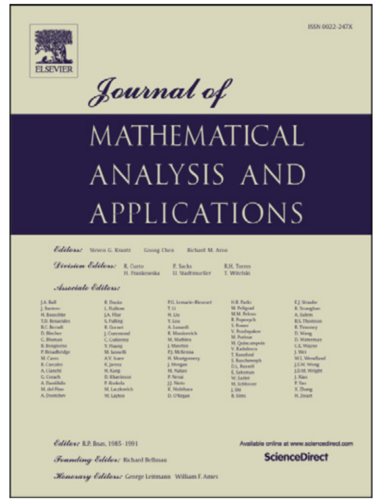

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A pre-order principle and set-valued Ekeland variational principle[1]

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Abstract. We establish a pre-order principle. From the principle, we obtain a very general set-valued Ekeland variational principle, where the objective function is a set-valued map taking values in a quasi ordered linear space and the perturbation contains a family of set-valued maps satisfying certain property. From this general set-valued Ekeland variational principle, we deduce a number of particular versions of set-valued Ekeland variational principle, which include many known Ekeland variational principles, their improvements and some new results.

Key words: Pre-order Principle, Ekeland variational principle, Set-valued map, Perturbation, Locally convex space, Vector optimization

Mathematics Subject Classifications (2000) 49J53 · 90C48 · 65K10 · 46A03

1. Introduction

In 1972, Ekeland [13] (see also [14, 15]) gave a variational principle, now known as Ekeland variational principle (for short, EVP), which says that for any lower semicontinuous function f bounded from below on a complete metric space, a slightly perturbed function has a strict minimum. In the last four decades, the famous EVP emerged as one of the most important results of nonlinear analysis and it has significant applications in optimization, optimal control theory, game theory, fixed point theory, nonlinear equations, dynamical systems, etc; see for example [3,

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10, 14, 15, 21, 36, 51]. Motivated by this wide usefulness, many authors have been interested in extending EVP to vector-valued maps or set-valued maps with values in a vector space quasi ordered by a convex cone, see, for example, [2, 4-10, 12, 16, 17, 21-25, 27-31, 33-35, 40-42, 44, 45, 47, 48] and the references therein.

Recently, there are many new and interesting results of EVP for set-valued maps. Here, we only mention some results which are related to this paper. In [24], Ha introduced a strict minimizer of a set-valued map by virtue of Kuroiwa's set optimization criterion (see [32]). Using the concept of cone extensions and Dancs-Hegedus-Medvegyev theorem (see [11]) she established a new version (see [24, Theorem 3.1]) of EVP for set-valued maps, which is expressed by the existence of a strict minimizer for a perturbed set-valued optimization problem. Inspired by Ha's work and using the Gerstewitz's function (see, for example, [18-20]), the author [41] obtained an improvement of Ha's version of EVP by relaxing several assumptions. In the above Ha's and Qiu's versions, the perturbation is given by a nonzero element k_0 of the ordering cone multiplied by the distance function $d(\cdot,\cdot)$, i.e., its form is as $d(\cdot,\cdot)k_0$ (disregarding a constant coefficient); and the objective functions are set-valued maps. Bednarczuk and Zagrodny [7] proved a vectorial EVP for a sequentially lower monotone vector-valued map (which is called a monotonically semicontinuous map in [7]), where the perturbation is given by a convex subset H of the ordering cone multiplied by the distance function $d(\cdot,\cdot)$, i.e., its form is as $d(\cdot,\cdot)H$. This generalizes the case where directions of the perturbations are singletons k_0. More generally, Gutiérrez, Jiménez and Novo [23] introduced a set-valued metric, which takes values in the set family of all subsets of the ordering cone and satisfies the triangle inequality. By using it they gave an original approach to extending the scalar-valued EVP to a vector-valued map, where the perturbation contains a set-valued metric. They also deduced several special versions of EVP involving approximate solutions for vector optimization problems and discussed their interesting applications in optimization. In the above EVPs given by Bednarczuk and Zagrodny [7] and by Gutiérrez, Jiménez and Novo [23], the objective maps are all a vector-valued (single-valued) map and the perturbations contain a convex subset of the ordering cone and a set-valued metric with values in the ordering cone, respectively.

Very recently, Liu and Ng [33], Tammer and Zălinescu [48] and Flores-Baźan, Gutiérrez and Novo [17] further considered more general versions of EVP, where not only the objective map is a set-valued map, but also the perturbation is a set-valued map, even a family of set-valued maps satisfying certain property. In particular, Liu and Ng [33] established several set-valued EVPs, where the objective map is a set-valued map and the perturbation is as the form $\gamma d(\cdot,\cdot)H$ or $\gamma' d(\cdot,\cdot)H$, $\gamma' \in (0,\gamma)$, where $\gamma > 0$ is a constant, $d(\cdot,\cdot)$ is the metric on the domain space and H is a closed convex subset of the ordering cone. Using the obtained EVPs, they provided some sufficient conditions ensuring the existence of error bounds for inequality systems. Tammer and Zălinescu [48] presented new minimal point theorems in product spaces and the corresponding set-valued EVPs. As special cases, they derived many of the previous EVPs and their extensions, for example, extensions of EVPs of Isac-Tammer's (see [28]) and Ha's versions (see [24]). Through an extension of Brézis-Browder principle, Flores-Baźan, Gutiérrez and Novo [17] established a general strong minimal point existence theorem on quasi ordered spaces and deduced several very general set-valued EVPs, where the objective map is a set-valued map and the perturbation even involves a family of set-valued maps satisfying "triangle inequality" property. As we have seen, these general set-valued EVPs extend and improve the previous EVPs and imply many new interesting results.

On the other hand, Bao and Mordukhovich (see [4,5]) proposed the limiting monotonicity condition on objective maps and established some enhanced versions of EVP for Pareto minimizers of set-valued maps. By using minimal element theorems for product orders in locally convex spaces, Khanh and Quy [31] generalized and improved the above enhanced versions of EVP. Particularly, they extended the direction of the perturbation from a single positive vector to a convex subset of the positive cone and removed the assumption in [4, 5] that the objective map is level closed.

In this paper, we first establish a pre-order principle, which consists of a pre-order set $(X, \preceq)$ and an extended real-valued function η which is monotone with respect to $\preceq$. The pre-order principle states that there exists a strong minimal point dominated by any given point provided that the monotone function η satisfies three general conditions. From the pre-order principle, we obtain a very general set-valued EVP, where the objective function is a set-valued map taking values in a quasi ordered linear space and the perturbation contains a family of set-valued maps satisfying certain property. Our assumption is accurate and weaker than ones appeared in the previous EVPs. And our proof is clear and concise. The key to the proof is to distinguish two different points by scalarizations. From the general EVP, we can deduce all of the above mentioned set-valued EVPs, their improvements and some new versions. In particular, our pre-order principle also implies generalizations of Khanh and Quy's minimal element theorems for product orders and hence we obtain several versions of EVP for Pareto minimizers, which generalize and improve the corresponding results of Bao and Mordukhovich ([4, 5]) and of Khanh and Quy ([31]).

The structure of this paper is as follows. In Section 2, we establish a pre-order principle. In Section 3, we give a general set-valued EVP and deduce a number of corollaries. In Section 4, we discuss set-valued EVPs, where perturbations contain a convex subset of the ordering cone. Moreover, we give several set-valued EVPs for approximately efficient solutions. In Section 5, we discuss minimal points for product orders and present several versions of EVP for Pareto minimizers.

2. A pre-order principle

Let X be a nonempty set. As in [17], a binary relation $\preceq$ on X is called a pre-order if it satisfies the transitive property; a quasi order if it satisfies the reflexive and transitive properties; a partial order if it satisfies the antisymmetric, reflexive and transitive properties. Let $(X, \preceq)$ be a pre-order set. An extended real-valued function $\eta : (X, \preceq) \to R \cup \{\pm\infty\}$ is called monotone with respect to $\preceq$ if for any x_1, $x_2 \in X$,

$$x_1 \preceq x_2 \implies \eta(x_1) \leq \eta(x_2).$$

For any given $x_0 \in X$, denote $S(x_0)$ the set $\{x \in X : x \preceq x_0\}$. First we give a pre-order principle as follows.

Theorem 2.1. *Let $(X, \preceq)$ be a pre-order set, $x_0 \in X$ such that $S(x_0) \neq \emptyset$ and $\eta : (X, \preceq) \to R \cup \{\pm\infty\}$ be an extended real-valued function which is monotone with respect to $\preceq$.*

Suppose that the following conditions are satisfied:

(A) $-\infty < \inf\{\eta(x) : x \in S(x_0)\} < +\infty$.

(B) *For any $x \in S(x_0)$ with $-\infty < \eta(x) < +\infty$ and $x' \in S(x) \backslash \{x\}$, one has $\eta(x) > \eta(x')$.*

(C) *For any sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1})$, $\forall n$, such that $\eta(x_n) - \inf\{\eta(x) : x \in S(x_{n-1})\} \to 0 \quad (n \to \infty)$, there exists $u \in X$ such that $u \in$*

$S(x_n)$, $\forall n$.

Then there exists $\hat{x} \in X$ such that

(a) $\hat{x} \in S(x_0)$;

(b) $S(\hat{x}) \subset \{\hat{x}\}$.

Proof. For brevity, we denote $\inf\{\eta(x) : x \in S(x_0)\}$ by $\inf \eta \circ S(X_0)$. By (A), we have

$$-\infty < \inf \eta \circ S(x_0) < +\infty. \tag{2.1}$$

So, there exists $x_1 \in S(x_0)$ such that

$$\eta(x_1) < \inf \eta \circ S(x_0) + \frac{1}{2}. \tag{2.2}$$

By the transitive property of $\preceq$, we have

$$S(x_1) \subset S(x_0). \tag{2.3}$$

If $S(x_1) \subset \{x_1\}$, then we may take $\hat{x} := x_1$ and clearly $\hat{x}$ satisfies (a) and (b). If not, by (2.1), (2.2) and (2.3) we conclude that

$$-\infty < \inf \eta \circ S(x_1) < +\infty.$$

So, there exists $x_2 \in S(x_1)$ such that

$$\eta(x_2) < \inf \eta \circ S(x_1) + \frac{1}{2^2}.$$

In general, if $x_{n-1} \in X$ has been chosen, we may choose $x_n \in S(x_{n-1})$ such that

$$\eta(x_n) < \inf \eta \circ S(x_{n-1}) + \frac{1}{2^n}.$$

If there exists n such that $S(x_n) \subset \{x_n\}$, then we may take $\hat{x} := x_n$ and clearly $\hat{x}$ satisfies (a) and (b). If not, we can obtain a sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1})$, $\forall n$, such that

$$\eta(x_n) < \inf \eta \circ S(x_{n-1}) + \frac{1}{2^n},\ \forall n. \tag{2.4}$$

Obviously, $\eta(x_n) - \inf \eta \circ S(x_{n-1}) \to 0$ when $n \to \infty$. By (C), there exists $\hat{x} \in X$ such that

$$\hat{x} \in S(x_n), \quad \forall n. \tag{2.5}$$

Obviously, $\hat{x} \in S(x_0)$, i.e., $\hat{x}$ satisfies (a). Next we show that $\hat{x}$ satisfies (b), i.e., $S(\hat{x}) \subset \{\hat{x}\}$. If it is not, there exists $\bar{x} \in S(\hat{x})$ and $\bar{x} \neq \hat{x}$. By (B),

$$\eta(\hat{x}) > \eta(\bar{x}). \tag{2.6}$$

On the other hand, by $\bar{x} \in S(\hat{x})$ and (2.5) we have

$$\bar{x} \in S(x_n) \quad \forall n. \tag{2.7}$$

Since η is monotone with respect to $\preceq$, by (2.5), (2.4) and (2.7) we have

$$\begin{aligned} \eta(\hat{x}) \leq \eta(x_n) \quad &< \quad \inf \eta \circ S(x_{n-1}) + \frac{1}{2^n} \\ &\leq \quad \eta(\bar{x}) + \frac{1}{2^n}, \ \forall n. \end{aligned}$$

Letting $n \to \infty$, we have $\eta(\hat{x}) \leq \eta(\bar{x})$, which contradicts (2.6). □

Remark 2.1. The pre-order principle given by Theorem 2.1 consists of a pre-order set $(X, \preceq)$ and a monotone extended real-valued function η on $(X, \preceq)$. It states that there exists a strong minimal point dominated by any given point provided that the monotone function η satisfies three general conditions (A), (B) and (C). First, condition (A) is fundamental and a starting point for constructing recurrently a decreasing sequence (x_n) with $\eta(x_n) - \inf \eta \circ S(x_{n-1}) \to 0$ $(n \to \infty)$. Now that we have such a sequence (x_n), condition (C) plays a key role, which says that there exists $\hat{x} \in X$ such that $\hat{x} \in S(x_n)$, $\forall n$. Particularly, $\hat{x} \in S(x_0)$ and conclusion (a) holds. The role of condition (B) is to distinguish points x and non-x in $S(x)$. Condition (B) together with the transitivity of $\preceq$ and the condition that $\eta(x_n) - \inf \eta \circ S(x_{n-1}) \to 0$ ensures that there is no $x' \neq \hat{x}$ such that $x' \in S(\hat{x})$. That is, conclusion (b) holds. We realize that although the proofs of various versions of EVP may be different, but their outlines are all similar to the above process. We shall see that the pre-order principle indeed includes many versions of EVP and their improvements. It should be noted also that in [49, 50] various kinds of ordering principles were established and many important applications were given. Our pre-order principle is different from them. It is specially made for deriving EVPs.

3. A general set-valued EVP and its corollaries

Let Y be a real linear space. If A, $B \subset Y$ and $\alpha \in R$, the sets $A+B$ and $\alpha\, A$ are defined as follows:

$$A+B := \{z \in Y :\, \exists x \in A,\, \exists y \in B \text{ such that } z = x+y\},$$

$$\alpha A := \{z \in Y :\, z = \alpha x,\, x \in A\}.$$

A nonempty subset D of Y is called a cone if $\alpha D \subset D$ for any $\alpha \geq 0$. And D is called a convex cone if $D + D \subset D$ and $\alpha D \subset D$ for any $\alpha \geq 0$. A convex cone D can specify a quasi order on Y as follows:

$$y_1,\, y_2 \in Y, \quad y_1 \leq_D y_2 \quad \Longleftrightarrow \quad y_1 - y_2 \in -D.$$

In this case, D is also called the ordering cone or positive cone. We always assume that D is nontrivial, i.e., $D \neq \{0\}$ and $D \neq Y$. An extended real function $\xi : Y \to R \cup \{\pm\infty\}$ is said to be D-monotone if $\xi(y_1) \leq \xi(y_2)$ whenever $y_1 \leq_D y_2$. For any nonempty subset M of Y, we put $\inf \xi \circ M = \inf\{\xi(y) :\, y \in M\}$. If $\inf \xi \circ M > -\infty$, we say that ξ is lower bounded on M. For any given $y \in Y$, sometimes we denote $\xi(y)$ by $\xi \circ y$. A family of set-valued maps $F_\lambda :\, X \times X \to 2^D \backslash \{\emptyset\},\ \lambda \in \Lambda$, is said to satisfy the "triangle inequality" property (briefly, denoted by property TI, see [17]) if for each $x_i \in X,\ i = 1, 2, 3$, and $\lambda \in \Lambda$ there exist $\mu, \nu \in \Lambda$ such that

$$F_\mu(x_1, x_2) + F_\nu(x_2, x_3) \subset F_\lambda(x_1, x_3) + D.$$

Let X be a nonempty set and let $f :\, X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map. For any nonempty set $A \subset X$, we put $f(A) := \cup\{f(x) :\, x \in A\}$. For any $x_1,\, x_2 \in X$, define $x_2 \preceq x_1$ iff

$$f(x_1) \subset f(x_2) + F_\lambda(x_2, x_1) + D, \quad \forall \lambda \in \Lambda.$$

Lemma 3.1. *"$\preceq$" is a pre-order on X, i.e., it is a binary relation satisfying transitive property.*

Proof. Let $x_2 \preceq x_1$ and $x_3 \preceq x_2$. We show below that $x_3 \preceq x_1$. By the definition of $\preceq$, we have

$$f(x_1) \subset f(x_2) + F_\lambda(x_2, x_1) + D, \quad \forall \lambda \in \Lambda; \tag{3.1}$$

and

$$f(x_2) \subset f(x_3) + F_\lambda(x_3, x_2) + D, \quad \forall \lambda \in \Lambda. \tag{3.2}$$

For the above x_1, x_2, $x_3 \in X$ and any given $\lambda \in \Lambda$, there exists μ, $\nu \in \Lambda$ such that

$$F_\mu(x_3, x_2) + F_\nu(x_2, x_1) \subset F_\lambda(x_3, x_1) + D. \tag{3.3}$$

By (3.1),

$$f(x_1) \subset f(x_2) + F_\nu(x_2, x_1) + D. \tag{3.4}$$

By (3.2),

$$f(x_2) \subset f(x_3) + F_\mu(x_3, x_2) + D. \tag{3.5}$$

Combining (3.4), (3.5) and (3.3), we have

$$\begin{aligned} f(x_1) &\subset f(x_2) + F_\nu(x_2, x_1) + D \\ &\subset f(x_3) + F_\mu(x_3, x_2) + D + F_\nu(x_2, x_1) + D \\ &\subset f(x_3) + F_\lambda(x_3, x_1) + D + D + D \\ &= f(x_3) + F_\lambda(x_3, x_1) + D. \end{aligned}$$

Since λ is arbitrary, we conclude that $x_3 \preceq x_1$. □

Theorem 3.1. *Let X be a nonempty set, Y be a real linear space, $D \subset Y$ be a convex cone specifying a quasi order $\leq_D$ on Y, $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map and $F_\lambda : X \times X \to 2^D \backslash \{\emptyset\}$, $\lambda \in \Lambda$, be a family of set-valued maps satisfying the property TI. Let $x_0 \in X$ such that*

$$S(x_0) := \{x \in X : f(x_0) \subset f(x) + F_\lambda(x, x_0) + D, \ \forall \lambda \in \Lambda\} \neq \emptyset.$$

Suppose that there exists a D-monotone extended real function $\xi : Y \to R \cup \{\pm\infty\}$ satisfying the following assumptions:

(D) $-\infty < \inf \xi \circ f(S(x_0)) < +\infty$.

