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The roles of diffusivity and curvature in patterns on surfaces of revolution

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ABSTRACT

We address the question of finding sufficient conditions for existence as well as nonexistence of nonconstant stable stationary solution to the diffusion equation $u_t = \mathrm{div}(a\nabla u) + f(u)$ on a surface of revolution with and without boundary. Conditions found relate the diffusivity function a and the geometry of the surface where diffusion takes place. In the case where f is a bistable function, necessary conditions for the development of inner transition layers are given.

1. Introduction

The main concern in this paper is to find sufficient conditions for existence as well as nonexistence of nonconstant stable stationary solutions (herein referred to as *patterns*, for short) to the diffusion problem

$$u_t = \mathrm{div}_g\big(a(x)\nabla_g u\big) + f(u), \quad (t,x) \in \mathbb{R}^+ \times \mathcal{M} \tag{1.1}$$

where $\mathcal{M} \subset \mathbb{R}^3$ is a surface of revolution without boundary with metric g. The case where $\mathcal{M}$ has boundary will also be treated. The function a is smooth and positive and f is a function in $C^1(\mathbb{R})$, sometimes considered of bistable type.

This kind of problem appears as a mathematical model in many distinct areas and, roughly speaking, a solution models the time evolution of the concentration of a diffusing substance in a heterogeneous medium whose diffusivity is given by a positive function a, under the effect of a source or sink term f. Usually the diffusivity is a property of the material which the surface is made of.

Our concern herein is to find mechanisms of interaction between the diffusivity function a and the geometry of the domain so as to produce patterns to the problem (1.1) as well as those which do not produce patterns.

There is a vast literature addressing the question of nonexistence as well as existence of patterns to (1.1) in bounded domains of $\mathbb{R}^n$ when diffusivity is constant. It seems to have been first addressed in [3] and [15] for problems with Neumann boundary condition where it was proved that, for the case of constant diffusivity, no pattern exists if the domain is convex. If a is a constant function, nonexistence of patterns to (1.1) on a Riemannian manifold without boundary with nonnegative Ricci curvature was proved in [7], thus generalizing a similar result for surface of revolution found in [17]. In particular, if $\mathcal{M}$ is a surface of revolution the authors in [1] show that there are no patterns if the sum of the Gaussian curvature in every point p and the square of the geodesic curvature of the parallel passing through p is nonnegative.

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For bounded domains in $\mathbb{R}^N$ the question of how the diffusivity function can give rise to patterns, or not, has been considered by some authors.

For one-dimensional domains, i.e., when $\mathcal{M}$ is an interval, subjected to zero Neumann boundary condition, a sufficient condition for nonexistence of patterns was found to be $a'' < 0$ in [4] and $(\sqrt{a})'' < 0$ in [19]. In domains with dimension $N \geqslant 2$ this remains an open problem.

Still regarding one-dimensional domains, existence of a diffusivity function a which gives rise to patterns to (1.1) was addressed in [10,12]. These results were generalized to two-dimensional domains in [5] and for any dimension in [6].

Let us briefly mention our main results. To this end consider a smooth curve C in $\mathbb{R}^3$ parametrized by $x = (x_1, x_2, x_3) = (\psi(s), 0, \chi(s))$, $s \in [0, l]$ with $\psi(0) = \psi(l) = 0$ and the borderless surface of revolution $\mathcal{M}$ generated by C. We suppose that the diffusivity function does not depend on the angular variable θ, so that, abusing notation, we set $a(x(s,\theta)) = a(s)$.

Then regarding nonexistence of patterns to (1.1) a sufficient condition is found to be

$$K + (K_g)^2 \geqslant \frac{(a'\psi)'}{2a\psi} \quad \text{in } (0, l)$$

where K stands for the Gaussian curvature and K_g for the geodesic curvature of $\mathcal{M}$.

Note that this generalizes [4] since $\mathcal{M}$ with border under zero Neumann boundary condition is also allowed. This can be seen by taking $\psi \equiv 1$, which would correspond to a finite right circular cylinder, and then the nonexistence condition for patterns would read $a'' \leqslant 0$, as found in [4].

As for existence of patterns, after introducing a positive small parameter in the equation, we found that a sufficient condition is that the function

- $\sqrt{a}\psi$ has a isolated local minimum somewhere in $(0, l)$,

provided f is of bistable type and satisfies the equal-area condition ($f(u) = u - u^3$, for instance). In particular, if $a \equiv$ constant then the sufficient condition is satisfied as long as, roughly speaking, $\mathcal{M}$ has a neck.

The geometric profile of these patterns are also given. All these results remain true for a surface of revolution with border under Neumann boundary condition.

Many examples of surfaces satisfying both conditions, namely, for existence as well as for nonexistence of patterns are given.

This paper is divided as follows. In Section 2 we recall some material from stability of solution, differential geometry and function of bounded variation. In Section 3 we will extend the nonexistence result of patterns given in [1] to the case where a is nonconstant (see Remark 3.3(i)). In Section 4 we introduce a parameter $\epsilon > 0$ in the problem (1.1) and give sufficient conditions for existence of a family of stable stationary solution $\{v_\epsilon\}_{0<\epsilon<\epsilon_0}$, for some $\epsilon_0 > 0$, using Γ-convergence techniques.

In order to utilize Γ-convergence results f has to be a function of bistable type that satisfies the equal-area condition, sometimes also referred to as f being balanced. In Section 6 we prove that this condition is actually necessary in our approach.

2. Preliminaries

We begin with some definitions and known results from Differential Geometry which will be used in the following sections.

