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Delta-shocks and vacuum states in the vanishing pressure limit of solutions to the isentropic Euler equations for modified Chaplygin gas [☆]

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ABSTRACT

The phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states in solutions to the isentropic Euler equations for a modified Chaplygin gas are analyzed as the double parameter pressure vanishes. Firstly, the Riemann problem of the isentropic Euler equations for a modified Chaplygin gas is solved analytically. Secondly, it is rigorously shown that, as the pressure vanishes, any two-shock Riemann solution to the isentropic Euler equations for a modified Chaplygin gas tends to a δ-shock solution to the transport equations, and the intermediate density between the two shocks tends to a weighted δ-measure that forms the δ-shock; any two-rarefaction-wave Riemann solution to the isentropic Euler equations for a modified Chaplygin gas tends to a two-contact-discontinuity solution to the transport equations, the nonvacuum intermediate state between the two rarefaction waves tends to a vacuum state. Finally, some numerical results exhibiting the formation of δ-shocks and vacuum states are presented as the pressure decreases.

1. Introduction

As to our knowledge, the investigation of delta shock waves has been increasingly active in the past over two decades. The delta shock wave, as a generalization of an ordinary shock wave, is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. Physically, the delta shock wave represents the process of concentration of the mass, and may be interpreted as the galaxies in the universe. See the results in [18,19,33,34,23,38,13,25,40] and the references cited therein.

In related researches of the delta-shocks, one very interesting topic is to explore the phenomena of concentration and cavitation and the formation of δ-shock waves and vacuum states in solutions. In earlier paper [9], Chen and Liu considered the Euler equations of isentropic gas dynamics in Eulerian coordinates which read

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$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + P\right)_x = 0, \end{cases} \tag{1.1}$$

where $\rho \geqslant 0$, u, P denote the density, the velocity and the pressure, respectively. The scalar pressure $P(\rho, \epsilon)$ satisfies

$$\lim_{\epsilon \to 0} P(\rho, \epsilon) = 0, \tag{1.2}$$

where $\epsilon > 0$ is a small parameter.

When $\epsilon \to 0$, obviously, system (1.1)–(1.2) formally becomes the transport equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x = 0, \end{cases} \tag{1.3}$$

which are also called the zero-pressure gas dynamics, or Euler equations for pressureless fluids, and can be obtained from Boltzmann equations [5] and the flux-splitting scheme of the full compressible Euler equations [1,21]. The transport equations (1.3) are also used to model the motion of free particles which stick under collision [7] and the formation of large-scale structures in the universe [37,29].

Since 1994, a great many authors have extensively studied the transport equations (1.3). Bouchut [5] first established the existence of measure solutions of the Riemann problem. E, Rykov and Sinai [37] studied the existence of global weak solution and the behavior of such global solution with random initial data. Sheng and Zhang [31] solved the 1-D and 2-D Riemann problems with the characteristic analysis and the vanishing viscosity method. Huang and Wang [17] proved the uniqueness of the weak solution for the case when the initial data is a Radon measure. Also see [33,22,11,39] for related results. In these papers it has been proved that δ-shock waves and vacuum states do occur in the Riemann solutions.

In their works, in (1.1) Chen and Liu [9] took the prototypical pressure functions for polytropic gas

$$P(\rho, \epsilon) = \epsilon p(\rho), \qquad p(\rho) = \rho^\gamma/\gamma, \quad \gamma > 1. \tag{1.4}$$

They identified and analyzed the phenomena of concentration and cavitation and the formation of δ-shock waves and vacuum states in solutions to the system (1.1) with (1.4) as $\epsilon \to 0$. Further, they also obtained this same results for the Euler equations for nonisentropic fluids in [10]. Specially, for $\gamma = 1$ in (1.4), the pressure vanishing limit had been studied by Li [20]. Besides, the results were extended to the relativistic Euler equations for polytropic gases by Yin and Sheng [41], the perturbed Aw–Rascle model by Shen and Sun [30], etc.

In 2002, Benaoum [2] proposed the modified Chaplygin gas

$$P = A\rho - \frac{B}{\rho^\alpha}, \quad 0 \leqslant \alpha \leqslant 1, \tag{1.5}$$

where two parameters $A, B > 0$. As an exotic fluid, such a gas can describe the current accelerated expansion of the universe. In this regard, just for their discovery of the accelerating expansion of the universe through observations of distant supernovae, Saul Perlmutter, Brian P. Schmidt and Adam G. Riess won the 2011 Nobel Prize in Physics. Usually, it is thought that the source of this acceleration is attributed to an exotic type of fluid with negative pressure called commonly dark energy. With the introduction of dark energy, the search began for different candidates that can effectively play the role of dark energy. Up to now, various kinds of theoretical models have been proposed to interpret the behavior of dark energy. Among them, the Chaplygin gas and modified Chaplygin gas are plausible. See [3,26,14,24] for more demonstrations.

Especially, one can easily see that, in (1.5), when $B = 0$, $P = A\rho$ is just the standard equation of state of perfect fluid. Whereas when $A = 0$, $P = -B/\rho^\alpha$ called as the generalized Chaplygin gas [27]. Further, when

$\alpha = 1$, it corresponds to an exotic background fluid $P = -B/\rho$ called as the (pure) Chaplygin gas which was introduced by Chaplygin [8], Tsien [35] and von Karman [36] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. Such a gas owns a negative pressure and occurs in certain theories of cosmology. It has been also advertised as a possible model for dark energy [4,12,15,28].

With regard to the isentropic Euler equations for a Chaplygin gas (1.1) and (1.5), Brenier [6] considered the one-dimensional Riemann problem and obtained the solutions with concentration when initial data belong to a certain domain in phase plane. Guo, Sheng and Zhang [16] studied the one-dimensional Riemann problems and obtained the general solutions. They also systematically solved the two-dimensional Riemann problem. It has been shown that, in their results, δ-shock waves do occur in the Riemann solutions, and vacuum states do not.

In view of the above analysis, one can observe that, in (1.5), when two parameters $A, B \to 0$, system (1.1) with (1.5) formally also changes into the transport equations (1.3). In the present paper, we focus on the double parameter pressure function for modified Chaplygin gas

$$P = A\rho - \frac{B}{\rho}, \tag{1.6}$$

as $\alpha = 1$. The rest cases will be further discussed in the future. The system (1.1) with (1.6) is called as the isentropic Euler equations for a modified Chaplygin gas which shows that the matter is taken to be a perfect fluid obeying an exotic equation of state. In the sense the system (1.1) and (1.6) can be used to understand the mysterious nature of the dark sector of the universe (i.e., dark energy and dark matter) (see [3]).

The main objective of this paper is to analyze the phenomena of concentration and cavitation and the formation of δ-shock waves and vacuum states in solutions to the isentropic Euler equations for a modified Chaplygin gas as the double parameter pressure vanishes, which corresponds to a two-parameter limit of solutions in contrast to previous works in [9,10,20,41,30].

We first solve the Riemann problem of (1.1) and (1.6) with Riemann initial data. By means of the analysis method in phase plane, we establish the existence and uniqueness of Riemann solutions with four different structures: $\overleftarrow{R} + \overrightarrow{R}$, $\overleftarrow{R} + \overrightarrow{S}$, $\overleftarrow{S} + \overrightarrow{R}$ and $\overleftarrow{S} + \overrightarrow{S}$.

Then we rigorously prove that, as the two-parameter pressure vanishes, any two-shock Riemann solution to the isentropic Euler equations for a modified Chaplygin gas tends to a δ-shock solution to the transport equations, and the intermediate density between the two shocks tends to a weighted δ-measure that forms the δ-shock; any two-rarefaction-wave Riemann solution to the isentropic Euler equations for a modified Chaplygin gas tends to a two-contact-discontinuity solution to the transport equations, the nonvacuum intermediate state between the two rarefaction waves tends to a vacuum state, even when the initial data stays away from the vacuum. As a result, the δ-shocks for the transport equations result from a phenomenon of concentration, while the vacuum states result from a phenomenon of cavitation in the vanishing pressure limit process. These results are in completely coincident with that in [9].

In addition, employing the fifth-order weighted essentially non-oscillatory scheme and third-order Runge–Kutta method [32], we produce some numerical experiments to examine the phenomena of concentration and cavitation and the formation process of δ-shocks and vacuum states in the level of the isentropic Chaplygin Euler equations (1.1) and (1.6) as the pressure decreases, which completely confirm the theoretical analysis.

The plan of this paper is organized as follows. In Section 2, we restate the δ-shocks and vacuums for (1.3). In Section 3, we solve the Riemann problem and examine the dependence of the Riemann solutions on the two parameters $A, B > 0$ for system (1.1) and (1.6). Sections 4 and 5 analyze the limit behavior of a two-shock solution and a two-rarefaction solution of system (1.1) and (1.6) in the process of the vanishing pressure limit. In Section 6, we present some representative numerical results to demonstrate the validity of the theoretical analysis in Sections 4 and 5.