(E) *For any $x \in S(x_0)$ with $-\infty < \inf \xi \circ f(x) < +\infty$ and for any $x' \in S(x) \backslash \{x\}$, one has $\inf \xi \circ f(x) > \inf \xi \circ f(x')$.*

(F) *For any sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1})$, $\forall n$, such that $\inf \xi \circ f(x_n) - \inf \xi \circ f(S(x_{n-1})) \to 0$, $n \to \infty$, there exists $u \in X$ such that $u \in S(x_n)$, $\forall n$.*

Then there exists $\hat{x} \in X$ such that

(a) *$f(x_0) \subset f(\hat{x}) + F_\lambda(\hat{x}, x_0) + D$, $\forall \lambda \in \Lambda$;*

(b) $\forall x \in X\backslash\{\hat{x}\}$, $\exists \lambda \in \Lambda$ *such that* $f(\hat{x}) \not\subset f(x) + F_\lambda(x, \hat{x}) + D$.

Proof. By Lemma 3.1, we can define a pre-order $\preceq$ on X as follows: for any $x_1, x_2 \in X$,

$$x_2 \preceq x_1 \quad \text{iff} \quad f(x_1) \subset f(x_2) + F_\lambda(x_2, x_1) + D, \ \forall \lambda \in \Lambda.$$

Thus, $S(x_0) = \{x \in X : x \preceq x_0\}$. Define an extended real-valued function $\eta : (X, \preceq) \to R \cup \{\pm\infty\}$ as follows

$$\eta(x) := \inf \xi \circ f(x), \quad \forall x \in X.$$

Let $x' \preceq x$. Then

$$f(x) \subset f(x') + F_\lambda(x', x) + D, \quad \forall \lambda \in \Lambda.$$

For any $y \in f(x)$, there exists $y' \in f(x')$, $q_\lambda(x', x) \in F_\lambda(x', x)$, $d_{\lambda,x',x} \in D$ such that

$$y = y' + q_\lambda(x', x) + d_{\lambda,x',x}.$$

Since

$$y - y' = q_\lambda(x', x) + d_{\lambda,x',x} \in D,$$

we have

$$\xi(y) \geq \xi(y') \geq \inf \xi \circ f(x').$$

As $y \in f(x)$ is arbitrary, we have

$$\inf \xi \circ f(x) \geq \inf \xi \circ f(x'), \quad \text{i.e.,} \quad \eta(x) \geq \eta(x').$$

Thus, η is monotone with respect to $\preceq$. It is easy to see that assumptions (D), (E) and (F) are exactly assumptions (A), (B) and (C) in Theorem 2.1. Now, applying Theorem 2.1, we know that there exists $\hat{x} \in X$ such that $\hat{x} \in S(x_0)$ and $S(\hat{x}) \subset \{\hat{x}\}$. This means that

$$f(x_0) \subset f(\hat{x}) + F_\lambda(\hat{x}, x_0) + D, \ \forall \lambda \in \Lambda$$

and

$$\forall x \in X\backslash\{\hat{x}\}, \exists \lambda \in \Lambda \text{ such that } f(\hat{x}) \not\subset f(x) + F_\lambda(x, \hat{x}) + D.$$

That is, (a) and (b) are satisfied. $\square$

For a real linear space Y, denote the algebraic dual of Y by $Y^{\#}$ and denote the positive polar cone of D in $Y^{\#}$ by $D^{+\#}$, i.e., $D^{+\#} = \{l \in Y^{\#} : l(d) \geq 0, \forall d \in D\}$. Obviously, every $\xi \in D^{+\#}$ is a D-monotone real function. Hence, the ξ in Theorem 3.1 may be an element of $D^{+\#}\backslash\{0\}$. In this case, assumptions (D) and (E) become more concise. And the expression of assumption (F) is the same as in Theorem 3.1.

Theorem 3.1$'$. *let X, Y, D, f, F_λ, $\lambda \in \Lambda$, and $x_0 \in X$ be the same as in Theorem 3.1. Suppose that there exists $\xi \in D^{+\#}\backslash\{0\}$ satisfying the following assumptions:*

(D) *ξ is lower bounded on $f(S(x_0))$, i.e., $-\infty < \inf \xi \circ f(S(x_0))$.*

(E) *For any $x \in S(x_0)$ and any $x' \in S(x)\backslash\{x\}$, one has $\inf \xi \circ f(x) > \inf \xi \circ f(x')$.*

(F) *See Theorem 3.1.*

Then the result of Theorem 3.1 remains true.

From Theorem 3.1$'$, we can deduce [17, Theorem 6.5], a general version of set-valued EVP, which extends EVPs in [7, 23].

Corollary 3.1 (see [17, Theorem 6.5]). *The result of Theorem 3.1$'$ remains true if assumption* (E) *is replaced by the following assumption*

(E_1) *For any $x, x' \in S(x_0)$ with $x \neq x'$, there exists $\lambda_0 \in \Lambda$ such that $\inf\{\xi \circ q_{\lambda_0} : q_{\lambda_0} \in F_{\lambda_0}(x', x)\} > 0$.*

Proof. By Theorem 3.1$'$ we only need to prove that (E_1) $\Rightarrow$ (E). Let $x \in S(x_0)$ and $x' \in S(x)\backslash\{x\}$. We shall show that $\inf \xi \circ f(x) > \inf \xi \circ f(x')$. Obviously,

$$x' \neq x \quad \text{and} \quad f(x) \subset f(x') + F_\lambda(x', x) + D, \quad \forall \lambda \in \Lambda.$$

By (E_1), there exists $\lambda_0 \in \Lambda$ such that

$$\eta := \inf\{\xi \circ q_{\lambda_0} : q_{\lambda_0} \in F_{\lambda_0}(x', x)\} > 0.$$

Clearly, there exists $y \in f(x)$ such that

$$\xi \circ y < \inf \xi \circ f(x) + \frac{1}{2}\eta. \tag{3.6}$$

As $y \in f(x) \subset f(x') + F_{\lambda_0}(x', x) + D$, there exists $y' \in f(x')$, $q_{\lambda_0} \in F_{\lambda_0}(x', x)$ and $d \in D$ such that

$$y = y' + q_{\lambda_0} + d.$$

Since $\xi \in D^{+\#}\backslash\{0\}$, we have

$$\xi \circ y \;=\; \xi \circ y' + \xi \circ q_{\lambda_0} + \xi \circ d \;\geq\; \xi \circ y' + \eta. \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\inf \xi \circ f(x) + \frac{1}{2}\eta \;>\; \xi \circ y \;\geq\; \xi \circ y' + \eta \;\geq\; \inf \xi \circ f(x') + \eta.$$

Thus,

$$\inf \xi \circ f(x) \;>\; \inf \xi \circ f(x') + \frac{1}{2}\eta \;>\; \inf \xi \circ f(x').$$

□

If ξ attains its infimum on $f(x)$ for any $x \in S(x_0)$, then assumption (E$_1$) in Corollary 3.1 can be weakened.

Corollary 3.2. *Suppose that ξ in Theorem 3.1$'$ attains its infimum on $f(x)$ for any $x \in S(x_0)$. Then the result of Theorem 3.1$'$ remains true if assumption* (E) *is replaced by the following assumption*

(E$_2$) *For any x, $x' \in S(x_0)$ with $x \neq x'$, there exists $\lambda_0 \in \Lambda$ such that $\xi \circ q_{\lambda_0} > 0$, $\forall q_{\lambda_0} \in F_{\lambda_0}(x', x)$.*

Proof. By Theorem 3.1$'$ we only need to prove that (E$_2$) $\Rightarrow$ (E). Let $x \in S(x_0)$ and $x' \in S(x)\backslash\{x\}$. Then $x' \neq x$ and

$$f(x) \subset f(x') + F_\lambda(x', x) + D, \quad \forall \lambda \in \Lambda. \tag{3.8}$$

By assumption (E$_2$), there exists

$$y_x \in f(x) \tag{3.9}$$

such that

$$\xi \circ y_x \;=\; \inf \xi \circ f(x). \tag{3.10}$$

Since $x \neq x'$, by (E$_2$) there exists $\lambda_0 \in \Lambda$ such that

$$\xi \circ q_{\lambda_0} > 0, \quad \forall q_{\lambda_0} \in F_{\lambda_0}(x', x).$$

By (3.9) and (3.8),

$$y_x \;\in\; f(x') + F_{\lambda_0}(x', x) + D.$$

Hence, there exists $y' \in f(x')$, $q_{\lambda_0} \in F_{\lambda_0}(x', x)$ and $d \in D$ such that

$$y_x = y' + q_{\lambda_0} + d.$$

As $\xi \in D^{+\#}$ and $\xi \circ q_{\lambda_0} > 0$, we have

$$\begin{aligned} \xi \circ y_x &= \xi \circ y' + \xi \circ q_{\lambda_0} + \xi \circ d \\ &\geq \xi \circ y' + \xi \circ q_{\lambda_0} \\ &> \xi \circ y' \\ &\geq \inf \xi \circ f(x'). \end{aligned}$$

Combining this with (3.10), we conclude that $\inf \xi \circ f(x) > \inf \xi \circ f(x')$ and hence (E) holds.

□

In particular, if $f(x)$ is a singleton for any $x \in X$, i.e., $f : X \to Y$ is a vector-valued map, then from Corollary 3.2 we can obtain [42, Theorem 3.15]. If Y is a separated topological vector space, we denote Y^* the topological dual of Y and denote D^+ the positive polar cone of D in Y^*, i.e., $D^+ := \{l \in Y^* : l(d) \geq 0, \forall d \in D\}$. It is possible that $Y^* = \{0\}$, i.e., there is no non-trivial continuous linear functional on Y. However, if Y is a locally convex separated topological vector space (briefly, denoted by a locally convex space), then Y^* is large enough so that Y^* can separate points in Y. Suppose that $K \subset Y$ is a weakly countably compact set (particularly, a weakly compact set) in the locally convex space Y. Then for any $l \in Y^*$, l attains its infimum on K. Thus, from Corollary 3.2, we have the following.

Corollary 3.3. *In Theorem 3.1′, we further assume that Y is a locally convex space and the set-valued map $f : X \to 2^Y \backslash \{\emptyset\}$ satisfies that $f(x)$ is a weakly countably compact set (particularly, a weakly compact set) in Y for any $x \in S(x_0)$. Then the result of Theorem 3.1′ remains true if assumption* (E) *is replaced by* (E$_2$)

Definition 3.1 (see [42, Definition 3.4]). Let X be a topological space and let $S(\cdot) : X \to 2^X \backslash \{\emptyset\}$ be a set-valued map. The set-valued map $S(\cdot)$ is said to be dynamically closed at $x \in X$ if $(x_n) \subset S(x)$, $S(x_{n+1}) \subset S(x_n) \subset S(x)$ for all n and $x_n \to \bar{x}$ then $\bar{x} \in S(x)$. In this case, we also say that $S(x)$ is dynamically closed. Moreover, let (X, d) be a metric space and $x \in X$. Then (X, d) is said to be $S(x)$-dynamically complete if every Cauchy sequence $(x_n) \subset S(x)$ such that

$S(x_{n+1}) \subset S(x_n) \subset S(x)$ for all n, is convergent in X.

The following corollary generalizes [48, Theorems 4.1 and 6.1].

Corollary 3.4. *Let (X, d) be a complete metric space, Y be a real linear space, $D \subset Y$ be a convex cone specifying a quasi order $\leq_D$ on Y, $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map and $F_\lambda : X \times X \to 2^D \backslash \{\emptyset\}$, $\lambda \in \Lambda$, be a family of set-valued maps satisfying the property TI. Let $x_0 \in X$ such that $S(x_0) := \{x \in X : f(x_0) \subset f(x) + F_\lambda(x, x_0) + D,\ \forall \lambda \in \Lambda\} \neq \emptyset$. Suppose that for any $x \in S(x_0)$, $S(x)$ is dynamically closed and that there exists $\xi \in D^{+\#}\backslash\{0\}$ satisfying the following assumptions:*

(D) *ξ is lower bounded on $f(S(x_0))$.*

(E$_3$) *There exists $\lambda_0 \in \Lambda$ such that for any $\delta > 0$, $\inf \xi \circ F_{\lambda_0 \delta} > 0$, where $F_{\lambda_0 \delta}$ denotes the set $\cup_{d(x,x') \geq \delta} F_{\lambda_0}(x, x')$.*

Then there exists $\hat{x} \in X$ such that

(a) $f(x_0) \subset f(\hat{x}) + F_\lambda(\hat{x}, x_0) + D, \quad \forall \lambda \in \Lambda;$

(b) $\forall x \in X \backslash \{\hat{x}\},\ \exists \lambda \in \Lambda$ *such that* $f(\hat{x}) \not\subset f(x) + F_\lambda(x, \hat{x}) + D$.