2.1. Surface of revolution

Consider $M = (\mathcal{M}, g)$ an n-dimensional Riemannian manifold with a metric given in local coordinates $x = (x^1, x^2, \ldots, x^n)$ given by (using Einstein summation convention)

$$ds^2 = g_{ij}\, dx^i\, dx^j, \qquad \left(g^{ij}\right) = \left(g_{ij}^{-1}\right), \qquad |g| = \det(g_{ij}).$$

Given a smooth vector field X on $\mathcal{M}$, the divergence operator of X is defined as

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|} X^i\right)$$

and the Riemannian gradient, denoted by ∇_g, of a sufficiently smooth real function ϕ defined on $\mathcal{M}$, as the vector field

$$(\nabla_g \phi)^i = g^{ij} \partial_j \phi.$$

We will see how the operator $\operatorname{div}_g(a(x)\nabla_g u)$ can be expressed for the particular case where $\mathcal{M}$ is a surface of revolution. Let C be the curve of $\mathbb{R}^3$ parametrized by

$$\begin{cases} x_1 = \psi(s) \\ x_2 = 0 \\ x_3 = \chi(s) \end{cases} \quad \bigl(s \in I := [0, l]\bigr)$$

where $\psi, \chi \in C^2(I)$, $\psi > 0$ in $(0, l)$ and $(\psi')^2 + (\chi')^2 = 1$ in I. Moreover,

$$\psi(0) = \psi(l) = 0, \tag{2.1}$$

and

$$\psi'(0) = -\psi'(l) = 1. \tag{2.2}$$

Let $\mathcal{M}$ be the surface of revolution parametrized by

$$\begin{cases} x_1 = \psi(s)\cos(\theta) \\ x_2 = \psi(s)\sin(\theta) \\ x_3 = \chi(s) \end{cases} \quad (s, \theta) \in [0, l] \times [0, 2\pi). \tag{2.3}$$

Set $x^1 = s$, $x^2 = \theta$, then the surface of revolution in $\mathbb{R}^3$ with the above parametrization is a 2-dimensional Riemannian manifold with metric

$$g = ds^2 + \psi^2(s)\, d\theta^2.$$

It follows from (2.1) and (2.2) that $\mathcal{M}$ has no boundary and we always assume that $\mathcal{M}$ and the Riemannian metric g on it are smooth (see [2], for instance). The area element on $\mathcal{M}$ is $d\sigma = \psi\, d\theta\, ds$ and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(\partial_s u, \frac{1}{\psi^2}\partial_\theta u\right).$$

Although the diffusivity function a may depend on (s, θ), throughout this work we suppose that it depends just on the variable s. Thus abusing notation, for simplicity sake, we set

$$a\bigl(x(s, \theta)\bigr) = a(s), \quad \text{for } x = \bigl(\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)\bigr) \in \mathcal{M} \tag{2.4}$$

and therefore

$$\operatorname{div}_g\bigl(a(x)\nabla_g u\bigr) = a u_{ss} + \frac{(\psi a)_s}{\psi} u_s + \frac{a}{\psi^2} u_{\theta\theta}. \tag{2.5}$$

Hence throughout this text problem (1.1) on $\mathcal{M}$ reduces to

$$u_t = a u_{ss} + \frac{(\psi a)_s}{\psi} u_s + \frac{a}{\psi^2} u_{\theta\theta} + f(u), \quad (s, \theta) \in (0, l) \times [0, 2\pi). \tag{2.6}$$

Note that the Gaussian curvature of $\mathcal{M}$ is given by

$$K(s) = \frac{-\psi''(s)}{\psi(s)} \quad \bigl(s \in (0, l)\bigr) \tag{2.7}$$

and, for future reference,

$$K_g(s) = \frac{\pm\psi'}{\psi} \quad \bigl(s \in (0, l)\bigr)$$

represents the *geodesic curvature* of the parallel circles $s = \text{constant}$ on $\mathcal{M}$. Here the sign depends on the orientation of the parametrization.

2.2. Stability analysis

By a stationary solution of problem (1.1) we mean a solution to the problem

$$\operatorname{div}_g\bigl(a(x)\nabla_g u\bigr) + f(u) = 0, \quad x \in \mathcal{M} \tag{2.8}$$

or equivalently, in our setting, a solution to (2.6) which does not depend on time. A stationary solution U of (2.6) is called *stable* (in the sense of Lyapunov) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|u(\cdot, t) - U\|_\infty < \epsilon$ for all $t > 0$, whenever $\|u_0 - U\|_\infty < \delta$, where $\|\cdot\|_\infty$ stands for the norm of the space $L^\infty(\mathcal{M})$. If there exists $\delta_1 > 0$ such that $\|u_0 - U\|_\infty < \delta_1$ implies that $\|u(\cdot, t) - U\|_\infty \to 0$, as $t \to \infty$, then U is called *asymptotically stable*. We say that U is *unstable* if it is not stable.

Regarding the linearized eigenvalue problem

$$\operatorname{div}_g(a\nabla_g \phi) + f'(U)\phi + \lambda\phi = 0 \quad \text{in } \mathcal{M}, \tag{2.9}$$

the first eigenvalue λ_1 is given by

$$\lambda_1 = \min\{R_U(\phi): \phi \in H^1(\mathcal{M}),\ \|\phi\|_{L^2(\mathcal{M})} = 1\} \tag{2.10}$$

where

$$R_U(\phi) = \int_{\mathcal{M}} \{a|\nabla_g\phi|^2 - f'(U)\phi^2\}\,d\sigma.$$

It is well known that if $\lambda_1 > 0$ then U is asymptotically stable and if $\lambda_1 < 0$ then U is unstable. If $\lambda_1 = 0$ then stability or instability can occur.

2.3. BV-functions and Γ-convergence

We say that v is a function of *essential bounded variation in an interval* $I \subset \mathbb{R}$ (and write $v \in BV(I)$) if its partial derivatives in the sense of distributions are measures with finite total variation in I. In the sense of distribution Dv is a vector valued Radon measure with finite total variation in I given by

$$|Dv| = \sup\left\{\int_I v\sigma'\,ds: \sigma \in C_0^\infty(I),\ |\sigma| \leqslant 1\right\}.$$

The total variation $|Dv|$ is a Radon measure itself. We denote by $BV(I;\{\alpha,\beta\})$ the class of all $v \in BV(I)$ which take values α and β only. If $v \in BV(I)$, the integral of any positive continuous function h with respect to the measure $|Dv|$ can be expressed as

$$\int_I h|Dv| = \sup\left\{\int_I v\sigma'\,ds: \sigma \in C_0^\infty(I),\ |\sigma| \leqslant h\right\}.$$

Given $u \in L^1_{loc}(I)$, the *jump set of* u, denoted by S_u, is the complement of the set of Lebesgue points of u, i.e., the set of points where the upper and lower approximate limits of u differ or are not finite. If $u \in BV(I,\{\alpha,\beta\})$ then $\mathcal{H}^0(S_u) < \infty$ and $(\beta-\alpha)\mathcal{H}^0(S_u)$ agrees with the total variation $|Du|$ of the derivative Du. Here $\mathcal{H}^0$ stands for the Hausdorff counting measure.

