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Bifurcations of limit cycles in equivariant quintic planar vector fields

Liqin Zhao [1]

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

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ABSTRACT

In this paper, we obtain 23 limit cycles for a Z_3-equivariant near-Hamiltonian system of degree 5 which is the perturbation of a Z_6-equivariant quintic Hamiltonian system. The configuration of these limit cycles is new and different from the configuration obtained by H.S.Y. Chan, K.W. Chung and J. Li, where the unperturbed system is a Z_3-equivariant quintic Hamiltonian system. Our unperturbed system is different from the unperturbed systems studied by Y. Wu and M. Han. The limit cycles are obtained by Poincaré–Pontryagin theorem and Poincaré–Bendixson theorem.

1. Introduction and the main result

Consider the following polynomial vector fields of degree n:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y). \tag{1.1}$$

It is well known that the second part of Hilbert's 16th problem is to ask the exact upper bound of the number of limit cycles and to determine their relative positions for system (1.1). Up to now, we only know that a specific planar polynomial system always has a finite number of limit cycles, see [4] and [8]. However, it is still an open problem to find the uniform upper bounds, even for $n = 2$.

Attentions are also paid to the lower bounds of $H(n)$ and $M(n)$ for $n \geq 2$, where $H(n)$ stands for the number of all the limit cycles for system (1.1), and $M(n)$ stands for the number of small amplitude limit cycles for system (1.1). There are many references for specific n, for instance, see [1,2,9–15,17,18,20–31,33] etc. The lower bound of $H(n)$ for general n was first considered in [3]. Recently, it was found in [6] that $H(n)$ grows at least as rapidly as $\frac{1}{2\ln 2}(n+2)^2\ln(n+2)$. It was obtained that $M(5) \geq 25$ in [15] and [28].

E-mail address: zhaoliqin@bnu.edu.cn.
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For general n, it was proved that $M(n) \geq n^2$ if $n \geq 23$ in [32]. It was also proved in [32] that $M(n)$ grows at least as rapidly as $\frac{18}{25} \cdot \frac{1}{2\ln 2}(n+2)^2 \ln(n+2)$ for all large n. One can obtain a detailed list of references with great ease via e-resources.

Let $z = x + Iy$, $\bar{z} = x - Iy$, then system (1.1) becomes

$$\dot{z} = F(z, \bar{z}), \qquad \dot{\bar{z}} = \bar{F}(z, \bar{z}), \tag{1.2}$$

where $I^2 = -1$ and $F(z, \bar{z}) = P_n(x, y) + IQ_n(x, y)$. Let Z_q be a cyclic group which is generated by a planar counterclockwise rotation through $2\pi/q$ about the origin. By Corollary 7.3 in [13], we get

Lemma 1.1. *(See [13].) For system (1.1) with $n = 5$, all non-trivial Z_q-equivariant vector fields have the following forms:*

(i) *For $q = 6$: $F(z, \bar{z}) = (\Lambda_0 + \Lambda_1|z|^2 + \Lambda_2|z|^4)z + \Lambda_3\bar{z}^5$;*
(ii) *For $q = 5$: $F(z, \bar{z}) = (\Lambda_0 + \Lambda_1|z|^2 + \Lambda_2|z|^4)z + \Lambda_3\bar{z}^4$;*
(iii) *For $q = 4$: $F(z, \bar{z}) = (\Lambda_0 + \Lambda_1|z|^2 + \Lambda_2|z|^4)z + (\Lambda_3 + \Lambda_4|z|^2)\bar{z}^3 + \Lambda_5 z^5$;*
(iv) *For $q = 3$: $F(z, \bar{z}) = (\Lambda_0 + \Lambda_1|z|^2 + \Lambda_2|z|^4)z + (\Lambda_3 + \Lambda_4|z|^2)\bar{z}^2 + \Lambda_5 z^4 + \Lambda_6\bar{z}^5$;*
(v) *For $q = 2$: $F(z, \bar{z}) = (\Lambda_0 + \Lambda_1|z|^2 + \Lambda_2|z|^4)z + (\Lambda_3 + \Lambda_4|z|^2 + \Lambda_5|z|^4)\bar{z} + (\Lambda_6 + \Lambda_7|z|^2)z^3 + (\Lambda_8 + \Lambda_9|z|^2)\bar{z}^3 + \Lambda_{10}z^5 + \Lambda_{11}\bar{z}^5$,*

where $\Lambda_i = A_i + IB_i$ with $A_i, B_i \in \mathcal{R}$ for $i = 0, \cdots, 11$. The above $F(z, \bar{z})$ define Z_q-equivariant Hamiltonian vector fields if and only if $A_0 = A_1 = A_2 = 0$ and for $q = 4$, $\Lambda_4 = -5\bar{\Lambda}_5$; for $q = 3$, $\Lambda_4 = -5\bar{\Lambda}_5$; for $q = 2$, $\Lambda_4 = -3\bar{\Lambda}_6$, $\Lambda_5 = -2\bar{\Lambda}_7$ and $\Lambda_9 = -5\bar{\Lambda}_{10}$.

In recent years, more attentions were paid to the lower bounds and distribution of limit cycles for system (1.1) with $n = 5$, for instance, see [2,9,11,12,14,15,17,18,20,21,24–31,33] and references therein. Some interesting results are listed as follows.

(i) For Z_2-equivariant system: it was shown in [24] that there were Z_2-equivariant quintic systems with at least 25 limit cycles for each of them, and it was obtained in [25] that there was a Z_2-equivariant quintic planar vector field having 28 limit cycles with four different configurations.

(ii) For Z_3-equivariant system: it was proved in [18] that there were at least 15 limit cycles for a Z_3-equivariant near-Hamiltonian system of degree 5 which was a perturbation of a Z_3-equivariant cubic Hamiltonian system. It was shown in [2] that there were at least 23 limit cycles for a Z_3-equivariant near-Hamiltonian system of degree 5 which was a perturbation of a Z_3-equivariant quintic Hamiltonian system. It was obtained in [26] that there were at least 24 limit cycles with two different configurations for a Z_3-equivariant near-Hamiltonian system of degree 5 which was the perturbations of a Z_6-equivariant quintic Hamiltonian system.

(iii) For Z_4-equivariant system: it was achieved in [27] that there were at least 28 limit cycles with two different configurations.

(iv) For Z_5-equivariant system: it was proved dependently in [15] and [30] that there were Z_5-equivariant planar polynomial vector fields of degree 5 having at least 25 small limit cycles for each of them.

(v) For Z_6-equivariant system: it was obtained that there were Z_6-equivariant planar polynomial vector fields of degree 5 having at least 24 small limit cycles for each of them in [15], and there were Z_6-equivariant planar polynomial vector fields of degree 5 having at least 24 limit cycles for each of them in [12].

Motivated by [2,9,11,12,14,15,17,18,20,21,24–31] and [33], in this paper, we intend to study the number and the distribution of limit cycles of the following Z_3-equivariant quintic systems:

$$\dot{x} = H_y(x, y) + \epsilon\mathcal{P}_5(x, y), \qquad \dot{y} = -H_x(x, y) + \epsilon\mathcal{Q}_5(x, y), \tag{1.3}$$

where

$$H(x,y) = -\frac{1}{2}(x^2+y^2) + \frac{7}{10}(x^4+y^4) + \frac{7}{5}x^2y^2 - \frac{3}{10}x^6 + \frac{3}{5}x^4y^2 - \frac{19}{10}x^2y^4 - \frac{2}{15}y^6, \tag{1.4}$$

$$\begin{aligned}\mathcal{P}_5(x,y) = {} & x(x^4-10x^2y^2+5y^4)A_6 + y(5x^4-10x^2+y^4)B_6 \\ & + x(x^2+y^2)^2A_2 - y(x^2+y^2)^2B_2 + (x^2-y^2)A_3 + 2xyB_3 \\ & + (x^4-y^4)A_4 + (x^4-6x^2y^2+y^4)A_5 + 2xy(x^2+y^2)B_4 - 4xy(x^2-y^2)B_5 \\ & + x(x^2+y^2)A_1 - y(x^2+y^2)B_1 + xA_0 - yB_0,\end{aligned} \tag{1.5}$$

$$\begin{aligned}\mathcal{Q}_5(x,y) = {} & -y(5x^4-10x^2y^2+y^4)A_6 + x(x^4-10x^2y^2+5y^4)B_6 \\ & + y(x^2+y^2)^2A_2 + x(x^2+y^2)^2B_2 - 2xyA_3 + (x^2-y^2)B_3 \\ & - 2xy(x^2+y^2)A_4 + 4xy(x^2-y^2)A_5 + (x^4-y^4)B_4 + (x^4-6x^2y^2+y^4)B_5 \\ & + y(x^2+y^2)A_1 + x(x^2+y^2)B_1 + yA_0 + xB_0.\end{aligned} \tag{1.6}$$

System (1.3) with $\epsilon = 0$ is Z_6-equivariant (see Lemma 1.1). The expressions of $\mathcal{P}_5(x,y)$ and $\mathcal{Q}_5(x,y)$ are obtained by Lemma 1.1(iv). System (1.3) is Hamiltonian if and only if

$$A_0 = A_1 = A_2 = 0, \qquad A_4 = -4A_5, \qquad B_4 = 4B_5. \tag{1.7}$$

Denote

$$\Delta = (A_0,\cdots,A_6,B_0,\cdots,B_6), \qquad \Delta^* = (A_0^*,\cdots,A_6^*,B_0^*,\cdots,B_6^*).$$

Our main result is stated as follows.

Theorem 1.1. *For $0 < \epsilon \ll 1$,*

(i) there exists a Δ^* with $A_2^* \neq 0$ and $A_4^* = -4A_5^*$ such that system (1.3) has at least 23 or 20 limit cycles. The configurations of these limit cycles are shown in Figs. 1(a) and 1(b);

(ii) there exists a Δ^* with $A_2^* \neq 0$ and $A_i^* = B_i^* = 0$ $(i = 3,4,5)$ such that system (1.3) has at least 24 or 18 limit cycles. The configurations of these limit cycles are shown in Figs. 1(c) and 1(d).

Remark 1.1. (i) System (1.3) with $A_i^* = B_i^* = 0$ $(i = 3,4,5)$ is Z_6-equivariant.

(ii) The configuration of the 23 limit cycles in Theorem 1.1 is new and different from that obtained in [2], where the unperturbed system is a Z_3-equivariant quintic Hamiltonian system. Our unperturbed system is different from that in [26], see Figs. 2(a) and 2(b). The configurations of limit cycles in Figs. 1(b) and 1(d) are also new.

This paper is organized as follows. In Section 2, we will give some preliminaries. We will study the expansion of the Abelian integral $I(h)$ near a homoclinic loop and give some criterions for determining the zeros of $I(h)$. In Section 3, we will obtain the specific expressions of $I(h)$ for system (1.3) near homoclinic loops. In Section 4, we shall give the proof of Theorem 1.1.

