $CQ_n$  below, we can write the NCO with the assurance that the multiplier  $\lambda$  (the scalar associated with the objective function) can always be chosen to be positive. First we provide the more general result and then we provide a corollary for which it is easier to verify whether the problem satisfies the conditions for the result to be applied.

$CQ_n$  (Constraint Qualification for Normality)

There exist positive constants  $\epsilon, \delta, K_u$ , and a control  $\hat{u} \in \mathcal{U}$  such that

$$\|f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))\| \leq K_u, \quad (7)$$

and

$$\zeta \cdot [f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] < -\delta, \quad (8)$$

for all  $\zeta \in \partial_x^> h(s, \bar{x}(s))$ , a.e.  $t, s \in (\tau - \epsilon, \tau] \cap [0, 1]$  where  $\tau$  is defined as

$$\tau = \inf \left\{ t \in [0, 1] : \int_{[t, 1]} \mu(ds) = 0 \right\}.$$

**Remark 4.1.** Condition (7) is satisfied for all problems with  $\Omega$  bounded and  $u \mapsto f(\cdot, \cdot, u)$  continuous. The last condition (8) in  $CQ_n$  says that there is a control that can pull the trajectory away from the boundary (faster than the optimal control) near to the last instant,  $\tau$ , in which the measure  $\mu$  is active.

**Theorem 4.2.** Assume hypotheses H1–H5 and H2'. Assume also that the constraint qualifications CQ and  $CQ_n$  hold. Then, the conditions of Theorem 3.1 are satisfied with  $\lambda = 1$ .

A somewhat stronger but easier to verify constraint qualification is the following

$CQ'_n$  There exist positive constants  $\epsilon, \delta, K_u$  and a control  $\hat{u} \in \mathcal{U}$  such that

$$\|f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))\| \leq K_u, \quad (9)$$

and

$$\zeta \cdot [f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] < -\delta, \quad (10)$$

for all  $\zeta \in \partial_x^> h(s, \bar{x}(s))$ , a.e.  $t, s \in (\tau - \epsilon, \tau] \cap [0, 1]$ , and for all

$$\tau \in \{\sigma \in [0, 1] : h(\sigma, \bar{x}(\sigma)) = 0\}.$$

Note that in this case, the constraint qualification becomes independent of the multipliers, making the condition  $CQ'_n$  much easier to verify *a priori* than  $CQ_n$ . Because the measure  $\mu$  is supported on the set of points where the constraint is active, the value of  $\tau$  defined in  $CQ_n$  must be in the set  $\{\sigma \in [0, 1] : h(\sigma, \bar{x}(\sigma)) = 0\}$ . Furthermore, assuming that the trajectory does not enter and leave the boundary of the state constraints an infinite number of times in finite time,  $CQ'_n$  can be even more simplified by considering  $\epsilon = 0$ . We have proved the following corollary.

**Corollary 4.3.** Assume hypotheses H1–H5 and H2'. Assume also that the constraint qualifications  $CQ$  and  $CQ'_n$  hold. Then, the conditions of Theorem 3.1 are satisfied with  $\lambda = 1$ .

## 5. Normality in calculus of variations

The main result in this section is a normal form of the NCO for the calculus of variations problem (CVP) with pathwise inequality constraints. It improves on the result of [16] by allowing the data to be nonsmooth. It is valid under a condition – a constraint qualification – that is much easier to verify than in the general case of optimal control problems.

Consider the problem

$$(CVP_1) \begin{cases} \text{Minimize} & J[x] = \int_0^1 L(x(t), \dot{x}(t)) dt \\ \text{subject to} & \\ & x(0) = x_0 \\ & h(x(t)) \leq 0 \quad \text{for all } t \in [0, 1]. \end{cases}$$

Assume that the following hypotheses are satisfied:

H1<sub>CV</sub> The function  $x \rightarrow L(x, u)$  is locally Lipschitz continuous for all  $u \in \mathbb{R}^n$ .

H2<sub>CV</sub> The function  $u \rightarrow L(x, u)$  is convex and bounded for all  $x \in \mathbb{R}^n$ .