Proof. Obviously, (E$_3$) $\Rightarrow$ (E$_1$). By Corollary 3.1, we only need to prove that assumption (F) in Theorem 3.1$'$ is satisfied. Let $(x_n) \subset S(x_0)$ such that $x_n \in S(x_{n-1})$ and

$$\inf \xi \circ f(x_n) - \inf \xi \circ f(S(x_{n-1})) \to 0, \quad n \to \infty.$$

We may take a positive real sequence (ϵ_n) such that $\epsilon_n \to 0$ $(n \to \infty)$ and such that

$$\inf \xi \circ f(x_n) < \inf \xi \circ f(S(x_{n-1})) + \epsilon_n, \quad \forall n.$$

For each n, choose $y_n \in f(x_n)$ such that

$$\xi \circ y_n < \inf \xi \circ f(S(x_{n-1})) + \epsilon_n. \tag{3.11}$$

We assert that (x_n) is a Cauchy sequence in (X, d). If not, there exists $\delta > 0$ such that for any $k \in N$, there exists $n_k > k$ such that

$$d(x_{n_k}, x_k) \geq \delta. \tag{3.12}$$

For any $k \in N$, $x_{n_k} \in S(x_{n_k - 1}) \subset S(x_k) \subset S(x_{k-1})$. Thus,

$$y_k \in f(x_k) \subset f(x_{n_k}) + F_{\lambda_0}(x_{n_k}, x_k) + D.$$

Hence, there exists $y'_{n_k} \in f(x_{n_k})$, $q_{\lambda_0,n_k,k} \in F_{\lambda_0}(x_{n_k}, x_k)$ and $d_k \in D$ such that

$$y_k \;=\; y'_{n_k} + q_{\lambda_0,n_k,k} + d_k.$$

As $\xi \in D^{+\#}\backslash\{0\}$, we have

$$\xi \circ y_k \;=\; \xi \circ y'_{n_k} + \xi \circ q_{\lambda_0,n_k,k} + \xi \circ d_k \;\geq\; \xi \circ y'_{n_k} + \xi \circ q_{\lambda_0,n_k,k}. \tag{3.13}$$

Remarking that $y'_{n_k} \in f(x_{n_k}) \subset f(S(x_{k-1}))$, we have

$$\xi \circ y'_{n_k} \;\geq\; \inf \xi \circ f(S(x_{k-1})). \tag{3.14}$$

By (3.13), (3.14) and (3.11), we have

$$\begin{aligned} \xi \circ q_{\lambda_0,n_k,k} &\leq \xi \circ y_k - \xi \circ y'_{n_k} \\ &\leq \xi \circ y_k - \inf \xi \circ f(S(x_{k-1})) \\ &< \epsilon_k. \end{aligned}$$

Letting $k \to \infty$, we have

$$\xi \circ q_{\lambda_0,n_k,k} \;\to\; 0 \;\;(k \to \infty). \tag{3.15}$$

On the other hand, by (3.12), every $q_{\lambda_0,n_k,k} \in F_{\lambda_0}(x_{n_k}, x_k) \subset F_{\lambda_0\delta}$. By ($E_3$),

$$\xi \circ q_{\lambda_0,n_k,k} \;\geq\; \inf \xi \circ F_{\lambda_0\delta} \;>\; 0, \;\; \forall k,$$

which contradicts (3.15). Thus, we have shown that (x_n) is a Cauchy sequence. Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$. By the assumption, $S(x_n)$ is dynamically closed for any n. Since $(x_{n+p})_{p\in N} \subset S(x_n)$, $x_{n+p+1} \in S(x_{n+p})$ and $x_{n+p} \to u$ $(p \to \infty)$, we have $u \in S(x_n)$, $\forall n$. This means that assumption (F) is satisfied. □

Let X be a metric space, Y be a locally convex space and $D \subset Y$ be a closed convex cone. As in [24], a set-valued map $f : X \to 2^Y\backslash\{\emptyset\}$ is said to be D-lower semicontinuous (briefly, D-l.s.c.) on X if for any $y \in Y$, the set $\{x \in X : f(x) \cap (y - D) \neq \emptyset\}$ is closed. And $f : X \to 2^Y\backslash\{\emptyset\}$ is said to have D-closed values if for any $x \in X$, $f(x) + D$ is closed. $f(X)$ is said to be D-bounded if there exists a bounded set $M \subset Y$ such that $f(X) \subset M + D$. In [48], $f(X)$ being D-bounded is also called $f(X)$ being quasibounded from below. In [41], a set-valued map $f : X \to 2^Y\backslash\{\emptyset\}$ is said to be D-sequentially lower monotone (briefly,

D-s.l.m.) if $f(x_n) \subset f(x_{n+1})+D,\ \forall n$, and $x_n \to \bar{x}$ imply $f(x_n) \subset f(\bar{x})+D,\ \forall n$. It is easy to show that a D-l.s.c. set-valued map is D-s.l.m. But the converse is not true (see [41]). From Corollary 3.4 we can deduce an improvement of Ha's version of set-valued EVP (see [24, Theorem 3.1]) as follows.

Corollary 3.5. *Let (X, d) be a complete metric space, Y be a locally convex space pre-ordered by a closed convex cone D and $k_0 \in D \backslash -D$. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be D-s.l.m., have D-closed values and $f(X)$ be D-bounded. Suppose that $x_0 \in X$ and $\epsilon > 0$ such that $f(x_0) \not\subset f(x) + \epsilon k_0 + D,\ \forall x \in X$. Then for any $\lambda > 0$, there exists $\hat{x} \in X$ such that*

(a) *$f(x_0) \subset f(\hat{x}) + (\epsilon/\lambda)d(\hat{x}, x_0)k_0 + D$;*

(b) *$\forall x \in X\backslash\{\hat{x}\},\ f(\hat{x}) \not\subset f(x)+(\epsilon/\lambda)d(x,\hat{x})k_0+D$., i.e., $\hat{x}$ is a strict minimizer of the map $x \mapsto f(x)+(\epsilon/\lambda)d(x,\hat{x})k_0$* (concerning a strict minimizer of a map, see [24]);

(c) *$d(x_0, \hat{x}) \le \lambda$.*

Proof. For any $x,\ x' \in X$, define $F(x, x') := (\epsilon/\lambda)d(x, x')k_0$. Obviously, the family $\{F\}$ satisfies the property TI. Put

$$S(x_0) := \{x \in X : f(x_0) \subset f(x) + \frac{\epsilon}{\lambda}d(x, x_0)k_0 + D\}.$$

Then $x_0 \in S(x_0)$ and $S(x_0) \neq \emptyset$. Since $k_0 \notin -D$ and D is closed, by the separation theorem, there exists $\xi \in D^+\backslash\{0\}$ such that $\xi(k_0) = 1$. When $d(x, x') \ge \delta$, we have

$$\xi(\frac{\epsilon}{\lambda}d(x, x')k_0) \ \ge\ \frac{\epsilon}{\lambda} \cdot \delta \ >\ 0.$$

Hence

$$\inf \xi \circ F_\delta \ >\ 0, \quad \text{where} \quad F_\delta = \cup_{d(x,x')\ge\delta} F(x, x') = \{\frac{\epsilon}{\lambda}d(x, x')k_0 : d(x, x') \ge \delta\}.$$

Remarking that $f(X)$ is D-bounded and $\xi \in D^+\backslash\{0\}$, we see that ξ is lower bounded on $f(X)$ and lower bounded on $f(S(x_0))$. Thus, we have verified that assumptions (D) and (E_3) in Corollary 3.4 are satisfied.

It remains to show that for any $x \in S(x_0)$, $S(x)$ is dynamically closed. Let $(x_n) \subset S(x),\ x_{n+1} \in S(x_n) \subset S(x),\ \forall n$, and $x_n \to u$. We shall show that $u \in S(x)$. Since $x_{n+1} \in S(x_n)$,

$$f(x_n) \ \subset\ f(x_{n+1}) + \frac{\epsilon}{\lambda}d(x_{n+1}, x_n)k_0 + D \ \subset\ f(x_{n+1}) + D, \quad \forall n.$$

Combining this with $x_n \to u$ and using the assumption that f is D-s.l.m., we have

$$f(x_n) \subset f(u) + D, \quad \forall n. \tag{3.16}$$

By $x_n \in S(x)$ and (3.16), we have

$$\begin{aligned} f(x) &\subset f(x_n) + \frac{\epsilon}{\lambda} d(x_n, x)k_0 + D \\ &\subset f(u) + D + \frac{\epsilon}{\lambda} d(x_n, x)k_0 + D \\ &= f(u) + \frac{\epsilon}{\lambda} d(x_n, x)k_0 + D. \end{aligned}$$

Thus,

$$f(x) - \frac{\epsilon}{\lambda} d(x_n, x)k_0 \subset f(u) + D. \tag{3.17}$$

Since $(\epsilon/\lambda)d(x_n, x)k_0 \to (\epsilon/\lambda)d(u, x)k_0$ $(n \to \infty)$ and $f(u) + D$ is closed, by (3.17) we have

$$f(x) - \frac{\epsilon}{\lambda} d(u, x)k_0 \subset f(u) + D,$$

which means that $u \in S(x)$.

Now applying Corollary 3.4, there exists $\hat{x} \in X$ such that (a) and (b) hold. It remains to show that (c) holds. If not, suppose that $d(\hat{x}, x_0) > \lambda$. Then by (a), we have

$$\begin{aligned} f(x_0) &\subset f(\hat{x}) + \frac{\epsilon}{\lambda} d(\hat{x}, x_0)k_0 + D \\ &= f(\hat{x}) + (\frac{\epsilon}{\lambda} d(\hat{x}, x_0) - \epsilon)k_0 + \epsilon k_0 + D \\ &\subset f(\hat{x}) + \epsilon k_0 + D. \end{aligned}$$

This contradicts the assumption that $f(x_0) \not\subset f(x) + \epsilon k_0 + D$, $\forall x \in X$. $\square$

In fact, the assumption that $f(X)$ is D-bounded in Corollary 3.5 can be replaced by a weaker assumption: there exists $\epsilon > 0$ such that $f(x_0) \not\subset f(X) + \epsilon k_0 + D$ (see [41]). For this, we need the following nonlinear scalarization function. The original version is due to Gerstewitz [18].

We present the concept in a general setting. Let Y be a real linear space and $A \subset Y$ be a nonempty set. We put (refer to [1])

$$\mathrm{vcl}(A) := \{y \in Y : \exists v \in Y, \exists \lambda_n \geq 0,\ \lambda_n \to 0 \text{ such that } y + \lambda_n v \in A,\ \forall n \in N\}.$$

For any given $v_0 \in Y$, put

$$\mathrm{vcl}_{v_0}(A) = \{y \in Y : \exists \lambda_n \geq 0,\ \lambda_n \to 0 \text{ such that } y + \lambda_n v_0 \in A,\ \forall n \in N\}.$$

Obviously,

$$A \subset \mathrm{vcl}_{v_0}(A) \subset \bigcup_{v \in Y} \mathrm{vcl}_v(A) = \mathrm{vcl}(A).$$

Moreover, if Y is a topological vector space, then $\mathrm{vcl}(A) \subset \mathrm{cl}(A)$ and the inclusion is proper. If $A = \mathrm{vcl}_{v_0}(A)$, then A is said to be v_0-closed. If $A = \mathrm{vcl}(A)$, then A is said to be vectorial closed. Obviously, v_0-closedness and vectorial closedness are both strictly weaker than topological closedness.

Let $D \subset Y$ be a convex cone specifying a quasi order $\leq_D$ on Y and $k_0 \in D \backslash -\mathrm{vcl}(D)$. For any $y \in Y$, if there exists $t \in R$ such that $y \in tk_0 - D$, then for any $t' > t$, $y \in t'k_0 - D$. We define a function $\xi_{k_0} : Y \to R \cup \{+\infty\}$ as follows: if there exists $t \in R$ such that $y \in tk_0 - D$, then define $\xi_{k_0}(y) = \inf\{t \in R : y \in tk_0 - D\}$; or else define $\xi_{k_0}(y) = +\infty$. As $k_0 \notin -\mathrm{vcl}(D)$, we can show that $\xi_{k_0}(y) \neq -\infty$. This function is called a Gerstewitz's function. Concerning the details of such a function and its properties, please refer to [18-20]. For brevity, we denote $D + (0, +\infty)k_0$ by $\mathrm{vint}_{k_0}(D)$. We list several properties of ξ_{k_0} as follows.

Lemma 3.2 (refer to [10, 21, 40, 41, 44]). *Let $y \in Y$ and $r \in R$. Then we have:*

(i) $\xi_{k_0}(y) < r \;\Leftrightarrow\; y \in r\,k_0 - \mathrm{vint}_{k_o}(D)$.

(ii) $\xi_{k_0}(y) \leq r \;\Leftrightarrow\; y \in rk_0 - \mathrm{vcl}_{k_0}(D)$.

(iii) $\xi_{k_0}(y) = r \;\Leftrightarrow\; y \in rk_0 - (\mathrm{vcl}_{k_0}(D) \backslash \mathrm{vint}_{k_0}(D))$. *In particular,* $\xi_{k_0}(k_0) = 1$ *and* $\xi_{k_0}(0) = 0$.

(iv) $\xi_{k_0}(y) \geq r \;\Leftrightarrow\; y \notin rk_0 - \mathrm{vint}_{k_0}(D)$.

(v) $\xi_{k_0}(y) > r \;\Leftrightarrow\; y \notin rk_0 - \mathrm{vcl}_{k_0}(D)$.

Moreover, we have:

(vi) $\xi_{k_0}(y_1 + y_2) \leq \xi_{k_0}(y_1) + \xi_{k_0}(y_2),\ \forall y_1,\, y_2 \in Y$.

(vii) $\xi_{k_0}(y + \lambda k_0) = \xi_{k_0}(y) + \lambda,\ \forall y \in Y, \forall \lambda \in R$.

(viii) $y_1 \leq_D y_2 \;\Rightarrow\; \xi_{k_0}(y_1) \leq \xi_{k_0}(y_2)$.

In Corollary 3.5, we need to assume that $f : X \to 2^Y \backslash \{\emptyset\}$ has D-closed values, i.e., $f(x) + D$ is closed for all $x \in X$. Here, we introduce a weaker property: a set-valued map f is said to have D-k_0-closed values if $f(x)+D$ is k_0-closed for all $x \in X$.

Corollary 3.6 (see [41, Theorem 3.1]). *Let (X, d) be a metric space, Y be a locally convex space quasi ordered by a convex cone D and $k_0 \in D \backslash -\mathrm{vcl}(D)$. Let*

$f : X \to 2^Y \backslash \{\emptyset\}$ be D-s.l.m. and have D-k_0-closed values. Suppose that $x_0 \in X$ and $\epsilon > 0$ such that $f(x_0) \not\subset f(X) + \epsilon k_0 + D$ and suppose that (X, d) is $S(x_0)$-dynamically complete, where $S(x_0) = \{x \in X : f(x_0) \subset f(x) + (\epsilon/\lambda)d(x, x_0)k_0 + D\}$. Then for any $\lambda > 0$, there exists $\hat{x} \in X$ such that

(a) $f(x_0) \subset f(\hat{x}) + (\epsilon/\lambda)d(\hat{x}, x_0)k_0 + D$;

(b) $\forall x \in X\backslash\{\hat{x}\},\ f(\hat{x}) \not\subset f(x) + (\epsilon/\lambda)d(x, \hat{x})k_0 + D$;

(c) $d(x_0, \hat{x}) \leq \lambda$.

Proof. We shall prove the result by using Theorem 3.1. Define $F : X \times X \to 2^D\backslash\{\emptyset\}$ as follows: $F(x, x') := (\epsilon/\lambda)d(x, x')k_0$. Then the family $\{F\}$ satisfies the property TI. Put

$$S(x_0) := \{x \in X : f(x_0) \subset f(x) + (\epsilon/\lambda)d(x, x_0)k_0 + D\}.$$

Obviously, $x_0 \in S(x_0)$ and $S(x_0) \neq \emptyset$. Since $f(x_0) \not\subset f(S(x_0)) + \epsilon k_0 + D$, there exists $y_0 \in f(x_0)$ such that $y_0 \notin f(S(x_0)) + \epsilon k_0 + D$, that is,

$$(f(S(x_0)) - y_0) \cap (-\epsilon k_0 - D) = \emptyset. \tag{3.18}$$

By (3.18) and Lemma 3.2(iv), we have

$$\xi_{k_0}(y - y_0) \geq -\epsilon, \quad \forall y \in f(S(x_0)).$$

Also, $\xi_{k_0}(y_0 - y_0) = 0$, where $y_0 \in f(x_0) \subset f(S(x_0))$. Put

$$\xi(y) = \xi_{k_0}(y - y_0), \quad \forall y \in Y.$$

Then ξ is a D-monotone extended real function such that

$$-\infty < -\epsilon \leq \inf \xi \circ f(S(x_0)) \leq 0 < +\infty.$$

That is, assumption (D) in Theorem 3.1 is satisfied. For any $x \in S(x_0)$, we remark that

$$\inf \xi \circ f(x) \leq \inf \xi \circ f(x_0) \leq \xi(y_0) = 0 < +\infty.$$

For any $x' \in S(x)\backslash\{x\}$, we have

$$f(x) \subset f(x') + \frac{\epsilon}{\lambda} d(x', x)k_0 + D. \tag{3.19}$$

Choose $y \in f(x)$ such that

$$\xi \circ y < \inf \xi \circ f(x) + \frac{\epsilon}{2\lambda} d(x', x). \tag{3.20}$$

By (3.19),

$$y - y_0 \in f(x') - y_0 + \frac{\epsilon}{\lambda} d(x', x) k_0 + D.$$

Thus, there exists $y' \in f(x')$ and $d \in D$ such that

$$y - y_0 = y' - y_0 + \frac{\epsilon}{\lambda} d(x', x) k_0 + d.$$

By Lemma 3.2, we have

$$\xi_{k_0}(y - y_0) \geq \xi_{k_0} \left(y' - y_0 + \frac{\epsilon}{\lambda} d(x', x) k_0 \right) = \xi_{k_0}(y' - y_0) + \frac{\epsilon}{\lambda} d(x', x),$$

that is,

$$\xi \circ y \geq \xi \circ y' + \frac{\epsilon}{\lambda} d(x', x). \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$\inf \xi \circ f(x) + \frac{\epsilon}{2\lambda} d(x', x) > \xi \circ y \geq \xi \circ y' + \frac{\epsilon}{\lambda} d(x', x) \geq \inf \circ f(x') + \frac{\epsilon}{\lambda} d(x', x).$$

From this,

$$\inf \xi \circ f(x) > \inf \xi \circ f(x') + \frac{\epsilon}{2\lambda} d(x', x) > \inf \xi \circ f(x').$$

That is, assumption (E) in Theorem 3.1 is satisfied.