2. Preliminaries

Consider the following near-Hamiltonian systems

$$\begin{cases} \dot{x} = H_y(x,y) + \epsilon P(x,y,\delta), \\ \dot{y} = -H_x(x,y) + \epsilon Q(x,y,\delta), \end{cases} \tag{2.1}$$

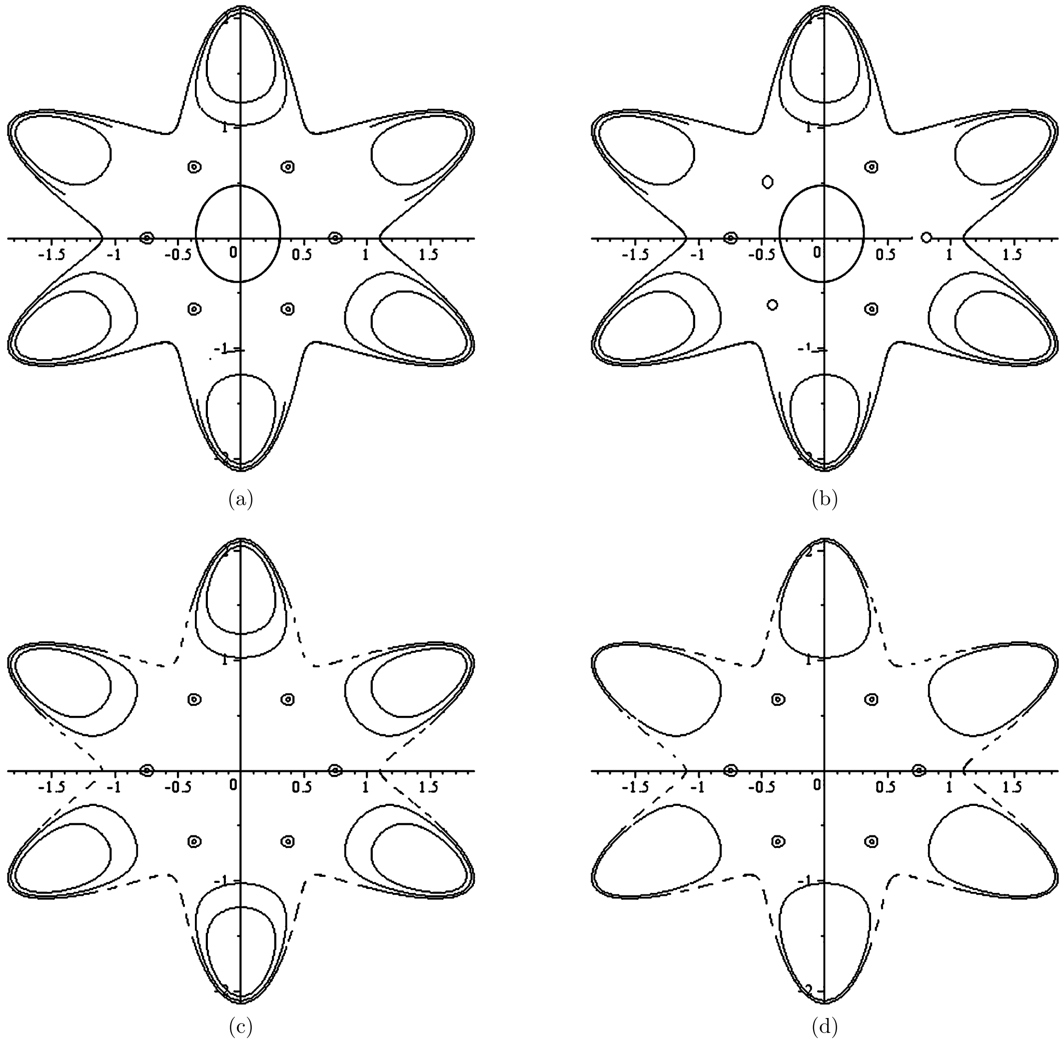

Fig. 1. The phase plane for Z_3-equivariant system (1.3) with (a) 23, (b) 20 limit cycles, and for Z_6-equivariant system (1.3) with (c) 24, (d) 18 limit cycles.

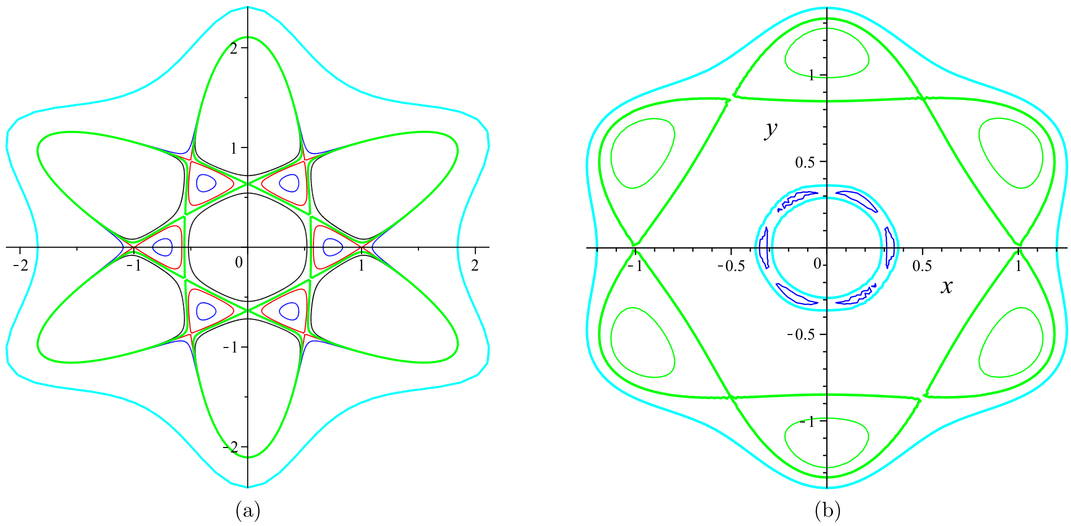

Fig. 2. (a) The phase portraits of system (3.1). (b) The phase portraits of the unperturbed system in [26].

where $0 < |\epsilon| \ll 1$ and H, P, Q are analytic in $(x, y, \delta) \in \mathcal{D} \subset R^2 \times R^m$ with $\mathcal{D}$ being bounded. We assume system $(2.1)_{\epsilon=0}$ has a continuous family of periodic orbits $\bigcup_{h \in \Sigma} \Gamma_h$, where $\Gamma_h = \{(x, y) : H(x, y) = h\}$. For the orientation of a periodic orbit Γ_h, we have

Lemma 2.1. *(See [5].) If the closed curve Γ_h is oriented clockwise, then it expands with h increasing. If the closed curve Γ_h is oriented counter clockwise, then it shrinks with h increasing.*

Let us take a segment σ, which is transversal to each of the ovals $\bigcup_{h \in \Sigma} \Gamma_h$, and parameterize σ by the values of $H(x, y)$. Denote by $\Gamma_\epsilon(h, \delta)$ a piece of orbit of system (2.1) between the starting point h on σ and the next intersection point $P_\epsilon(h, \delta)$ with σ. As usual, we call

$$d_\epsilon(h, \delta) = P_\epsilon(h, \delta) - h \tag{2.2}$$

a displacement function of system (2.1). Obviously, if there are $\epsilon^* \neq 0$ and $\delta^* \in R^m$ such that $d_{\epsilon^*}(h, \delta^*) \not\equiv 0$ and $d_{\epsilon^*}(h_0, \delta^*) = 0$, then $\Gamma_{\epsilon^*}(h_0, \delta^*)$ is a limit cycle of system (2.1) with $\epsilon = \epsilon^*$ and $\delta = \delta^*$.

Lemma 2.2 *(Poincaré–Pontryagin). (See [19].) The displacement function $d_\epsilon(h, \delta)$ has the following asymptotic expansion:*

$$d_\epsilon(h, \delta) = \epsilon\big(I(h, \delta) + \epsilon\varphi_\epsilon(h, \delta)\big), \quad \epsilon \to 0, \tag{2.3}$$

where

$$I(h, \delta) = \oint_{\Gamma_h} Q\,dx - P\,dy, \tag{2.4}$$

and $\varphi_\epsilon(h, \delta)$ is analytic and uniformly bounded for (h, ϵ) in a compact neighborhood of $(h, 0)$. $I(h, \delta)$ is called the Abelian integral of system (2.1).

The Poincaré–Pontryagin theorem has the following immediate corollaries.

Corollary 2.1. *(See [10].) (i) If an h_0 is the simple zero of $I(h)$, then for sufficiently small ϵ, $d_\epsilon(h)$ also has one simple zero close to h_0. Therefore, system (2.1) has one limit cycle close to $\Gamma(h_0)$.*

(ii) If $I(h)$ is well defined in (h_1, h_2) and $I(h_1^+)I(h_2^-) < 0$, then for sufficiently small ϵ, $d_\epsilon(h)$ has at least one zero. It follows that system (2.1) has at least one limit cycle.

Lemma 2.3. *(See [5,7,16].) Suppose that system $(2.1)_{\epsilon=0}$ has a homoclinic loop Γ_0 corresponding to $H(x, y) = 0$ through a hyperbolic saddle p. Assume the periodic orbits $\bigcup_{h \in \Sigma} \Gamma_h$ expand to Γ_0 as h decreases to 0. Then, for $0 < h \ll 1$, there exist analytical functions $a_1(h, \delta)$ and $b_1(h, \delta)$ with $b_1(0, \delta) = 0$ such that $I(h, \delta) = a_1(h, \delta) + b_1(h, \delta) \ln h$, that is,*

$$I(h, \delta) = a_1(0, \delta) + b_1'(0, \delta)h \ln h + a_1'(0, \delta)h + h.o.t.$$

where

$$a_1(0, \delta) = \int_{\Gamma_0} Q\,dx - P\,dy, \qquad b_1'(0, \delta) = -\frac{1}{\lambda}(P_x + Q_y)(p),$$

$$a_1'(0, \delta) = \int_{\Gamma_0} (P_x + Q_y)\,dt \quad \text{if } b_1'(0, \delta) = 0,$$

and $\lambda > 0$ is the eigenvalue of the saddle p.