2. δ-shocks and vacuums for the transport equations

For completeness, this section briefly recalls δ-shocks and vacuum states in the Riemann solutions to the transport equations (1.3). See [31] for more details.

System (1.3) has a double eigenvalue $\lambda = u$ with the associated eigenvector $r = (0,1)^T$ satisfying $\nabla\lambda \cdot r = 0$, which means the system (1.3) is nonstrictly hyperbolic and λ linearly degenerate.

Consider the Riemann problem of (1.3) with initial conditions

$$(u,\rho)(0,x) = \begin{cases} (u_-,\rho_-), & x<0, \\ (u_+,\rho_+), & x>0, \end{cases} \tag{2.1}$$

where $(u_\pm,\rho_\pm)$ are arbitrary constants and $\rho_\pm > 0$. By seeking self-similar solution $(u,\rho)(t,x) = (u,\rho)(\xi)$ $(\xi = x/t)$, it is easy to find that, besides the constant state and singular solution $u=\xi$, $\rho = 0$ (called vacuum states), the elementary waves of (1.3) are nothing but contact discontinuities. The Riemann problem (1.3) and (2.1) can be solved by the following two cases.

For the case $u_- < u_+$, the solution includes two contact discontinuities and a vacuum state besides constant states. That is,

$$(u,\rho)(\xi) = \begin{cases} (u_-,\rho_-), & -\infty < \xi \leqslant u_-, \\ (u(\xi),0), & u_- < \xi < u_+, \\ (u_+,\rho_+), & u_+ \leqslant \xi < +\infty. \end{cases} \tag{2.2}$$

For the case $u_- > u_+$, a solution containing a weighted δ-measure (i.e., δ-shock) supported on a line will develop in solutions due to the overlap of characteristic lines.

In order to define the measure solutions, a two-dimensional weighted δ-function $w(s)\delta_S$ supported on a smooth curve S parameterized as $t = t(s)$, $x = x(s)$ $(c \leqslant s \leqslant d)$ can be defined by

$$\bigl\langle w\bigl(t(s)\bigr)\delta_S, \varphi\bigl(t(s),x(s)\bigr)\bigr\rangle = \int_c^d w\bigl(t(s)\bigr)\varphi\bigl(t(s),x(s)\bigr)\sqrt{x'(s)^2+t'(s)^2}\,ds \tag{2.3}$$

for all the test functions $\varphi(t,x) \in C_0^\infty(R^+ \times R^1)$.

With this definition, a δ-shock solution can be introduced to construct the solution of (1.3), which is

$$\rho(t,x) = \rho_0(t,x) + w(t)\delta_S, \qquad u(t,x) = u_0(t,x), \tag{2.4}$$

where $S = \{(t,\sigma t)\colon 0 \leqslant t < \infty\}$,

$$\rho_0(t,x) = \rho_- + [\rho]\chi(x-\sigma t), \qquad u_0(t,x) = u_- + [u]\chi(x-\sigma t), \qquad w(t) = \frac{t}{\sqrt{1+\sigma^2}}\bigl(\sigma[\rho] - [\rho u]\bigr), \tag{2.5}$$

in which $[g] = g_+ - g_-$ denotes the jump of function g across the discontinuity, σ is the velocity of the δ-shock, and $\chi(x)$ the characteristic function that is 0 when $x<0$ and 1 when $x>0$.

As shown in [31], for any $\varphi(t,x) \in C_0^\infty(R^+ \times R^1)$, the δ-shock solution constructed above satisfies

$$\begin{aligned} &\langle \rho, \varphi_t\rangle + \langle \rho u, \varphi_x\rangle = 0, \\ &\langle \rho u, \varphi_t\rangle + \bigl\langle \rho u^2, \varphi_x\bigr\rangle = 0, \end{aligned} \tag{2.6}$$

where

$$\langle\rho,\varphi\rangle=\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho_0\varphi\,dx\,dt+\langle w\delta_S,\varphi\rangle,$$
$$\langle\rho u,\varphi\rangle=\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho_0u_0\varphi\,dx\,dt+\langle\sigma w\delta_S,\varphi\rangle. \tag{2.7}$$

Furthermore, substituting (2.4) and (2.5) into (2.6) under the condition (2.3) and (2.7), we can get the generalized Rankine–Hugoniot relation

$$\begin{cases}\dfrac{dx}{dt}=\sigma,\\ \dfrac{d(w(t)\sqrt{1+\sigma^2}\,)}{dt}=\sigma[\rho]-[\rho u],\\ \dfrac{d(w(t)\sigma\sqrt{1+\sigma^2}\,)}{dt}=\sigma[\rho u]-\left[\rho u^2\right],\end{cases} \tag{2.8}$$

which reflects the relationship between the limit states on two sides of a δ-shock wave and the relations among the location, weight and propagation speed of the δ-shock wave.

To guarantee the uniqueness, the entropy condition is supplemented as

$$u_->\sigma>u_+, \tag{2.9}$$

which means that all the characteristic lines on both sides of the discontinuity are not out-going. So it is a overcompressive condition.

Then solving the generalized Rankine–Hugoniot relation (2.8) with initial data $x(0)=0$ and $w(0)=0$ under the entropy condition (2.9) yields

$$\sigma=\frac{\sqrt{\rho_+}u_++\sqrt{\rho_-}u_-}{\sqrt{\rho_+}+\sqrt{\rho_-}}\quad\text{and}\quad w(t)=\frac{\sqrt{\rho_+\rho_-}(u_--u_+)t}{\sqrt{1+\sigma^2}}. \tag{2.10}$$

Therefore, a δ-shock solution defined by (2.4) with (2.5) and (2.10) is obtained.

3. Riemann solutions to system (1.1) and (1.6)

In this section, we first solve the elementary waves and construct solutions to the Riemann problem of (1.1), (1.6) with (2.1), and then examine the dependence of the Riemann solutions on the two parameters $A,B>0$ for the Euler equations (1.1) and (1.6).

The system has two eigenvalues

$$\lambda_1=u-\sqrt{A+\frac{B}{\rho^2}},\qquad\lambda_2=u+\sqrt{A+\frac{B}{\rho^2}}, \tag{3.1}$$

and the corresponding right eigenvectors

$$\vec{r}_1=\left(\sqrt{A+\frac{B}{\rho^2}},-\rho\right)^T,\qquad\vec{r}_2=\left(\sqrt{A+\frac{B}{\rho^2}},\rho\right)^T,$$

satisfying

$$\nabla\lambda_i\cdot\vec{r}_i=\frac{A}{\sqrt{A+\frac{B}{\rho^2}}}\neq 0\quad(i=1,2).\tag{3.2}$$

Therefore, both the characteristic fields are genuinely nonlinear and the elementary waves consist of shock waves and rarefaction waves.

Consider the self-similar solution $(u,\rho)(t,x)=(u,\rho)(\xi)(\xi=x/t)$. Then Riemann problem (1.1) and (1.6) with (2.1) is reduced to a two-point boundary value problem of first-order ordinary differential equations

$$\begin{cases}-\xi\rho_\xi+(\rho u)_\xi=0,\\ -\xi(\rho u)_\xi+\left(\rho u^2+P\right)_\xi=0,\quad P=A\rho-\dfrac{B}{\rho}\end{cases}\tag{3.3}$$

and

$$(u,\rho)(\pm\infty)=(u_\pm,\rho_\pm).\tag{3.4}$$

Any smooth solution of (3.3) satisfies

$$\begin{pmatrix}\rho & u-\xi\\ u-\xi & \frac{A+\frac{B}{\rho^2}}{\rho}\end{pmatrix}\begin{pmatrix}du\\ d\rho\end{pmatrix}=0.$$

It provides either the general solution (constant state)

$$(u,\rho)(\xi)=\text{constant},$$

or the backward rarefaction wave

$$\overleftarrow{R}\colon\quad\begin{cases}\xi=\lambda_1=u-\sqrt{A+\dfrac{B}{\rho^2}},\\ du+\dfrac{\sqrt{A+\frac{B}{\rho^2}}}{\rho}\,d\rho=0,\end{cases}\tag{3.5}$$

or the forward rarefaction wave

$$\overrightarrow{R}\colon\quad\begin{cases}\xi=\lambda_2=u+\sqrt{A+\dfrac{B}{\rho^2}},\\ du-\dfrac{\sqrt{A+\frac{B}{\rho^2}}}{\rho}\,d\rho=0.\end{cases}\tag{3.6}$$

From (3.5) and (3.6), we obtain that

$$\frac{d\lambda_1}{d\rho}=\frac{\partial\lambda_1}{\partial u}\frac{du}{d\rho}+\frac{\partial\lambda_1}{\partial\rho}=-\frac{\sqrt{A+\frac{B}{\rho^2}}}{\rho}+\frac{B}{\rho^3\sqrt{A+\frac{B}{\rho^2}}}=-\frac{A}{\rho\sqrt{A+\frac{B}{\rho^2}}}<0\tag{3.7}$$

and

$$\frac{d\lambda_2}{d\rho}=\frac{\partial\lambda_2}{\partial u}\frac{du}{d\rho}+\frac{\partial\lambda_2}{\partial\rho}=\frac{\sqrt{A+\frac{B}{\rho^2}}}{\rho}-\frac{B}{\rho^3\sqrt{A+\frac{B}{\rho^2}}}=\frac{A}{\rho\sqrt{A+\frac{B}{\rho^2}}}>0.\tag{3.8}$$

which imply the velocity of backward (forward) rarefaction wave λ_1 (λ_2) is monotonic decreasing (increasing) with respect to ρ.