For details the reader is referred to [8,11], for instance.

Definition 2.1. A family $\{E_\epsilon\}_{\epsilon>0}$ of real-extended functionals defined in $L^1(I)$ is said to *Γ-converge, as* $\epsilon \to 0$, *to a functional* E_0 and we write

$$\Gamma - \lim_{\epsilon\to 0} E_\epsilon(v) = E_0(v)$$

if:

(i) for each $v \in L^1(I)$ and for any sequence $\{v_\epsilon\}$ in $L^1(I)$ such that $v_\epsilon \to v$ in $L^1(I)$, as $\epsilon \to 0$, implies $E_0(v) \leqslant \lim_{\epsilon\to 0}\inf E_\epsilon(v_\epsilon)$;
(ii) for each $v \in L^1(I)$ there is a sequence $\{v_\epsilon\}$ in $L^1(I)$ such that $v_\epsilon \to v$ in $L^1(I)$, as $\epsilon \to 0$, and $E_0(v) \geqslant \lim_{\epsilon\to 0}\sup E_\epsilon(v_\epsilon)$.

Definition 2.2. We shall call $v_0 \in L^1(I)$ a L^1*-local minimizer of* E_0 if there is $\mu > 0$ such that $E_0(v_0) \leqslant E_0(v)$ whenever $0 < \|v - v_0\|_{L^1(I)} < \mu$. Moreover if $E_0(v_0) < E_0(v)$ for $0 < \|v - v_0\|_{L^1(I)} < \mu$, then v_0 is called an *isolated* L^1*-local minimizer of* E_0.

3. Sufficient conditions for nonexistence of patterns

We start this section with a lemma concerning the characterization of stationary solutions of (1.1) under our hypotheses. Next result was observed in [17] for $a \equiv 1$ and for convenience of the reader we will prove it in our case.

Lemma 3.1. *Every stationary solution u of problem (1.1) on $\mathcal{M}$, which depends on the angular variable θ, is unstable.*

Proof. By (2.5) u satisfies the equation

$$au_{ss} + \frac{(\psi a)_s}{\psi}u_s + \frac{a}{\psi^2}u_{\theta\theta} + f(u) = 0.$$

As the function a does not depend on θ we have that u_θ is an eigenfunction of (2.9) with corresponding eigenvalue $\lambda = 0$. Since u_θ must change sign it cannot be the eigenfunction corresponding to the lowest eigenvalue. Hence $\lambda_1 < 0$. □

Next we state and prove the main result of this section.

Theorem 3.2. *If*

$$-\left(\frac{\psi'}{\psi}\right)'(s) \geqslant \frac{(a'\psi)'(s)}{2(a\psi)(s)}, \quad \forall s \in (0,l) \tag{3.1}$$

then every nonconstant stationary solution of (1.1) is unstable.

Proof. If v is a nonconstant stationary solution by Lemma 3.1 we can assume that $v = v(s)$. Thus by (2.5) v satisfies

$$av_{ss} + \frac{(\psi a)_s}{\psi} v_s + f(v) = 0 \quad \text{in } (0,l). \tag{3.2}$$

Differentiating with respect to s and setting $' := \frac{d}{ds}$, we have

$$a'v'' + av''' + \left(\frac{(\psi a)'}{\psi}\right)' v' + \frac{(\psi a)'}{\psi} v'' + f'(v)v' = 0$$

in other words,

$$\operatorname{div}_g\big(a\nabla_g v'\big) + a'v'' + \left(\frac{(\psi a)'}{\psi}\right)' v' + f'(v)v' = 0.$$

Multiplying by v' and integrating over $\mathcal{M}$ we obtain

$$\int_{\mathcal{M}} v' \operatorname{div}_g\big(a\nabla_g v'\big) + a'v''v' + \left(\frac{(\psi a)'}{\psi}\right)' \big(v'\big)^2 + f'(v)\big(v'\big)^2\, d\sigma = 0.$$

It follows that

$$\int_{\mathcal{M}} a\big(v''\big)^2 - f'(v)\big(v'\big)^2\, d\sigma = \int_{\mathcal{M}} v''a'v'\, d\sigma + \int_{\mathcal{M}} \left(\frac{(\psi a)'}{\psi}\right)' \big(v'\big)^2\, d\sigma.$$

Now, note that

$$\begin{aligned}
\int_{\mathcal{M}} v''a'v'\, d\sigma &= \int_0^{2\pi}\int_0^{l} v''a'v'\psi\, ds\, d\theta \\
&= -\int_0^{2\pi}\int_0^{l} \big(v'\big)^2\big(a'\psi\big)'\, ds\, d\theta - \int_0^{2\pi}\int_0^{l} v''a'v'\psi\, ds\, d\theta \\
&= -\int_0^{2\pi}\int_0^{l} \big(v'\big)^2\big(a'\psi\big)'\, ds\, d\theta - \int_{\mathcal{M}} v''a'v'\, d\sigma
\end{aligned}$$

and thus

$$\begin{aligned}
R_v\big(v'\big) &= \int_{\mathcal{M}} a\big(v''\big)^2 - f'(v)\big(v'\big)^2\, d\sigma \\
&= \int_{\mathcal{M}} v''a'v'\, d\sigma + \int_{\mathcal{M}} \left(\frac{(\psi a)'}{\psi}\right)' \big(v'\big)^2\, d\sigma \\
&= \int_0^{2\pi}\int_0^{l} \left[-\frac{(a'\psi)'}{2} + \psi\left(\frac{(\psi a)'}{\psi}\right)'\right]\big(v'\big)^2\, ds\, d\theta.
\end{aligned}$$

Finally, a simple calculation shows that

$$-\left(\frac{\psi'}{\psi}\right)' \geqslant \frac{\psi'a' + \psi a''}{2a\psi} \quad \Rightarrow \quad -\frac{(a'\psi)'}{2} + \psi\left(\frac{(\psi a)'}{\psi}\right)' \leqslant 0.$$

Therefore $R_v(v') \leqslant 0$ and by (2.10) we have that $\lambda_1 \leqslant 0$. If $R_v(v') < 0$ then $\lambda_1 < 0$ and v is unstable. If $\lambda_1 = 0$, since λ_1 is a simple eigenvalue we have that $v' = c\phi$ where ϕ is the eigenfunction corresponding to the eigenvalue $\lambda_1 = 0$ and $c \neq 0$. It is well known that ϕ has no zeros in $[0,l]$ and this gives a contradiction since v is assumed to be nonconstant on $[0,l]$.