Lemma 2.4. *(See [5,7,16].) Assume system (2.1)$_{\epsilon=0}$ has a continuous family of periodic orbits $\bigcup_{h_1<h<h_2} \Gamma_h$ and Γ_{h_2} is a homoclinic loop through a hyperbolic saddle p. Denote λ (>0) the eigenvalue of the saddle p.*

(i) Suppose that $\bigcup_{h_1<h<h_2} \Gamma_h$ expand to Γ_{h_2} as h decreases to h_2. Then, for $0 < h - h_2 \ll 1$,

$$I(h) = c_0(\delta) + c_1(\delta)(h - h_2)\ln(h - h_2) + c_2(\delta)(h - h_2) + h.o.t., \tag{2.5}$$

where

$$c_0(\delta) = \int_{\Gamma_{h_2}} Q\,dx - P\,dy, \qquad c_1(\delta) = -\frac{1}{\lambda}(P_x + Q_y)_p, \tag{2.6}$$

$$c_2(\delta) = \int_{\Gamma_{h_2}} (P_x + Q_y)\,dt \quad \text{if } c_1(\delta) = 0. \tag{2.7}$$

(ii) Suppose that $\bigcup_{h_1<h<h_2} \Gamma_h$ expand to Γ_{h_2} as h increases to h_2. Then, for $0 < |h - h_2| \ll 1$ with $h < h_2$,

$$I(h) = c_0(\delta) + c_1(\delta)(h_2 - h)\ln(h_2 - h) + c_2(\delta)(h_2 - h) + h.o.t., \tag{2.8}$$

where $c_0(\delta)$ is given in (2.6), and

$$c_1(\delta) = \frac{1}{\lambda}(P_x + Q_y)(p), \qquad c_2(\delta) = -\int_{\Gamma_{h_2}} (P_x + Q_y)\,dt \quad \text{if } c_1(\delta) = 0. \tag{2.9}$$

3. The expressions of the related Abelian integrals

For $\epsilon = 0$, system (1.3) is given as follows:

$$\dot{x} = H_y(x, y) = yH_1(x, y), \qquad \dot{y} = -H_x(x, y) = -xH_2(x, y), \tag{3.1}$$

where

$$H_1(x, y) = \left[-5 + 14\left(x^2 + y^2\right) + 2\left(3x^4 - 19x^2y^2 - 2y^4\right)\right]/5, \tag{3.2}$$
$$H_2(x, y) = -\left[5 - 14\left(x^2 + y^2\right) + 9x^4 - 12x^2y^2 + 19y^4\right]/5. \tag{3.3}$$

The phase portraits of system (3.1) are shown in Fig. 2.

System (3.1) is a Z_6-equivariant quintic system with 4 saddles on the axes of coordinates:

$$S_1(1, 0), \qquad S_2(-1, 0), \qquad S_3\left(0, \frac{\sqrt{7 - \sqrt{29}}}{2}\right), \qquad S_4\left(0, -\frac{\sqrt{7 - \sqrt{29}}}{2}\right) \tag{3.4}$$

and 4 centers on the axes of coordinates:

$$C_1\left(\frac{\sqrt{5}}{3}, 0\right), \qquad C_2\left(-\frac{\sqrt{5}}{3}, 0\right), \qquad C_3\left(0, \frac{\sqrt{7 + \sqrt{29}}}{2}\right), \qquad C_4\left(0, -\frac{\sqrt{7 + \sqrt{29}}}{2}\right). \tag{3.5}$$

Let Γ_1 denote the homoclinic loop passing the saddle $S_1(1, 0)$. Since

$$H(C_1) \approx -0.1131687243, \qquad H(S_1) = -0.1,$$

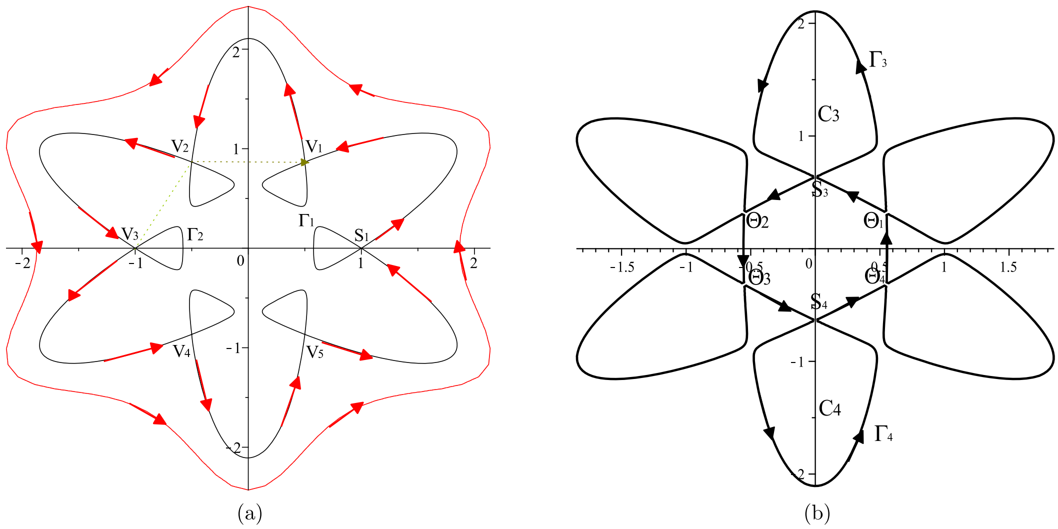

Fig. 3. (a) The homoclinic loops of Γ_1 and Γ_2. (b) The homoclinic loops of Γ_3 and Γ_4.

the periodic orbits inside the Γ_1 expand to Γ_1 as h increases to -0.1, and Γ_1 is orientated clockwise. Let Γ_2 denote the homoclinic loop passing the saddle $(-1, 0)$. We know that Γ_2 and Γ_1 are symmetric about y-axis, see Fig. 3(a).

Let Γ_3 denote the homoclinic loop passing the saddle $S_3(0, y_3)$ with $y_3 = \frac{\sqrt{7-\sqrt{29}}}{2}$ given in (3.4). Since

$$H(C_3) \approx 1.2050, \qquad H(S_3) \approx -0.096542,$$

the periodic orbits inside the Γ_3 expand to Γ_3 as h decreases to -0.096542, and Γ_3 is orientated counter-clockwise. Let Γ_4 denote the homoclinic loop passing the saddle $S_4(0, -y_3)$. We know that Γ_3 and Γ_4 are symmetric about x-axis, see Fig. 3(b).

We introduce the following notations. For $i = 1, \cdots, 4$, let Γ_i be expressed as $(x_i(t), y_i(t))$ for $t \in (-\infty, +\infty)$. Denote D_i the region enclosed by Γ_i and set

$$\gamma_i(t) = \big(x_i(t), y_i(t)\big), \tag{3.6}$$

$$\omega(x,y) := \frac{\partial \mathcal{P}_5}{\partial x} + \frac{\partial \mathcal{Q}_5}{\partial y} = \sum_{i=0,\neq 3}^{5} \omega_i(x,y)A_i + \nu_4(x,y)B_4 + \nu_5(x,y)B_5, \tag{3.7}$$

$$\begin{aligned} \omega_0(x,y) &= 2, \qquad \omega_1(x,y) = 4\big(x^2+y^2\big), \\ \omega_2(x,y) &= 6\big(x^2+y^2\big)^2, \qquad \omega_4(x,y) = 2x\big(x^2-3y^2\big), \qquad \omega_5(x,y) = 4\omega_4(x,y), \\ \nu_4(x,y) &= 2y\big(3x^2-y^2\big), \qquad \nu_5(x,y) = -4\nu_4(x,y). \end{aligned} \tag{3.8}$$

Theorem 3.1. *The homoclinic loop Γ_1 can be expressed as $y = y_+(x)$ (> 0) and $y = y_-(x)$ with $y_-(x) = -y_+(x)$. It intersects x-axis at the points $(\frac{\sqrt{3}}{3}, 0)$ and $(1, 0)$. We express Γ_2 as $y = y_+^l(x)$ (> 0) and $y = y_-^l(x)$ (< 0). Then $y_-^l(x) = -y_+^l(x)$ and $y_+^l(-x) = y_+(x)$, see Fig. 4. Suppose the Abelian integral of system (1.3) near Γ_i is expressed as*

$$I^{(i)}(h) = c_{i0} + c_{i1}(h_1 - h)\ln(h_1 - h) + c_{i2}(h_1 - h) + h.o.t., \quad i = 1, 2, \tag{3.9}$$

where $h_1 = H(S_1)$. Then

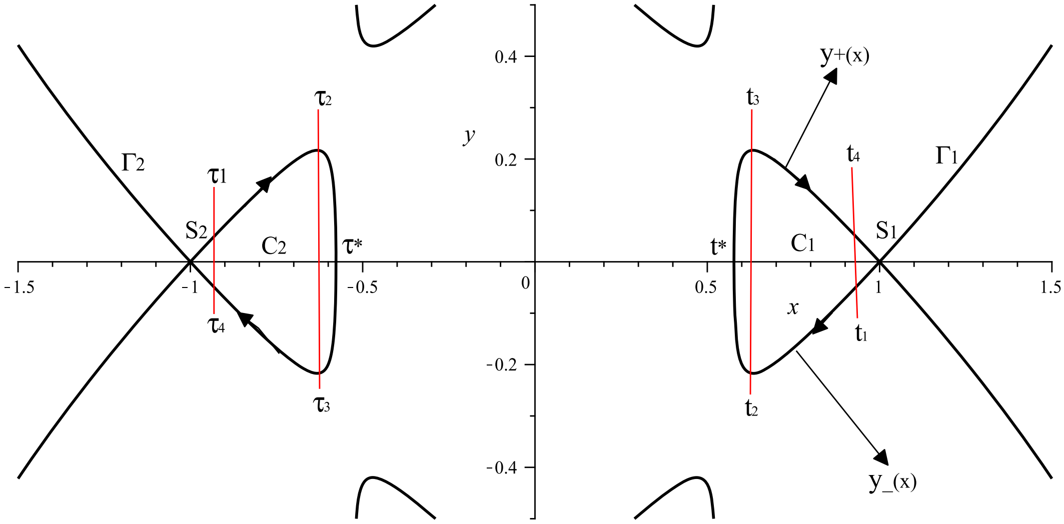

Fig. 4. About the calculation of c_{12} and c_{22}: $x(t^*) = \frac{\sqrt{3}}{3}$, $x(t_2) = x_2^*$, $x(t_1) = x_1^*$, $C_1 = (\frac{\sqrt{5}}{3}, 0)$.

(i)
$$\begin{cases} c_{10} = (\alpha_0, \alpha_1, \alpha_2, \alpha_4, 4\alpha_4) \cdot (A_0, A_1, A_2, A_4, A_5)^T, \\ c_{20} = (\alpha_0, \alpha_1, \alpha_2, -\alpha_4, -4\alpha_4) \cdot (A_0, A_2, A_2, A_4, A_5)^T, \\ \alpha_i = \iint\limits_{D_1} \omega_i(x,y)\,dx\,dy; \end{cases}$$

(ii)
$$\begin{cases} c_{11} = \dfrac{2}{\lambda_1}(A_0 + 2A_1 + 3A_2 + A_4 + 4A_5), \\ c_{21} = \dfrac{2}{\lambda_1}(A_0 + 2A_1 + 3A_2 - A_4 - 4A_5), \end{cases}$$

where $\lambda_1 = \sqrt{\frac{24}{5}}$ *is the eigenvalue of system (3.1) at the saddle* $S_1(1,0)$.