Finally, we show that assumption (F) in Theorem 3.1 is satisfied. Let a sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1}),\ \forall n$, such that

$$\inf\ \xi \circ f(x_n) - \inf\ \xi \circ f(S(x_{n-1})) \to 0,\ n \to \infty.$$

Put

$$\epsilon_n := \inf\ \xi \circ f(x_n) - \inf\ \xi \circ f(S(x_{n-1})) + \frac{1}{2^n}.$$

Then

$$0 \leq \inf\ \xi \circ f(x_n) - \inf\ \xi \circ f(S(x_{n-1})) < \epsilon_n \to 0 \quad (n \to \infty).$$

For each n, take $y_n \in f(x_n)$ such that

$$\xi \circ y_n - \inf\ \xi \circ f(S(x_{n-1})) < \epsilon_n. \tag{3.22}$$

Since $x_n \in S(x_{n-1}),\ \forall n$, it is easy to see that

$$S(x_0) \supset S(x_1) \supset S(x_2) \supset \cdots.$$

When $m > n$, $x_m \in S(x_{m-1}) \subset S(x_n)$. Thus,

$$y_n \in f(x_n) \subset f(x_m) + \frac{\epsilon}{\lambda} d(x_m, x_n) k_0 + D.$$

Hence, there exists $y'_m \in f(x_m)$ such that

$$y_n - y_0 \ \geq_D \ y'_m - y_0 + \frac{\epsilon}{\lambda} d(x_m, x_n) k_0. \tag{3.23}$$

From (3.23) and using Lemma 3.2, we have

$$\xi_{k_0}(y_n - y_0) \ \geq \ \xi_{k_0}(y'_m - y_0) + \frac{\epsilon}{\lambda} d(x_m, x_n).$$

Remarking that $y'_m \in f(x_m) \subset f(S(x_{n-1}))$ and using (3.22), we have

$$\begin{aligned} \frac{\epsilon}{\lambda} d(x_m, x_n) &\leq \xi_{k_0}(y_n - y_0) - \xi_{k_0}(y'_m - y_0) \\ &= \xi(y_n) - \xi(y'_m) \\ &\leq \xi \circ y_n - \inf \xi \circ f(S(x_{n-1})) \\ &< \epsilon_n \ \rightarrow \ 0 \ \ (m > n \rightarrow \infty). \end{aligned}$$

From this, we conclude that (x_n) is a Cauchy sequence which satisfies that $S(x_{n+1}) \subset S(x_n) \subset \cdots \subset S(x_0)$, $\forall n$. As (X, d) is $S(x_0)$-dynamically complete, there exists $u \in X$ such that $x_n \rightarrow u$. Let n be given. Then for any $i \in N$, $x_{n+i} \in S(x_n)$ and

$$f(x_n) \subset f(x_{n+i}) + \frac{\epsilon}{\lambda} d(x_{n+i}, x_n) k_0 + D. \tag{3.24}$$

Since f is D-s.l.m. and f has D-k_0-closed values, i.e., $f(x) + D$ is k_0-closed for any $x \in X$, from (3.24) we can deduce that

$$f(x_n) \subset f(u) + \frac{\epsilon}{\lambda} d(u, x_n) k_0 + D.$$

That is, $u \in S(x_n)$ and assumption (F) is satisfied. Thus, we can apply Theorem 3.1 and obtain $\hat{x} \in X$ such that (a) and (b) hold. As done in the proof of Corollary 3.5, we can also deduce that (c) holds. □

Remark 3.1. In order to compare Corollary 3.6 with [41, Theorem 3.1], let us recall the notion of (f, D)-lower completeness. A metric space (X, d) is said to be (f, D)-lower complete if every Cauchy sequence $(x_n) \subset X$ satisfying $f(x_n) \subset f(x_{n+1}) + D$ for each n, is convergent. If a Cauchy sequence $(x_n) \subset S(x_0)$ satisfies that $x_{n+1} \in S(x_n)$, i.e., $f(x_n) \subset f(x_{n+1}) + (\epsilon/\lambda) d(x_{n+1}, x_n) k_0 + D$, then

$f(x_n) \subset f(x_{n+1}) + D$. Hence, it is obvious that (X,d) being (f,D)- lower complete implies that (X,d) is $S(x_0)$-dynamically complete. Besides, in Corollary 3.6 we only require that $k_0 \in D\backslash -\mathrm{vcl}(D)$ and f has D-k_0-closed values, which are respectively weaker than the conditions that $k_0 \in D\backslash -\mathrm{cl}(D)$ and that f has D-closed values. Hence, Corollary 3.6 is indeed a generalization of [41, Theorem 3.1].

Similarly, in Corollaries 3.4 and 3.5, the assumption that (X,d) is complete can also be replaced by a weaker assumption: (X,d) is $S(x_0)$-dynamically complete.

4. Set-valued EVPs with perturbations containing a set

In this section, by using the results in Section 3 we give several set-valued EVPs, where perturbations containing a convex subset of the ordering cone. First we give a generalization of [33, Theorem 3.4].

Theorem 4.1. *Let (X,d) be a complete metric space. Y be a locally convex space pre-ordered by a convex cone D, $H \subset D\backslash\{0\}$ be a convex set and $f : X \to 2^Y\backslash\{\emptyset\}$ be a set-valued map.*

Suppose that the following assumptions are satisfied:

(B$_1$) $0 \notin \mathrm{cl}(H+D)$.

(B$_2$) $f(X)$ *is D-bounded.*

(B$_3$) f *is D-s.l.m.*

Then for any $x_0 \in X$ and any $\gamma > 0$, there exists $\hat{x} \in X$ such that

(a) $f(x_0) \subset f(\hat{x}) + \gamma' d(\hat{x}, x_0)H + D, \quad \forall \gamma' \in (0,\gamma)$;

(b) $\forall x \in X\backslash\{\hat{x}\}, \ \exists \gamma' \in (0,\gamma)$ *such that* $f(\hat{x}) \not\subset f(x) + \gamma' d(x,\hat{x})H + D$.

Proof. Put $F_{\gamma'}(x,x') := \gamma' d(x,x')H$, $\forall \gamma' \in (0,\gamma)$. Clearly, the family $\{F_{\gamma'}\}_{\gamma' \in (0,\gamma)}$ satisfies the property TI. Put

$$S(x_0) := \{x \in X : \ f(x_0) \subset f(x) + \gamma' d(x,x_0) + D, \ \forall \gamma' \in (0,\gamma)\}.$$

Obviously, $x_0 \in S(x_0)$ and $S(x_0) \neq \emptyset$. By assumption (B$_1$) and the separation theorem, there exists $\xi \in Y^*$ and $\alpha > 0$ such that $\xi(H+D) \geq \alpha > 0$. Thus, $\xi \in D^+\backslash\{0\}$ and $\xi(H) \geq \alpha > 0$. By assumption (B$_2$), ξ is lower bounded on $f(X)$ and lower bounded on $f(S(x_0))$. Take any fixed $\gamma_0 \in (0,\gamma)$. For any $\delta > 0$, put

$F_{\gamma_0\delta} := \bigcup\{\gamma_0 d(x,x')H : d(x,x') \geq \delta\}$. Obviously,

$$\inf \xi \circ F_{\gamma_0\delta} \geq \gamma_0 \, \delta \, \alpha > 0.$$

Hence, assumption (E_3) in Corollary 3.4 is satisfied. In order to apply Corollary 3.4 we need to show that for any $x \in S(x_0)$, $S(x)$ is dynamically closed. Let $(x_n) \subset S(x)$ such that $x_{n+1} \in S(x_n) \subset S(x)$, $\forall n$, and $x_n \to u$. For any $n \in N$ and any $\gamma' \in (0,\gamma)$, we have

$$f(x_n) \subset f(x_{n+1}) + \gamma' d(x_{n+1}, x_n)H + D \subset f(x_{n+1}) + D.$$

By assumption (B_3) we have

$$f(x_n) \subset f(u) + D. \tag{4.1}$$

For any $k \geq n$, $x_k \in S(x_n)$. By this and (4.1), for any $\gamma'' \in (0,\gamma)$, we have

$$f(x_n) \subset f(x_k) + \gamma'' d(x_k, x_n)H + D \subset f(u) + \gamma'' d(x_k, x_n)H + D. \tag{4.2}$$

For any fixed $\gamma' \in (0,\gamma)$, take any $\gamma'' \in (\gamma', \gamma)$. Since $d(x_k, x_n) \to d(u, x_n)$ $(k \to \infty)$, there exists $k' \geq n$ such that

$$d(x_{k'}, x_n) \geq \frac{\gamma'}{\gamma''} d(u, x_n), \quad \text{i.e.,} \quad \gamma'' d(x_{k'}, x_n) \geq \gamma' d(u, x_n). \tag{4.3}$$

By (4.2) and (4.3), we have

$$\begin{aligned} f(x_n) &\subset f(u) + \gamma'' d(x_{k'}, x_n)H + D \\ &\subset f(u) + \gamma' d(u, x_n)H + D. \end{aligned}$$

Thus, $u \in S(x_n) \subset S(x)$ and $S(x)$ is dynamically closed. Now, applying Corollary 3.4 we obtain the result. $\square$

Remark 4.1. If we denote the set $\{l \in Y^* : \inf l \circ H > 0\}$ by H^{+s}, then (B_1) is equivalent to the following assumption:

(B_1') $H^{+s} \cap D^+ \neq \emptyset$.

In this expression, (B_2) can be relaxed to the following weaker assumption:

(B_2') $\exists \xi \in H^{+s} \cap D^+$ such that ξ is lower bounded on $f(X)$.

Remark 4.2. When Y is a Banach space, [33, Theorem 3.4] gave the same result under the following assumptions (A1)-(A4):

(A1) $H \subset D \backslash \{0\}$ is a closed convex set (thus we have $\kappa := d(0, H) > 0$).

(A2) $\zeta := \inf\{d(h/\|h\|, -D) : h \in H\} > 0$.

(A3) $f(X)$ is D-bounded.

(A4) epi$f$ is closed in $X \times Y$, where epi$f = \{(x, y) \in X \times Y : y \in f(x) + D\}$.

Obviously, assumption (A3) is exactly (B_2) here. It is easy to show that assumptions (A1) and (A2) imply (B_1). Here, we needn't assume that H is closed. The essential assumption is $0 \notin \mathrm{cl}(H + D)$. Next we show that assumption (B_3) is strictly weaker than (A4). It is easy to see that (A4) $\Rightarrow$ (B_3). In fact, assume that (A4) holds. Let $(x_n) \subset X$ such that $f(x_n) \subset f(x_{n+1}) + D$, $\forall n$, and $x_n \to u$. Let n be given. For any $k \geq n$, we have $f(x_n) \subset f(x_k) + D$. Take any $y_n \in f(x_n)$. Then the points $(x_k, y_n) \in$ epi$f$. Since $(x_k, y_n) \to (u, y_n)$ ($k \to \infty$) in $X \times Y$ and epi$f$ is closed, we have $y_n \in f(u) + D$ and hence $f(x_n) \subset f(u) + D$. That is, f is D-s.l.m. The following example shows that there exists a D-s.l.m. set-valued map f such that epi$f$ is not closed, i.e., (B_3)$\not\Rightarrow$ (A4). Hence, even in the case that Y is a Banach space, Theorem 4.1 is also an improvement of [33, Theorem 3.4].

Example 4.1. A D-s.l.m. set-valued map f such that epi$f$ is not closed. Let $X = R$, $Y = R$ and $D = [0, +\infty)$. Define a set-valued map $f : X \to 2^Y \backslash \{\emptyset\}$ as follows:

$$f(x) = \begin{cases} x + 1 + D, & \text{if } x \in [0, +\infty), \\ \{x\}, & \text{if } x \in (-\infty, 0). \end{cases}$$

It is easy to verify that f is D-s.l.m. However, epi$f$ is not closed in $X \times Y$. For example, take a sequence (x_n) in the interval $(-\infty, 0)$ such that $x_n \to 0$. Obviously, $(x_n, x_n) \to (0, 0)$ in $X \times Y = R^2$. Here, every $(x_n, x_n) \in$ epi$f$. But $0 \notin f(0) + D$, i.e., $(0, 0) \notin$ epi$f$. Thus, epi$f$ is not closed.

Next we further introduce the following assumption:

(B'_3) f is D-s.l.m. and has D-closed values.

It is easy to see that (A4) $\Rightarrow$ (B'_3) $\Rightarrow$ (B_3). In Example 4.1, f also has D-closed values. Hence Example 4.1 shows that (B'_3) $\not\Rightarrow$ (A4). The following theorem is a set-valued extension of both [48, Theorem 6.2] and [42, Theorem 6.8], and it also generalizes and improves [33, Theorem 3.5] in the case that (ii) holds, where (ii) means that H is bounded. First let us recall some concepts (see [37, Definition 2.1.4]). Let H be a subset of a topological vector space. A convex series of points in H is a series of the form $\sum_{n=1}^{\infty} \lambda_n x_n$, where $x_n \in H$, $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. H is said to be σ-convex if every convex series of its points converges to a point of H. Sometimes, a σ-convex set is called a cs-complete set, see, e.g. [48, 51]. In [33, Theorem 3.5], $H \subset D \backslash \{0\}$ is assumed to be a closed convex subset of a Banach space Y and (ii) means that H is bounded, so H there is a σ-convex set. But a σ-convex set may be non-closed. For example, an open ball in a Banach space is a σ-convex set but non-closed. For details, see e.g., [43] and the references therein. As every singleton is σ-convex, if we take $H = \{k_0\}$, where $k_0 \in D \backslash - \mathrm{cl}(D)$, then we can also deduce Corollary 3.5 from the following theorem.

Theorem 4.2. *Let (X, d) be a complete metric space, $x_0 \in X$, Y be a locally convex space quasi ordered by a convex cone D, $H \subset D \backslash \{0\}$ be a convex set and $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map. Suppose that the following assumptions are satisfied:*

(B_1') *$H^{+s} \cap D^+ \neq \emptyset$, or equivalently, $0 \notin \mathrm{cl}(H + D)$.*

(B_2') *$\exists \xi \in H^{+s} \cap D^+$ such that ξ is lower bounded on $f(S(x_0))$, where $S(x_0) = \{x \in X : f(x_0) \subset f(x) + \gamma d(x, x_0)H + D\}$.*

(B_3') *f is D-s.l.m. and has D-closed values.*

Moreover, suppose that H is σ-convex, or, Y is locally complete and H is locally closed, bounded.

Then for any $\gamma > 0$, there exists $\hat{x} \in X$ such that

(a) *$f(x_0) \subset f(\hat{x}) + \gamma d(\hat{x}, x_0)H + D$;*

(b) *$\forall x \in X \backslash \{\hat{x}\}$, $f(\hat{x}) \not\subset f(x) + \gamma d(x, \hat{x})H + D$.*

Concerning local completeness and local closedness, see [37, Chapter 5] and [38, 39, 46]

Proof. Put $F(x, x') = \gamma d(x, x')H$. Obviously, the family $\{F\}$ satisfies the property TI. Also, it is obvious that $x_0 \in S(x_0)$ and $S(x_0) \neq \emptyset$. By (B_1') and (B_2'), we can easily show that assumptions (D) and (E_3) in Corollary 3.4 are satisfied. In order to apply Corollary 3.4, we only need to show that for any $x \in S(x_0)$, $S(x)$ is dynamically closed. Let $(x_n) \subset S(x)$, $x_{n+1} \in S(x_n) \subset S(x)$, $\forall n$, and $x_n \to u$. Take any fixed $n_0 \in N$ and put $z_1 := x_{n_0}$. As $d(x_k, u) \to 0$ $(k \to \infty)$, we may choose a sequence (z_n) from (x_k) such that $d(z_{n+1}, u) < 1/(n+1)$ and $z_{n+1} \in S(z_n)$, $\forall n$.