H3<sub>CV</sub> There exists an increasing function  $\theta : [0, \infty) \rightarrow [0, \infty)$  such that

$$\lim_{\alpha \rightarrow \infty} \frac{\theta(\alpha)}{\alpha} = +\infty,$$

and a constant  $\beta$  such that  $L(x, v) > \theta(\|v\|) - \beta\|v\|$  for all  $x \in \mathbb{R}^n$ ,  $v \in \mathbb{R}^n$ .

H4<sub>CV</sub> There exists a scalar  $K_h > 0$  such that the function  $x \rightarrow h(x)$  is Lipschitz continuous of rank  $K_h$ .

Consider also the following constraint qualification:

CQ<sub>CV</sub> There exist positive constants  $\delta$ , and  $\varepsilon$  such that

- If  $h(\bar{x}(0)) = 0$ , then for all  $x_1, x_2 \in \{x_0\} + \varepsilon \mathbb{B}$ 

$$\gamma_1 \cdot \gamma_2 > \delta, \quad (11)$$
for all  $\gamma_1 \in \partial_x^> h(x_1)$  and all  $\gamma_2 \in \partial_x^> h(x_2)$ .

- For all  $\tau \in \{s : h(\bar{x}(s)) = 0\}$  and for all  $x_1, x_2 \in \{\bar{x}(s) : s \in (\tau - \varepsilon, \tau] \cap [0, 1]\}$ 

$$\gamma_1 \cdot \gamma_2 > \delta, \quad (12)$$
for all  $\gamma_1 \in \partial_x^> h(x_1)$  and all  $\gamma_2 \in \partial_x^> h(x_2)$ .

**Theorem 5.1.** Let  $(\bar{x}, \bar{u})$  be a local minimizer for (CVP<sub>1</sub>). Assume that hypotheses H1<sub>CV</sub>–H4<sub>CV</sub> and CQ<sub>CV</sub> are satisfied. Then, there exist  $p \in W^{1,1}([0, 1] : \mathbb{R}^n)$ , a measurable function  $\gamma$  and a nonnegative Radon measure  $\mu \in C^*([0, 1], \mathbb{R})$  such that

$$\dot{p}(t) \in \text{co } \partial_x^L L(\bar{x}(t), \dot{\bar{x}}(t)) \quad \text{and} \quad q(t) \in \text{co } \partial_u^L L(\bar{x}(t), \dot{\bar{x}}(t)), \quad (13)$$

$$q(1) = 0, \quad (14)$$

$$\gamma(t) \in \partial_x^> h(\bar{x}(t)) \quad \mu\text{-a.e.}, \quad (15)$$

$$\text{supp}\{\mu\} \subset \{t \in [0, 1] : h(\bar{x}(t)) = 0\}, \quad (16)$$

where

$$q(t) = \begin{cases} p(t) + \int_{[0,t)} \gamma(s) \mu(ds), & t \in [0, 1) \\ p(1) + \int_{[0,1]} \gamma(s) \mu(ds), & t = 1. \end{cases}$$

**Remark 5.2.** A major feature of this result is the nonexistence of a multiplier  $\lambda$  associated with the objective function (i.e.  $\lambda = 1$ ). In the case when  $h$  is continuously differentiable, the set  $\partial_x^> h(\bar{x}(s))$  is a singleton. Therefore, the constraint qualification reduces to  $h_x(\bar{x}(s)) \neq 0$ , confirming the CQ and the result in [16].

## 6. Comparison of normal forms of NCO applied to CVP

The normal form of NCO for CVP ([Theorem 5.1](#)) is a consequence of the normal form of the MP for OCP, using [Corollary 4.3](#) and the fact that CVP can be seen as a particular case of OCP.

For that it is enough to consider a new absolutely continuous state variable

$$z(t) = \int_0^t L(x(s), \dot{x}(s)) ds$$

and a change of variable  $\dot{x}(t) = u$ .

The (CVP<sub>1</sub>) can then be written as:

$$(OCP_2) \left\{ \begin{array}{l} \text{Minimize } z(1) \\ \text{subject to} \\ \dot{\eta}(t) = f(\eta(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\ (x(0), z(0)) = (x_0, 0) \\ u(t) \in \mathbb{R}^n \\ h(x(t)) \leq 0 \quad \text{for all } t \in [0, 1] \end{array} \right.$$

with  $\eta(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$  and  $f(\eta(t), u(t)) = \begin{pmatrix} u(t) \\ L(x(t), u(t)) \end{pmatrix}$ .