(iii) *For* x_1^* *and* x_2^* *satisfying* $\frac{\sqrt{3}}{3} < x_2^* < \frac{\sqrt{5}}{3} < x_1^* < 1$, *and* $y_2^* = -y_+(x_2^*)$ *(see Fig. 4),* c_{12} *and* c_{22} *can be expressed as*

$$c_{12} = -\big(J_1^{(1)} + J_2^{(1)} + J_3^{(1)}\big), \qquad c_{22} = -\big(J_1^{(2)} + J_2^{(2)} + J_3^{(2)}\big),$$

where

$$\begin{cases} J_1^{(1)} = 4\displaystyle\int_{x_1^*}^{1} \frac{2yA_1 + 3y(2x^2+y^2)A_2 - 3xyA_4 - 12xyA_5}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \\ \qquad + 4\displaystyle\int_{x_1^*}^{1} \frac{x-1}{y} \cdot \frac{\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5}{H_1(x,y)}\bigg|_{y=y_+(x)} dx, \\ J_1^{(2)} = 4\displaystyle\int_{x_1^*}^{1} \frac{2yA_1 + 3y(2x^2+y^2)A_2 + 3xyA_4 + 12xyA_5}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \\ \qquad + 4\displaystyle\int_{x_1^*}^{1} \frac{x-1}{y} \cdot \frac{\beta_1 A_1 + \beta_1 A_2 - \beta_4 B_4 - 4\beta_4 B_5}{H_1(x,y)}\bigg|_{y=y_+(x)} dx, \end{cases} \tag{3.10}$$

$$\begin{cases} J_2^{(1)} = 2\displaystyle\int_{x_2^*}^{x_1^*} \frac{\widetilde{\omega}_1 A_1 + \widetilde{\omega}_2 A_2 + \widetilde{\omega}_4 A_4 + 4\widetilde{\omega}_4 A_5}{H_y}(x,y)\bigg|_{y=y_+(x)} dx, \\ J_2^{(2)} = 2\displaystyle\int_{x_2^*}^{x_1^*} \frac{\widetilde{\omega}_1 A_1 + \widetilde{\omega}_2 A_2 - \widetilde{\omega}_4 A_4 - 4\widetilde{\omega}_4 A_5}{H_y}(x,y)\bigg|_{y=y_+(x)} dx; \end{cases} \tag{3.11}$$

$$\begin{cases} J_3^{(1)} = 2\displaystyle\int_{0}^{-y_2^*} \frac{\widetilde{\omega}_1 A_1 + \widetilde{\omega}_2 A_2 + \widetilde{\omega}_4 A_4 + 4\widetilde{\omega}_4 A_5}{-H_x}(x,y)\bigg|_{x=x(y)} dy, \\ J_3^{(2)} = 2\displaystyle\int_{0}^{-y_2^*} \frac{\widetilde{\omega}_1 A_1 + \widetilde{\omega}_2 A_2 - \widetilde{\omega}_4 A_4 - 4\widetilde{\omega}_4 A_5}{-H_x}(x,y)\bigg|_{x=x(y)} dy, \end{cases} \tag{3.12}$$

$$\beta_1(x) = 2(x+1), \qquad \beta_2 = 3\left(x^2+1\right)(x+1), \qquad \beta_4(x) = x^2+x+1, \tag{3.13}$$

$$\widetilde{\omega}_1 = 4\left(x^2+y^2-1\right), \qquad \widetilde{\omega}_2 = 6\left[\left(x^2+y^2\right)^2-1\right], \qquad \widetilde{\omega}_4 = 2x\left(x^2-3y^2\right)-2, \tag{3.14}$$

$H_2(x,y)$ and $H_1(x,y)$ are defined in (3.2) and (3.3) respectively, and $f(x,y)|_{y=g(x)}$ is defined as $f(x,y)|_{y=g(x)} = f(x,g(x))$.

Proof. First, let us consider the expressions of c_{1i} $(i = 0,1,2)$. The expression of c_{10} follows from Lemma 2.4(ii) and Green's formula. By (2.9) and (3.7), we get $c_{11} = \frac{1}{\lambda_1}\omega(1,0)$, which gives the first formula in (ii). It follows from $c_{11} = 0$ that

$$A_0 = -2A_1 - 3A_2 - A_4 - 4A_5. \tag{3.15}$$

Let us give the expression of c_{12}. For some $-\infty < t_1 < t_2 < t^* < t_3 < t_4 < +\infty$ such that

$$x\left(t^*\right) < x(t_2) < \frac{\sqrt{5}}{3} < x(t_1) < 1, \qquad x(t_2) = x(t_3), x(t_1) = x(t_4), \qquad y_+\left(x\left(t^*\right)\right) = 0,$$

see Fig. 4, let $x(t_1) = x_1^*$, $x(t_2) = x_2^*$, $y_2^* = -y(x(t_2)) > 0$, and

$$J_1^{(1)} = \left(\int_{-\infty}^{t_1} + \int_{t_4}^{+\infty}\right)\omega\left(\gamma_1(t)\right) dt,$$

$$J_2^{(1)} = \left(\int_{t_1}^{t_2} + \int_{t_3}^{t_4}\right)\omega\left(\gamma_1(t)\right) dt,$$

$$J_3^{(1)} = \int_{t_2}^{t_3} \omega\left(\gamma_1(t)\right) dt.$$

Then, by (2.9), we get $c_{12} = -\oint_{\Gamma_1} \omega(\gamma_1(t))\, dt = -(J_1^{(1)} + J_2^{(1)} + J_3^{(1)})$. By system (3.1), we get

$$J_1^{(1)} = \int_{1}^{x(t_1)} \frac{\omega}{H_y}\left(x, y_-(x)\right) dx + \int_{x(t_4)}^{1} \frac{\omega}{H_y}\left(x, y_+(x)\right) dx$$

$$= \int_{x(t_4)}^{1} \frac{\omega(x, y_+(x)) + \omega(x, y_-(x))}{H_y(x, y_+(x))}\, dx.$$

It is easy to know that $\omega(x, y)$ under (3.15) can be expressed as

$$\omega(x, y) = f(x, y) + 2(x-1)(\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5)$$

where β_i $(i = 1, \cdots, 4)$ are given by (3.13) and

$$f(x, y) = 2y\big[2yA_1 + 3y\big(2x^2 + y^2\big)A_2 - 3xyA_4 - 12xyA_5 + \big(3x^2 - y^2\big)B_4 - 4\big(3x^2 - y^2\big)B_5\big]. \tag{3.16}$$

Noting that $y_-(x) = -y_+(x)$, we get

$$J_1^{(1)} = \int_{x(t_1)}^{1} \frac{4y[2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5]}{yH_1(x, y)}\bigg|_{y=y_+(x)} dx$$
$$+ \int_{x(t_1)}^{1} \frac{4(x-1)(\beta_1 A_1 + \beta_2 A_2 + \beta_4 B_4 + 4\beta_4 B_5)}{yH_1(x, y)}\bigg|_{y=y_+(x)} dx$$

which gives the first formula of (3.10). Similarly, we can get

$$J_2^{(1)} = \int_{x(t_2)}^{x(t_1)} \frac{\omega(x, y_+(x)) + \omega(x, y_-(x))}{H_y(x, y_+(x))}\, dx. \tag{3.17}$$

We know that $\omega(x, y)$ under (3.15) can also be expressed as

$$\omega(x, y) = \sum_{i=1,\neq 3}^{5} \omega_i(x, y)A_i + \sum_{i=4}^{5} \nu_i(x, y)B_i - 4A_1 - 6A_2 - 2A_4 - 8A_5$$
$$= \sum_{i=1,\neq 3}^{5} \widetilde{\omega}_i(x, y)A_i + \nu_4(x, y)B_4 + \nu_5(x, y)B_5, \tag{3.18}$$

where $\widetilde{\omega}_i(x, y)$ are given by (3.14) and $\widetilde{\omega}_5(x, y) = 4\widetilde{\omega}_4(x, y)$. Substituting (3.18) into (3.17), we can get the first formula of (3.11). Noting that $x(-y) = x(y)$, we have

$$J_3^{(1)} = \left(\int_{t_2}^{t_2^*} + \int_{t_2^*}^{t_3}\right)\omega\big(\gamma_1(t)\big)\, dt = \left(\int_{y(t_2)}^{0} + \int_{0}^{-y(t_2)}\right)\frac{\omega}{-H_x}\big(x(y), y\big)\, dy.$$

Let $u = -y$, then

$$\int_{y(t_2)}^{0} \frac{\omega(x(y), y)}{-x(y)H_2(x(y), y)}\, dy = -\int_{-y(t_2)}^{0} \frac{\omega(x(-u), -u)}{-x(-u)H_2(x(-u), -u)}\, du.$$

Since $H_2(x, -u) = H_2(x, u)$, thus

$$J_3^{(1)} = \int_0^{-y(t_2)} \frac{\omega(x,y)+\omega(x,-y)}{-H_x(x,y)}\bigg|_{x=x(y)} dy.$$

By $\nu(x,-y) = -\nu(x,y)$ and $\widetilde{\omega}_i(x,-y) = \widetilde{\omega}_i(x,y)$, we can get the first formula of (3.12).