Take any $y_1 \in f(z_1)$. As $z_2 \in S(z_1)$, we have

$$y_1 \in f(z_1) \subset f(z_2) + \gamma d(z_2, z_1)H + D.$$

Hence, there exists $y_2 \in f(z_2)$, $h_1 \in H$ and $d_1 \in D$ such that

$$y_1 = y_2 + \gamma d(z_2, z_1)h_1 + d_1.$$

In general, if $y_n \in f(z_n)$ is given, then

$$y_n \in f(z_n) \subset f(z_{n+1}) + \gamma d(z_{n+1}, z_n)H + D,$$

so there exists $y_{n+1} \in f(z_{n+1})$, $h_n \in H$ and $d_n \in D$ such that

$$y_n = y_{n+1} + \gamma d(z_{n+1}, z_n)h_n + d_n.$$

Adding two sides of the above n equalities, we have

$$\sum_{i=1}^{n} y_i = \sum_{i=2}^{n+1} y_i + \gamma \sum_{i=1}^{n} d(z_{i+1}, z_i)h_i + \sum_{i=1}^{n} d_i.$$

From this,

$$y_1 = y_{n+1} + \gamma \sum_{i=1}^{n} d(z_{i+1}, z_i)h_i + \sum_{i=1}^{n} d_i. \tag{4.4}$$

As $\xi \in H^{+s} \cap D^+$, $\xi(D) \geq 0$ and there exists $\alpha > 0$ such that $\xi(H) \geq \alpha$. Acting on two sides of (4.4) by ξ, we have

$$\begin{aligned} \xi \circ y_1 &= \xi \circ y_{n+1} + \gamma \sum_{i=1}^{n} d(z_{i+1}, z_i)\, \xi(h_i) + \xi \left(\sum_{i=1}^{n} d_i \right). \\ &\geq \xi \circ y_{n+1} + \gamma\alpha \left(\sum_{i=1}^{n} d(z_{i+1}, z_i) \right). \end{aligned}$$

From this and by (B_2'),

$$\begin{aligned} \sum_{i=1}^{n} d(z_{i+1}, z_i) &\leq \frac{1}{\gamma\alpha}(\xi \circ y_1 - \xi \circ y_{n+1}) \\ &\leq \frac{1}{\gamma\alpha}(\xi \circ y_1 - \inf \xi \circ f(S(x_0))) \\ &< +\infty. \end{aligned}$$

Hence, $\sum\limits_{i=1}^{\infty} d(z_{i+1}, z_i) < +\infty$. By the assumption that H is σ-convex, we conclude that

$$\frac{\sum\limits_{i=1}^{\infty} d(z_{i+1}, z_i)h_i}{\sum\limits_{j=1}^{\infty} d(z_{j+1}, z_j)}$$

is convergent to some point $\bar{h} \in H$. Put

$$h'_n := \frac{\sum_{i=1}^{n} d(z_{i+1}, z_i) h_i}{\sum_{j=1}^{n} d(z_{j+1}, z_j)}.$$

Then every $h'_n \in H$ and $h'_n \to \bar{h}$. From (4.4), we have

$$y_1 \in y_{n+1} + \gamma \left(\sum_{i=1}^{n} d(z_{i+1}, z_i) \right) h'_n + D. \tag{4.5}$$

Remark that

$$\sum_{i=1}^{n} d(z_{i+1}, z_i) \geq d(z_1, u) - d(z_{n+1}, u) \ \text{ and } \ d(z_{n+1}, u) < 1/(n+1). \tag{4.6}$$

Also, By (B'_3),

$$y_{n+1} \in f(z_{n+1}) \subset f(u) + D. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we have

$$\begin{aligned} y_1 &\in y_{n+1} + \gamma(d(z_1, u) - d(z_{n+1}, u)) h'_n + D \\ &\subset y_{n+1} + \gamma(d(z_1, u) - 1/(n+1)) h'_n + D \\ &\subset f(u) + \gamma(d(z_1, u) - 1/(n+1)) h'_n + D. \end{aligned} \tag{4.8}$$

Since $\gamma(d(z_1, u) - 1/(n+1)) h'_n \to \gamma d(z_1, u) \bar{h}$ and $f(u) + D$ is closed by (B'_3), from (4.8) we have

$$y_1 \in f(u) + \gamma d(z_1, u) \bar{h} + D, \quad \text{where } \ \bar{h} \in H.$$

Thus, we have shown that

$$f(z_1) \subset f(u) + \gamma d(z_1, u) H + D, \quad \text{that is, } \ u \in S(z_1) \subset S(x).$$

Now, we can apply Corollary 3.4 and the result follows. Finally, we point out that Y is locally complete iff it is l^1-complete (see [46]). Hence it is easy to see that Y being locally complete and H being locally closed bounded imply that H is σ-convex. □

In Theorem 4.2, if we strengthen assumption (B'_3) to the following (B''_3): f is D-s.l.m. and has (H, D)-closed values, i.e., for any $x \in X$ and any $\lambda \geq 0$, $f(x) + \lambda H + D$ is closed, then the assumption that H is σ-convex can be weakened to that H is bounded.

Theorem 4.2′. *Let (X, d) be a complete metric space, $x_0 \in X$, Y be a locally convex space pre-ordered by a convex cone D, $H \subset D\backslash\{0\}$ be a convex set and $f : X \to 2^Y\backslash\{\emptyset\}$ be a set-valued map. Suppose that the following assumptions are satisfied:*

(B_1') *$H^{+s} \cap D^+ \neq \emptyset$, or equivalently, $0 \notin \mathrm{cl}(H + D)$.*

(B_2') *$\exists \xi \in H^{+s} \cap D^+$ such that ξ is lower bounded on $f(S(x_0))$.*

(B_3'') *f is D-s.l.m. and has (H, D)-closed values.*

Moreover, suppose that H is bounded.

Then for any $\gamma > 0$, there exists $\hat{x} \in X$ such that

(a) *$f(x_0) \subset f(\hat{x}) + \gamma d(\hat{x}, x_0)H + D$;*

(b) *$\forall x \in X\backslash\{\hat{x}\}$, $f(\hat{x}) \not\subset f(x) + \gamma d(x, \hat{x})H + D$.*

Proof. As shown in the proof of Theorem 4.2, we easily see that assumption (E_3) in Corollary 3.4 is satisfied. In order to apply Corollary 3.4, we only need to show that for any $x \in S(x_0)$, $S(x)$ is dynamically closed. Let $(x_n) \subset S(x)$, $x_{n+1} \in S(x_n) \subset S(x)$, $\forall n$, and $x_n \to u$. Take any fixed $n_0 \in N$ and put $z_1 := x_{n_0}$. As $d(x_k, u) \to 0$ $(k \to \infty)$, we may choose a sequence (z_n) from (x_k) such that $d(z_{n+1}, u) < 1/(n + 1)$ and $z_{n+1} \in S(z_n)$, $\forall n$. Take any $y_1 \in f(z_1)$. As done in the proof of Theorem 4.2, for every n, we can choose $y_{n+1} \in f(z_{n+1})$, $h_n \in H$ and $d_n \in D$ such that

$$y_1 \;=\; y_{n+1} + \gamma\left(\sum_{i=1}^{n} d(z_{i+1}, z_i)h_i\right) + \sum_{i=1}^{n} d_i \;=\; y_{n+1} + \gamma\left(\sum_{i=1}^{n} d(z_{i+1}, z_i)\right) h_n' + \sum_{i=1}^{n} d_i,$$

where

$$h_n' \;=\; \frac{\sum_{i=1}^{n} d(z_{i+1}, z_i)h_i}{\sum_{j=1}^{n} d(z_{j+1}, z_j)} \;\in\; H.$$

Combining this with the assumption that $d(z_{n+1}, u) < 1/(n+1)$, we can also deduce (4.8), i.e.,

$$y_1 \in f(u) + \gamma\left(d(z_1, u) - \frac{1}{n+1}\right) h_n' + D.$$

From this,

$$y_1 + \frac{\gamma}{n+1} h_n' \;\in\; f(u) + \gamma d(z_1, u)H + D.$$

Since $f(u) + \gamma d(z_1, u)H + D$ is closed by (B_3''), letting $n \to \infty$ we have

$$y_1 \in f(u) + \gamma d(z_1, u)H + D.$$

Thus,

$$f(z_1) \subset f(u) + \gamma d(z_1, u)H + D \quad \text{and} \quad u \in S(z_1) = S(x_{n_0}) \subset S(x).$$

Now, applying Corollary 3.4, we obtain the result. □

Theorem 4.3. *Let (X, d) be a complete metric space, Y be a locally convex space whose topology is determined by a saturated family $\{p_\alpha\}_{\alpha\in\Lambda}$ of semi-norms* (concerning saturated family of semi-norms, see [26, p.96]), *$D \subset Y$ be a convex cone and $H \subset D\backslash\{0\}$ be a closed convex set. Suppose that the following assumptions are satisfied:*

(B_2) *$f(X)$ is D-bounded.*

(B_3') *f is D-s.l.m. and has D-closed values.*

Moreover, assume that Y is l^∞-complete (see [37, 38]) *and for each $\alpha \in \Lambda$ there exists $\xi_\alpha \in D^+\backslash\{0\}$ and $\lambda_\alpha > 0$ such that $\lambda_\alpha p_\alpha(h) \le \xi_\alpha(h),\ \forall h \in H$.*

Then for any $x_0 \in X$ and any $\gamma > 0$, there exists $\hat{x} \in X$ such that

(a) *$f(x_0) \subset f(\hat{x}) + \gamma d(\hat{x}, x_0)H + D$;*

(b) *$\forall x \in X\backslash\{\hat{x}\},\ f(\hat{x}) \not\subset f(x) + \gamma d(x, \hat{x})H + D$.*

Proof. Put $F(x, x') = \gamma d(x, x')H$. Obviously, the family $\{F\}$ satisfies the property TI. Also, $x_0 \in S(x_0)$ and $S(x_0) \neq \emptyset$, where $S(x_0)$ is the same as one in the proof of Theorem 4.2. Since H is closed and $0 \notin H$, there exists $\alpha_0 \in \Lambda$ and $\eta > 0$ such that $p_{\alpha_0}(h) \ge \eta,\ \forall h \in H$. By the assumption, there exists $\xi_{\alpha_0} \in D^+\backslash\{0\}$ and $\lambda_{\alpha_0} > 0$ such that

$$\lambda_{\alpha_0}\eta \le \lambda_{\alpha_0} p_{\alpha_0}(h) \le \xi_{\alpha_0}(h), \quad \forall h \in H. \tag{4.9}$$

Thus, $\xi_{\alpha_0} \in D^+ \cap H^{+s}$. Combining this with ($B_2$), we can show that assumption (E_3) in Corollary 3.4 is satisfied. By Corollary 3.4, it is sufficient to show that for any $x \in S(x_0)$, $S(x)$ is dynamically closed. Let $(x_n) \subset S(x)$, $x_{n+1} \in S(x_n) \subset S(x),\ \forall n$, and $x_n \to u$. As done in the proof of Theorem 4.2, we can obtain a sequence (z_n) from (x_k) such that $d(z_{n+1}, u) < 1/(n+1)$ and $z_{n+1} \in S(z_n),\ \forall n$.

Take any $y_1 \in f(z_1)$. We may choose $y_{n+1} \in f(z_{n+1})$, $h_n \in H$ and $d_n \in D$ such that (see (4.4))

$$y_1 \;=\; y_{n+1} + \gamma \sum_{i=1}^{n} d(z_{i+1}, z_i) h_i + \sum_{i=1}^{n} d_i. \tag{4.10}$$

For each $\alpha \in \Lambda$, acting on two sides of (4.10) by ξ_α, we have

$$\begin{aligned}
\xi_\alpha \circ y_1 \;&=\; \xi_\alpha \circ y_{n+1} + \gamma \sum_{i=1}^{n} d(z_{i+1}, z_i)\, \xi_\alpha \circ h_i + \sum_{i=1}^{n} \xi_\alpha \circ d_i \\
&\geq\; \xi_\alpha \circ y_{n+1} + \gamma \sum_{i=1}^{n} d(z_{i+1}, z_i)\, \xi_\alpha \circ h_i \\
&\geq\; \xi_\alpha \circ y_{n+1} + \gamma \lambda_\alpha \sum_{i=1}^{n} d(z_{i+1}, z_i) p_\alpha(h_i).
\end{aligned}$$

From this,

$$\begin{aligned}
\sum_{i=1}^{n} d(z_{i+1}, z_i) p_\alpha(h_i) \;&\leq\; \frac{1}{\gamma\lambda_\alpha}(\xi_\alpha \circ y_1 - \xi_\alpha \circ y_{n+1}) \\
&\leq\; \frac{1}{\gamma\lambda_\alpha}(\xi_\alpha \circ y_1 - \inf \xi_\alpha \circ f(S(x_0))) \\
&<\; +\infty.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{\infty} d(z_{i+1}, z_i) p_\alpha(h_i) \;<\; +\infty, \quad \forall \alpha \in \Lambda.$$

Since Y is l^∞-complete, we know that $\sum\limits_{i=1}^{\infty} d(z_{i+1}, z_i) h_i$ is convergent in Y. Combining this with (4.9), we have

$$\lambda_{\alpha_0} \eta \sum_{i=1}^{n} d(z_{i+1}, z_i) \;\leq\; \sum_{i=1}^{n} d(z_{i+1}, z_i)\, \xi_{\alpha_0}(h_i) \;\leq\; \xi_{\alpha_0}\left(\sum_{i=1}^{\infty} d(z_{i+1}, z_i) h_i\right), \quad \forall n \in N.$$

Thus,

$$\sum_{i=1}^{\infty} d(z_{i+1}, z_i) \;<\; +\infty.$$

Put

$$h'_n := \frac{\sum\limits_{i=1}^{n} d(z_{i+1}, z_i) h_i}{\sum\limits_{j=1}^{n} d(z_{j+1}, z_j)}.$$

Then

$$h'_n \in H \quad \text{and} \quad h'_n \to \bar{h} := \frac{\sum\limits_{i=1}^{\infty} d(z_{i+1}, z_i) h_i}{\sum\limits_{j=1}^{\infty} d(z_{j+1}, z_j)} \;\in\; H.$$

From (4.10), we have

$$y_1 \in y_{n+1} + \gamma \left(\sum_{i=1}^{n} d(z_{i+1}, z_i) \right) h'_n + D.$$

This is exactly (4.5). The remains of the proof is the same as one in the proof of Theorem 4.2 and we omit the details. □

Remark 4.3. Here we needn't assume that (B'_1) (equivalently, (B_1)) holds in advance. It can be deduced from H being closed and the existence of $\xi_\alpha \in D^+ \backslash \{0\}$ and $\lambda_\alpha > 0$ such that $\lambda_\alpha p_\alpha(h) \le \xi_\alpha(h), \ \forall h \in H$. In fact, in [33, Theorem 3.5], assumption (A2) can be removed when(i) holds, where Y is a Banach space and (i) means that there exists $\xi \in D^+ \backslash \{0\}$ and $\lambda > 0$ such that $\lambda \|h\| \le \xi(h), \ \forall h \in H$. Obviously, Theorem 4.3 is a generalization of the part (where (i) holds) of [33, Theorem 3.5].

Similarly, by Corollary 3.4 we can also obtain the part (where (iii) holds, i.e., Y is reflexive) of [33, Theorem 3.5]. In this case, assumption (A4) (i.e., epi$f$ is closed in $X \times Y$) in [33] can also be relaxed to (B'_3) (i.e., f is D-s.l.m. and has D-closed values). We already presented several set-valued EVPs, where perturbations are given by a convex subset multiplied by the distance. Next we further consider more general version of set-valued EVP, where the perturbation is given by a convex subset multiplied by a real function which is more general than the distance.