The theorem is proved. □

Remark 3.3.

(i) The left-hand side of inequality (3.1) has a geometrical meaning (see (2.7)) in the sense that

$$-\left(\frac{\psi'}{\psi}\right)' = -\frac{\psi''}{\psi} + \left(\frac{\psi'}{\psi}\right)^2 = K + (K_g)^2.$$

Also (3.1) generalizes the condition in [1], namely $-(\frac{\psi'}{\psi})' \geqslant 0$, where the case $a \equiv$ constant has been addressed.

(ii) Note that Theorem 3.2 is valid for any $f \in C^1(\mathbb{R})$. In the case the domain $\mathcal{M}$ has boundary with the Neumann boundary condition (see (6.1)), $\psi > 0$ in $[0,l]$ and $a \equiv$ constant, the authors in [1], based on [19], showed that if $-(\frac{\psi'}{\psi})'(s_0) < 0$ for some $s_0 \in (0,l)$ then there exists $f \in C^1(\mathbb{R})$ such that (1.1) possesses patterns. In the next section we take $f \in C^1(\mathbb{R})$ bistable and give sufficient conditions for existence of patterns.

Example 3.4. If we take $\mathcal{M}$ to be the unit sphere then $\psi(s) = \sin(s)$, $\chi(s) = \cos(s)$ and $I = (0,\pi)$. In this case if $a(s) = \sin^2(s) + 1$, i.e., $a(x) = x_1^2 + x_2^2 + 1$, $x \in \mathcal{M}$, a simple calculation shows that condition (3.1) is satisfied and therefore (1.1) has no patterns.

On the other hand, if $a(s) = \sin^2(2s) + 1$, i.e., $a(x) = 4x_3^2(x_1^2 + x_2^2) + 1$, $x \in \mathcal{M}$ then condition (3.1) is not satisfied on the unit sphere. Hence there might exist f such that problem (1.1) would possess patterns. In fact, in the next section we will see that this is actually the case when f is of bistable type.

4. Sufficient conditions for existence of patterns

The main concern in this section is to find sufficient conditions for existence of patterns for (1.1) with a small positive parameter ϵ introduced

$$\partial_t u_\epsilon = \epsilon^2 \operatorname{div}_g\bigl(a(x)\nabla_g u_\epsilon\bigr) + f(u_\epsilon), \quad (t,x) \in \mathbb{R}^+ \times \mathcal{M}, \tag{4.1}$$

with $\mathcal{M}$ and a as in (2.6).

Here we assume that the function f satisfies:

(f_1) f has three consecutive zeros α, θ and β, $\alpha < \theta < \beta$, satisfying $f(\alpha) = f(\theta) = f(\beta) = 0$ and $f'(\alpha) < 0$, $f'(\beta) < 0$.
(f_2) $\int_\alpha^\beta f(\xi)\,d\xi = 0$ (the equal-area condition).
(f_3) There exist positives constants c_1, c_2, s_0 and a number $p \geqslant 2$ such that $c_1|s|^p \leqslant F(s) \leqslant c_2|s|^p$ for $|s| \geqslant s_0$, where

$$F(s) = -\int_\alpha^s f(\xi)\,d\xi. \tag{4.2}$$

The method used in this section to find stable stationary solutions using Theorem 4.1 below has been used in other articles ([5,14], for instance) and it also provides the geometric qualitative structure of such solutions.

Note that under our hypotheses (4.1) becomes

$$\psi\partial_t u_\epsilon = \epsilon^2 \partial_s(\psi a \partial_s u_\epsilon) + \psi f(u_\epsilon), \quad s \in (0,l). \tag{4.3}$$

The family of functionals $E_\epsilon : L^1(I) \to \mathbb{R} \cup \{\infty\}$ with $I = (0,l)$, whose critical points are stationary solutions to (4.3) is given by

$$E_\epsilon(u) = \begin{cases} \int_0^l [\frac{\epsilon a(s)\psi(s)}{2}(u')^2 + \frac{\psi(s)}{\epsilon}F(u)]\,ds, & u \in H^1(I), \\ \infty, & \text{otherwise}. \end{cases} \tag{4.4}$$

Remark that due to (f_1) and (f_2), the potential F satisfies:

- $F \in C^2$ and $F \geqslant 0$,
- F has exactly two roots α and β ($\alpha < \beta$) and
- $F'(\alpha) = F'(\beta) = 0$ and $F''(\alpha) > 0$, $F''(\beta) > 0$.

These conditions are necessary in order to use the Γ-convergence technique below. As for (f_3), it is necessary in a compactness argument, as will be explained later. Now our goal is to find local minimizers of E_ϵ and for this purpose we will use the following theorem which can be found in [14].