With the requirement $\lambda_1(u,\rho) > \lambda_1(u_-,\rho_-)$ and $\lambda_2(u,\rho) > \lambda_2(u_-,\rho_-)$, noticing (3.7) and (3.8), we integrate the second equations of (3.5) and (3.6), respectively, and get that

$$\overleftarrow{R}\colon\quad \begin{cases} \xi = \lambda_1 = u - \sqrt{A + \dfrac{B}{\rho^2}}, \\ u = u_- - \displaystyle\int_{\rho_-}^{\rho} \frac{\sqrt{A+\frac{B}{s^2}}}{s}\,ds, \quad \rho < \rho_-, \end{cases} \tag{3.9}$$

and

$$\overrightarrow{R}\colon\quad \begin{cases} \xi = \lambda_2 = u + \sqrt{A + \dfrac{B}{\rho^2}}, \\ u = u_- + \displaystyle\int_{\rho_-}^{\rho} \frac{\sqrt{A+\frac{B}{s^2}}}{s}\,ds, \quad \rho > \rho_-, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} \int_{\rho_-}^{\rho} \frac{\sqrt{A+\frac{B}{s^2}}}{s}\,ds = {} & -\sqrt{A+\frac{B}{\rho^2}} + \sqrt{A}\ln\left(\sqrt{A+\frac{B}{\rho^2}}+\sqrt{A}\right) + \sqrt{A}\ln\rho \\ & + \sqrt{A+\frac{B}{\rho_-^2}} - \sqrt{A}\ln\left(\sqrt{A+\frac{B}{\rho_-^2}}+\sqrt{A}\right) - \sqrt{A}\ln\rho_-. \end{aligned} \tag{3.11}$$

The curves defined by the second equations of (3.9) and (3.10) in the phase plane (u,ρ) are respectively called the backward rarefaction wave curve and forward rarefaction wave curve. Further, consider the properties of these curves. From the second equation of (3.5), a regular calculation gives $u_\rho < 0$ and $u_{\rho\rho} > 0$, which mean that the backward rarefaction wave curve is monotonic decreasing and convex. Similarly, from the second equation of (3.6), we have $u_\rho > 0$ and $u_{\rho\rho} < 0$, which mean that the forward rarefaction wave curve is monotonic increasing and concave. In addition, we prove that $\lim_{\rho\to 0} u = +\infty$ for the backward rarefaction wave curve and $\lim_{\rho\to+\infty} u = +\infty$ for the forward rarefaction wave curve.

Now we consider the discontinuous solution. A bounded discontinuity at $\xi = \sigma$ should satisfy the Rankine–Hugoniot compatibility condition

$$\begin{cases} \sigma[\rho] - [\rho u] = 0, \\ \sigma[\rho u] - \left[\rho u^2 + P\right] = 0, \quad P = A\rho - \dfrac{B}{\rho}. \end{cases} \tag{3.12}$$

Then eliminating σ in (3.12), a regular calculation gives

$$(u-u_-)^2 = \frac{\rho_- - \rho}{\rho_-\rho}\left(A(\rho_- - \rho) + B\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right)\right).$$

Thus, we have

$$
\begin{aligned}
u-u_- &= \pm\sqrt{\frac{\rho_- - \rho}{\rho_-\rho}\left(A(\rho_- - \rho) + B\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right)\right)} \\
&= \pm\sqrt{\frac{1}{\rho_-\rho}\left(A + \frac{B}{\rho_-\rho}\right)(\rho - \rho_-)^2}.
\end{aligned}
\tag{3.13}
$$

In order to identify the admissible solution, the discontinuity associating with λ_1 has to be satisfied with the stability condition

$$
\sigma < \lambda_1(u_-, \rho_-) < \lambda_2(u_-, \rho_-), \qquad \lambda_1(u, \rho) < \sigma < \lambda_2(u, \rho),
\tag{3.14}
$$

and the discontinuity associating with λ_2 should satisfy

$$
\lambda_1(u_-, \rho_-) < \sigma < \lambda_2(u_-, \rho_-), \qquad \lambda_1(u, \rho) < \lambda_2(u, \rho) < \sigma.
\tag{3.15}
$$

Furthermore, from the first equation of (3.12), it is easy to find

$$
\sigma = \frac{\rho_- u_- - \rho u}{\rho_- - \rho} = u_- + \frac{\rho(u_- - u)}{\rho_- - \rho} = u + \frac{\rho_-(u_- - u)}{\rho_- - \rho}.
\tag{3.16}
$$

Thus, by a simple calculation, (3.14) is equivalent to

$$
-\sqrt{A\rho^2 + B} < \frac{\rho_-\rho}{\rho_- - \rho}(u_- - u) < -\sqrt{A\rho_-^2 + B},
\tag{3.17}
$$

and (3.15) is equivalent to

$$
\sqrt{A\rho^2 + B} < \frac{\rho_-\rho}{\rho_- - \rho}(u_- - u) < \sqrt{A\rho_-^2 + B}.
\tag{3.18}
$$

(3.17) and (3.18) imply that $\rho > \rho_-$, $u < u_-$ and $\rho < \rho_-$, $u < u_-$, respectively.

Therefore, associating with λ_1, we obtain the following backward shock denoted by $\overleftarrow{S}$

$$
\overleftarrow{S}: \quad
\begin{cases}
\sigma = u_- - \rho\sqrt{\dfrac{1}{\rho_-\rho}\left(A + \dfrac{B}{\rho_-\rho}\right)}, \\[2ex]
u = u_- - \sqrt{\dfrac{1}{\rho_-\rho}\left(A + \dfrac{B}{\rho_-\rho}\right)}(\rho - \rho_-), \quad \rho > \rho_-,
\end{cases}
\tag{3.19}
$$

and associating with λ_2, we get the forward shock denoted by $\overrightarrow{S}$

$$
\overrightarrow{S}: \quad
\begin{cases}
\sigma = u_- + \rho\sqrt{\dfrac{1}{\rho_-\rho}\left(A + \dfrac{B}{\rho_-\rho}\right)}, \\[2ex]
u = u_- + \sqrt{\dfrac{1}{\rho_-\rho}\left(A + \dfrac{B}{\rho_-\rho}\right)}(\rho - \rho_-), \quad \rho < \rho_-.
\end{cases}
\tag{3.20}
$$

The curves expressed by the second equations of (3.19) and (3.20) in the (u, ρ)-plane are called the backward shock curve and forward shock curve, respectively. Similar with the analysis of the rarefaction

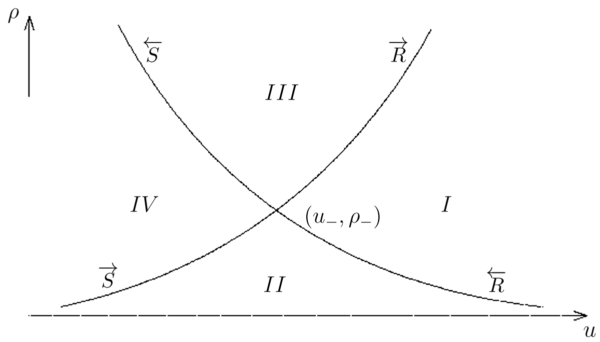

Fig. 1. Curves of elementary waves.

waves, we can get $u_\rho < 0$ and $u_{\rho\rho} > 0$ for the backward shock curve, which mean that this curve is monotonic decreasing and convex. While we can get $u_\rho > 0$ and $u_{\rho\rho} < 0$ for the forward shock curve, which mean that this curve is monotonic increasing and concave. In addition, we verify that $\lim_{\rho\to+\infty} u = -\infty$ for the backward shock curve and $\lim_{\rho\to 0} u = -\infty$ for the forward shock curve.