Next, let us consider the expressions of c_{2i} $(i = 0,1,2)$. By Lemma 2.4(ii), we get

$$c_{20} = \oint_{\Gamma_2} \mathcal{Q}_5\,dx - \mathcal{P}_5\,dy = \sum_{i=1,\neq 3}^{5} A_i \iint_{D_2} \omega_i\big(x,y^l\big)\,dx\,dy + \sum_{i=4}^{5} B_i \iint_{D_2} \nu_i\big(x,y^l\big)\,dx\,dy.$$

Noting that

$$\iint_{D_2} \omega_i\big(x,y^l\big)\,dx\,dy = \iint_{D_1} \omega_i(x,y)\,dx\,dy, \quad i = 1,3,6;$$
$$\iint_{D_2} \omega_4\big(x,y^l\big)\,dx\,dy = -\iint_{D_1} \omega_4(x,y)\,dx\,dy;$$

we get

$$c_{20} = (\alpha_0,\alpha_1,\alpha_2,-\alpha_4,-4\alpha_4)\cdot(A_0,A_1,A_2,A_4,A_5)^T,$$

where α_i $(i = 1,\cdots,4)$ are given in Theorem 3.1(i). By Lemma 2.4(ii) and (3.7), we have

$$c_{21} = \frac{1}{\lambda_1}\omega(-1,0) = \frac{2}{\lambda_1}(A_0 + 2A_1 + 3A_2 - A_4 - 4A_5).$$

It follows from $c_{21} = 0$ that

$$A_0 = -2A_1 - 3A_2 + A_4 + 4A_5. \tag{3.19}$$

Let us give the expression of c_{22}. For $-\infty < \tau_1 < \tau_2 < \tau^* < \tau_3 < \tau_4 < +\infty$ such that

$$x(\tau_1) = x(\tau_4) = -x(t_1), \qquad x(\tau_2) = x(\tau_3) = -x(t_2), \qquad x\big(\tau^*\big) = -x\big(t^*\big),$$

see Fig. 4, let

$$J_1^{(2)} = \left(\int_{-\infty}^{\tau_1} + \int_{\tau_4}^{+\infty}\right)\omega\big(\gamma_2(t)\big)\,dt,$$
$$J_2^{(2)} = \left(\int_{\tau_1}^{\tau_2} + \int_{\tau_3}^{\tau_4}\right)\omega\big(\gamma_2(t)\big)\,dt,$$
$$J_3^{(2)} = \int_{\tau_2}^{\tau_3} \omega\big(\gamma_2(t)\big)\,dt.$$

Then, by Lemma 2.4(ii), we get $c_{22} = -\oint_{\Gamma_2} \omega(\gamma_2(t))\,dt = -(J_1^{(2)} + J_2^{(2)} + J_3^{(2)})$. By (3.1), we get

$$J_1^{(2)} = \int_{-1}^{x(\tau_1)} \frac{\omega}{H_y}\big(x, y_+^l(x)\big)\,dx + \int_{x(\tau_4)}^{-1} \frac{\omega}{H_y}\big(x, y_-^l(x)\big)\,dx$$

$$= \int_{-1}^{x(\tau_1)} \frac{\omega(x, y_+^l(x)) + \omega(x, y_-^l(x))}{H_y(x, y_+^l(x))}\,dx.$$

It is easy to know that $\omega(x, y)$ under (3.19) can be expressed as

$$\omega = f(x, y) + 2(x + 1)(\tilde{\beta}_1 A_1 + \tilde{\beta}_2 A_2 + \tilde{\beta}_4 B_4 + 4\tilde{\beta}_4 B_5), \tag{3.20}$$

where $f(x, y)$ is given by (3.16) and

$$\tilde{\beta}_1(x) = 2(x - 1), \qquad \tilde{\beta}_2(x) = 3(x - 1)\big(x^2 + 1\big), \qquad \tilde{\beta}_4(x) = x^2 - x + 1.$$

Hence,

$$J_1^{(2)} = \int_{-1}^{x(\tau_1)} \frac{4y[2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5]}{yH_1(x, y)}\bigg|_{y=y_+^l(x)}\,dx$$

$$+ \int_{-1}^{x(\tau_1)} \frac{4(x + 1)(\tilde{\beta}_1 A_1 + \tilde{\beta}_2 A_2 + \tilde{\beta}_4 B_4 + 4\tilde{\beta}_4 B_5)}{yH_1(x, y)}\bigg|_{y=y_+^l(x)}\,dx$$

$$= 4\int_{-1}^{x(\tau_1)} \frac{2yA_1 + 3y(2x^2 + y^2)A_2 - 3xyA_4 - 12xyA_5}{H_1(x, y)}\bigg|_{y=y_+^l(x)}\,dx$$

$$+ 4\int_{-1}^{x(\tau_1)} \frac{x + 1}{y_+(x)} \cdot \frac{\tilde{\beta}_1 A_1 + \tilde{\beta}_2 A_2 + \tilde{\beta}_4 B_4 + 4\tilde{\beta}_4 B_5}{H_1(x, y)}\bigg|_{y=y_+^l(x)}\,dx.$$

Since $y_+^l(-x) = y_+(x)$ and

$$\tilde{\beta}_1(-x) = -\beta_1(x), \qquad \tilde{\beta}_2(-x) = -\beta_2(x), \qquad \tilde{\beta}_4(-x) = \beta_4(x),$$

we get

$$J_1^{(2)} = 4\int_{x(t_1)}^{1} \frac{2yA_1 + 3y(2x^2 + y^2)A_2 + 3xyA_4 + 12xyA_5}{H_1(x, y)}\bigg|_{y=y_+(x)}\,dx$$

$$+ 4\int_{x(t_1)}^{1} \frac{x - 1}{y_+(x)} \cdot \frac{\beta_1 A_1 + \beta_2 A_2 - \beta_4 B_4 - 4\beta_4 B_5}{H_1(x, y)}\bigg|_{y=y_+(x)}\,dx,$$

which yields the second formula of (3.10). Similarly, we have

$$J_2^{(2)} = \int_{x(\tau_1)}^{x(\tau_2)} \frac{\omega(x, y_+^l(x)) + \omega(x, y_-^l(x))}{H_y(x, y_+^l(x))}\,dx.$$

We can check that $\omega(x,y)$ under (3.19) can also be expressed as

$$\omega(x,y)=\sum_{i=1,\neq 3}^{5}\bar{\omega}_i(x,y)A_i+\nu_4(x,y)B_4+\nu_5(x,y)B_5,$$
$$\bar{\omega}_i(x,y)=\widetilde{\omega}_i(x,y),\quad i=1,2;$$
$$\bar{\omega}_4(x,y)=2x\big(x^2-3y^2\big)+2,\qquad \bar{\omega}_5(x,y)=4\bar{\omega}_4.$$

Since $\bar{\omega}_i(-x,y)=\widetilde{\omega}_i(x,y)$ for $i=1,2$ and $\bar{\omega}_4(-x,y)=-\widetilde{\omega}_4(x,y)$, thus,

$$J_2^{(2)}=2\sum_{i=1,2}A_i\int_{x(t_2)}^{x(t_1)}\frac{\widetilde{\omega}_i}{H_y}\big(x,y_+(x)\big)\,dx-2(A_4+4A_5)\int_{x(t_2)}^{x(t_1)}\frac{\widetilde{\omega}_4}{H_y}\big(x,y_+(x)\big)\,dx,$$

which yields the second formula of (3.11). For $J_3^{(2)}$, we have

$$J_3^{(2)}=\left(\int_{\tau_2}^{\tau^*}+\int_{\tau^*}^{\tau_3}\right)\omega\big(\gamma_2(t)\big)\,dt=\left(\int_{y(\tau_2)}^{0}+\int_{0}^{y(\tau_3)}\right)\frac{\omega}{-H_x}\big(x^l(y),y\big)\,dy.$$

Let $u=-y$, then

$$\int_0^{y(\tau_3)}\frac{\omega(x^l(y),y)}{-x(y)H_2(x^l(y),y)}\,dy=-\int_0^{-y(\tau_3)}\frac{\omega(x^l(-u),-u)}{-x^l(-u)H_2(x^l(-u),-u)}\,du.$$

Since $x^l(-y)=x^l(y)$ and $H_2(x,-u)=H_2(x,u)$, we have

$$J_3^{(2)}=\int_{-y(\tau_3)}^{0}\frac{\omega(x^l(y),y)+\omega(x^l(y),-y)}{-H_x(x^l(y),y)}\,dy.$$

Noting that $x^l(y)=-x^r(y)$, we get

$$\begin{aligned}J_3^{(2)}&=\sum_{i=1,\neq 3}^{5}A_i\int_{-y(t_2)}^{0}\frac{\bar{\omega}_i(x^l(y),y)}{-H_x(x^l(y),y)}\,dy\\&=2\sum_{i=1,2}A_i\int_0^{-y(t_2)}\frac{\widetilde{\omega}_i}{-H_x}\big(x(y),y\big)\,dy-2(A_4+4A_5)\int_0^{-y(t_2)}\frac{\widetilde{\omega}_4}{-H_x}\big(x(y),y\big)\,dy,\end{aligned}$$

which yields the second formula of (3.12). This completes the proof. $\square$

Theorem 3.2. *Suppose the Abelian integral of system (1.3) near the homoclinic loop Γ_i is expressed as*

$$I^{(i)}(h)=c_{i0}+c_{i1}(h-h_3)\ln(h-h_3)+c_{i2}(h-h_3)+h.o.t.,\quad i=3,4,\tag{3.21}$$

where $h_3=H(S_3)$. Then

(i) $$\begin{cases} c_{30} = (\eta_0, \eta_1, \eta_2, \eta_4, -4\eta_4) \cdot (A_0, A_1, A_2, B_4, B_5)^T, \\ c_{40} = (\eta_0, \eta_1, \eta_2, -\eta_4, 4\eta_4) \cdot (A_0, A_1, A_2, B_4, B_5)^T, \\ \eta_i = -\iint\limits_{D_3} \omega_i(x,y)\,dx\,dy, \quad i = 0,1,2; \qquad \eta_4 = -\iint\limits_{D_3} \nu_4(x,y)\,dx\,dy, \end{cases}$$

where $\omega_i(x,y)$ are given in (3.8).

(ii) $$\begin{cases} c_{31} = -\dfrac{1}{\lambda_3}\left[2A_0 + (7-\sqrt{29})A_1 + \dfrac{3}{8}(7-\sqrt{29})^2 A_2 - \dfrac{1}{4}(7-\sqrt{29})^{\frac{3}{2}} B_4 + (7-\sqrt{29})^{\frac{3}{2}} B_5\right], \\ c_{41} = -\dfrac{1}{\lambda_3}\left[2A_0 + (7-\sqrt{29})A_1 + \dfrac{3}{8}(7-\sqrt{29})^2 A_2 + \dfrac{1}{4}(7-\sqrt{29})^{\frac{3}{2}} B_4 - (7-\sqrt{29})^{\frac{3}{2}} B_5\right], \end{cases}$$

where $\lambda_3 = \frac{1}{10}\sqrt{-19140 + 3570\sqrt{29}}$ is the eigenvalue of system (3.1) at the saddle S_3.