Since the special structure of CVP permits the derivation of CQ that can be much easier to verify than in the optimal control case, the following question arises: if we apply the normal form of the MP, valid under a CQ that no longer involves the minimizing  $\bar{u}$ , for a CVP, do we have weaker CQ?

Normal forms of MP for OCP valid under a CQ that no longer involves the minimizing  $\bar{u}$  appear in [\[17,19\]](#). Such CQ are typically of the form<sup>1</sup>:

$CQ''_{\eta} \exists \epsilon > 0$  and  $\hat{u}(t) \in \mathcal{U}$ :

$$h_x(\bar{x}(t)) \cdot f(\bar{x}(t), \hat{u}(t)) < -\delta,$$

for  $t \in (s - \epsilon, s + \epsilon)$  where  $s \in \{t \in [0, 1] : h(\bar{x}(t)) = 0\}$ .

Applying the constraint qualification  $CQ''_{\eta}$  to (OCP<sub>2</sub>), we conclude that  $\exists \hat{u} \in \mathbb{R}^n$  such that

$$h_{\eta}(\bar{x}) \cdot f((\bar{x}, z), \hat{u}) < -\delta,$$

for a constant  $\delta > 0$ .

Consequently,

$$(h_x(\bar{x}), 0) \cdot \begin{pmatrix} \hat{u} \\ L(\bar{x}, \hat{u}) \end{pmatrix} < -\delta.$$

Considering  $\hat{u}(t) = -h_x(\bar{x}(t))$ , we have  $h_x(\bar{x}) \cdot (-h_x(\bar{x})) = -\|h_x(\bar{x})\|^2$ .

It follows that, for CVP, the constraint qualification  $CQ''_{\eta}$  reduces to

$$h_x(\bar{x}) \neq 0.$$

Comparing this CQ with the  $CQ_{CV}$ , we conclude that the latter is more general; it can be applied to problems with less regularity on the data.

In summary, we can say that, in the case of optimal control problems, the NCO of [\[17,19\]](#), when compared with the results here, do not involve the control function explicitly, and therefore are easier to verify. However, in the special case of calculus of variations problems, the  $CQ_{CV}$  proposed here and the corresponding result (obtained from the results in [Corollary 4.3](#) for OCP) can be applied to a wider class of problems, requiring less regularity.

## 7. Proof of Theorem 4.2

Expanding the internal product and applying a well-known nonsmooth calculus rule (see [\[23, Proposition 2.3.3\]](#)) to the adjoint inclusion (2), we obtain

$$-\dot{p}(t) \in \text{co } \partial_x^L \left( \sum_{i=1}^n q_i(t) f_i(t, \bar{x}(t), \bar{u}(t)) \right) \subset \sum_{i=1}^n q_i(t) \text{co } \partial_x^L f_i(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t \in [0, 1].$$

<sup>1</sup> In [\[17\]](#), this CQ also has to be satisfied on a neighborhood of the state constraint boundary.

Define the matrix  $\xi(t) = \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_n(t) \end{bmatrix}$  for some  $\xi_i(t) \in \text{co } \partial_{x_i}^L f_i(t, \bar{x}(t), \bar{u}(t))$  conveniently selected such that

$$-\dot{p}(t) = q(t) \cdot \xi(t) \quad \text{a.e. } t \in [0, 1].$$

It follows that

$$p(t) = p(1) + \int_t^1 q(s) \xi(s) ds$$

or equivalently

$$q(t) = q(1) + \int_t^1 q(s) \xi(s) ds - \int_{[t,1]} \gamma(s) \mu(ds).$$

We can establish the following necessary conditions of optimality:

if  $(\bar{x}, \bar{u})$  is an optimal process, then there exist a function  $q$  of bounded variation and continuous from left, a scalar  $\lambda \geq 0$ , and a nonnegative Radon measure  $\mu \in \mathcal{C}^*([0, 1], \mathbb{R})$  such that