Theorem 4.4. *Let (X, d) be a metric space, Y be a locally convex space, $D \subset Y$ be a convex cone, $H \subset D \backslash \{0\}$ be a convex set and $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map. Let a real function $p : X \times X \to R^+ := [0, +\infty)$ satisfy the following properties:*

(p_1) *for any $x_1, x_2, x_3 \in X$, $p(x_1, x_3) \le p(x_1, x_2) + p(x_2, x_3)$;*

(p_2) *every sequence (x_n) with $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ is a Cauchy sequence, where $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ means that for any $\epsilon > 0$, there exists $n_0 \in N$ such that $p(x_n, x_m) < \epsilon$ for all $m > n \ge n_0$;*

(p_3) *$p(x, x') > 0, \ \forall x \ne x'$.*

Let $x_0 \in X$ such that

$$S(x_0) := \{x \in X : \ f(x_0) \subset f(x) + p(x_0, x)H + D\} \ne \emptyset$$

and (X, d) be $S(x_0)$-dynamically complete. Suppose that for any $x \in S(x_0)$, $S(x)$ is dynamically closed and the following assumptions are satisfied:

(B'_1) $H^{+s} \cap D^+ \neq \emptyset$, *or equivalently,* $0 \notin \mathrm{cl}(H + D)$.

(B'_2) $\exists \xi \in H^{+s} \cap D^+$ *such that* ξ *is lower bounded on* $f(S(x_0))$.

Then there exists $\hat{x} \in X$ *such that*

(a) $f(x_0) \subset f(\hat{x}) + p(x_0, \hat{x})H + D$;

(b) $\forall x \in X \backslash \{\hat{x}\},\ f(\hat{x}) \not\subset f(x) + p(\hat{x}, x)H + D$.

Proof. Put $F(x, x') := p(x', x)H,\ \forall x, x' \in X$. Obviously, the family $\{F\}$ satisfies the property TI. By assumptions (B'_1) and (B'_2), there exists $\xi \in D^+\backslash\{0\}$ and $\alpha > 0$ such that $\xi(H) \geq \alpha > 0$ and ξ is lower bounded on $f(S(x_0))$. Clearly, ξ satisfies assumption (D) in Theorem 3.1′. For any $x \in S(x_0)$ and any $x' \in S(x)\backslash\{x\}$, we have

$$f(x) \subset f(x') + p(x, x')H + D. \tag{4.11}$$

By (p_3), $p(x, x')\alpha > 0$, thus we may take $y \in f(x)$ such that

$$\xi \circ y < \inf \xi \circ f(x) + \frac{1}{2} p(x, x')\alpha. \tag{4.12}$$

By (4.11), we have

$$y \in f(x') + p(x, x')H + D.$$

Thus, there exists $y' \in f(x'),\ h' \in H$ and $d' \in D$ such that

$$y = y' + p(x, x')h' + d'.$$

Acting two sides of the above equality by ξ, we have

$$\begin{aligned} \xi \circ y &= \xi \circ y' + p(x, x')\, \xi \circ h' + \xi \circ d' \\ &\geq \xi \circ y' + p(x, x')\, \alpha \\ &\geq \inf \xi \circ f(x') + p(x, x')\, \alpha. \end{aligned}$$

Combining this with (4.12), we have

$$\inf \xi \circ f(x) > \inf \xi \circ f(x') + \frac{1}{2} p(x, x')\, \alpha > \inf \xi \circ f(x').$$

Thus, ξ satisfies assumption (E) in Theorem 3.1′.

Next we show that assumption (F) is satisfied. Let a sequence $(x_n) \subset S(x_0)$ such that $x_n \in S(x_{n-1})$ and

$$\inf \xi \circ f(x_n) < \inf \xi \circ f(S(x_{n-1})) + \epsilon_n,\ \forall n,$$

where $\epsilon_n > 0$ and $\epsilon_n \to 0$. For each n, take $y_n \in f(x_n)$ such that

$$\xi \circ y_n \ < \ \inf \xi \circ f(S(x_{n-1})) + \epsilon_n. \tag{4.13}$$

When $m > n$, $x_m \in S(x_n)$. Hence

$$y_n \ \in \ f(x_n) \ \subset \ f(x_m) + p(x_n, x_m)H + D.$$

Thus, there exists $y_{m,n} \in f(x_m)$, $h_{m,n} \in H$ and $d_{m,n} \in D$ such that

$$y_n \ = \ y_{m,n} + p(x_n, x_m)h_{m,n} + d_{m,n}.$$

Acting two sides of the above equality by ξ, we have

$$\xi \circ y_n \ = \ \xi \circ y_{m,n} + p(x_n, x_m)\, \xi \circ h_{mn} + \xi \circ d_{mn} \ \geq \ \xi \circ y_{mn} + p(x_n, x_m)\, \alpha. \tag{4.14}$$

Observe that $y_{m,n} \in f(x_m) \subset f(S(x_{n-1}))$. From (4.14) and (4.13), we have

$$\begin{aligned} p(x_n, x_m) \ &\leq \ \frac{1}{\alpha}(\xi \circ y_n - \xi \circ y_{m,n}) \\ &\leq \ \frac{1}{\alpha}(\xi \circ y_n - \inf \xi \circ f(S(x_{n-1}))) \\ &< \ \frac{1}{\alpha}\epsilon_n. \end{aligned}$$

Hence $p(x_n, x_m) \to 0$ $(m > n \to \infty)$. By property (p_2), (x_n) is a Cauchy sequence. As (X, d) is $S(x_0)$-dynamically complete, there exists $u \in X$ such that $x_n \to u$. Since $S(x_n)$ is dynamically closed, we easily see that $u \in S(x_n)$, $\forall n$. Thus, assumption (F) in Theorem 3.1$'$ is satisfied. By Theorem 3.1$'$, we obtain the result. □

At the end of this section, we consider EVPs for approximately efficient solutions. Németh [35] gave the concept of approximately efficient solutions for vector-valued maps. Here, we extend the concept to set-valued maps.

Let X be a nonempty set, Y be a real linear space, $D \subset Y$ be a convex pointed cone and $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map. We consider the following vector optimization problem:

$$\mathrm{Min}\{f(x) : \ x \in X\}. \tag{4.15}$$

Moreover, let $\epsilon > 0$ and $H \subset D \backslash \{0\}$ be a convex set.

Definition 4.1. A point $x_0 \in X$ is called an efficient solution of (4.15) if $f(x_0) \not\subset f(X) + D \backslash \{0\}$, where $f(X)$ denotes the set $\cup_{x \in X} f(x)$. A point $x_0 \in X$ is called an (ϵ, H)-efficient solution of (4.15) if $f(x_0) \not\subset f(X) + \epsilon\, H + D$, or equivalently, there exists $y_0 \in f(x_0)$ such that $(y_0 - \epsilon\, H - D) \cap f(X) = \emptyset$.

Theorem 4.5. *In Theorem 4.1, moreover assume that x_0 is an (ϵ, H)-efficient solution of (4.15). Then there exists $\hat{x} \in X$ such that*
(a) $f(x_0) \subset f(\hat{x}) + \gamma' d(\hat{x}, x_0)H + D, \quad \forall \gamma' \in (0, \gamma)$;
(b) $\forall x \in X \backslash \{\hat{x}\}, \ \exists \gamma' \in (0, \gamma)$ *such that* $f(\hat{x}) \not\subset f(x) + \gamma' d(x, \hat{x})H + D$;
(c) $d(\hat{x}, x_0) \leq \epsilon/\gamma$.

Proof. By Theorem 4.1, we conclude that (a) and (b) hold. Hence, we only need to show that (c) holds. If not, assume that $d(\hat{x}, x_0) > \epsilon/\gamma$. Then $\epsilon/d(\hat{x}, x_0) < \gamma$. By (a),

$$\begin{aligned} f(x_0) &\subset f(\hat{x}) + (\epsilon/d(\hat{x}, x_0))d(\hat{x}, x_0)H + D \\ &= f(\hat{x}) + \epsilon H + D \\ &\subset f(X) + \epsilon H + D. \end{aligned}$$

This contradicts the assumption that x_0 is an (ϵ, H)-efficient solution of (4.15). □

Similarly we can prove the following:

Theorem 4.6. *In Theorems 4.2, 4.2′ and 4.3, moreover assume that x_0 is an (ϵ, H)-efficient solution of (4.15). Then there exists $\hat{x} \in X$ such that*
(a) $f(x_0) \subset f(\hat{x}) + \gamma d(\hat{x}, x_0)H + D$;
(b) $\forall x \in X \backslash \{\hat{x}\}, \ f(\hat{x}) \not\subset f(x) + \gamma d(x, \hat{x})H + D$;
(c) $d(\hat{x}, x_0) < \epsilon/\gamma$.

5. Pre-orders and minimal points in product spaces

In this section, by using Theorem 2.1 we discuss pre-orders and minimal points in product spaces. Particularly, we obtain several versions of EVP for Pareto minimizers, which generalize the corresponding results in [4, 5, 31, 48].

First we recall some concepts on Pareto minimum. Let Y be a real linear space with a quasi order $\leq_D$ defined by a convex cone D in Y. Let $B \subset Y$ be nonempty.

A point $\bar{y} \in B$ is called a Pareto minimum of B if $y \in B$ and $y \leq_D \bar{y}$ implies that $\bar{y} \leq_D y$. And $\bar{y} \in B$ is called a strict Pareto minimum of B if $y \not\leq_D \bar{y}$ for all $y \in B \backslash \{\bar{y}\}$, i.e., $(B-\bar{y}) \cap (-D) = \{0\}$. We denote by $\mathrm{Min}^D B$ (resp., $\mathrm{SMin}^D B$) the sets of all Pareto minima (resp., strict Pareto minima) with respect to the order $\leq_D$. In general, we have $\mathrm{SMin}^D B \subset \mathrm{Min}^D B$. If D is pointed, i.e., $D \cap (-D) = \{0\}$, then $\mathrm{SMin}^D B = \mathrm{Min}^D B$. A subset B of Y is said to have the domination (resp., strict domination) property if, for any $y \in B$, there exists $\bar{y} \in \mathrm{Min}^D B$ (resp., $\bar{y} \in \mathrm{SMin}^D B$) such that $\bar{y} \leq_D y$.

Moreover, let (X, d) be a metric space and let $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map. A point $\bar{x} \in X$ is called a Pareto minimizer (resp., strict Pareto minmizer) of f if there exists $\bar{y} \in f(\bar{x})$ such that $\bar{y} \in \mathrm{Min}^D f(X)$ (resp., $\bar{y} \in \mathrm{SMin}^D f(X)$).

Let $F : X \times X \to 2^D \backslash \{\emptyset\}$ satisfy the following conditions (see [48]):

(F1) $0 \in F(x, x),\ \forall x \in X$;

(F2) $F(x_1, x_2) + F(x_2, x_3) \subset F(x_1, x_3) + D,\ \forall x_1,\ x_2,\ x_3 \in X$;

(F3) there exists a D-monotone extended real function $\xi : Y \to (-\infty, +\infty]$ such that for any $y \in Y$ and any $z \in F(X \times X)$, $\xi(y + z) = \xi(y) + \xi(z)$ and such that for any $\delta > 0$,

$$\zeta(\delta) := \inf\{\xi(y) :\ y \in \cup_{d(x,x') \geq \delta} F(x, x')\} > 0.$$

Obviously, $\xi(0) = \xi(0 + 0) = \xi(0) + \xi(0) = 2\xi(0)$, so $\xi(0) = 0$.

As in [48], define a quasi-order $\preceq_F$ on $X \times Y$ as follows:

$$(x_2, y_2) \preceq_F (x_1, y_1) \iff y_1 \in y_2 + F(x_2, x_1) + D.$$

Consider a nonempty set $A \subset X \times Y$ and a point $(x_0, y_0) \in A$. Put

$$S_F(x_0, y_0) := \{(x, y) \in A :\ (x, y) \preceq_F (x_0, y_0)\}.$$

Moreover, define a partial order $\preceq_{F^*}$ on $X \times Y$ as follows:

$$(x_2, y_2) \preceq_{F^*} (x_1, y_1) \iff \begin{cases} (x_2, y_2) = (x_1, y_1) \ \text{ or} \\ (x_2, y_2) \preceq_F (x_1, y_1) \text{ and } \xi(y_2 - y_0) < \xi(y_1 - y_0). \end{cases}$$

Put

$$S_{F^*}(x_0, y_0) := \{(x, y) \in A :\ (x, y) \preceq_{F^*} (x_0, y_0)\}.$$

If ξ is strict D-monotone, i.e., $y_2 \leq_D y_1$ and $y_2 \neq y_1$ imply that $\xi(y_2) < \xi(y_1)$, then the orders $\preceq_F$ and $\preceq_{F^*}$ are coincident.

By using Theorem 2.1, we can deduce the following minimal point theorem in product spaces.

Theorem 5.1. *Let (X, d) be a metric space, Y be a real linear space, $D \subset Y$ be a convex cone and a set-valued map $F: X \times X \to 2^D \backslash \{\emptyset\}$ satisfy (F1)-(F3). Let $A \subset X \times Y$ be a nonempty set and $(x_0, y_0) \in A$ be given. Suppose that the following conditions are satisfied:*

(i) *for any $\preceq_F$-decreasing sequence $\{(x_n, y_n)\}$ in $S_F(x_0, y_0)$, if $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ is convergent in X;*

(ii) *ξ* (from in (F3)) *is lower bounded on $P_Y(S_F(x_0, y_0)) - y_0$, where P_Y is the projection from $X \times Y$ on Y;*

(iii) *for every $\preceq_F$-decreasing sequence $\{(x_n, y_n)\}$ in $S_F(x_0, y_0)$, if $\{x_n\}$ converges to x, then there exists $y \in Y$ such that $(x, y) \in A$ and $(x, y) \preceq_F (x_n, y_n)$, $\forall n$* (in [48], (iii) is called (H1)).

Then, there exists $(\hat{x}, \hat{y}) \in A$ such that

(a) $(\hat{x}, \hat{y}) \preceq_{F^*} (x_0, y_0)$;

(b) $(x, y) \preceq_{F^*} (\hat{x}, \hat{y}) \implies (x, y) = (\hat{x}, \hat{y})$.

From this, we conclude that $(\hat{x}, \hat{y}) \in A$ satisfies the following

(a$'$) $y_0 \in \hat{y} + F(\hat{x}, x_0) + D$;

(b$'$) *for any $(x, y) \in A$ with $x \neq \hat{x}$. $\hat{y} \notin y + F(x, \hat{x}) + D$.*

Proof. Obviously, $(A, \preceq_{F^*})$ is a partial order set and $S_{F^*}(x_0, y_0) \neq \emptyset$. Define $\eta: (A, \preceq_{F^*}) \to (-\infty, +\infty]$ as follows: $\eta \circ (x, y) = \xi(y - y_0)$, $\forall (x, y) \in A$. Clearly, η is monotone with respect to $\preceq_{F^*}$. By (ii), ξ is lower bounded on $P_Y(S_F(x_0, y_0)) - y_0$, Also, $0 \in P_Y(S_F(x_0, y_0)) - y_0$ and $\xi(0) = 0$. Thus,

$$-\infty < \inf\{\eta \circ (x, y) = \xi(y - y_0) : (x, y) \in S_{F^*}(x_0, y_0)\} < +\infty.$$

This means that assumption (A) is satisfied if we regard $(A, \preceq_{F^*})$ as a pre-order set $(X, \preceq)$ in Theorem 2.1. For any $(x, y) \in S_{F^*}(x_0, y_0)$ and any $(x', y') \in S_{F^*}(x, y) \backslash \{(x, y)\}$, we have $y \in y' + F(x', x) + D$ and $\xi(y' - y_0) < \xi(y - y_0)$, that is, $\eta \circ (x', y') < \eta \circ (x, y)$. Thus, assumption (B) in Theorem 2.1 is satisfied.