Theorem 4.1. *(See [14].) Suppose that a sequence of real-extended functionals $\{E_\epsilon\}$, Γ-converges to a real-extended functional E_0 and also that the following hypotheses are satisfied:*

(i) *Any sequence $\{v_\epsilon\}_{\epsilon>0}$ such that $E_\epsilon \leqslant C < \infty$ for all $\epsilon > 0$, is compact in L^1.*
(ii) *There exists an isolated L^1-local minimizer u_0 of E_0.*

Then there exist an $\epsilon_0 > 0$ and a family $\{v_\epsilon\}_{0<\varepsilon<\epsilon_0}$ such that

- *v_ϵ is an L^1-local minimizer of E_ϵ and*
- *$\|v_\epsilon - v_0\|_{L^1} \to 0$, as $\epsilon \to 0$.*

The computation of the Γ-limit of the family of functionals $\{E_\varepsilon\}$ for N-dimensional domains ($N \geqslant 2$) and a, ψ constant functions can be found in [18] or [16], for instance. Essentially the same proof can be used to our case (the presence of the positive functions a and ψ adds no additional difficulty) thus yielding

Theorem 4.2. *Let $E_0 : L^1(I) \to \mathbb{R} \cup \{\infty\}$ be defined as*

$$E_0(u) = \begin{cases} \gamma \int_0^l \sqrt{a(s)}\psi(s)|D\chi_{\{u=\alpha\}}|, & u \in BV(I, \{\alpha, \beta\}), \\ \infty, & \text{otherwise,} \end{cases}$$

where $\gamma = \int_\alpha^\beta \sqrt{F(s)}\,ds$. Then

$$\Gamma - \lim_{\epsilon\to 0} E_\epsilon(u) = E_0(u)$$

where E_ϵ is given by (4.4).

Remark 4.3. By a result of Federer, Theorem 4.5.9 in [9], for $u \in BV(I, \{\alpha, \beta\})$ above functional E_0 can be written, except for a multiplicative constant taken to be 1 for simplicity, in the form

$$E_0(u) = \gamma \int_0^l \sqrt{a(s)}\psi(s)|Du|.$$

In order to apply Theorem 4.1 we need to find an isolated $L^1(I)$-local minimizer of E_0. Indeed condition (i) of Theorem 4.1 was proved in [18] and at this point the condition (f_3) on f is essential. In what follows χ_A denotes the characteristic function of the set A.

Theorem 4.4. *If the function $\sqrt{a}\psi : [0, l] \to \mathbb{R}$ assumes an isolated local minimum at $s_0 \in (0, l)$ then*

$$u_0 = \alpha\chi_{(0,s_0)} + \beta\chi_{(s_0,l)} \tag{4.5}$$

is an isolated $L^1(I)$-local minimizer of E_0.

Proof. By hypothesis there exists $\delta_0 > 0$ such that

$$(\sqrt{a}\psi)(s_0) < (\sqrt{a}\psi)(s)$$

for $0 < |s - s_0| < \delta_0$.

Take $\delta = \frac{1}{2}\delta_0|\alpha - \beta|$ and $u \in BV(I; \{\alpha, \beta\})$ such that

$$0 < \|u - u_0\|_{L^1(I)} < \delta. \tag{4.6}$$

Note that if $u \notin BV(I, \{\alpha, \beta\})$ then $E_0(u) > E_0(u_0)$. Let $S_u \subset (0, l)$ be the jump set of the function u. If $S_u \cap (s_0 - \delta_0, s_0 + \delta_0) = \emptyset$, then

$$\|u - u_0\|_{L^1(I)} = \int_I |u - u_0|\,ds \geqslant |\alpha - \beta|\delta_0 > \delta$$

which contradicts (4.6). Thus $S_u \cap (s_0 - \delta_0, s_0 + \delta_0) \neq \emptyset$ and there exists $s_1 \in S_u \cap (s_0 - \delta_0, s_0 + \delta_0)$.

If $s_1 \neq s_0$ then by virtue of Remark 4.3 and using the co-area formula

$$
\begin{aligned}
E_0(u) &= \gamma \int_I \sqrt{a(s)}\psi(s)|D\chi_{\{u=\beta\}}| = \gamma \int_I \sqrt{a(s)}\psi(s)|Du| \\
&= \gamma \int_{-\infty}^{\infty} \left\{ \int_{I\cap\partial\{u>\xi\}} \sqrt{a(s)}\psi(s)\, d\mathcal{H}^0 \right\} d\xi \\
&= \gamma|\beta-\alpha| \int_{S_u} \sqrt{a(s)}\psi(s)\, d\mathcal{H}^0 \\
&= \gamma|\beta-\alpha| \sum_{s\in S_u} \sqrt{a(s)}\psi(s) \\
&\geqslant \gamma|\beta-\alpha|(\sqrt{a}\psi)(s_1) \\
&> \gamma|\beta-\alpha|(\sqrt{a}\psi)(s_0) = E_0(u_0)
\end{aligned}
$$

as desired.

If $s_1 = s_0$, there are two possibilities: either $u \equiv u_0$ or $u = \beta\chi_{(0,s_0)} + \alpha\chi_{(s_0,l)}$. The former one is ruled out since it is required that $0 < \|u-u_0\|_{L^1(I)} < \delta$ and the latter one based on the fact that in this case $\|u-u_0\|_{L^1(I)} \geqslant 2\delta_0|\alpha-\beta| > \delta$, thus contradicting (4.6).

The theorem is proved. □

Theorem 4.5. *Let $\{u_\epsilon\}_{0<\epsilon\leqslant\epsilon_0}$ be the family of local minimizers of E_ϵ provided by Theorems 4.1, 4.2 and 4.4. Then every u_ϵ is a stable stationary solution to (4.1).*

Proof. By (2.5) each stationary solution to (4.3) is also a stationary solution to (4.1). Consider the following eigenvalue problem obtained by linearizing problem (4.1) around u_ϵ

$$\epsilon^2 a\partial_s^2\phi + \epsilon^2\frac{\partial_s(\psi a)}{\psi}\partial_s\phi + \epsilon^2\frac{a}{\psi^2}\partial_\theta^2\phi + f'(u_\epsilon)\phi + \lambda\phi = 0. \tag{4.7}$$