$$\mu\{(0, 1]\} + \|q\|_\infty + \lambda > 0, \quad (17)$$

$$q(t) = q(1) + \int_t^1 q(s) \xi(s) ds - \int_{[t,1]} \gamma(s) \mu(ds),$$

where  $\gamma(t) \in \partial_x^> h(t, \bar{x}(t))$   $\mu$ -a.e.,

$$-q(1) \in \lambda \partial_x^L g(\bar{x}(1)),$$

$$\text{supp}\{\mu\} \subset \{t \in [0, 1] : h(t, \bar{x}(t)) = 0\},$$

and

$$q(t) \cdot [f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))] \leq 0, \quad (18)$$

for all  $u \in \Omega(t)$ , a.e.  $t \in [0, 1]$ .

Now suppose in contradiction that  $\lambda = 0$ . In this case we can write  $q(1) = 0$  and

$$q(t) = \int_t^1 q(s) \xi(s) ds - \int_{[t,1]} \gamma(s) \mu(ds).$$

Let  $\tau = \inf\{t \in [0, 1] : \int_{[t,1]} \mu(ds) = 0\}$ . If  $\tau = 0$ , then  $\int_{(0,1]} \mu(ds) = 0$ . This implies that  $q(t) = 0$  for all  $t \in [0, 1]$ . Hence  $\mu\{(0, 1]\} + \|q\|_\infty + \lambda = 0$  and we arrive at a contradiction with the nontriviality condition (17).

It remains to consider the case when  $\tau > 0$ . We show that when  $\lambda = 0$  and  $\text{CQ}_n$  is verified, the maximization condition (18) cannot be satisfied.

Defining  $\Phi(t, s)$  as the transition matrix for the linear system  $\dot{z}(t) = \xi(t)z(t)$ , the function  $q$  can be written as

$$q(t) = - \int_{[t,1]} \gamma(s) \Phi(s, t) \mu(ds).$$

Let  $\Delta f(t, \bar{x}(t)) = f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))$ , where  $\hat{u}$  is the control function chosen in  $\text{CQ}_n$  for  $t \in (\tau - \epsilon, \tau] \cap [0, 1]$  and is equal to  $\bar{u}$  a.e. on  $[\tau, 1]$ . We have

$$\begin{aligned} q(t) \cdot \Delta f(t, \bar{x}(t)) &= - \int_{[t,1]} \gamma(s) \Phi(s, t) \Delta f(t, \bar{x}(t)) \mu(ds) \\ &= - \int_{[t,\tau]} \gamma(s) \Phi(s, t) \Delta f(t, \bar{x}(t)) \mu(ds) \\ &= - \int_{[t,\tau]} \gamma(s) \Delta f(t, \bar{x}(t)) \mu(ds) - \int_{[t,\tau]} \gamma(s) [\Phi(s, t) - \Phi(\tau, \tau)] \Delta f(t, \bar{x}(t)) \mu(ds) \\ &> \delta \mu\{[t, \tau]\} - \int_{[t,\tau]} \gamma(s) [\Phi(s, t) - \Phi(\tau, \tau)] \Delta f(t, \bar{x}(t)) \mu(ds). \end{aligned}$$

As  $\Phi$  is continuous we can assure the existence of a positive scalar  $\delta_1$  such that  $\|\Phi(s, t) - \Phi(\tau, \tau)\| < \frac{\delta}{2K_u K_h}$  for all  $(s, t)$  satisfying  $\|(s, t) - (\tau, \tau)\| < \delta_1$ . Hence, for a.e.  $t \in (\tau - \epsilon, \tau] \cap (\tau - \delta_1, \tau]$  we have

$$q(t) \cdot \Delta f(t, \bar{x}(t)) > \delta \mu\{[t, \tau]\} - \frac{\delta}{2} \mu\{[t, \tau]\} > 0$$

contradicting the maximization condition (18).  $\square$

## 8. Proof of Theorem 5.1

As mentioned before, here we discuss the normality results of OCP, in the particular case of CVP. Therefore, we start by seeing  $(CVP_1)$  as a special case of  $(OCP_1)$ . For that, it is enough to consider a new absolutely continuous state variable

$$z(t) = \int_0^t L(x(s), \dot{x}(s)) ds$$

and the dynamics  $\dot{x}(t) = u$ .