Starting from a constant state (u_-,ρ_-), we draw the elementary wave curves passing through this point in the phase-plane (u,ρ). Then the phase plane can be divided into four domains I, II, III and IV(u_-,ρ_-) (see Fig. 1).

By the analysis method in phase plane, it is easy to construct Riemann solutions for any given (u_+,ρ_+) as follows:

(1) $(u_+,\rho_+) \in \mathrm{I}(u_-,\rho_-)$: $\overleftarrow{R}+\overrightarrow{R}$;
(2) $(u_+,\rho_+) \in \mathrm{II}(u_-,\rho_-)$: $\overleftarrow{R}+\overrightarrow{S}$;
(3) $(u_+,\rho_+) \in \mathrm{III}(u_-,\rho_-)$: $\overleftarrow{S}+\overrightarrow{R}$;
(4) $(u_+,\rho_+) \in \mathrm{IV}(u_-,\rho_-)$: $\overleftarrow{S}+\overrightarrow{S}$.

Then we directly have the following result.

Theorem 1. *For the Riemann problem of (1.1) and (1.6) with (2.1), there exists a unique entropy solution, which consists of rarefaction waves, shock waves, and constant states.*

4. Formation of δ-shocks

In this section, we describe the formation of δ-shock waves in the Riemann solutions of system (1.1) and (1.6) in the case $(u_+,\rho_+) \in \mathrm{IV}(u_-,\rho_-)$ with $u_- > u_+$ as the pressure vanishes.

4.1. Limit behavior of the Riemann solutions as $A, B \to 0$

When $(u_+,\rho_+) \in \mathrm{IV}(u_-,\rho_-)$, for each pair of fixed $A > 0$ and $B > 0$, suppose that (u_*^{AB},ρ_*^{AB}) is the intermediate state connected with (u_-,ρ_-) by $\overleftarrow{S}$ with speed σ_1^{AB} and (u_+,ρ_+) by $\overrightarrow{S}$ with speed σ_2^{AB}, then it follows

$$\overleftarrow{S}:\quad \begin{cases} \sigma_1^{AB} = u_- - \rho_*^{AB}\sqrt{\dfrac{1}{\rho_-\rho_*^{AB}}\left(A + \dfrac{B}{\rho_-\rho_*^{AB}}\right)}, \\ u_*^{AB} = u_- - \sqrt{\dfrac{1}{\rho_-\rho_*^{AB}}\left(A + \dfrac{B}{\rho_-\rho_*^{AB}}\right)}(\rho_*^{AB} - \rho_-), \quad \rho_*^{AB} > \rho_-, \end{cases} \tag{4.1}$$

and

$$\overrightarrow{S}\colon\quad \begin{cases} \sigma_2^{AB} = u_*^{AB} + \rho_+\sqrt{\dfrac{1}{\rho_*^{AB}\rho_+}\left(A + \dfrac{B}{\rho_*^{AB}\rho_+}\right)}, \\ u_+ = u_*^{AB} + \sqrt{\dfrac{1}{\rho_*^{AB}\rho_+}\left(A + \dfrac{B}{\rho_*^{AB}\rho_+}\right)}\left(\rho_+ - \rho_*^{AB}\right), \quad \rho_+ < \rho_*^{AB}. \end{cases} \tag{4.2}$$

In what follows, we give some lemmas to show the limit behavior of the Riemann solutions of system (1.1) and (1.6) as $A, B \to 0$.

Lemma 1. $\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} = +\infty$.

Proof. Eliminating u_*^{AB} in the second equations of (4.1) and (4.2) gives

$$u_- - u_+ = \sqrt{\frac{1}{\rho_-\rho_*^{AB}}\left(A + \frac{B}{\rho_-\rho_*^{AB}}\right)}\left(\rho_*^{AB} - \rho_-\right) + \sqrt{\frac{1}{\rho_*^{AB}\rho_+}\left(A + \frac{B}{\rho_*^{AB}\rho_+}\right)}\left(\rho_*^{AB} - \rho_+\right). \tag{4.3}$$

If $\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} = M \in (\max(\rho_-, \rho_+), +\infty)$, then by taking the limit of (4.3) as $A, B \to 0$, we obtain that $u_- - u_+ = 0$, which contradicts with $u_- > u_+$. Therefore we must have $\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} = +\infty$. $\square$

By Lemma 1, from (4.3) we immediately have the following lemma.

Lemma 2. $\lim_{\substack{A\to 0\\ B\to 0}} A\rho_*^{AB} = \frac{\rho_+\rho_-}{(\sqrt{\rho_+}+\sqrt{\rho_-})^2}(u_- - u_+)^2$.

Lemma 3. *Set* $\lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB} = \sigma$, *then* $\lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB} = \lim_{\substack{A\to 0\\ B\to 0}} \sigma_1^{AB} = \lim_{\substack{A\to 0\\ B\to 0}} \sigma_2^{AB} = \sigma \in (u_+, u_-)$.

Proof. From the first equations of the Rankine–Hugoniot relation (3.12) for $\overleftarrow{S}$ and $\overrightarrow{S}$, by Lemma 1, it gives

$$\lim_{\substack{A\to 0\\ B\to 0}} \sigma_1^{AB} = \lim_{\substack{A\to 0\\ B\to 0}} \frac{\rho_- u_- - \rho_*^{AB} u_*^{AB}}{\rho_- - \rho_*^{AB}} = \lim_{\substack{A\to 0\\ B\to 0}} \frac{\frac{\rho_- u_-}{\rho_*^{AB}} - u_*^{AB}}{\frac{\rho_-}{\rho_*^{AB}} - 1} = \lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB},$$

and

$$\lim_{\substack{A\to 0\\ B\to 0}} \sigma_2^{AB} = \lim_{\substack{A\to 0\\ B\to 0}} \frac{\rho_*^{AB} u_*^{AB} - \rho_+ u_+}{\rho_*^{AB} - \rho_+} = \lim_{\substack{A\to 0\\ B\to 0}} \frac{u_*^{AB} - \frac{\rho_+ u_+}{\rho_*^{AB}}}{1 - \frac{\rho_+}{\rho_*^{AB}}} = \lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB},$$

which immediately lead to $\lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB} = \lim_{\substack{A\to 0\\ B\to 0}} \sigma_1^{AB} = \lim_{\substack{A\to 0\\ B\to 0}} \sigma_2^{AB} = \sigma$.

From the second equation of (4.1), observing Lemmas 1–2, we get

$$\begin{aligned} \lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB} &= u_- - \lim_{\substack{A\to 0\\ B\to 0}} \sqrt{\frac{1}{\rho_-\rho_*^{AB}}\left(A + \frac{B}{\rho_-\rho_*^{AB}}\right)\left(\rho_*^{AB} - \rho_-\right)^2} \\ &= u_- - \lim_{\substack{A\to 0\\ B\to 0}} \sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*^{AB}}\right)\left(A\left(\rho_*^{AB} - \rho_-\right) + B\left(\frac{1}{\rho_-} - \frac{1}{\rho_*^{AB}}\right)\right)} \\ &= u_- - \lim_{\substack{A\to 0\\ B\to 0}} \sqrt{\frac{1}{\rho_-}A\rho_*^{AB}} \\ &< u_-. \end{aligned}$$

Performing the similar analysis as above, from the second equation of (4.2) and Lemmas 1–2, we obtain

$$\begin{aligned}\lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB} &= u_+ + \lim_{\substack{A\to 0\\ B\to 0}} \sqrt{\frac{1}{\rho_+\rho_*^{AB}}\left(A+\frac{B}{\rho_+\rho_*^{AB}}\right)\left(\rho_*^{AB}-\rho_+\right)^2}\\ &= u_+ + \lim_{\substack{A\to 0\\ B\to 0}} \sqrt{\left(\frac{1}{\rho_+}-\frac{1}{\rho_*^{AB}}\right)\left(A\left(\rho_*^{AB}-\rho_+\right)+B\left(\frac{1}{\rho_+}-\frac{1}{\rho_*^{AB}}\right)\right)}\\ &= u_+ + \lim_{\substack{A\to 0\\ B\to 0}} \sqrt{\frac{1}{\rho_+}A\rho_*^{AB}}\\ &> u_+.\end{aligned}$$

The proof is completed. □

Lemma 4.