Proof. We express the homoclinic loop Γ_3 as $x = x_+(y)$ (>0) and $x = x_-(y) = -x_+(y)$, and express Γ_4 as $x = x_+^d(x)$ (>0) and $x = x_-^d(y)$ (<0). Then $x_-^d(y) = -x_+^d(y)$ and $x_-^d(-y) = x_+(y)$. By Lemma 2.4(i) and (3.7), we have

$$c_{30} = \oint\limits_{\Gamma_3} \mathcal{Q}_5\,dx - \mathcal{P}_5\,dy$$
$$= -\sum_{i=0,1,2} A_i \iint\limits_{D_3} \omega_i(x,y)\,dx\,dy - (B_4 - 4B_5)\iint\limits_{D_3} \nu_4(x,y)\,dx\,dy,$$

which gives the first formula in (i). By Lemma 2.4(i), we get $c_{31} = -\frac{1}{\lambda_3}\omega(S_3)$. Noting the orbits of system (3.1) are symmetric about both of x-axis and y-axis, we can obtain the formula of c_{4i} by the similar arguments to that of c_{2i} $(i = 0,1)$. This ends the proof. □

For system (3.1), there exists a heteroclinic-polycycle $S^{(6)}$ through 6 saddles (see Fig. 3(b)). It is orientated counter-clockwise and corresponds to $H(x,y) = H(S_3)$. It follows from [7] and [16] that for $0 < h - H(S_3) \ll 1$, $I(h)$ can be expressed as

$$I^{(6)}(h) = \int\limits_{S^{(6)}} \mathcal{Q}_5(x,y)\,dx - \mathcal{P}_5(x,y)\,dy + \text{h.o.t.}$$

Denote $c_{60} = \int_{S^{(6)}} \mathcal{Q}_5\,dx - \mathcal{P}_5\,dy$. Since system (1.3) is Z_3-equivariant, we get

$$c_{60} = -3\Bigg(\int\limits_{\widehat{\Theta_2 S_3}} + \int\limits_{\widehat{S_3\Theta_1}}\Bigg)\big[\mathcal{Q}_5(x,y)\,dx - \mathcal{P}_5(x,y)\,dy\big], \tag{3.22}$$

where $\Theta_2(x_{3l}, y_{3l})$ is obtained by rotating S_3 counter-clockwise with $\frac{\pi}{3}$, and $\Theta_1(x_{3r}, y_{3r})$ is obtained by rotating S_3 clockwise with $\frac{\pi}{3}$. By symmetry, we get $x_{3l} = -x_{3r}$ and $y_{3r} = y_{3l}$. Hence,

$$-\frac{1}{3}c_{60} = \Bigg(\int_{x_{3l}}^{0} + \int_{0}^{x_{3r}}\Bigg)\big[\mathcal{Q}_5\big(x, y(x)\big) - \mathcal{P}_5\big(x, y(x)\big)y'(x)\big]\,dx.$$

Let $u = -x$. Since along $\widehat{\Theta_2 S_3}$ and $\widehat{S_3\Theta_1}$, we have $y(-x) = y(x)$ and $y'(-x) = -y'(x)$, then

$$\int_0^{x_{3r}} \left[\mathcal{Q}_5\big(x, y(x)\big) - \mathcal{P}_5\big(x, y(x)\big)y'(x)\right] dx$$

$$= \int_0^{-x_{3r}} \left[\mathcal{Q}_5\big(-u, y(-u)\big) - \mathcal{P}_5\big(-u, y(-u)\big)y'(-u)\right] d(-u)$$

$$= \int_{x_{3l}}^{0} \left[\mathcal{Q}_5\big(-u, y(u)\big) + \mathcal{P}_5\big(-u, y(u)\big)y'(u)\right] du.$$

So,

$$-\frac{1}{3}c_{60} = \int_{x_{3l}}^{0} \left[\mathcal{Q}_5\big(x, y(x)\big) + \mathcal{Q}_5\big(-x, y(x)\big)\right] dx$$

$$- \int_{x_{3l}}^{0} \left[\mathcal{P}_5\big(x, y(x)\big) - \mathcal{P}_5\big(-x, y(x)\big)\right] y'(x)\, dx$$

$$= 2\int_{x_{3l}}^{0} \left\{y\big(x^2+y^2\big)^2 A_2 + y\big(x^2+y^2\big)A_1 + yA_0\right\}\Big|_{y=y(x)} dx$$

$$- 2\int_{x_{3l}}^{0} \left\{x\big(x^2+y^2\big)^2 A_2 + x\big(x^2+y^2\big)A_1 + xA_0\right\}\Big|_{y=y(x)} y'(x)\, dx. \qquad (3.23)$$

Remark 3.1. In the calculations of c_{i2} $(i = 1, 2)$, we find that $H_y(x, y)$ has the factor y, which leads to the integrals cannot be calculated directly by Matlab even though we know that c_{i2} exists by Lemma 2.4. We deal with this problem by the way of (3.16) and (3.20).

4. Proof of the main results

Set $x_1^* = 0.95$, $x_2^* = \frac{\sqrt{3}}{3} + 0.1$ and $\frac{x-1}{y_+(x)} \approx -\lambda_1^{-1}$ for $x \in (0.95, 1)$. Then by Matlab 7.5, we get
(i) The values in Theorems 3.1(i) and 3.2(i):

$$\alpha_0 \approx 0.22126, \qquad \alpha_1 \approx 0.24487, \qquad \alpha_2 \approx 0.21722, \qquad \eta_0 \approx 1.981659229;$$

$$\eta_1 \approx 7.62447, \qquad \eta_2 \approx 27.60503, \qquad \eta_4 \approx -5.21282, \qquad \eta_5 \approx 4 \times 5.2128.$$

(ii) The values for (3.10)–(3.12):

$$
\begin{cases}
\displaystyle\int_{x_1^*}^{1} \frac{2y}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \approx 6.806860858 \times 10^{-4}, \\
\displaystyle\int_{x_1^*}^{1} \frac{3y(2x^2+y^2)}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \approx 1.907295061 \times 10^{-3}, \\
\displaystyle\int_{x_1^*}^{1} \frac{\beta_3}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \approx 9.192955257 \times 10^{-1}, \\
\displaystyle\int_{x_1^*}^{1} \frac{\beta_6}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \approx 2.108043101 \times 10^{-1}, \\
\displaystyle\int_{x_1^*}^{1} \frac{\beta_4}{H_1(x,y)}\bigg|_{y=y_+(x)} dx \approx 5.337516448 \times 10^{-2},
\end{cases}
$$

$$
\begin{cases}
\displaystyle\int_{x_2^*}^{x_1^*} \frac{\widetilde{\omega}_3}{H_y}\big(x, y_+(x)\big)\, dx \approx -2.363016631, \\
\displaystyle\int_{x_2^*}^{x_1^*} \frac{\widetilde{\omega}_6}{H_y}\big(x, y_+(x)\big)\, dx \approx -5.808452011, \\
\displaystyle\int_{0}^{-y_2^*} \frac{\widetilde{\omega}_3}{H_y}\big(x(y), y\big)\, dy \approx -4.828232524, \\
\displaystyle\int_{0}^{-y_2^*} \frac{\widetilde{\omega}_6}{H_y}\big(x(y), y\big)\, dy \approx -9.852007882.
\end{cases}
$$

(iii) The values for (3.23):

$$
\begin{cases}
\displaystyle\int_{x_{3l}}^{0} y\big(x^2+y^2\big)^2\big|_{y=y(x)}\, dx \approx 3.092091723 \times 10^{-2}, \\
\displaystyle\int_{x_{3l}}^{0} \big(x^4-y^4\big)\big|_{y=y(x)}\, dx \approx -2.580407313 \times 10^{-2}, \\
\displaystyle\int_{x_{3l}}^{0} y\big(x^2+y^2\big)\big|_{y=y(x)}\, dx \approx 9.015509722 \times 10^{-2}, \\
\displaystyle\int_{x_{3l}}^{0} y(x)\, dx \approx 2.646601805 \times 10^{-1},
\end{cases}
$$

$$\begin{cases} \displaystyle\int_{x_{3l}}^{0} x\left(x^2+y^2\right)^2\big|_{y=y(x)} y'(x)\,dx \approx -1.303820399 \times 10^{-2}, \\ \displaystyle\int_{x_{3l}}^{0} 2xy\left(x^2+y^2\right)\big|_{y=y(x)} y'(x)\,dx \approx -3.064747205 \times 10^{-2}, \\ \displaystyle\int_{x_{3l}}^{0} x\left(x^2+y^2\right)\big|_{y=y(x)} y'(x)\,dx \approx -3.064747205 \times 10^{-2}, \\ \displaystyle\int_{x_{3l}}^{0} xy'(x)\,dx \approx -1.088923565 \times 10^{-1}. \end{cases}$$

Remark 4.1. It is crucial for us to obtain the expressions of the homoclinic loops Γ_1 and Γ_3. To do this, we solve the equations $H(x,y) = H(S_1)$ and $H(x,y) = H(S_3)$ by the formula of root for the equation $s^3 + as^2 + bs + c = 0$, where $s \in R$ is variable. There are three branches for each of $H(x,y) = H(S_1)$ and $H(x,y) = H(S_3)$. Then, we plot the image for each of the branches to make sure that the expressions are right.

Let $A_4 = -4A_5$, then we have $\omega(O) = 2A_0$, and

$$\begin{cases} c_{10} = c_{20} = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \\ c_{11} = c_{21} = 2(A_0 + 2A_1 + 3A_2)/\lambda_1, \\ c_{12} \approx 16.05817189A_1 + 31.69816485A_2 + 0.09744927200B_4 + 0.3897970879B_5, \\ c_{22} \approx 16.05817189A_1 + 31.69816485A_2 - 0.09744927200B_4 - 0.3897970879B_5, \\ \omega(C_1) = \omega(C_2) = \dfrac{2}{27}[27A_0 + 30A_1 + 25A_2] \approx 2.A_0 + 2.2222A_1 + 1.8519A_2. \end{cases} \tag{4.1}$$

$$\begin{cases} c_{30} = \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 + \eta_4 B_4 - 4\eta_4 B_5, \\ c_{31} = -\dfrac{1}{\lambda_3}\left[2A_0 + (7-\sqrt{29})A_1 + \dfrac{3}{8}(7-\sqrt{29})^2 A_2 - \dfrac{1}{4}(7-\sqrt{29})^{\frac{3}{2}} B_4 + (7-\sqrt{29})^{\frac{3}{2}} B_5\right] \\ \qquad \approx -\dfrac{1}{\lambda_3}[2.A_0 + 1.614835193A_1 + 0.9778847629A_2 - 0.5130176625B_4 + 2.052070650B_5], \\ \omega(C_3) = 2A_0 + (7+\sqrt{29})A_1 + \dfrac{3}{8}(7+\sqrt{29})^2 A_2 - \dfrac{1}{4}(7+\sqrt{29})^{3/2} B_4 + (7+\sqrt{29})^{3/2} B_5 \\ \qquad \approx 2.A_0 + 12.385A_1 + 57.521A_2 - 10.896B_4 + 43.584B_5; \end{cases} \tag{4.2}$$

$$\begin{cases} c_{40} = \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 - \eta_4 B_4 + 4\eta_4 B_5, \\ \omega(C_4) \approx 2.A_1 + 12.385A_1 + 57.521A_2 + 10.896B_4 - 43.584B_5, \\ c_{60} \approx -0.9346069440A_0 - 0.3570457510A_1 - 0.1072962794A_2 \\ \qquad + 0.3387092711B_4 + 1.354837084B_5, \end{cases} \tag{4.3}$$

where $\lambda_1 > 0$ and $\lambda_3 > 0$ are respectively the eigenvalues of system (3.1) at the saddles S_1 and S_3.