The  $(CVP_1)$  can then be written as  $(OCP_2)$ . Recalling the problem formulation

$$(OCP_2) \begin{cases} \text{Minimize} & z(1) \\ \text{subject to} & \dot{\eta}(t) = f(\eta(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\ & (x(0), z(0)) = (x_0, 0) \\ & u(t) \in \mathbb{R}^n \\ & h(x(t)) \leq 0 \quad \forall t \in [0, 1] \end{cases}$$

with  $\eta(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$  and  $f(\eta(t), u(t)) = \begin{pmatrix} u(t) \\ L(x(t), u(t)) \end{pmatrix}$ .

In order to apply Corollary 4.3, we need to verify CQ and  $CQ'_\eta$ . In Step 1 of the proof, below, we show that these constraint qualifications are implied by  $CQ_{CV}$ . In Step 2, we apply Corollary 4.3 to  $(OCP_2)$  and thereby obtain the assertions (13)–(16) of Theorem 5.1.

**Step 1.** We start by establishing regularity of the minimizers for  $(CVP_1)$ . Although we consider that the minimizers are arcs, we can conclude that they are actually Lipschitz continuous functions.

**Lemma 8.1** (See Theorem 11.5.1 [26]). *Let  $\bar{x}$  be a local minimizer for  $(CVP_1)$ . Assume that hypotheses  $H1_{CV}$ – $H4_{CV}$  are satisfied. Then  $\bar{x}$  is a Lipschitz continuous function.*

The proof of this lemma can be found in [26, pp. 422–425].

So, we have that the minimizer  $\bar{x}$  for  $(CVP_1)$  is a Lipschitz continuous function, or equivalently

$$\|\dot{\bar{x}}(t)\| \leq K_1 \quad \text{a.e. } t \in [0, 1], \quad (19)$$

where  $K_1$  is any number strictly greater than  $\|\dot{\bar{x}}\|_\infty$ .

By hypothesis, we have that the function  $u \rightarrow \bar{L}(\cdot, u)$  is bounded and having in consideration condition (19), we conclude that condition (4) is satisfied.

Let us define the following function and sets:

$$\begin{aligned} \Delta f_{\eta, \bar{u}}(t) &= [f(\eta(t), \hat{u}(t)) - f(\eta(t), \bar{u}(t))], \\ A_\varepsilon &= \{x_0 + \varepsilon \mathbb{B} : h(x_0) = 0\}, \\ \Gamma &= \{\tau \in [0, 1] : h(\bar{x}(\tau)) = 0\}, \\ B_\varepsilon(\tau) &= \{\bar{x}(s) : s \in (\tau - \varepsilon, \tau] \cap [0, 1]\}. \end{aligned}$$

Note that  $CQ_{CV}$  can be decomposed into the following conditions:

$CQ_{CV1}$  There exist positive scalars  $\delta$  and  $\varepsilon$  such that for all  $x_1, x_2 \in A_\varepsilon$ ,

$$\gamma_1 \cdot \gamma_2 > \delta, \quad (20)$$

$$\forall \gamma_1 \in \partial_x^> h(x_1) \text{ and } \forall \gamma_2 \in \partial_x^> h(x_2).$$

$CQ_{CV2}$  There exist positive scalars  $\delta$  and  $\varepsilon$  such that for all  $\tau \in \Gamma$  and all  $x_1, x_2 \in B_\varepsilon(\tau)$ ,

$$\gamma_1 \cdot \gamma_2 > \delta, \quad (21)$$

$$\forall \gamma_1 \in \partial_x^> h(x_1) \text{ and } \forall \gamma_2 \in \partial_x^> h(x_2).$$

On the other hand,  $CQ_{CV1}$  implies that

$CQ'_{CV1}$   $\exists \delta > 0, \varepsilon > 0$  and  $\exists x_2 \in A_\varepsilon$  such that  $\forall x_1 \in A_\varepsilon$ ,

$$\gamma_1 \cdot \gamma_2 > \delta, \quad (22)$$

$$\forall \gamma_1 \in \partial_x^> h(x_1) \text{ and } \forall \gamma_2 \in \partial_x^> h(x_2).$$



We would like to stress that  $\text{CQ}_{\text{CV1}}$  and  $\text{CQ}'_{\text{CV1}}$  are trivially satisfied in the case of  $h(x_0) \neq 0$ , and so is  $\text{CQ}$ . We now show that  $\text{CQ}'_{\text{CV1}}$  implies (6) when  $h(x_0) = 0$ .