Let a sequence $\{(x_n, y_n)\} \subset S_{F^*}(x_0, y_0)$ satisfy $(x_n, y_n) \preceq_{F^*} (x_{n-1}, y_{n-1})$ $\forall n$. We shall show that there exists $(\bar{x}, \bar{y}) \in A$ such that $(\bar{x}, \bar{y}) \preceq_{F^*} (x_n, y_n)$, $\forall n$, i.e., assumption (C) in Theorem 2.1 satisfied. If there exists a sequence $n_1 < n_2 < \cdots$ such that $(x_{n_i}, y_{n_i}) = (x_{n_{i+1}}, y_{n_{i+1}})$, $\forall i$, then we have $(x_k, y_k) = (x_{n_1}, y_{n_1})$ for all $k \geq$

n_1 and the result is trivial. Hence, we may assume that $(x_n, y_n) \neq (x_{n-1}, y_{n-1})$, $\forall n$. From the definition of $\preceq_{F^*}$, we have $y_{n-1} \in y_n + F(x_n, x_{n-1}) + D$ and

$$\xi(y_n - y_0) \; < \; \xi(y_{n-1} - y_0), \quad \forall n. \tag{5.1}$$

As ξ is lower bounded on $P_Y(S_F(x_0, y_0)) - y_0$, $\{\xi(y_n - y_0)\}$ is a lower bounded, decreasing real sequence, so $\{\xi(y_n - y_0)\}$ is convergent. We assume that

$$\gamma := \lim_{n\to\infty} \xi(y_n - y_0) \; \in \mathbf{R}. \tag{5.2}$$

First, we assert that $\{x_n\}$ is a Cauchy sequence. If not, there exists $\delta > 0$ and a sequence $n_1 < n_2 < \cdots$ such that $d(x_{n_i}, x_{n_{i+1}}) \geq \delta$. Since $(x_{n_{i+1}}, y_{n_{i+1}}) \preceq_{F^*} (x_{n_i}, y_{n_i})$, we have $y_{n_i} \in y_{n_{i+1}} + F(x_{n_{i+1}}, x_{n_i}) + D$. Thus,

$$\xi(y_{n_i} - y_0) - \xi(y_{n_{i+1}} - y_0) \; \geq \; \zeta(\delta) \; > \; 0, \quad \forall i,$$

which contradicts (5.2). Thus, $\{x_n\}$ is a Cauchy sequence and by assumption (i), there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. By assumption (iii), there exists $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in A$ and $(\bar{x}, \bar{y}) \preceq_F (x_n, y_n)$, $\forall n$. From this, we have $\xi(\bar{y} - y_0) \leq \xi(y_n - y_0)$ and by (5.1), we have $\xi(\bar{y} - y_0) < \xi(y_n - y_0)$ for all n. Thus, $(\bar{x}, \bar{y}) \preceq_{F^*} (x_n, y_n)$, $\forall n$. Now, we can apply Theorem 2.1 and obtain the result, i.e., there exists $(\hat{x}, \hat{y}) \in A$ such that (a) and (b) hold. Obviously, $(a) \Rightarrow (a')$. Next, we show that $(b) \Rightarrow (b')$. Assume that (b′) is not true. That is, there exists $(x, y) \in A$ with $x \neq \hat{x}$ such that

$$\hat{y} \in y + F(x, \hat{x}) + D. \tag{5.3}$$

Thus, there exists $v \in F(x, \hat{x})$ such that $\hat{y} \in y + v + D$ and hence $\hat{y} - y_0 \in y - y_0 + v + D$. As $d(x, \hat{x}) > 0$, we have $\xi(v) > 0$ and

$$\xi(\hat{y} - y_0) \geq \xi(y - y_0) + \xi(v) > \xi(y - y_0).$$

Combining this with (5.3), we have $(x, y) \preceq_{F^*} (\hat{x}, \hat{y})$ and $x \neq \hat{x}$, which contradicts (b). $\square$

Remark 5.1. If ξ appeared in (F3) is a positive linear functional, i.e., $\xi \in D^+ \backslash \{0\}$, then the case becomes easier. This time, we needn't use y_0. For example, we may define $\preceq_{F^*}$ on $X \times Y$ as follows:

$$(x_2, y_2) \preceq_{F^*} (x_1, y_1) \iff \begin{cases} (x_2, y_2) = (x_1, y_1) \;\; \text{or} \\ (x_2, y_2) \preceq_F (x_1, y_1) \text{ and } \xi(y_2) < \xi(y_1). \end{cases}$$

Moreover, assumption (ii) in Theorem 5.1 can be written as: ξ is lower bounded on $P_Y(S_F(x_0, y_0))$.

Obviously, Theorem 5.1 generalizes [48, Theorem 2.1]. In fact, it also includes [31, Theorem 4.2]. In [31, Theorem 4.2], we assume that Y is a locally convex space and $H \subset D$ is a convex set such that $0 \notin \mathrm{cl}(H + D)$. By the Hahn-Banach separation theorem, there exists $\xi \in D^+$ such that $\alpha := \inf\{\xi(y) : y \in H\} > 0$. Put $F(x, x') := d(x, x')H, \ \forall x, x' \in X$. Then for any $\delta > 0$, $\zeta(\delta) := \inf\{\xi(y) : y \in \cup_{d(x,x') \geq \delta} d(x, x')H\} \geq \alpha\delta > 0$. It is clear that F satisfies (F1)-(F3). Now, applying Theorem 5.1 we can obtain [31, Theorem 4.2]. In order to obtain a result on strict minimal elements in product orders, we need to strengthen assumption (ii) in Theorem 5.1 (see [31, Theorem 4.3]).

Theorem 5.2. *Impose the assumptions of Theorem 5.1 with* (ii) *being strengthened as*

(ii′) *ξ is lower bounded on $P_Y(S_F(x_0, y_0)) - y_0$ and for all $x \in P_X(S_F(x_0, y_0))$, the set $\{y' \in Y : (x, y') \in A\}$ has the strict domination property.*

Then, there exists $(\hat{x}, \hat{y}) \in A$ such that

(a) *$y_0 \in \hat{y} + F(\hat{x}, x_0) + D$ and $\hat{y} \in \mathrm{SMin}^D\{y' : (\hat{x}, y') \in A\}$;*

(b) *for any $(x, y) \in A \backslash \{(\hat{x}, \hat{y})\}$, $\hat{y} \notin y + F(x, \hat{x}) + D$.*

Proof. By Theorem 5.1, there exists $(\hat{x}, \tilde{y}) \in A$ such that

$$y_0 \in \tilde{y} + F(\hat{x}, x_0) + D \tag{5.4}$$

and such that

$$\tilde{y} \notin y + F(x, \hat{x}) + D \text{ for any } (x, y) \in A \text{ with } x \neq \hat{x}. \tag{5.5}$$

By the imposed strict domination property, there is $\hat{y} \in \mathrm{SMin}^D\{y' : (\hat{x}, y') \in A\}$ such that $\hat{y} \leq_D \tilde{y}$. Next, we show that $(\hat{x}, \hat{y})$ is a desired element. By (5.4) and $\hat{y} \leq_D \tilde{y}$, we have

$$y_0 \in \hat{y} + D + F(\hat{x}, x_0) + D = \hat{y} + F(\hat{x}, x_0) + D.$$

Hence, $(\hat{x}, \hat{y})$ satisfies (a).

Let $(x, y) \in A \backslash \{(\hat{x}, \hat{y})\}$. Assume that

$$\hat{y} \in y + F(x, \hat{x}) + D. \tag{5.6}$$

Then $\tilde{y} \in \hat{y} + D \subset y + F(x, \hat{x}) + D$. By (5.5), we have $x = \hat{x}$. Thus, we have $\hat{y}, y \in \{y' : (\hat{x}, y') \in A\}$. By (5.6), $y \leq_D \hat{y}$. Since $\hat{y} \in \mathrm{SMin}^D\{y' : (\hat{x}, y') \in A\}$, we have $y = \hat{y}$. This leads to $(x, y) = (\hat{x}, \hat{y})$, a contradiction! □

Let $f : X \to 2^Y \backslash \{\emptyset\}$ and $(x_0, y_0) \in \mathrm{gr}f$, where $\mathrm{gr}f$ denotes the set $\{(x, y) \in X \times Y : x \in X, y \in f(x)\}$. For $(x, y), (x', y') \in \mathrm{gr}f$, we define $(x', y') \preceq_F (x, y)$ iff $y \in y' + F(x', x) + D$. Denote the set $\{(x, y) \in \mathrm{gr}f : (x, y) \preceq_F (x_0, y_0)\}$ by $S_F(x_0, y_0)$. By taking $A = \mathrm{gr}f$ in Theorem 5.2, we obtain the following.

Corollary 5.1. *Let X, Y, D, F be the same as in Theorem 5.2. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map and $(x_0, y_0) \in \mathrm{gr}f$. Suppose that the following conditions are satisfied:*

(i) *for any $\preceq_F$-decreasing sequence $\{(x_n, y_n)\}$ in $S_F(x_0, y_0)$, if $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ is convergent in X;*

(ii) ξ (from (F3)) *is lower bounded on $P_Y(S_F(x_0, y_0)) - y_0$ and $f(x)$ has the strict domination property for any $x \in X$;*

(iii) *for any $\preceq_F$-decreasing sequence $\{(x_n, y_n)\}$ in $S_F(x_0, y_0)$, if $\{x_n\}$ converges to x, then there exists $y \in f(x)$ such that $(x, y) \preceq_F (x_n, y_n)$, $\forall n$.*

Then, there exists $(\hat{x}, \hat{y}) \in \mathrm{gr}f$ such that

(a) *$y_0 \in \hat{y} + F(\hat{x}, x_0) + D$ and $\hat{y} \in \mathrm{SMin}^D f(\hat{x})$;*

(b) *for any $(x, y) \in \mathrm{gr}f \backslash \{(\hat{x}, \hat{y})\}$, $\hat{y} \notin y + F(x, \hat{x}) + D$.*

Remark 5.2. Condition (i) in Corollary 5.1 can be replaced by the following stronger condition

(i′) for any sequence $\{(x_n, y_n)\} \subset \mathrm{gr}f$, if $y_{n+1} \leq_D y_n$, $\forall n$, and $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ is convergent.

Compare (i′) with the following (f, D)-lower completeness (see Remark 3.1 or [41]): every Cauchy sequence $\{x_n\} \subset X$ satisfying $f(x_n) \subset f(x_{n+1}) + D$ for every n, is convergent. We see that condition (i′) is stronger than (f, D)-lower completeness. Let's call (i′) strong (f, D)-lower completeness. Certainly, if f is a vector-valued map, then the above two kinds of lower completeness are coincident.

For set-valued maps, Khanh and Quy [31] introduced the following concepts.

Definition 5.1. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be a set-valued map.

(i) f is said to be D-lower semi-continuous from above(briefly, denoted by D-lsca) at $\bar{x}$ if, for any convergent sequence $x_n \to \bar{x}$ and any sequence $y_n \in f(x_n)$ with $y_{n+1} \leq_D y_n$ $\forall n$, there exists $\bar{y} \in f(\bar{x})$ such that $\bar{y} \leq_D y_n$, $\forall n$.

(ii) f is said to be weak D-lower semi-continuous from above (briefly, denoted by w.D-lsca) at $\bar{x}$ if, for each sequence $x_n \to \bar{x}$ with $f(x_n) \subset f(x_{n+1}) + D$. $\forall n$, one has $f(x_n) \subset f(\bar{x}) + D$.

As pointed out in [31], D-lsca implies w.D-lsca. We see that f being w.D-lsca is exactly that f is D-s.l.m. Hence, D-lsca maps can also be called strongly D-s.l.m. maps. If f is a vector-valued map, then D-s.l.m. and strong D-s.l.m. are coincident.

Corollary 5.2. *Let (X, d) be a metric space, Y be a locally convex space, $D \subset Y$ be a convex cone, $H \subset D$ be a convex set such that $0 \notin \text{cl}(H + D)$ (i.e., $H^{+s} \cap D^+ \neq \emptyset$) and $H + D$ be h_0-closed for some $h_0 \in H$.*

Let $f : X \to 2^Y \backslash \{\emptyset\}$ be strongly D-s.l.m. (i.e., D-lsca) and $(x_0, y_0) \in \text{gr}f$.

For $(x, y), (x', y') \in \text{gr}f$, define

$$(x', y') \preceq_F (x, y) \;\text{ iff }\; y \in y' + d(x', x)H + D.$$

Suppose that the following conditions are satisfied:

(i) *for any $\preceq_F$-decreasing sequence $\{(x_n, y_n)\}$ in $S_F(x_0, y_0)$, if $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ is convergent in X (particularly, (X, d) is strongly (f, D)-lower complete);*

(ii) *there exists $\xi \in H^{+s} \cap D^+$ such that ξ is lower bounded on $P_Y(S_F(x_0, y_0))$ and $f(x)$ has the strict domination property for every $x \in X$.*

Then the result of Corollary 5.1 holds.

Proof. It suffices to check assumption (iii) in Corollary 5.1. Let a sequence $\{(x_n, y_n)\} \subset \text{gr}f$ satisfy $(x_{n+1}, y_{n+1}) \preceq_F (x_n, y_n)$ and $x_n \to \bar{x}$. Clearly, $y_{n+1} \leq_D y_n$, $\forall n$. By the assumption, f is strongly D-s.l.m., hence there exists $\bar{y} \in f(\bar{x})$ such that $\bar{y} \leq_D y_n$, $\forall n$. Next, we show that $(\bar{x}, \bar{y}) \preceq_F (x_n, y_n)$, $\forall n$.

If $d(x_n, \bar{x}) = 0$, then $x = x_n$. In this case, $\bar{y} \leq_D y_n$ is equivalent to that $y_n \in \bar{y} + d(\bar{x}, x_n)H + D$. Certainly, we have $(\bar{x}, \bar{y}) \preceq_F (x_n, y_n)$.

If $d(x_n, \bar{x}) > 0$, take $i \in \mathbf{N}$ such that $d(x_n, \bar{x}) - (1/i) > 0$. Since $d(x_m, x_n) \to d(\bar{x}, x_n)$ $(m \to \infty)$, we may take $m > n$ such that $d(x_m, x_n) \geq d(\bar{x}, x_n) - (1/i)$. As

$(x_m, y_m) \preceq_F (x_n, y_n)$ and $\bar{y} \leq_D y_m$, we have

$$\begin{aligned} y_n &\in y_m + d(x_m, x_n)H + D \\ &\subset y_m + (d(\bar{x}, x_n) - \frac{1}{i})H + D \\ &\subset \bar{y} + (d(\bar{x}, x_n) - \frac{1}{i})H + D. \end{aligned}$$

Thus,

$$\begin{aligned} y_n + \frac{1}{i}h_0 &\in \bar{y} + (d(\bar{x}, x_n) - \frac{1}{i})H + \frac{1}{i}H + D \\ &= \bar{y} + d(\bar{x}, x_n)H + D. \end{aligned}$$

From this,

$$\frac{y_n - \bar{y} + (1/i)h_0}{d(\bar{x}, x_n)} \in H + D.$$

Letting $i \to \infty$ and remarking that $H + D$ is h_0-closed, we have

$$\frac{y_n - \bar{y}}{d(\bar{x}, x_n)} \in H + D \quad \text{and hence} \quad y_n \in \bar{y} + d(\bar{x}, x_n)H + D.$$

That is, $(\bar{x}, \bar{y}) \preceq_F (x_n, y_n)$. □

As we have seen, the assumption that $\alpha H + D$ is closed for all $\alpha > 0$ (see [31, Theorem 5.2]) is not necessary. Here, we only assume that $H + D$ is h_0-closed for some $h_0 \in H$. In particular, if H is a singleton $\{k_0\}$ with $k_0 \in D \backslash -\mathrm{cl}(D)$, then we only need to assume that D is k_0-closed. As pointed out in [31, Section 5], the condition that f is strongly D-s.l.m. (i.e., D-lsca) and $f(x)$ has the strict domination property imposed in Corollary 5.2 is equivalent to the limiting monotonicity condition assumed in [5, Theorem 3.4]. Obviously, Corollary 5.2 improves [31, Theorem 5.2] and [5, Theorem 3.4]. Certainly, it also includes properly [4, Theorem 1].

As done in the proof of [31, Theorem 5.3], from Corollary 5.2 we can obtain the following.

Corollary 5.3. *Let X, Y, D and H be the same as in Corollary 5.2 and additionally, D be pointed and closed. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be D-lsc(D-lower semi-continuous), compact-valued and D-bounded (i.e., there exists a bounded set M such that $f(X) \subset M + D$), and $(x_0, y_0) \in \mathrm{gr} f$. Suppose that assumption (i)* *of Corollary 5.2 is satisfied (particularly, (X, d) is strongly (f, D)-lower complete). Then, the result of Corollary 5.2 holds.*

Proof. Since D is a closed convex pointed cone and f is compact-valued, f has the strict domination property (see [31]). As f is D-bounded, for any $\xi \in H^{+s} \cap D^+$, ξ is lower bounded on $P_Y(S_F(x_0, y_0))$. Thus, assumption (ii) in Corollary 5.2 is satisfied. Also, f being D-lsc and having compact-valued implies that f is D-lsca (see [31]). Now, we can apply Corollary 5.2 and obtain the result. □

Concerning the strict domination property, there have been many interesting results, for example, refer to [21, 43] and the references therein. In fact, in Corollaries 5.1 and 5.2, the condition that f has the strict domination property can be replaced by any one which implies f having the strict domination property. For example, from Corollary 5.2 we can obtain the following Corollaries 5.4 and 5.5. First we recall some related notions.