4.1. The 23 limit cycles for Z_3-equivariant quintic systems

Let

$$\begin{cases} \mu_1 = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \\ \mu_2 = \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 + \eta_4 B_4 - 4\eta_4 B_5, \\ \mu_3 = -\Big[2A_0 + (7-\sqrt{29})A_1 + \frac{3}{8}(7-\sqrt{29})^2 A_2 - \frac{1}{4}(7-\sqrt{29})^{\frac{3}{2}} B_4 + (7-\sqrt{29})^{\frac{3}{2}} B_5\Big]. \end{cases} \tag{4.4}$$

Then we get

$$\begin{cases} A_0 \approx 2.447341576A_2 - 3.448106551\mu_1 - 0.09612190380\mu_2 - 0.9767044561\mu_3, \\ A_1 \approx -3.098455496A_2 + 7.199444833\mu_1 + 0.08685397331\mu_2 + 0.8825320703\mu_3, \\ B_4 \approx 1.694040964A_2 + 4B_5 + 9.219378069\mu_1 - 0.1013395027\mu_2 + 0.9195296149\mu_3, \\ \omega(C_1) \approx -0.138921654A_2 + 9.102553198\mu_1 + 0.0007650220\mu_2 + 0.007773466\mu_3, \\ \omega(C_3) \approx 5.58255808A_2 - 18.19016710\mu_1 + 1.987717275\mu_2 - 1.04288905\mu_3, \\ \omega(C_4) \approx 42.50127484A_2 + 182.7303625\mu_1 - 0.220803343\mu_2 + 18.99668151\mu_3, \\ \omega(O) \approx 2(2.447341576A_2 - 3.448106551\mu_1 - 0.09612190380\mu_2 - 0.9767044561\mu_3), \end{cases} \tag{4.5}$$

and

$$\begin{cases} I^{(1)}(h) = I^{(2)}(h) = \mu_1 + c_{11}^*(h_1 - h)\ln(h_1 - h) + \text{h.o.t.}, \\ I^{(3)}(h) = \mu_2 + \frac{\mu_3}{\lambda_3}(h - h_3)\ln(h - h_3) + \text{h.o.t.}, \\ I^{(4)}(h) = c_{40}^* + \text{h.o.t.}, \\ I^{(6)}(h) = c_{60}^* + \text{h.o.t.}, \end{cases} \tag{4.6}$$

with

$$\begin{cases} c_{11}^* \approx \frac{2}{\lambda_1}(-0.749569416A_2 + 10.95078312\mu_1 + 0.0775860428\mu_2 + 0.7883596849\mu_3), \\ c_{40}^* \approx 17.66147727A_2 + 96.11800402\mu_1 - 0.0565301320\mu_2 + 9.586693437\mu_3, \\ c_{60}^* \approx -0.7145209599A_2 + 2.709674167B_5 \\ \qquad + 3.774781965\mu_1 + 0.02450072754\mu_2 + 0.9091836462\mu_3. \end{cases} \tag{4.7}$$

Denote $\mu = (\mu_1, \mu_2, \mu_3)$. Since

$$\begin{cases} \lim_{\mu\to 0} c_{11}^* \approx \frac{2}{\lambda_1}(-0.749569416)A_2, \\ \lim_{\mu\to 0} c_{40}^* \approx 17.66147727A_2, \\ \lim_{\mu\to 0} c_{60}^* \approx -0.7145209599A_2 + 2.709674167B_5, \\ \lim_{\mu\to 0} \omega(C_1) = \lim_{\mu\to 0} \omega(C_2) \approx -0.138921654A_2, \\ \lim_{\mu\to 0} \omega(C_3) \approx 5.58255808A_2, \\ \lim_{\mu\to 0} \omega(C_4) \approx 42.50127484A_2, \\ \lim_{\mu\to 0} \omega(O) \approx 2 \times 2.447341576A_2, \end{cases}$$

for some A_2^* and B_5^* such that $A_2^* > 0$ and $-0.7145209599A_2^* + 2.709674167B_5^* > 0$, there exists a neighborhood U of $\mu = 0$ such that for $\mu \in U$ we have

$$c_{11}^* < 0, \quad \omega(C_1) < 0; \quad \omega(C_3) > 0; \quad c_{40}^* > 0, \quad \omega(C_4) > 0; \quad c_{60}^* > 0, \quad \omega(O) > 0.$$

Set some $\mu^* \in U$ such that $\mu_i^* < 0$ $(i = 1, 2, 3)$ with $|\mu_2| \ll |\mu_3|$. For $\mu = \mu^*$ and $\Delta = \Delta^*$ with A_0^*, A_1^* and B_4^* determined by (4.5), we get simultaneously the following facts.

(1) For $j = 1, 2, 3$, $I^{(j)}(h)$ has 1 zero.

(2) For $j = 1, 2, 3$, the stability of limit cycles (generated by homoclinic bifurcation) and the stability of centers C_j yields 1 limit cycle in the region enclosed by Γ_j in view of Poincaré–Bendixson theorem.

(3) The broken position of the separatrixes of S_4 and the stability of the center C_4 yields one limit cycle.

(4) The broken position of the separatrixes of the polycycle $S^{(6)}$ and the stability of the center O yields one limit cycle.

From (1)–(4), we obtain $9 + 9 + 3 + 1 = 22$ limit cycles altogether. Finally, let us consider the limit cycles bifurcating from the unbounded period annulus. Let V_i be the singular points obtained by rotating the saddle S_1 around the origin counter-clockwise with $\frac{2\pi}{6}i$ $(i = 1, 2, 3, 4, 5)$ and $V_3 = S_2$. Then, we obtain a heteroclinic polycycle through the saddles S_1, V_1, $\cdots$, V_5, denoted as $V^{(6)}$, see Fig. 3(a). The unperturbed system has an unbounded period annulus with the boundary $V^{(6)}$ for $h \in (-\infty, H(S_1))$ see Fig. 2(a). For A_0^*, A_1^*, A_2^*, B_4^* and B_5^* defined above and $h \in (-\infty, H(S_1))$, we have

$$\begin{aligned} I(h) &= \int\limits_{\Gamma_h} \mathcal{Q}_5(x,y)\,dx - \mathcal{P}_5(x,y)\,dy \\ &= -\iint\limits_{D} \left(\frac{\partial \mathcal{P}_5}{\partial x} + \frac{\partial \mathcal{Q}_5}{\partial y}\right) dx\,dy \\ &= -\iint\limits_{D} \left\{2A_0^* + 4\left(x^2+y^2\right)A_1^* + 6\left(x^2+y^2\right)^2 A_2^*\right. \\ &\qquad \left. + 2y\left(3x^2-y^2\right)B_4^* - 8y\left(3x^2-y^2\right)B_5^*\right\} dx\,dy, \end{aligned} \tag{4.8}$$

where D is the region enclosed by Γ_h. It follows from (4.8) that $I(h)$ has the opposite sign with A_2^* for $|h|$ sufficiently large. Hence, there exists an h^* with $h^* < H(S_1)$ and $|h^*| \gg 1$ such that $I(h^*) < 0$. In the meanwhile, by (1.5), (1.6) and (3.7),

$$\begin{aligned} I\big(H(S_1)\big) &= 3\Bigg(\int\limits_{\widehat{V_1V_2}} + \int\limits_{\widehat{V_2V_3}}\Bigg) \mathcal{Q}_5(x,y)\,dx - \mathcal{P}_5(x,y)\,dy \\ &= -3\int\limits_{\widehat{V_1V_2}\cup\overline{V_2V_1}} \omega(x,y)\,dx\,dy - 3\int\limits_{\widehat{V_2V_3}\cup\overline{V_3V_2}} \omega(x,y)\,dx\,dy \\ &\quad + 3\Bigg(\int\limits_{\overline{V_2V_1}} + \int\limits_{\overline{V_3V_2}}\Bigg) \mathcal{Q}_5(x,y)\,dx - \mathcal{P}_5(x,y)\,dy. \end{aligned}$$

It is easy to see that both $\int_{\widehat{V_1V_2}\cup\overline{V_2V_1}} \omega(x,y)\,dx\,dy$ and $\int_{\widehat{V_2V_3}\cup\overline{V_3V_2}} \omega(x,y)\,dx\,dy$ are linear functions of A_0^*, A_1^*, A_2^*, B_4^*, B_5^* and μ_i^* $(i = 1, 2, 3)$, and

$$\begin{aligned} &\Bigg(\int\limits_{\overline{V_2V_1}} + \int\limits_{\overline{V_3V_2}}\Bigg) \mathcal{Q}_5(x,y)\,dx - \mathcal{P}_5(x,y)\,dy \\ &= \int\limits_{-1/2}^{1/2} \mathcal{Q}_5\left(x, \frac{\sqrt{3}}{2}\right) dx + \int\limits_{-1}^{-1/2} \left(\mathcal{Q}_5\big(x, \sqrt{3}(x+1)\big) - \sqrt{3}\mathcal{P}_5\big(x, \sqrt{3}(x+1)\big)\right) dx \\ &= \sqrt{3}A_0^* + \frac{5}{6}\sqrt{3}A_1^* + \frac{7}{10}\sqrt{3}A_2^* + \frac{77}{64}B_6. \end{aligned}$$

Hence, there exists a $B_6^* > 0$ such that for $B_6 = B_6^*$, we have $I(H(S_1)) > 0$. Then, it follows from Corollary 2.1(ii) that there exists at least one "big" limit cycle around all 25 singular points for system (1.3). To sum up, we have obtained 23 limit cycles for system (1.3). The distribution of the 23 limit cycles is shown in Fig. 1(a).