Let  $\gamma_\eta$  be an element of the hybrid partial subdifferential

$$\gamma_\eta \in \partial_\eta^> h(x_1).$$

From the definition, we have

$$\partial_\eta^> h(x_1) = \text{co} \{(\varepsilon_1, \varepsilon_2) : \text{exist } x_i \rightarrow x_1 \text{ such as } h(x_i) > 0 \forall i, h(x_i) \rightarrow h(x_1) \text{ and } h_\eta(x_i) \rightarrow (\varepsilon_1, \varepsilon_2)\}$$

with  $x_1 \in \{x_0\} + \varepsilon\mathbb{B}$ .

Since  $h_z(x_i) = 0$ , we have

$$(h_x(x_i), 0) \rightarrow (\varepsilon_1, \varepsilon_2).$$

It follows that,

$$\gamma_\eta \in (\text{co} \{\varepsilon_1 : \text{exist } x_i \rightarrow x_1 \text{ such as } h(x_i) > 0 \forall i, h(x_i) \rightarrow h(x_1) \text{ and } h_x(x_i) \rightarrow \varepsilon_1\}, 0).$$

We conclude that

$$\gamma_\eta = (\gamma_1, 0), \quad \text{where } \gamma_1 \in \partial_x^> h(x_1)$$

with  $x_1 \in \{x_0\} + \varepsilon\mathbb{B}$ .

Therefore

$$\gamma_\eta \cdot \Delta f_{\eta, \hat{u}}(t) = \gamma_1 \cdot (\hat{u}(t) - \dot{\bar{x}}(t)).$$

So, in the context of  $(\text{OCP}_2)$ , condition (6) is equivalent to: if  $h(x_0) = 0$ , then there exist positive constants  $\varepsilon, \varepsilon_1, \delta$  and a control  $\hat{u} : [0, \varepsilon] \rightarrow \mathbb{R}^m$  such that

$$\gamma_1 \cdot [\hat{u}(t) - \dot{\bar{x}}(t)] < -\delta, \quad (23)$$

for all  $\gamma_1 \in \partial_x^> h(x_1)$ ,  $x_1 \in x_0 + \varepsilon_1\mathbb{B}$ .

If inequality (22) holds, then we can consider the control function  $\hat{u}(t) = \dot{\bar{x}}(t) - \varrho_1 \gamma_2$ , where  $\gamma_2 \in \partial_x^> h(x_2)$  and  $\varrho_1$  is a positive number chosen small enough to ensure that  $\|\hat{u}(t)\| \leq K_2$ , for all  $t \in [0, 1]$ , where  $K_2$  is constant. We have

$$\gamma_\eta \cdot \Delta f_{\eta, \hat{u}}(t) = \gamma_1 \cdot (-\varrho_1 \gamma_2).$$

From the inequality (22),

$$\gamma_\eta \cdot \Delta f_{\eta, \hat{u}}(t) = -\varrho_1 \gamma_1 \cdot \gamma_2 < -\delta$$

for some  $\delta > 0$  and for any  $x_1 \in \{x_0\} + \varepsilon\mathbb{B}$ , we conclude that the inequality (6) is confirmed.

In a similar way, we prove that if  $\text{CQ}_{\text{CV2}}$  holds, then condition (10) is satisfied.

Let  $T_\varepsilon(\tau) = \{s : s \in (\tau - \varepsilon, \tau] \cap [0, 1]\}$ . Condition  $\text{CQ}_{\text{CV2}}$  is then equivalent to:

$\text{CQ}'_{\text{CV2}} \exists \delta > 0, \varepsilon > 0$  such that  $\forall \tau \in \Gamma$  and  $\forall t_1, t_2 \in T_\varepsilon(\tau)$

$$\gamma_1 \cdot \gamma_2 > \delta, \quad (24)$$

$$\forall \gamma_1 \in \partial_x^> h(\bar{x}(t_1)) \text{ and } \forall \gamma_2 \in \partial_x^> h(\bar{x}(t_2)).$$