We claim that if ϕ_1 is an eigenfunction corresponding to the first eigenvalue λ_1 of problem (4.7) then ϕ_1 is independent of θ. It is easy to see that for any $\theta_0 > 0$, $\phi_1(s, \theta+\theta_0)$ is also an eigenfunction corresponding to λ_1. Moreover we have that ϕ_1 is 2π-periodic in θ and

$$\int_0^l \int_0^{2\pi} \phi_1^2(s,\theta)\psi\, d\theta\, ds = 1. \tag{4.8}$$

Since λ_1 is a simple eigenvalue, there exist a constant k such that $\phi_1(s,\theta) = k\phi_1(s,\theta+\theta_0)$.
It follows that

$$\int_0^l \int_0^{2\pi} \phi_1^2(s,\theta+\theta_0)\psi\, d\theta\, ds = \int_0^l \int_{\theta_0}^{2\pi+\theta_0} \phi_1^2(s,\theta)\psi\, d\theta\, ds = \int_0^l \int_0^{2\pi} \phi_1^2(s,\theta)\psi\, d\theta\, ds = 1,$$

then

$$1 = \int_0^l \int_0^{2\pi} \phi_1^2(s,\theta)\psi\, d\theta\, ds = k^2 \int_0^l \int_0^{2\pi} \phi_1^2(s,\theta+\theta_0)\psi\, d\theta\, ds = k^2.$$

Hence $k = \pm 1$ for any $\theta_0 > 0$, $0 \leqslant s \leqslant 1$ and $0 < \theta < 2\pi$ which proves the claim.
Recall that u_ϵ is a local minimizer of E_ϵ, then for all $\phi \in H^1(I)$

$$E''_\epsilon(u_\epsilon)(\phi) = \int_0^l \left\{ \epsilon a\psi(\phi')^2 - \frac{\psi f'(u_\epsilon)\phi^2}{\epsilon} \right\} ds \geqslant 0. \tag{4.9}$$

Therefore, if ϕ_1 is the eigenfunction corresponding to the first eigenvalue λ_1, we have that

$$\lambda_1 = R_{u_\epsilon}(\phi_1) = 2\pi E''_\epsilon(u_\epsilon)(\phi_1) \geqslant 0.$$

If $\lambda_1 > 0$ then u_ϵ is stable. If $\lambda_1 = 0$ then stability also occurs. Indeed in this case 0 is a simple eigenvalue and therefore there is a local one-dimensional critical invariant manifold $W(u_\epsilon)$ tangent to the eigenfunction ϕ_1 such that if u_ϵ is stable in $W(u_\epsilon)$ then it also stable in $H^1(\mathcal{M})$ (see [13, Theorem 6.2.1], for instance). Now the claimed stability of u_ϵ in $W(u_\epsilon)$ follows from the existence of a Lyapunov functional and the fact that $W(u_\epsilon)$ is one-dimensional. $\square$

Summing up, combining Theorems 4.1, 4.2, 4.4 and 4.5, we can state the main result of this section.

Theorem 4.6. *Suppose that f satisfies (f_1), (f_2), (f_3) and that*

- *the function $\sqrt{a}\psi$ assumes an isolated local minimum in $(0,l)$.*

Then

- *$\exists\epsilon_0 > 0$ and a family $\{u_\epsilon\}_{0<\epsilon\leqslant\epsilon_0}$ of nonconstant stable stationary solution to (4.1).*
- *Moreover $\|u_\epsilon - u_0\|_{L^1(I)} \to 0$, as $\varepsilon \to 0$, where u_0 is given by (4.5).*

In particular this is the case if $a \equiv$ constant and ψ assumes an isolated local minimum at some point, say, $s_m \in (0,l)$ (this corresponds to a surface of revolution having a neck at height $\chi(s_m)$).

Remark 4.7.

(i) Standard bootstrap arguments ensure that each solution $u_\epsilon \in C^2(\mathcal{M})$.

(ii) As expected, if $\sqrt{a}\psi$ assumes an isolated local minimum in $(0,l)$ then the condition (3.1) is not satisfied. Indeed, a simple computation shows that if $(\sqrt{a}\psi)'(s_0) = 0$ and $(\sqrt{a}\psi)''(s_0) > 0$ for some $s_0 \in (0,l)$ then

$$-\left(\frac{\psi'}{\psi}\right)'(s_0) < \frac{a'(s_0)\psi'(s_0) + a''(s_0)\psi(s_0)}{2a(s_0)\psi(s_0)}.$$

(iii) For the case $a \equiv$ constant the authors in [1] proved that if $(\frac{\psi'}{\psi})'(s_0) > 0$, for some $s_0 \in (0,l)$, then there exists a function f (depending on ψ) such that (1.1) has a nonconstant stable solution. Instead we fix f, e.g., $f(u) = u - u^3$, and prove the same existence result provided ψ assumes an isolated local minimum at $s_0 \in (0,l)$. Note that the latter assumption, by its turn, implies that $(\frac{\psi'}{\psi})'(s_0) > 0$.

Example 4.8. As stated in Example 3.4 if $\mathcal{M}$ is the unit sphere and $a(s) = \sin^2(2s) + 1$, $s \in [0,\pi]$, then the function $\sqrt{a}\psi$ assumes an isolated local minimum in $s = \frac{\pi}{2}$ and therefore there exists a family $\{u_\epsilon\}_{0<\epsilon\leqslant\epsilon_0}$ of nonconstant stable stationary solutions to (4.1).

5. The Neumann boundary condition case

In this section we discuss the case where the domain is a surface of revolution with boundary. Let the surface of revolution $\mathcal{M}$ be as before and let $D \subset \mathcal{M}$ be the domain delimited by two circles C_{s_1} and C_{s_2}, $0 < s_1 < s_2 < l$, parametrized in the local coordinates (s,θ) as follows:

$$C_{s_1}\colon \begin{cases} s(t) = s_1 \\ \theta(t) = t \end{cases} \quad \text{and} \quad C_{s_2}\colon \begin{cases} s(t) = s_2 \\ \theta(t) = t \end{cases}$$

with $t \in [0, 2\pi)$.

Let ν be the outer normal vector of ∂D lying in the tangent space $T_p(\mathcal{M})$ for any $p \in \partial D$. We shall assume that ∂D is orientable so that the outer normal is well-defined and continuous.

The derivative of u in the direction of ν at ∂D is given by

$$\frac{\partial u}{\partial \nu} = \langle \nabla_g u, \nu \rangle,$$

where $\nu = \nu_1 \frac{\partial}{\partial s} + \nu_2 \frac{\partial}{\partial \theta}$ and $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\}$ is the basis of $T_p(\mathcal{M})$.