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \rho_*^{AB}\, d\xi = \sigma[\rho]-[\rho u], \tag{4.4}$$

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \rho_*^{AB}u_*^{AB}\, d\xi = \sigma[\rho u]-\left[\rho u^2\right]. \tag{4.5}$$

Proof. The first equations of the Rankine–Hugoniot relation (3.12) for $\overleftarrow{S}$ and $\overrightarrow{S}$ read

$$\begin{cases}\sigma_1^{AB}\left(\rho_- - \rho_*^{AB}\right) = \rho_-u_- - \rho_*^{AB}u_*^{AB},\\ \sigma_2^{AB}\left(\rho_*^{AB}-\rho_+\right) = \rho_*^{AB}u_*^{AB}-\rho_+u_+,\end{cases} \tag{4.6}$$

from which we have

$$\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB}\left(\sigma_2^{AB}-\sigma_1^{AB}\right) = \lim_{\substack{A\to 0\\ B\to 0}}\left(-\sigma_1^{AB}\rho_- + \sigma_2^{AB}\rho_+ + \rho_-u_- - \rho_+u_+\right) = \sigma[\rho]-[\rho u]. \tag{4.7}$$

Similarly, from the second equations of the Rankine–Hugoniot relation (3.12) for $\overleftarrow{S}$ and $\overrightarrow{S}$

$$\begin{cases}\sigma_1^{AB}\left(\rho_-u_- - \rho_*^{AB}u_*^{AB}\right) = \left(\rho_-u_-^2 - \rho_*^{AB}\left(u_*^{AB}\right)^2\right) + A\left(\rho_- - \rho_*^{AB}\right) + B\left(\frac{1}{\rho_*^{AB}}-\frac{1}{\rho_-}\right),\\ \sigma_2^{AB}\left(\rho_*^{AB}u_*^{AB} - \rho_+u_+\right) = \left(\rho_*^{AB}\left(u_*^{AB}\right)^2 - \rho_+u_+^2\right) + A\left(\rho_*^{AB}-\rho_+\right) + B\left(\frac{1}{\rho_+}-\frac{1}{\rho_*^{AB}}\right),\end{cases} \tag{4.8}$$

we obtain

$$\begin{aligned}&\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB}u_*^{AB}\left(\sigma_2^{AB}-\sigma_1^{AB}\right)\\ &\quad= \lim_{\substack{A\to 0\\ B\to 0}}\left(-\sigma_1^{AB}\rho_-u_- + \sigma_2^{AB}\rho_+u_+ + \rho_-u_-^2 - \rho_+u_+^2 + A(\rho_- - \rho_+) + B\left(\frac{1}{\rho_+}-\frac{1}{\rho_-}\right)\right)\\ &\quad= \sigma[\rho u]-\left[\rho u^2\right].\end{aligned} \tag{4.9}$$

Thus, from (4.7) and (4.9) it follows that

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \rho_*^{AB}\, d\xi = \lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB}\big(\sigma_2^{AB}-\sigma_1^{AB}\big) = \sigma[\rho]-[\rho u]$$

and

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \rho_*^{AB} u_*^{AB}\, d\xi = \lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} u_*^{AB}\big(\sigma_2^{AB}-\sigma_1^{AB}\big) = \sigma[\rho u]-\big[\rho u^2\big].$$

The proof is finished. □

Lemma 5. *For σ mentioned above, we have*

$$\sigma = \frac{[\rho u]+\sqrt{[\rho u]^2-[\rho][\rho u^2]}}{[\rho]} = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+}+\sqrt{\rho_-}} \tag{4.10}$$

when $\rho_- \neq \rho_+$, and

$$\sigma = \frac{u_- + u_+}{2} \tag{4.11}$$

when $\rho_- = \rho_+$.

Proof. By Lemmas 3–4, one can find that

$$\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} u_*^{AB}\big(\sigma_2^{AB}-\sigma_1^{AB}\big) = \lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB}\big(\sigma_2^{AB}-\sigma_1^{AB}\big) \cdot \lim_{\substack{A\to 0\\ B\to 0}} u_*^{AB},$$

then we have

$$\sigma[\rho u]-\big[\rho u^2\big] = \big(\sigma[\rho]-[\rho u]\big)\sigma, \tag{4.12}$$

from which, noticing $\sigma \in (u_+, u_-)$, the sought results in (4.10) and (4.11) are easily reached. □

Remark 1. As $A, B \to 0$, the velocities of shocks $\overleftarrow{S}$ and $\overrightarrow{S}$ and the intermediate velocity u_*^{AB} of (1.1) and (1.6) approach to σ, which determines the δ-shock solution of (1.3), and the intermediate density ρ_*^{AB} becomes singular simultaneously.

4.2. δ-shocks and concentration

Now, we present the following conclusion characterizing the vanishing pressure limit for the case $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$.

Theorem 2. *Let $u_- > u_+$ and $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$. For any fixed $A, B > 0$, assume that (ρ^{AB}, u^{AB}) is the two-shock Riemann solution of system (1.1) and (1.6) with Riemann data (2.1) constructed in Section 3.*

Then as $A, B \to 0$, ρ^{AB} and $\rho^{AB}u^{AB}$ converge in the sense of distributions, and the limit functions of ρ^{AB} and $\rho^{AB}u^{AB}$ are all the sums of a step function and a Dirac delta function with weights

$$\frac{t}{\sqrt{1+\sigma^2}}\big(\sigma[\rho]-[\rho u]\big) \quad \text{and} \quad \frac{t}{\sqrt{1+\sigma^2}}\big(\sigma[\rho u]-\big[\rho u^2\big]\big), \tag{4.13}$$

respectively, which just form the δ-shock solution of (1.3) with the same Riemann data (2.1).

Proof. 1. Set $\xi = x/t$. Then for any $A > 0$ and $B > 0$, the two-shock Riemann solution can be expressed as

$$\big(u^{AB}, \rho^{AB}\big)(\xi) = \begin{cases} (u_-, \rho_-), & \xi < \sigma_1^{AB}, \\ (u_*^{AB}(\xi), \rho_*^{AB}(\xi)), & \sigma_1^{AB} < \xi < \sigma_2^{AB}, \\ (u_+, \rho_+), & \xi > \sigma_2^{AB}, \end{cases} \tag{4.14}$$

which satisfies the weak formulations

$$-\int_{-\infty}^{+\infty} \rho^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi + \int_{-\infty}^{+\infty} \rho^{AB}\phi\,d\xi = 0, \tag{4.15}$$

and

$$-\int_{-\infty}^{+\infty} \rho^{AB}u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi - \int_{-\infty}^{+\infty}\left(A\rho^{AB} - \frac{B}{\rho^{AB}}\right)\phi'\,d\xi + \int_{-\infty}^{+\infty} \rho^{AB}u^{AB}\phi\,d\xi = 0, \tag{4.16}$$

for any $\phi \in C_0^1(-\infty, +\infty)$.

2. Consider the limits of $\rho^{AB}u^{AB}$ and ρ^{AB} depending on ξ. The first integral on the left hand side of (4.16) can be rewritten as

$$\int_{-\infty}^{+\infty} \rho^{AB}u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi = \left(\int_{-\infty}^{\sigma_1^{AB}} + \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} + \int_{\sigma_2^{AB}}^{+\infty}\right)\rho^{AB}u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi. \tag{4.17}$$

With the regular calculation, we get the following fact that

$$\begin{aligned} &\lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{\sigma_1^{AB}} \rho^{AB}u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi + \lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_2^{AB}}^{+\infty} \rho^{AB}u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi \\ &= \lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{\sigma_1^{AB}} \rho_-u_-(u_- - \xi)\phi'\,d\xi + \lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_2^{AB}}^{+\infty} \rho_+u_+(u_+ - \xi)\phi'\,d\xi \\ &= \big(\sigma[\rho u] - \big[\rho u^2\big]\big)\phi(\sigma) + \int_{-\infty}^{+\infty} H(\xi-\sigma)\phi\,d\xi, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
&\lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \rho^{AB} u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \rho_*^{AB} u_*^{AB}\big(u_*^{AB}-\xi\big)\phi'\,d\xi \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} u_*^{AB}\big(\sigma_2^{AB}-\sigma_1^{AB}\big)\left(\frac{\phi(\sigma_2^{AB})-\phi(\sigma_1^{AB})}{\sigma_2^{AB}-\sigma_1^{AB}}u_*^{AB} - \frac{\sigma_2^{AB}\phi(\sigma_2^{AB})-\sigma_1^{AB}\phi(\sigma_1^{AB})}{\sigma_2^{AB}-\sigma_1^{AB}}\right. \\
&\qquad \left. + \frac{1}{\sigma_2^{AB}-\sigma_1^{AB}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \phi\,d\xi\right) \\
&= \big(\sigma[\rho u]-\big[\rho u^2\big]\big)\big(\sigma\phi'(\sigma)-\sigma\phi'(\sigma)-\phi(\sigma)+\phi(\sigma)\big) \\
&= 0,
\end{aligned} \tag{4.19}$$

then, a combination of (4.18) and (4.19) immediately yields

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{+\infty} \rho^{AB} u^{AB}\big(u^{AB}-\xi\big)\phi'\,d\xi = \big(\sigma[\rho u]-\big[\rho u^2\big]\big)\phi(\sigma) + \int_{-\infty}^{+\infty} H(\xi-\sigma)\phi\,d\xi, \tag{4.20}$$

where

$$H(\xi-\sigma) = \begin{cases} \rho_- u_-, & \xi<\sigma, \\ \rho_+ u_+, & \xi>\sigma. \end{cases}$$