Let Y be a locally convex space, $A \subset Y$ be nonempty, $\Theta \subset Y$ be a bounded convex set and $D \subset Y$ be a convex cone specifying a quasi-order $\leq_D$. Put $\Theta_0 := \cup_{0 \leq \lambda \leq 1} \lambda\Theta$. Then $\cap_{\epsilon > 0}(A - \epsilon\Theta_0)$ is called the Θ-closure of A and denoted by $\mathrm{cl}_\Theta(A)$. If $\mathrm{cl}_\Theta(A) = A$, then A is said to be Θ-closed. It is easy to see that A is locally closed iff A is Θ-closed for every bounded convex set Θ. And A is vectorial closed iff for every singleton Θ, A is Θ-closed. The following implications are obvious:

$$\text{closedness} \implies \text{localclosedness} \implies \Theta - \text{closedness}.$$

But neither of two converses is true, for details, see [43].

A nonempty subset $A \subset Y$ is said to be D-complete (resp., D-locally complete) iff every Cauchy sequence (resp., locally Cauchy sequence) $\{y_n\} \subset A$ with $y_{n+1} \leq_D y_n$, is convergent (resp., locally convergent) to some point $\bar{y} \in A$. It is easy to show that every D-complete set is D locally complete.

Corollary 5.4. *Let X, Y, D and H be the same as in Corollary 5.2 and, additionally, D have a σ-convex base Θ. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be strong D-s.l.m. (i.e., D-lsca), D-bounded and $f(x)$ be Θ-closed (particularly, locally closed or closed) for all $x \in X$ and $(x_0, y_0) \in \mathrm{gr} f$. Suppose that assumption (i) of Corollary 5.2 is satisfied. Then, the result of Corollary 5.2 holds.*

Proof. Since D has a σ-convex base Θ and $f(x)$ is Θ-closed, by [43, Corollary 5.2], $f(x)$ has the strict domination property. Also, $f(X)$ being D-bounded implies that assumption (ii) in Corollary 5.2 is satisfied. Thus, the result follows from Corollary 5.2. □

Corollary 5.5. *Let X, Y, D and H be the same as in Corollary 5.2 and, additionally, D be locally closed and have a bounded base. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be strong D-s.l.m. (i.e., D-lsca), D-bounded and $f(x)$ be D-locally complete (particularly, D-complete or complete) for all $x \in X$ and $(x_0, y_0) \in \mathrm{gr} f$. Suppose that assumption (i) of Corollary 5.2 is satisfied. Then, the result of Corollary 5.2 holds.*

Proof. As done in the proof of Corollary 5.4, we only show that $f(x)$ has the strict domination property. For any $x \in X$, $f(x)$ is D-bounded and $f(x)$ is D-locally complete. Also, D has a bounded base and D is locally closed. By [43, Theorem 5.2], $f(x)$ has the strict domination property and the proof is completed.

□

Finally, as an application of Corollary 5.1, we give a Pareto minimizer's version of Corollary 3.6.

Corollary 5.6. *Let (X, d) be a metric space, Y be a locally convex space, $D \subset Y$ be a convex cone, $k_0 \in D \backslash -\mathrm{vcl}(D)$ and D be k_0-closed. Let $f : X \to 2^Y \backslash \{\emptyset\}$ be strongly s.l.m. (i.e., D-lsca) and $f(x)$ have the strict domination property for all $x \in X$ and let (X, d) be strongly (f, D)-lower complete.*

Suppose the $(x_0, y_0) \in \mathrm{gr} f$ and $\epsilon > 0$ such that $y_0 \notin f(X) + \epsilon k_0 + D$.

Then for any $\lambda > 0$, there exists $(\hat{x}, \hat{y}) \in \mathrm{gr} f$ such that

(a) *$y_0 \in \hat{y} + (\epsilon/\lambda) d(\hat{x}, x_0) k_0 + D$ and $\hat{y} \in \mathrm{SMin}^D f(\hat{x})$;*

(b) *for any $(x, y) \in \mathrm{gr} f \backslash \{(\hat{x}, \hat{y})\}$, $\hat{y} \notin y + (\epsilon/\lambda) d(x, \hat{x}) k_0 + D$;*

(c) *$d(x_0, \hat{x}) \leq \lambda$.*

Proof. We shall apply Corollary 5.1 to prove the conclusion. Put

$$F(x, x') := (\epsilon/\lambda) d(x, x') k_0, \quad \forall x,\, x' \in X.$$

Obviously, F satisfies (F1) and (F2). Since $k_0 \in D \backslash -\mathrm{vcl}(D)$, we know that

$\xi_{k_0}(y) \neq -\infty,\ \forall y \in Y$. By Lemma 3.2, ξ_{k_0} is D-monotone and satisfies that

$$\xi_{k_0}(y+z) = \xi_{k_0}(y) + \xi_{k_0}(z), \quad \forall y \in Y,\ \forall z \in F(X \times X).$$

Besides, for any $\delta > 0$,

$$\zeta(\delta) := \inf\{\xi_{k_0}(y) : y \in \cup_{d(x,x')\geq\delta} F(x,x')\} \geq (\epsilon/\lambda)\delta > 0.$$

Hence, F satisfies (F3) for ξ_{k_0}. For (x,y), $(x',y') \in \mathrm{gr} f$, we define

$$(x',y') \preceq_F (x,y) \iff y \in y' + (\epsilon/\lambda)d(x',x)k_0 + D.$$

Denote the set $\{(x,y) \in \mathrm{gr} f : (x,y) \preceq_F (x_0,y_0)\}$ by $S_F(x_0,y_0)$.

Let $\{(x_n,y_n)\}$ be a $\preceq_F$-decreasing sequence in $S_F(x_0,y_0)$ and let $\{x_n\}$ be a Cauchy sequence. From $(x_{n+1},y_{n+1}) \preceq_F (x_n,y_n)$, we have

$$y_n \in y_{n+1} + (\epsilon/\lambda)d(x_{n+1},x_n)k_0 + D \subset y_{n+1} + D \quad \text{and} \quad y_{n+1} \leq_D y_n.$$

By the assumption that (X,d) is strongly (f,D)-lower complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$ $(n \to \infty)$, that is, condition (i) in Corollary 5.1 is satisfied.

Let $\{(x_n,y_n)\}$ be a $\preceq_F$-decreasing sequence in $S_F(x_0,y_0)$ and let $x_n \to \bar{x}$. Since $y_{n+1} \leq_D y_n$, $x_n \to \bar{x}$ and f is strongly D-l.s.m. (i.e., D-lsca), there exists $\bar{y} \in f(\bar{x})$ such that $\bar{y} \leq_D y_n,\ \forall n$. For any given n, when $m > n$, we have

$$\begin{aligned} y_n &\in y_m + (\epsilon/\lambda)d(x_m,x_n)k_0 + D \\ &\subset \bar{y} + (\epsilon/\lambda)d(x_m,x_n)k_0 + D. \end{aligned} \tag{5.7}$$

Since $d(x_m,x_n) \to d(\bar{x},x_n)$ $(m \to \infty)$ and D is k_0-closed, from (5.7) we can deduce that $y_n \in \bar{y} + (\epsilon/\lambda)d(\bar{x},x_n)k_0 + D$, and hence $(\bar{x},\bar{y}) \preceq_F (x_n,y_n)$. Thus, condition (iii) in Corollary 5.1 is satisfied.

Since $y_0 \notin f(X) + \epsilon k_0 + D$, by Lemma 3.2, ξ_{k_0} is lower bounded on $P_Y(S_F(x_0,y_0)) - y_0$. And by the assumption that $f(x)$ has the strict domination property, we know that condition (ii) in Corollary 5.1 holds.

Now, applying Corollary 5.1, we obtain $(\hat{x},\hat{y}) \in \mathrm{gr} f$ such that (a) and (b) hold. Finally, from (a), we have $y_0 \in \hat{y} + (\epsilon/\lambda)d(\hat{x},x_0)k_0 + D$. Combining this with $y_0 \notin f(X) + \epsilon k_0 + D$, we conclude that $d(\hat{x},x_0) \leq \lambda$. □

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References

[1] M. Adan, V. Novo, Proper efficiency in vector optimization on real linear spaces, J. Optim. Theory Appl., 121 (2004), 515-540.

[2] Y. Araya, Ekeland's variational principle and its equivalent theorems in vector optimization, J. Math. Anal. Appl., 346 (2008), 9-16.

[3] J. -P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, 1990.

[4] T. Q. Bao, B. S. Mordukhovich, Variational principles for set-valued mappings with applications to multiobjective optimization, Control Cybern., 36 (2007), 531-562.

[5] T. Q. Bao, B. S. Mordukhovich, Relative Pareto minimizers for multiobjective problems: existence and optimality conditions, Math. Program, Ser.A, 122 (2010), 301-347.

[6] E. M. Bednarczuk, M. J. Przybyla, The vector-valued variational principle in Banach spaces ordered by cones with nonempty interiors, SIAM J. Optim. 18 (2007), 907-913.

[7] E. M. Bednarczk, D. Zagrodny, Vector variational principle, Arch. Math. (Basel), 93 (2009), 577-586.

[8] G. Y. Chen, X. X. Huang, A unified approach to the existing three types of variational principle for vector valued functions, Math. Methods Oper. Res., 48 (1998), 349-357.

[9] G. Y. Chen, X. X. Huang, S. H. Hou, General Ekeland's variational principle for set-valued mappings, J. Optim. Theory Appl., 106 (2000), 151-164.

[10] G. Y. Chen, X. X. Huang, X. G. Yang, Vector Optimization, Set-Valued and Variational Analysis, Springer-Verlag, Berlin, 2005.

[11] S. Dancs, M. Hegedus, P. Medvegyev, A general ordering and fixed point principle in complete metric space, Acta Sci. Math. (Szeged), 46 (1983), 381-388.

[12] D. Dentcheva, S. Helbig, On variational principles, level sets, well-posedness, and ϵ-solutions in vector optimization, J. Optim. Theory Appl., 89 (1996), 325-349.

[13] I. Ekeland, Sur les probèmes variationnels, C. R. Acad. Sci. Paris 275 (1972), 1057-1059.

[14] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324-353.

[15] I. Ekeland, Nonconvex minimization problems, Bull. Amer. Math. Soc. (N.S.) 1 (1979), 443-474.

[16] C. Finet, L. Quarta, C. Troestler, Vector-valued variational principles, Nonlinear Anal. 52 (2003), 197-218.

[17] F. Flores-Bazán, C. Gutiérrez, V. Novo, A Brézis-Browder principle on partially ordered spaces and related ordering theorems, J. Math. Anal. Appl., 375 (2011), 245-260.

[18] Chr. Gerstewitz (Tammer), Nichtkonvexe Dualität in der Vektoroptimierung, Wiss. Z. TH Leuna-Merseburg 25 (1983), 357-364.

[19] Chr. Gerstewitz (Tammer), E. Iwanow, Dualität für nichtkonvexe Vektoroptimierungsprobleme, Wiss. Z. TH llmenau 31 (1985), 61-81.

[20] Chr. Gerth (Tammer), P. Weidner, Nonconvex separation theorems and some applications in vector optimization, J. Optim. Theory Appl., 67 (1990), 297-320.

[21] A. Göpfert, H. Riahi, Chr. Tammer, C. Zălinescu, Variational Methods in Partially Ordered Spaces, Springer-Verlag, New York, 2003.

[22] A. Göpfert, C. Tammer and C. Zălinescu, On the vectorial Ekeland's variational principle and minimal point theorems in product spaces, Nonlinear Anal. 39 (2000), 909-922.

[23] C. Gutiérrez, B. Jiménez, V. Novo, A set-valued Ekeland's variational principle in vector optimization, SIAM J. Control. Optim., 47 (2008), 883-903.

[24] T. X. D. Ha, Some variants of the Ekeland variational principle for a set-valued map, J. Optim. Theory Appl., 124 (2005), 187-206.

[25] A. H. Hamel, Equivalents to Ekeland's variational principle in uniform spaces, Nonlinear Anal. 62 (2005), 913-924.

[26] J. Horváth, Topological Vector Spaces and Distributions, vol. 1, Addison-Wesley, Reading, MA, 1966.

[27] G. Isac, The Ekeland's principle and the Pareto ϵ-efficiency, In: M. Tamiz (ed.) Multi-Objective Programming and Goal Programming: Theories and Applications, Lecture Notes in Econom. and Math. Systems, vol. 432, Springer-Verlag, Berlin, 1996, 148-163.

[28] G, Isac, Chr. Tammer, Nuclear and full nuclear cones in product spaces: Pareto efficiency and an Ekeland type variational principle, Positivity, 9 (2005), 511-539.

[29] P. Q. Khanh, D. N. Quy, On generalized Ekeland's variational principle and equivalent formulations for set-valued mappings, J. Glob. Optim., 49 (2011), 381-396.

[30] P. Q. Khanh, D. N. Quy, On Ekeland's variational principle for Pareto minima of set-valued mappings, J. Optim. Theory Appl., 153 (2012), 280-297.

[31] P. Q. Khanh, D. N. Quy, Versions of Ekeland's variational principle involving set perturbations, J. Glob. Optim. DOI 10.1007/s10898-012-9983-3.

[32] D. Kuroiwa, On set-valued optimization, Nonlinear Anal., 47 (2001), 1395-1400.

[33] C. G. Liu, K. F. Ng, Ekeland's variational principle for set-valued functions, SIAM J. Optim., 21 (2011), 41-56.

[34] P. Loridan, ϵ-Solutions in vector minimization problems, J. Optim. Theory Appl., 43 (1984), 265-276.

[35] A. B. Németh, A nonconvex vector minimization problem, Nonlinear Anal. 10 (1986), 669-678.

[36] D. Pallaschke, S. Rolewicz, Foundations of Mathematical Optimization, Math. Appl. 388, Kluwer, Dordrecht, 1997.

[37] P. Pérez Carreras, J. Bonet, Barrelled Locally Convex Spaces, North-Holland, Amsterdam, 1987.

[38] J. H. Qiu, Local completeness and dual local quasi-completeness, Proc. Amer. Math. Soc. 129 (2001) 1419-1425.

[39] J. H. Qiu, Local completeness and drop theorem, J. Math. Anal. Appl. 266 (2002), 288-297.

[40] J. H. Qiu, A generalized Ekeland vector variational principle and its applications in optimization, Nonlinear Anal., 71 (2009), 4705-4717.

[41] J. H. Qiu, On Ha's version of set-valued Ekeland's variational principle, Acta Math. Sinica, English Series, 28 (2012), 717-726.

[42] J. H. Qiu, Set-valued quasi-metrics and a general Ekeland's variational principle in vector optimization, SIAM J. Control Optim., 51 (2013), 1350-1371.

[43] J. H. Qiu, The domination property for efficiency and Bishop-Phelps theorem in locally convex spaces, J. Math. Anal. Appl., 402 (2013), 133-146.

[44] J. H. Qiu, F. He, A general vectorial Ekeland's variational principle with a p-distance, Acta Math. Sinica, 29 (2013), 1655-1678.

[45] J. H. Qiu, B. Li, F. He, Vectorial Ekeland's variational principle with a w-distance and its equivalent theorems, Acta Math. Sci., 32B (2012), 2221-2236.

[46] S. A. Saxon, L. M. Sánchez Ruiz, Dual local completeness, Proc. Amer. Math. Soc., 125 (1997), 1063-1070.

[47] C. Tammer, A generalization of Ekeland's variational principle, Optimization, 25 (1992) 129-141.

[48] C. Tammer, C. Zălinescu, Vector variational principle for set-valued functions, Optimization, 60 (2011), 839-857.

[49] M. Turinici, Maximal elements in a class of order complete metric spaces, Math. Japon, 25 (1980), 511-517.

[50] M. Turinici, Maximality principles and mean-value theorems, Anais Acad. Brasil. Ciencias, 53 (1981), 653-655.

[51] C. Zălinescu, Convex Analysis in General Vector Spaces, World Sci., Singapore, 2002.