Let  $\gamma_\eta$  be an element of the hybrid partial subdifferential

$$\gamma_\eta \in \partial_\eta^> h(\bar{x}(t_1)).$$

From the definition, we have

$$\begin{aligned} \partial_\eta^> h(\bar{x}(t_1)) = \text{co} \{(\varepsilon_1, \varepsilon_2) : \text{exist } s_i \rightarrow t_1 \text{ such as } h(\bar{x}(s_i)) > 0 \forall i, \\ h(\bar{x}(s_i)) \rightarrow h(\bar{x}(t_1)) \text{ and } h_\eta(\bar{x}(s_i)) \rightarrow (\varepsilon_1, \varepsilon_2)\} \end{aligned}$$

with  $t_1 \in (\tau - \varepsilon, \tau] \cap [0, 1]$ , such that  $h(\bar{x}(\tau)) = 0$ .

Since  $h_z(\bar{x}(s_i)) = 0$ , we have

$$(h_x(\bar{x}(s_i)), 0) \rightarrow (\varepsilon_1, \varepsilon_2).$$

It follows that,

$$\gamma_\eta \in (\text{co} \{\varepsilon_1 : \text{exist } s_i \rightarrow t_1 \text{ such as } h(\bar{x}(s_i)) > 0 \forall i, h(\bar{x}(s_i)) \rightarrow h(\bar{x}(t_1)) \text{ and } h_x(\bar{x}(s_i)) \rightarrow \varepsilon_1\}, 0).$$

We conclude that

$$\gamma_\eta = (\gamma_1, 0), \quad \text{for all } \gamma_1 \in \partial_x^> h(\bar{x}(t_1))$$

with  $t_1 \in (\tau - \varepsilon, \tau] \cap [0, 1]$  such that  $h(\bar{x}(\tau)) = 0$ .

Therefore,

$$\gamma_\eta \cdot \Delta f_{\eta, \hat{u}}(t) = \gamma_1 \cdot (\hat{u}(t) - \dot{\bar{x}}(t)).$$

So, in the context of (OCP<sub>2</sub>), condition (10) is equivalent to: there exist positive constants  $\varepsilon$ ,  $\delta$  and a control function  $\hat{u}$  such that

$$\gamma_1 \cdot [\hat{u}(t_2) - \dot{\bar{x}}(t_2)] < -\delta, \quad \forall \gamma_1 \in \partial_x^> h(\bar{x}(t_1)) \quad (25)$$

for a.e.  $t_1, t_2 \in T_\varepsilon(\tau)$  and all  $\tau \in \Gamma$ .

If inequality (24) holds, then for any  $\tau \in \Gamma$  and any  $t_2 \in T_\varepsilon(\tau)$  we can consider the control  $\hat{u}(t_2) = \dot{\bar{x}}(t_2) - \varrho_2 \gamma_2$ , for some  $\gamma_2 \in \partial_x^> h(\bar{x}(t_2))$  where  $\varrho_2$  is a positive number chosen small enough to ensure that  $\|\hat{u}(t_2)\| \leq K_3$ , with  $K_3$  constant. We have

$$\gamma_\eta(t_1) \cdot \Delta f_{\eta, \hat{u}}(t_2) = \gamma_1 \cdot (-\varrho_2 \gamma_2).$$

From inequality (24),

$$\gamma_\eta(t_1) \cdot \Delta f_{\eta, \hat{u}}(t_2) = -\varrho_2 \gamma_1 \cdot \gamma_2 < -\delta$$

for some  $\delta > 0$  and any  $t_1, t_2 \in T_\varepsilon$ , we conclude that inequality (10) holds.

Since  $\|\hat{u}(t)\| \leq K$  for some positive, sufficiently small  $\varrho_1$  and  $\varrho_2$ , and the function  $L(x(t), \dot{x}(t))$  is bounded, the conditions (5) and (9) are satisfied.

Consequently, we can apply Corollary 4.3.