Compute the second term on the left hand side of (4.16), noticing Lemmas 1–3, we obtain that

$$\begin{aligned}
&\lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{+\infty} \left(A\rho^{AB} - \frac{B}{\rho^{AB}}\right)\phi'\,d\xi \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \left(\int_{-\infty}^{\sigma_1^{AB}} + \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} + \int_{\sigma_2^{AB}}^{+\infty}\right)\left(A\rho^{AB} - \frac{B}{\rho^{AB}}\right)\phi'\,d\xi \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{\sigma_1^{AB}} \left(A\rho_- - \frac{B}{\rho_-}\right)\phi'\,d\xi + \lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_1^{AB}}^{\sigma_2^{AB}} \left(A\rho_*^{AB} - \frac{B}{\rho_*^{AB}}\right)\phi'\,d\xi \\
&\quad + \lim_{\substack{A\to 0\\ B\to 0}} \int_{\sigma_2^{AB}}^{+\infty} \left(A\rho_+ - \frac{B}{\rho_+}\right)\phi'\,d\xi \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \left(\left(A\rho_- - \frac{B}{\rho_-}\right)\phi\big(\sigma_1^{AB}\big) - \left(A\rho_+ - \frac{B}{\rho_+}\right)\phi\big(\sigma_2^{AB}\big) + A\rho_*^{AB}\big(\phi\big(\sigma_2^{AB}\big)-\phi\big(\sigma_1^{AB}\big)\big)\right. \\
&\quad \left. - \frac{B}{\rho_*^{AB}}\big(\phi\big(\sigma_2^{AB}\big)-\phi\big(\sigma_1^{AB}\big)\big)\right) \\
&= 0.
\end{aligned} \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.16), it is easy to have

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{+\infty} \left(\rho^{AB} u^{AB} - H(\xi-\sigma)\right)\phi\, d\xi = \left(\sigma[\rho u] - \left[\rho u^2\right]\right)\phi(\sigma) \tag{4.22}$$

for any $\phi \in C_0^1(-\infty, +\infty)$.

Similarly, from (4.15) we can get

$$\lim_{\substack{A\to 0\\ B\to 0}} \int_{-\infty}^{+\infty} \left(\rho^{AB} - \widetilde{H}(\xi-\sigma)\right)\phi\, d\xi = \left(\sigma[\rho] - [\rho u]\right)\phi(\sigma) \tag{4.23}$$

for any $\phi \in C_0^1(-\infty, +\infty)$, where

$$\widetilde{H}(\xi-\sigma) = \begin{cases} \rho_-, & \xi < \sigma, \\ \rho_+, & \xi > \sigma. \end{cases}$$

3. Then take into account the limits of $\rho^{AB}u^{AB}$ and ρ^{AB} depending on t. For any $\psi \in C_0^\infty(R \times R^+)$, noticing (4.22), we have

$$\begin{aligned}
&\lim_{\substack{A\to 0\\ B\to 0}} \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho^{AB}(x/t)u^{AB}(x/t)\psi(x,t)\, dx\, dt \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho^{AB}(\xi)u^{AB}(\xi)\psi(\xi t,t) d(\xi t)\, dt \\
&= \lim_{\substack{A\to 0\\ B\to 0}} \int_0^{+\infty} t\left(\int_{-\infty}^{+\infty} \rho^{AB}(\xi)u^{AB}(\xi)\psi(\xi t,t)\, d\xi\right) dt \\
&= \int_0^{+\infty} t\left(\left(\sigma[\rho u] - \left[\rho u^2\right]\right)\psi(\sigma t,t) + \int_{-\infty}^{+\infty} H(\xi-\sigma)\psi(\xi t,t)\, d\xi\right) dt \\
&= \int_0^{+\infty} t\left(\int_{-\infty}^{+\infty} H(\xi-\sigma)\psi(\xi t,t)\, d\xi\right) dt + \int_0^{+\infty} t\left(\sigma[\rho u] - \left[\rho u^2\right]\right)\psi(\sigma t,t)\, dt \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} H(x-\sigma t)\psi(x,t)\, dx\, dt + \int_0^{+\infty} \left(\sigma[\rho u] - \left[\rho u^2\right]\right)t\psi(\sigma t,t)\, dt,
\end{aligned} \tag{4.24}$$

in which by definition (2.3), we obtain

$$\int_0^{+\infty} \left(\sigma[\rho u] - \left[\rho u^2\right]\right)t\psi(\sigma t,t)\, dt = \left\langle w_1(\cdot)\delta_S, \psi(\cdot,\cdot)\right\rangle$$

where

$$w_1(t) = \frac{t}{\sqrt{1+\sigma^2}}\big(\sigma[\rho u] - [\rho u^2]\big).$$

Similarly, from (4.23), it gives

$$\begin{aligned}
&\lim_{\substack{A\to 0\\ B\to 0}} \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho^{AB}(x/t)\psi(x,t)\,dx\,dt \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} \widetilde{H}(x-\sigma t)\psi(x,t)\,dx\,dt + \int_0^{+\infty} \big(\sigma[\rho] - [\rho u]\big)t\psi(\sigma t,t)\,dt, \qquad (4.25)
\end{aligned}$$

in which by definition (2.3), we have

$$\int_0^{+\infty} \big(\sigma[\rho] - [\rho u]\big)t\psi(\sigma t,t)\,dt = \big\langle w_2(\cdot)\delta_S, \psi(\cdot,\cdot)\big\rangle$$

where

$$w_2(t) = \frac{t}{\sqrt{1+\sigma^2}}\big(\sigma[\rho] - [\rho u]\big).$$

The proof of Theorem 2 is completed. □

5. Formation of vacuums

In this section, we analyze the formation of vacuum states in the Riemann solutions of system (1.1) and (1.6) in the case $(u_+,\rho_+) \in \mathrm{I}(u_-,\rho_-)$ with $u_- < u_+$ and $\rho_\pm > 0$ as the pressure decreases. At this moment, the Riemann solution consists of two rarefaction waves $\overleftarrow{R}$, $\overrightarrow{R}$, and an intermediate state (u_*^{AB},ρ_*^{AB}) besides two constant states $(u_\pm,\rho_\pm)$. They read

$$\overleftarrow{R}: \quad \begin{cases} \xi = \lambda_1 = u - \sqrt{A + \dfrac{B}{\rho^2}}, \\ u = u_- - \displaystyle\int_{\rho_-}^{\rho} \frac{\sqrt{A+\frac{B}{s^2}}}{s}\,ds, \quad \rho_*^{AB} \leqslant \rho \leqslant \rho_-, \end{cases} \tag{5.1}$$

and

$$\overrightarrow{R}: \quad \begin{cases} \xi = \lambda_2 = u + \sqrt{A + \dfrac{B}{\rho^2}}, \\ u = u_+ + \displaystyle\int_{\rho_+}^{\rho} \frac{\sqrt{A+\frac{B}{s^2}}}{s}\,ds, \quad \rho_*^{AB} \leqslant \rho \leqslant \rho_+. \end{cases} \tag{5.2}$$

Now, from the second equations of (5.1) and (5.2), using (3.11), it follows that the intermediate state (u_*^{AB},ρ_*^{AB}) satisfies

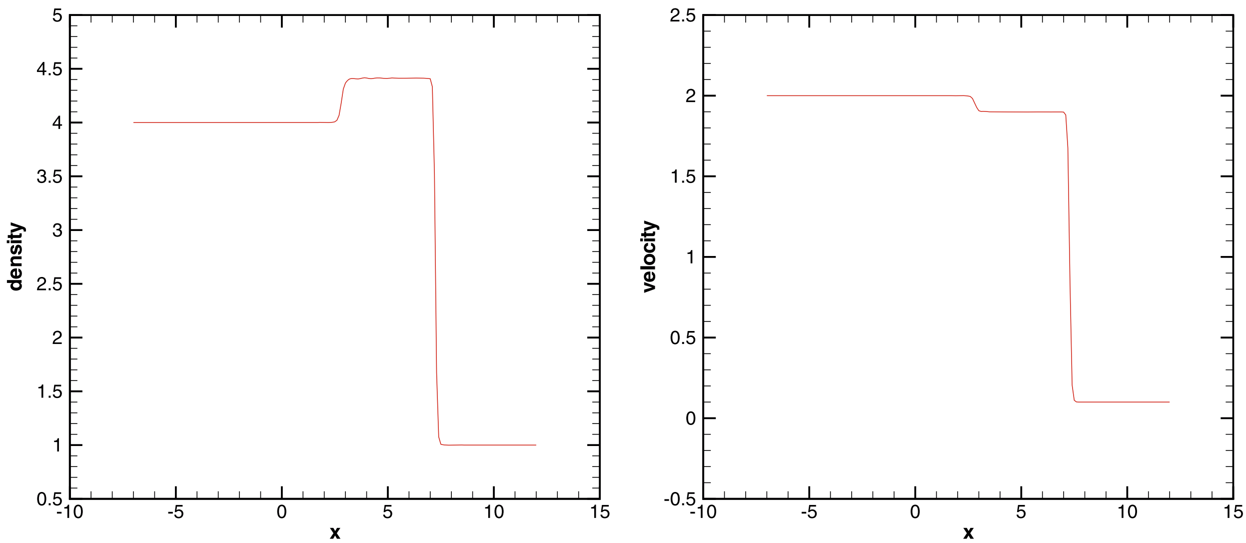

Fig. 2. Density (left) and velocity (right) of δ-shock wave for $A = 1.0$ and $B = 1.0$.