Moreover it is supposed that

$$\chi'(s) \geqslant 0, \quad s \in (s_1, s_1 + \delta) \cup (s_2 - \delta, s_2) \tag{5.1}$$

for some $\delta > 0$. Thus there holds $\nu = \frac{\partial}{\partial s}$ on C_{s_2} and $\nu = -\frac{\partial}{\partial s}$ on C_{s_1}.

The following problem on the domain $D \subset \mathcal{M}$, described above, with Neumann boundary condition is considered

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\big(a(x)\nabla u\big) + f(u), & (t,x) \in \mathbb{R}^+ \times D, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial D. \end{cases} \tag{5.2}$$

Except for a few natural changes the proofs of the following theorems are similar to those rendered for domains without boundary.

Theorem 5.1. *If*

$$-\left(\frac{\psi'}{\psi}\right)'(s) \geqslant \frac{(a'\psi)'(s)}{2(a\psi)(s)} \tag{5.3}$$

for all $s \in (s_1, s_2)$ then every nonconstant stationary solution to (5.2) is unstable. In particular this is the case if

- *D is convex and*
- *$a'\psi$ is a non-increasing function.*

Theorem 5.2. *If f satisfies (f_1), (f_2), (f_3) and $\sqrt{a}\psi$ assumes an isolated local minimum $s_0 \in (s_1, s_2)$, then there exists a family $\{v_\epsilon\}_{0<\epsilon\leqslant\epsilon_0}$, for some $\epsilon_0 > 0$, of nonconstant stable stationary solution to the problem*

$$\begin{cases} \partial_t v_\epsilon = \epsilon^2 \operatorname{div}\big(a(x)\nabla v_\epsilon\big) + f(v_\epsilon), & (t,x) \in \mathbb{R}^+ \times D, \\ \partial_\nu v_\epsilon = 0, & x \in \partial D. \end{cases} \tag{5.4}$$

Also in this case we will prove that the hypothesis (f_2) (the equal-area condition) is necessary for the conclusion of Theorem 5.2.

As a consequence of the two theorems above many examples of existence and nonexistence of patterns can be created. Here are some simple ones.

Example 5.3. For the sake o illustration let us consider the following surfaces

- D_1 a cylindrical surface given by $\psi_1(s) = 1$ and $\chi_1(s) = s + 1$, $s \in [0,1]$,
- D_2 a frustum of right circular cone given by $\psi_2(s) = -\frac{\sqrt{2}}{2}s + 1$ and $\chi_2(s) = \frac{\sqrt{2}}{2}s + 1$, $s \in [0,1]$,
- D_3 given by $\psi_3(s) = \frac{s^2}{4} + \frac{1}{2}$ and $\chi_3(s) = \frac{s}{4}\sqrt{4 - s^2} + \arcsin(\frac{s}{2})$, $s \in (0, \frac{1}{2})$, which resembles a frustum of a hyperboloid.

As a consequence of the above theorems the following conclusions hold.

(i) In D_1 if $a(s) = s + 1$ ($a(x) = x_3, x \in D_1$) or $a(s) = -s^2 - 2s + 4$ ($a(x) = -x_3^2 + 5, x \in D_1$), $s \in [0,1]$, then (5.3) is satisfied since $K \equiv 0$ in D_1 and $a'\psi_1$ is non-increasing in both cases. Hence in these cases there are no patterns.
(ii) If $a(x) = x_3^2$ then (5.3) is verified in D_2 ($K \geqslant 0$ in D_2 and $a'\psi_2$ is non-increasing) but is not verified in D_1. Hence there are no patterns in D_2 whereas the argument is inconclusive in D_1.
(iii) Still in D_1, if f satisfies (f_1), (f_2), (f_3) and $a(s) = s^2 - s + 2$ with $s \in [0,1]$ ($a(x) = x_3^2 - 3x_3 + 4, x \in D_1$) then problem (5.2) possesses patterns, since $(\psi_1\sqrt{a})(\cdot)$ assumes an isolated local minimum at $s = \frac{1}{2}$.
(iv) In D_3 the Gaussian curvature K is negative and if $a(s) = -s^2 + \frac{1}{2}$ ($a(x) = -4\sqrt{x_1^2 + x_2^2} + \frac{5}{2}, x \in D_3$) then (5.3) is verified, i.e., there are no patterns.

6. Necessity of the equal-area condition

Notice that the family of patterns to (1.1) found in Theorem 5.2 develops internal transition layer as $\epsilon \to 0$ (see definition below).

Our goal in this section is to prove that the equal-area condition (f_2) is in fact a necessary condition for the development of internal transition layer by the stationary solutions of (4.3), i.e.,

$$\epsilon\big(\psi a u_\epsilon'\big)' + \psi f(u_\epsilon) = 0, \quad s \in (0,l) \tag{6.1}$$

where f satisfies (f_1), $\psi(0) = \psi(l) = 0$ and, for the sake of simplicity, ϵ^2 has been replaced with ϵ since the scaling plays no role here.

This setting corresponds to the case where $\mathcal{M}$ has no boundary. However for the case $\mathcal{M}$ has boundary, supplied with Neumann boundary condition, the result still holds and the proof requires only a few minor and natural changes.

Definition 6.1. We will say that a family $\{v_\epsilon\}_{0<\epsilon\leqslant\epsilon_0}$ of solutions to (6.1) in $C^2(I)$ *develops inner transition layer, with interface at a point $p \in (0,l)$*, as $\epsilon \to 0$, if

$$v_\epsilon \to v_0 \quad \text{in } L^1(I), \text{ as } \epsilon \to 0,$$

where $v_0 = \alpha\chi_{(0,p)} + \beta\chi_{(p,l)}$.

We took just one point $p \in (0,l)$ for the sake of simplicity but the result holds for a finite number of points in $(0,l)$ as well.

The following theorem would still remain valid had the convergence in $L^1(I)$ in Definition 6.1 been replaced with uniform convergence on compacts sets in $I - \{p\}$. The proof still goes through without any change.