Step 2. Let  $((\bar{x}(t), \bar{z}(t)), \bar{u}(t))$  be a local minimizer for (OCP<sub>2</sub>). Then there exist two absolutely continuous functions,  $p_1 : [0, 1] \rightarrow \mathbb{R}^n$  and  $p_2 : [0, 1] \rightarrow \mathbb{R}$ , a measurable function  $\gamma$  and a nonnegative Radon measure  $\mu \in C^*([0, 1], \mathbb{R})$  such that

$$(-\dot{p}_1(t), -\dot{p}_2(t)) \in \text{co} \left( \partial_\eta^L((q_1(t), q_2(t)) \cdot f(\bar{\eta}(t), \bar{u}(t))) \right) \quad \text{a.e. } t \in [0, 1], \quad (26)$$

$$(-q_1(1), -q_2(1)) \in \partial_\eta^L \bar{z}(1), \quad (27)$$

$$\gamma(t) \in \partial_x^> h(\bar{x}(t)) \quad \mu\text{-a.e.}, \quad (28)$$

$$\text{supp}\{\mu\} \subset \{t \in [0, 1] : h(\bar{x}(t)) = 0\}, \quad (29)$$

a.e.  $t \in [0, 1]$ ,  $\bar{u}(t)$  maximize

$$u \rightarrow (q_1(t), q_2(t)) \cdot f(\bar{\eta}(t), u), \quad (30)$$

where

$$q_1(t) = \begin{cases} p_1(t) + \int_{[0,t)} \gamma(s) \mu(ds), & t \in [0, 1) \\ p_1(1) + \int_{[0,1]} \gamma(s) \mu(ds), & t = 1 \end{cases}$$

and

$$q_2(t) = p_2(t), \quad t \in [0, 1]. \quad (31)$$

Applying a nonsmooth rule to the transversality condition (27), we obtain

$$(-q_1(1), -q_2(1)) \in \partial_x^L \bar{z}(1) \times \partial_z^L \bar{z}(1). \quad (32)$$

that implies:

$$\begin{cases} q_1(1) = 0 \\ -q_2(1) = 1 \end{cases} \iff \begin{cases} p_1(1) + \int_{[0,1]} \gamma(s) \mu(ds) = 0 \\ p_2(1) = -1. \end{cases} \quad (33)$$

On the other hand, we can write condition (26), as

$$(-\dot{p}_1(t), -\dot{p}_2(t)) \in \text{co} \partial_\eta^L (q_1(t) \cdot \bar{u}(t) + q_2(t) \cdot L(\bar{x}(t), \bar{u}(t))).$$

Using a nonsmooth calculus propriety, it follows that

$$(-\dot{p}_1(t), -\dot{p}_2(t)) \in q_2(t) \text{co} \partial_\eta^L (L(\bar{x}(t), \bar{u}(t))).$$

Consequently, we have

$$(-\dot{p}_1(t), -\dot{p}_2(t)) \in \{q_2(t) \text{co} \partial_x^L (L(\bar{x}(t), \bar{u}(t))) \times 0\},$$

hence,

$$\begin{cases} -\dot{p}_1(t) \in q_2(t) \operatorname{co} \partial_x^L (L(\bar{x}(t), \bar{u}(t))) \\ \dot{p}_2(t) = 0. \end{cases} \quad (34)$$

By (31), (33) and (34), we know that  $q_2(t) = -1$ , then the condition (30) can be written as:  
 $\dot{\bar{x}}(t)$  maximize

$$u \rightarrow \left( p_1(t) + \int_{[0,t)} \gamma(s) \mu(ds) \right) \cdot u - L(\bar{x}(t), u)$$

a.e.  $t \in [0, 1]$ .

H2<sub>CV</sub> is satisfied, the “maximization of the Hamiltonian” property implies

$$0 \in \operatorname{co} \partial_u^L \left( \left( p_1(t) + \int_{[0,t)} \gamma(s) \mu(ds) \right) \cdot \dot{\bar{x}} - L(\bar{x}(t), \dot{\bar{x}}(t)) \right).$$

By a nonsmooth calculus property, we have

$$\left( p_1(t) + \int_{[0,t)} \gamma(s) \mu(ds) \right) \in \operatorname{co} \partial_u^L (L(\bar{x}(t), \dot{\bar{x}}(t))). \quad (35)$$

The conditions (28), (29) and (33)–(35) may be assembled to give the assertions (13)–(16) of the theorem.  $\square$

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