$$\begin{aligned}
u_+ - u_- = \lim_{\substack{A\to 0\\ B\to 0}} \Bigg\{ & 2\sqrt{A + \frac{B}{(\rho_*^{AB})^2}} - 2\sqrt{A}\ln\left(\sqrt{A + \frac{B}{(\rho_*^{AB})^2}} + \sqrt{A}\right) - 2\sqrt{A}\ln \rho_*^{AB} \\
& - \sqrt{A + \frac{B}{\rho_-^2}} + \sqrt{A}\ln\left(\sqrt{A + \frac{B}{\rho_-^2}} + \sqrt{A}\right) + \sqrt{A}\ln\rho_- \\
& - \sqrt{A + \frac{B}{\rho_+^2}} + \sqrt{A}\ln\left(\sqrt{A + \frac{B}{\rho_+^2}} + \sqrt{A}\right) + \sqrt{A}\ln\rho_+ \Bigg\},
\end{aligned} \tag{5.3}$$

which implies the following result.

Theorem 3. *When $u_- < u_+$ and $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$, the vacuum state occurs as $A, B \to 0$, and two rarefaction waves become two contact discontinuities connecting the constant states $(u_\pm, \rho_\pm)$ and the vacuum $(\rho = 0)$.*

Indeed, if $\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} = K \in (0, \min(\rho_-, \rho_+))$, then (5.3) leads to $u_+ - u_- = 0$, which contradicts with $u_- < u_+$. Thus we have $\lim_{\substack{A\to 0\\ B\to 0}} \rho_*^{AB} = 0$, which just means vacuum occurs. Moreover, as $A, B \to 0$, one can directly derive that $\lambda_1, \lambda_2 \to u$, and two rarefaction waves, $\overleftarrow{R}$ and $\overrightarrow{R}$, tend to two contact discontinuities $\xi = x/t = u_\pm$, respectively. These reach the desired conclusion.

In summary, when $A, B \to 0$, the limit of solution of Riemann problem (1.1), (1.6) and (2.1) in this case can be expressed as (2.2), which is just the solution of (1.3) and (2.1) containing two contact discontinuities $\xi = x/t = u_\pm$ and a vacuum state in between. Therefore, it is proved that the vacuum state of (1.3) is obtained as a limit of a two-rarefaction-wave solution of (1.1) and (1.6) as the perturbed pressure vanishes.

6. Numerical simulations

In order to verify the validity of the formation of δ-shocks and vacuums mentioned in Sections 4 and 5, we presents two selected groups of representative numerical simulations. Many more numerical tests have been performed to make sure that what are presented are not numerical artifacts. To discretize the system, we use the fifth-order weighted essentially non-oscillatory scheme and third-order Runge–Kutta method [32] with 150×150 cells.

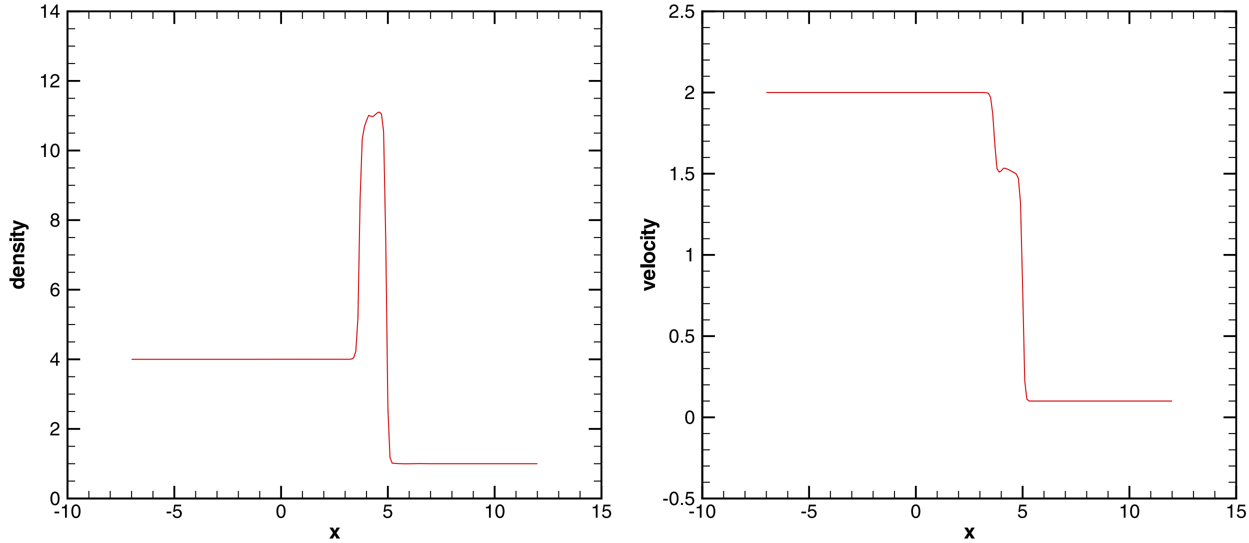

Fig. 3. Density (left) and velocity (right) of δ-shock wave for $A = 0.20$ and $B = 0.14$.

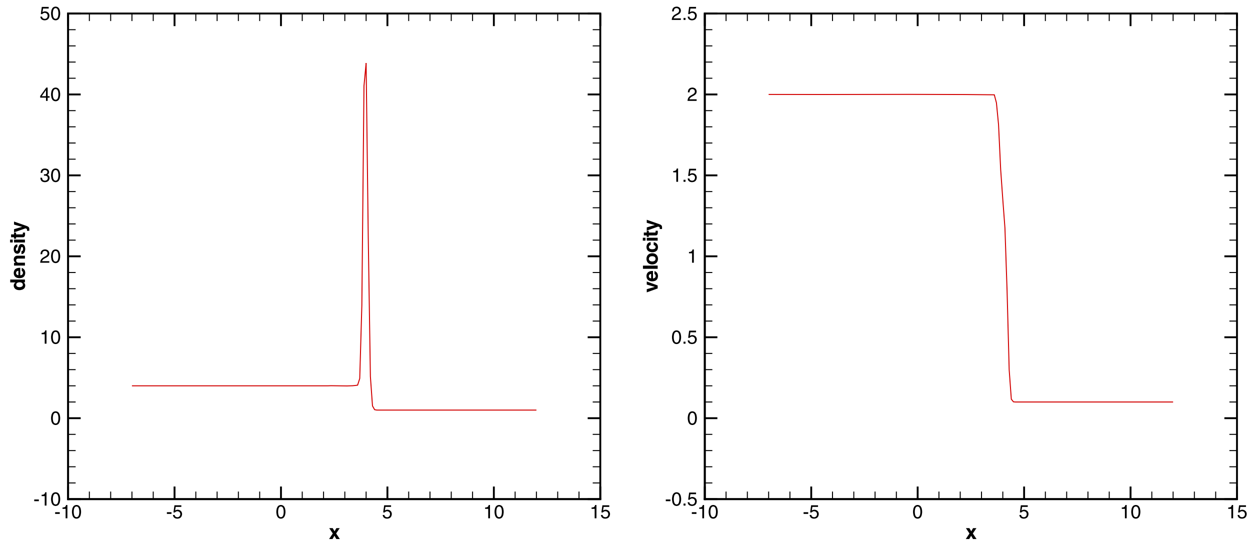

Fig. 4. Density (left) and velocity (right) of δ-shock wave for $A = 0.00014$ and $B = 0.0009$.