Theorem 6.2. *Let $\{v_\epsilon\}_{0<\epsilon\leqslant\epsilon_0}$ be a family of uniformly bounded (with respect to ϵ) solutions of (6.1) which develops inner transition layer with interfaces in p. Then*

$$\int_\alpha^\beta f(\xi)\,d\xi = 0.$$

Proof. We will drop the subindex ϵ in v_ϵ, in the next computations. Multiplying Eq. (6.1) by sv' we have

$$\epsilon sv'\big(a\psi v'\big)' + sv'\psi f(v) = 0.$$

Note that

$$\big(sv'a\psi v'\big)' = sv'\big(a\psi v'\big)' + a\psi v'\big(v' + sv''\big),$$

hence

$$sv'\big(a\psi v'\big)' = \big(sv'a\psi v'\big)' - a\psi\left(\big(v'\big)^2 + s\left[\frac{(v')^2}{2}\right]'\right). \tag{6.2}$$

Integrating over $(0,l)$, we obtain

$$\begin{aligned}
&-\epsilon\int_0^l \big(sv'a\psi v'\big)'\,ds + \epsilon\int_0^l a\psi\left[\big(v'\big)^2 + s\left(\frac{(v')^2}{2}\right)'\right]ds \\
&\quad = \int_0^l sv'\psi f(v)\,ds = \int_0^l s\big(\psi F(v)\big)'\,ds - \int_0^l s\psi' F(v)\,ds
\end{aligned}$$

where $F(v) = \int_\theta^v f(\xi)\,d\xi$ and θ is any constant such that $\alpha < \theta < \beta$.

We also have

$$\big(sa\psi\big(v'\big)^2\big)' = sa\psi\big[\big(v'\big)^2\big]' + \big(v'\big)^2\big(a\psi + s(a\psi)'\big)$$

hence

$$\begin{aligned}
&-\epsilon\int_0^l sa\psi\big[\big(v'\big)^2\big]' + \big(v'\big)^2\big(a\psi + s(a\psi)'\big)\,ds + \epsilon\int_0^l a\psi\big(v'\big)^2 + a\psi s\left[\frac{(v')^2}{2}\right]' ds \\
&\quad = \int_0^l \big(s\psi F(v)\big)'\,ds - \int_0^l \psi F(v)\,ds - \int_0^l s\psi' F(v)\,ds
\end{aligned}$$

on account that $(s\psi F(v))' = \psi F(v) + s(\psi F(v))'$. It follows that

$$\begin{aligned}
&-\frac{\epsilon}{2}\int_0^l sa\psi\big[\big(v'\big)^2\big]'\,ds - \epsilon\int_0^l \big(v'\big)^2 s(a\psi)'\,ds \\
&\quad = \int_0^l \big(s\psi F(v)\big)'\,ds - \int_0^l \psi F(v)\,ds - \int_0^l s\psi' F(v)\,ds.
\end{aligned}$$

Integrating by parts we obtain

$$-\frac{\epsilon}{2}sa\psi\left(v'\right)^2\Big|_0^l+\frac{\epsilon}{2}\int_0^l\left(v'\right)^2s(a\psi)'\,ds+\frac{\epsilon}{2}\int_0^l\left(v'\right)^2a\psi\,ds-\epsilon\int_0^l\left(v'\right)^2s(a\psi)'\,ds$$
$$=\int_0^l\left(s\psi F(v)\right)'ds-\int_0^l\psi F(v)\,ds-\int_0^l s\psi'F(v)\,ds.$$

Finally, recovering the sub-script ϵ and recalling that $\psi(0)=\psi(l)=0$, we have

$$-\frac{\epsilon}{2}\int_0^l\left(v'_\epsilon\right)^2s(a\psi)'\,ds+\frac{\epsilon}{2}\int_0^l\left(v'_\epsilon\right)^2a\psi\,ds$$
$$=\int_0^l\left(s\psi F(v_\epsilon)\right)'ds-\int_0^l\psi F(v_\epsilon)\,ds-\int_0^l s\psi'F(v_\epsilon)\,ds. \tag{6.3}$$

We claim that the two terms on the left-hand side of the equation above approach zero, as $\epsilon\to0$. Indeed, denote these terms by I_1, I_2, respectively.

Multiplying Eq. (6.1) by v_ϵ and integrating by parts on $(0,l)$, we obtain

$$\epsilon\int_0^l a\psi\left(v'_\epsilon\right)^2ds=\int_0^l v_\epsilon\psi f(v_\epsilon)\,ds,$$

i.e.,

$$2I_2=\epsilon\int_0^l a\psi\left(v'_\epsilon\right)^2ds=\int_0^l v_\epsilon\psi f(v_\epsilon)\,ds. \tag{6.4}$$

Since by hypothesis $|v_\epsilon|\leqslant M$, uniformly in ϵ, f is continuous and $v_\epsilon\to v_0$ a.e. in I, an application of the Lebesgue bounded convergence theorem yields that the right-hand side of (6.4) approach zero, as $\epsilon\to0$ (recall that $f(\alpha)=f(\beta)=0$). One now easily sees that I_1 also approaches zero, as $\epsilon\to0$.

From the fact that $\psi(0)=\psi(l)=0$ we obtain

$$\int_0^l\left(s\psi F(v_\epsilon)\right)'ds=0.$$

Moreover, $\{v_\epsilon\}$ is bounded uniformly in ϵ and F is C^1, then $\{F(v_\epsilon)\}$ is bounded in $(0,l)$, uniformly in ϵ and so we can use the Lebesgue bounded theorem to compute the limit, in $(0,p)$ and in (p,l), of the terms on the right-hand side of Eq. (6.3). Finally, as $\epsilon\to0$, we obtain

$$0=\int_0^p\psi F(\alpha)\,ds+\int_p^l\psi F(\beta)\,ds+\int_0^p s\psi'F(\alpha)\,ds+\int_p^l s\psi'F(\beta)\,ds$$
$$=\int_0^p F(\alpha)(s\psi)'\,ds+\int_p^l F(\beta)(s\psi)'\,ds$$
$$=\left(F(\alpha)-F(\beta)\right)p\psi(p).$$

On the account that $p\psi(p)\neq0$ we obtain $F(\alpha)=F(\beta)$, i.e.,

$$\int_\alpha^\beta f(\xi)\,d\xi=0$$

and the theorem is proved. □

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