4.2. The 20 limit cycles for Z_3-equivariant quintic systems

Set

$$\begin{cases} \mu_1 = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \\ \mu_2 = 2(A_0 + 2A_1 + 3A_2), \\ \mu_3 = \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 + \eta_4 B_4 - 4\eta_4 B_5. \end{cases}$$

Then we get

$$\begin{cases} A_0 \approx 1.518694660A_2 + 10.11889703\mu_1 - 1.238907158\mu_2; \\ A_1 \approx -2.259347330A_2 - 5.059448517\mu_1 + 1.119453579\mu_2; \\ B_4 \approx 2.568326255A_2 + 4.000000000B_5 - 3.553433097\mu_1 \\ \qquad + 1.166383357\mu_2 - 0.1918345717\mu_3, \end{cases} \tag{4.9}$$

and

$$\begin{cases} I^{(1)}(h) = \mu_1 + \mu_2/\lambda_1(h_1 - h)\ln(h_1 - h) + c_{12}^*(h_1 - h) + \text{h.o.t.}, \\ I^{(2)}(h) = \mu_1 + \mu_2/\lambda_1(h_1 - h)\ln(h_1 - h) + c_{22}^*(h_1 - h) + \text{h.o.t.}, \\ I^{(3)}(h) = \mu_3 + c_{31}^*(h - h_3)\ln(h - h_3) + \text{h.o.t.}, \\ I^{(4)}(h) = c_{40}^* + \text{h.o.t.}, \\ I^{(6)}(h) = c_{60}^* + \text{h.o.t.}, \end{cases}$$

where

$$\begin{cases} c_{12}^* \approx -4.33254141A_2 + 0.7795941759B_5 - 81.59177342\mu_1 + 18.09004120\mu_2 - 0.01869413936\mu_3, \\ c_{22}^* \approx -4.83310445A_2 - 0.7795941759B_5 - 80.89921450\mu_1 + 17.86271478\mu_2 + 0.01869413936\mu_3, \\ c_{31}^* \approx -\dfrac{1}{\lambda_3}[-0.950796228A_2 + 13.89059250\mu_1 - 1.268456544\mu_2 + 0.09841452355\mu_3], \\ c_{40}^* \approx 26.77646924A_2 - 37.04684784\mu_1 + 12.16030402\mu_2 - 0.9999999998\mu_3, \\ c_{60}^* \approx 0.1499274240A_2 + 2.709674168B_5 - 8.854317565\mu_1 + 1.153259946\mu_2 - 0.06497614795\mu_3 \end{cases}$$

and

$$\begin{cases} \omega(C_1) = \omega(C_2) \approx -0.131530670A_2 + 8.99457515\mu_1 + 0.009860304\mu_2, \\ \omega(C_3) \approx 4.59098313A_2 - 3.70382038\mu_1 - 1.32285945\mu_2 + 2.090352704\mu_3, \\ \omega(C_4) \approx 60.56324797A_2 - 81.14479894\mu_1 + 24.09646495\mu_2 - 2.090352704\mu_3, \\ \omega(O) \approx 2(1.518694660A_2 + 10.11889703\mu_1 - 1.238907158\mu_2). \end{cases}$$

Denote $\mu = (\mu_1, \mu_2, \mu_3)$. Then, we get

$$\begin{cases} \lim\limits_{\mu\to 0} c_{22}^* \approx -4.83310445A_2 - 0.7795941759B_5, \\ \lim\limits_{\mu\to 0} \omega(C_1) \approx -0.131530670A_2, \\ \lim\limits_{\mu\to 0} c_{60}^* \approx 0.0249879037A_2 + 0.4516123614B_5, \\ \lim\limits_{\mu\to 0} \omega(O) \approx -2.259347330A_2, \\ \lim\limits_{\mu\to 0} c_{31}^* \approx \dfrac{0.950796228}{\lambda_3}A_2, \\ \lim\limits_{\mu\to 0} \omega(C_3) \approx 4.59098313A_2, \\ \lim\limits_{\mu\to 0} c_{40}^* \approx 26.77646924A_2, \\ \lim\limits_{\mu\to 0} \omega(C_4) \approx 60.56324797A_2. \end{cases}$$

Choose A_2^*, B_5^* such that $A_2^* > 0$ and

$$-4.83310445A_2^* - 0.7795941759B_5^* > 0, \qquad 0.0249879037A_2^* + 0.4516123614B_5^* < 0.$$

It is easy to know that

$$\begin{aligned} -4.83310445A_2^* - 0.7795941759B_5^* > 0 \quad &\Longleftrightarrow \quad A_2^* < -18.07323923B_5^*, \\ 0.0249879037A_2^* + 0.4516123614B_5^* < 0 \quad &\Longleftrightarrow \quad A_2^* < -0.1613029853B_5^*. \end{aligned}$$

Then, there is a neighborhood U of $\mu = 0$ such that for $\mu \in U$ we have $c_{60}^* < 0$, $\omega(O) < 0$; and

$$c_{22}^* > 0, \qquad \omega(C_2) < 0; \qquad c_{31}^* > 0, \qquad \omega(C_3) > 0; \qquad c_{40}^* > 0, \qquad \omega(C_4) > 0.$$

Set some $\mu^* \in U$ such that $\mu_i^* < 0$ $(i = 1, 2, 3)$ with $\mu_i^* > 0$ and $|\mu_1| \ll |\mu_2|$. For $\mu = \mu^*$ and $\Delta = \Delta^*$ with A_0^*, A_1^* and B_4^* determined by (4.9) we get simultaneously the following facts.

(1) $I^{(1)}(h)$ has 1 zero, $I^{(2)}(h)$ has 2 zeros and $I^{(3)}(h)$ has 1 zero.

(2) The stability of limit cycles (generated by homoclinic bifurcation) and the stability of the center C_3 yields 1 limit cycle in the region enclosed by Γ_3.

(3) The broken position of the separatrixes of S_4 and the stability of the center C_4 yields one limit cycle.

(4) The broken position of the separatrixes of the polycycle $S^{(6)}$ and the stability of the center O yields one limit cycle.

(5) By the similar arguments as in Section 4.1, we can obtain a "big" limit cycle around all 25 singular points.

To sum up, we have obtained 20 limit cycles for system (1.3). The distribution of the 20 limit cycles is shown in Fig. 1(b).

4.3. The 24 limit cycles for Z_6-equivariant quintic systems

For $\mathcal{P}_5$, $\mathcal{Q}_5$ given in (1.5) and (1.6), let $A_i^* = B_i^* = 0$ $(i = 3, 4, 5)$. Then system (1.3) with $0 < \epsilon \ll 1$ is Z_6-equivariant. Let

$$\mu_1 = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \qquad \mu_2 = \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2. \tag{4.10}$$

Then we get

$$\begin{cases} A_0 \approx 4.246715224A_2 + 6.344517305\mu_1 - 0.2037625303\mu_2, \\ A_1 \approx -4.724336223A_2 - 1.648988847\mu_1 + 0.1841160513\mu_2, \\ \omega(C_1) \approx -0.15324264A_2 + 9.024614950\mu_1 + 0.0016217201\mu_2, \\ \omega(C_3) \approx 7.50386299A_2 - 7.73396403\mu_1 + 1.872782579\mu_2, \end{cases} \tag{4.11}$$

and

$$\begin{cases} I^{(1)}(h) = \mu_1 + \dfrac{c_{11}^*}{\lambda_1}(h_1 - h)\ln(h_1 - h) + \text{h.o.t.}, \\ I^{(3)}(h) = \mu_2 + c_{31}^*(h - h_3)\ln(h - h_3) + \text{h.o.t.}, \\ c_{11}^* \approx -2.010102733A_2 + 2.781097447\mu_1 + 0.1501394912\mu_2, \\ c_{31}^* \approx -\dfrac{1}{\lambda_3}(1.842290815A_2 + 10.02618939\mu_1 - 0.1102079814\mu_2). \end{cases} \tag{4.12}$$

Denote $\mu = (\mu_1, \mu_2)$. Since

$$\lim_{\mu\to 0} c_{11}^* \approx -2.010102733A_2,$$
$$\lim_{\mu\to 0} c_{31}^* \approx -\frac{1.842290815}{\lambda_3}A_2,$$
$$\lim_{\mu\to 0} \omega(C_1) \approx -0.15324264A_2,$$
$$\lim_{\mu\to 0} \omega(C_3) \approx 7.50386299A_2,$$

for $A_2^* > 0$, there exists a neighborhood U of $\mu = 0$ such that for $\mu \in U$ we have

$$c_{11}^* < 0, \omega(C_1) < 0; \qquad c_{31}^* < 0, \omega(C_3) > 0.$$

Set some $\mu^* \in U$ such that $\mu_i^* < 0$ $(i = 1, 2)$. For $\mu = \mu^*$ and $\Delta = \Delta^*$ with $A_4^* = A_5^* = B_4^* = B_5^* = 0$ and $A_{0,1}^*$ determined by (4.11), we get simultaneously the following facts.

(1) For $j = 1, 3$, $I^{(j)}(h)$ has 1 zero, which yields $2 \times 6 = 12$ limit cycles (generated by homoclinic bifurcation) for system (1.3) by Corollary 2.1 and rotating the vector fields around the origin by $2\pi/6$ since system (1.3) is Z_6-equivariant.

(2) For $j = 1, 3$, the stability of limit cycles (generated by homoclinic bifurcation) and the stability of the centers C_j yields 1 limit cycle in the region enclosed by Γ_j in view of Poincaré–Bendixson theorem. So, we obtain another $2 \times 6 = 12$ limit cycles for system (1.3). The distribution of the 24 limit cycles is shown in Fig. 1(c).

4.4. The 19 limit cycles for Z_6-equivariant quintic systems

Let

$$\mu_1 = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \qquad \mu_2 = \frac{2}{\lambda_1}(A_0 + 2A_1 + 3A_2). \tag{4.13}$$

Then we get

$$\begin{cases} A_0 \approx 1.518694660A_2 + 10.11889704\mu_1 - 1.357154793\mu_2, \\ A_1 \approx -2.259347330A_2 - 5.059448518\mu_1 + 1.226299954\mu_2, \end{cases} \tag{4.14}$$

and

$$\begin{cases} I^{(1)}(h) = \mu_1 + \mu_2(h_1 - h)\ln(h_1 - h) + c_{12}^*(h_1 - h) + \text{h.o.t.}, \\ I^{(3)}(h) = c_{30}^* + \text{h.o.t.}, \\ c_{12}^* \approx -4.58282293A_2 - 81.24549397\mu_1 + 19.69213545\mu_2, \\ c_{30}^* \approx 13.38823462A_2 - 18.52342392\mu_1 + 6.660472813\mu_2, \\ \omega(C_3) \approx 32.57711555A_2 - 42.42430966\mu_1 + 12.47361745\mu_2. \end{cases} \tag{4.15}$$

Denote $\mu = (\mu_1, \mu_2)$. Since

$$\lim_{\mu\to 0} c_{12}^* \approx -4.58282293A_2,$$
$$\lim_{\mu\to 0} c_{30}^* \approx 13.38823462A_2,$$
$$\lim_{\mu\to 0} \omega(C_3) \approx 32.57711555A_2,$$

for $A_2^* > 0$, there exists a neighborhood U of $\mu = 0$ such that for $\mu \in U$ we have

$$c_{12}^* < 0, \qquad c_{30}^* > 0, \qquad \omega(C_3) > 0.$$

Choose some $\mu^* \in U$ such that $\mu_2^* < 0$ and $\mu_1^* < 0$ with $|\mu_1| \ll |\mu_2|$. Then $I^{(1)}(h)$ with μ^*, A_2^*, $A_4^* = A_5^* = B_4^* = B_5^* = 0$ and $A_{0,1}^*$ determined by (4.14) has 2 zeros. Simultaneously, the broken position of the separatrixes of the saddle S_3 and the stability of center C_3 leads another 1 limit cycle in the region enclosed by Γ_3 according to Poincaré–Bendixson theorem. So, system (1.3) has $(2+1)\times 6 = 18$ limit cycles by Corollary 2.1 and rotating the vector fields around the origin by $2\pi/6$, see Fig. 1(d). This completes the proof of Theorem 1.1. □

Remark 4.2. In Section 4.1, we obtain a "big" limit cycle around all 25 singular points of system (1.3) and a limit cycle around only the origin. The configuration of the 23 limit cycles is new and different from the result obtained in [2].

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