For the formation of δ-shocks, we take the initial data

$$\rho_- = 4.0, \qquad u_- = 2.0, \qquad \rho_+ = 1.0, \qquad u_+ = 0.1,$$

and compute the solution up to $t = 3.0$, the numerical simulations for different choices of parameters A, B, starting with $A = B = 1.0$, then $A = 0.20$, $B = 0.14$, and finally $A = 0.00014$, $B = 0.0009$, are presented in Figs. 2–4 which show the process of concentration and formation of a δ-shock in the vanishing pressure limit of solutions containing two shocks.

From these numerical results, we can clearly observe that, as A, B decreases, the locations of the two shocks become closer, and the density of the intermediate state increases dramatically, while the velocity is closer to a step function. In the end, as $A, B \to 0$, along with the intermediate state, the two shocks coincide to form a δ-shock of the transport equations (1.3), while the velocity is a step function.

For the formation of vacuum states, the initial data is given as

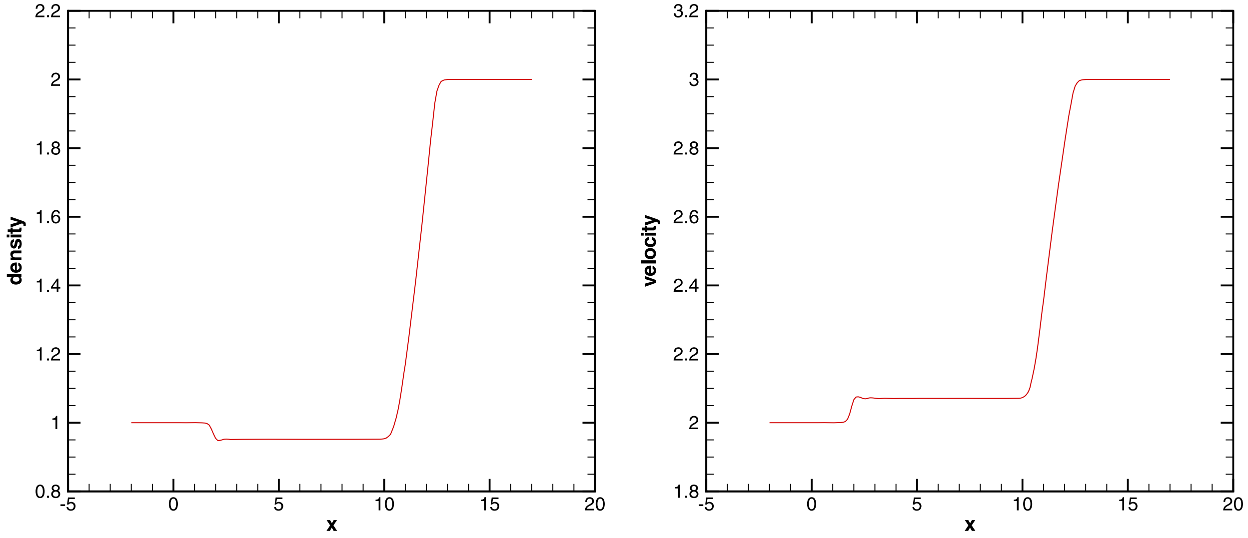

Fig. 5. Density (left) and velocity (right) of vacuum states for $A = 1.0$ and $B = 1.0$.

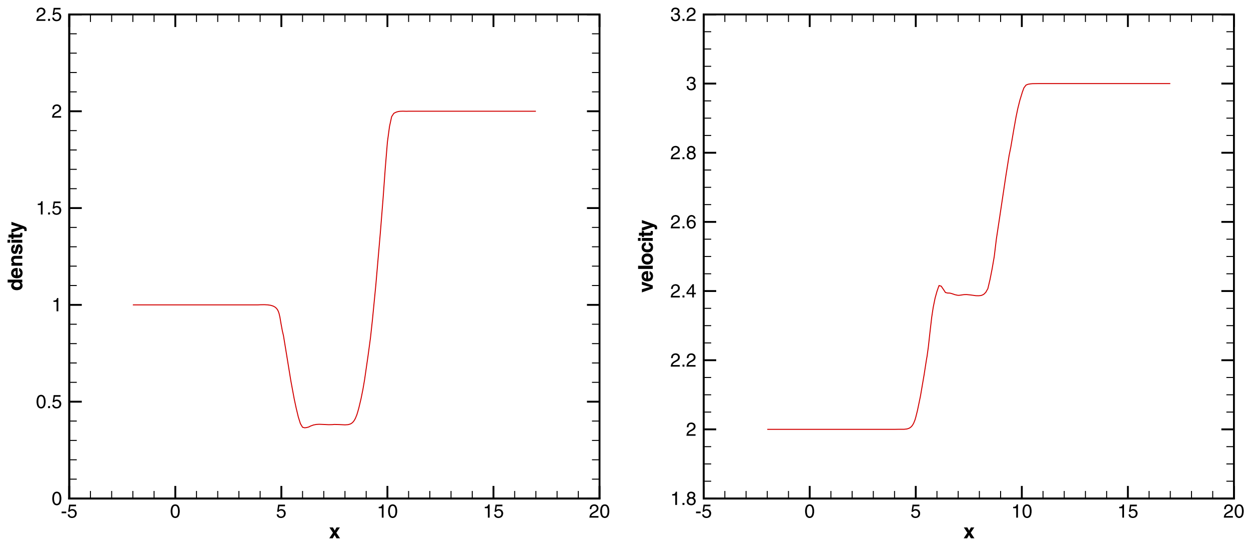

Fig. 6. Density (left) and velocity (right) of vacuum states for $A = 0.09$ and $B = 0.025$.

$$\rho_- = 1.0, \qquad u_- = 2.0, \qquad \rho_+ = 2.0, \qquad u_+ = 3.0.$$

The numerical simulations for $t = 3.0$ are listed in Figs. 5–7 which match different choices of parameters A, B starting with $A = B = 1.0$, then $A = 0.09$, $B = 0.025$, and finally $A = 0.00014$, $B = 0.00025$. These figures exhibit the process of cavitation and formation of a vacuum state in the vanishing pressure limit of solutions containing two rarefaction waves.

From these numerical results, we can clearly see that, as A, B decreases, the boundaries of two rarefaction waves become closer and closer, and along with the density of the intermediate state approaching zero, an inside vacuum state is forming, while the velocity is approximating to a linear function. As a matter of fact, as $A, B \to 0$, a two-rarefaction-wave solution tends to a two-contact-discontinuity solution including a vacuum state of the transport equations (1.3).

To sum up, all the numerical simulations given above are in completely coincident with the theoretical analysis.

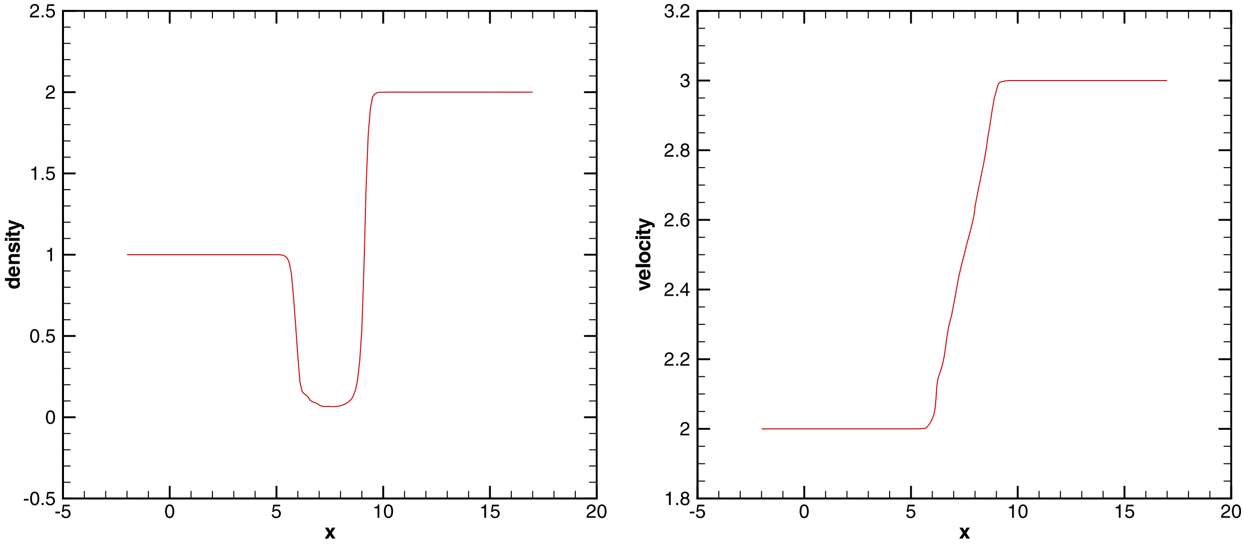

Fig. 7. Density (left) and velocity (right) of vacuum states for $A = 0.00014$ and $B = 0.00025$.

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