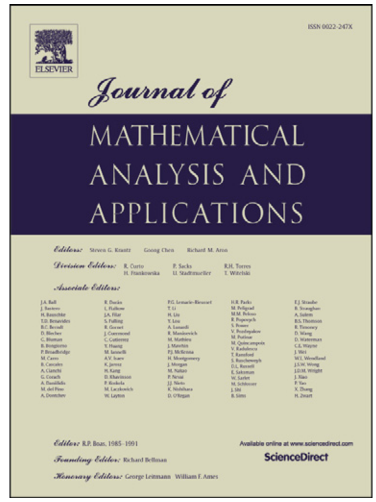

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PII: S0022-247X(14)00916-0
DOI: 10.1016/j.jmaa.2014.10.003
Reference: YJMAA 18899

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 9 June 2014

Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform

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Abstract

The quaternion Fourier transform - a generalized form of the classical Fourier transform - has been shown to be a powerful analyzing tool in image and signal processing. This paper investigates the Pitt's inequality and uncertainty principle associated with the two-sided quaternion Fourier transform. It is shown that by applying the symmetric form $f = f_1 + \mathbf{i}\, f_2 + f_3\, \mathbf{j} + \mathbf{i}\, f_4\, \mathbf{j}$ of quaternion from Hitzer and the novel module or L^p-norm of the quaternion Fourier transform $\widehat{f}$, then any nonzero quaternion signal and its quaternion Fourier transform cannot both be highly concentrated. Two part results are provided, one part is the Heisenberg-Weyl's uncertainty principle associated with the quaternion Fourier transform. It is formulated by using logarithmic estimates which may obtained from a sharp of Pitt's inequality; the other part is the uncertainty principle of Donoho and Stark associated with the quaternion Fourier transform.

Keywords: Quaternion Fourier transform, Pitt's inequality, logarithmic uncertainty estimate, uncertainty principle.

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2000 Mathematics Subject Classification. 42A05, 42B10, 58G35.

1. Introduction

The uncertainty principle of harmonic analysis states that a non-trivial function and its Fourier transform (FT) cannot both be sharply localized. The uncertainty principle plays an important role in signal processing [6, 7, 8, 10, 11, 17, 20, 21, 23, 27], and physics [1, 5, 14, 15, 16, 18, 19, 24, 28, 29]. In quantum mechanics an uncertainty principle asserts that one cannot make certain of the position and velocity of an electron (or any particle) at the same time. That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. In quaternionic analysis some papers combine the uncertainty relations and the quaternionic Fourier transform (QFT) [2, 12, 22, 37, 38]. Recently, the Heisenberg's uncertainty relations were extended to the quaternion linear canonical transform [37] - a generalized form of the QFT.

The QFT plays a vital role in the representation of (hypercomplex) signals. It transforms a real (or quaternionic) $\mathbb{R}^d$ signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex FT. Due to the noncommutative property of multiplication of quaternion, there are three different types of QFT: left-sided, right-sided and two-sided QFT. Hitzer [35] introduced these different types of QFT and investigated their important properties. In [25] the authors used the QFT to proceed color image analysis. The paper [4] implemented the QFT to design a color image digital watermarking scheme. The authors in [3] applied the QFT to image pre-processing and neural computing techniques for speech recognition.

There are different types of uncertainty relations, including entropy-based uncertainty relations, Heisenberg's uncertainty for time spread and frequency spread, the uncertainty relations for time-frequency distribution and so on. In this paper, we will mainly focus on the new inequalities for the two-sided QFT, including the Pitt's inequality [30] in QFT domains, the logarithmic uncertainty relations in QFT domains, the Heisenberg-Weyl's uncertainty relations in QFT domains, the uncertainty relations of Donoho and Stark [32] in QFT domains etc. To the best of our knowledge, the study of the Pitt's inequality and its Heisenberg-Weyl's uncertainty relations associate with two-sided QFT has not been carried out yet. The results in this paper are new in the literature. The main motivation of the present study is to develop further technical applications in the theory of partial differential equations [9]. We would like to apply these ideas to the existence and smoothness of solutions of PDE, construction of explicit fundamental solutions, and eigenvalues of Schrodinger operators in the Hamiltonian quaternionic algebra. Further investigations and extensions of this topic will be reported in a forthcoming paper.

The paper is organized as follows. Section 2 gives a brief introduction to some general definitions and basic properties of quaternion analysis. The QFT of $\mathbb{R}^d$ quaternionic signal in introduced and studied in Section 3. Some important properties such as Plancherel's and inversion theorems are obtained. The classical Pitt's inequality and logarithmic uncertainty principle are generalized in the quaternion Fourier domains in Section 4. In Section 5 the uncertainty principle of Donoho and Stark associated with two-sided quaternion Fourier transform is provided. Finally conclusions are summarized in Section 6.

2. Preliminaries

For convenience of further discussions, we briefly review some notions and terminology on quaternion. We write

$$x \cdot u = \sum_{i=1}^{d} x_i u_i$$

for the inner product on $\mathbb{R}^d$, and abbreviate $x^2 = x \cdot x$. The Euclidean norm is defined by $|x| := \sqrt{x \cdot x}$.

Let $\mathbb{H}$ denote the quaternion algebra over $\mathbb{R}$, which is an associative noncommutative four-dimensional algebra

$$\mathbb{H} := \{q = q_1 + \mathbf{i}\, q_2 + \mathbf{j}\, q_3 + \mathbf{k}\, q_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \tag{1}$$

where the elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ obey Hamilton's multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}, \quad \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}, \quad \mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}. \tag{2}$$

The conjugate of a quaternion q is defined by

$$\overline{q} := q_1 - \mathbf{i}\, q_2 - \mathbf{j}\, q_3 - \mathbf{k}\, q_4, \tag{3}$$

and the module $|q|$ of a quaternion q is given by

$$|q| := \sqrt{q\,\overline{q}} = (q_1^2 + q_2^2 + q_3^2 + q_4^2)^{1/2}. \tag{4}$$

In particular, when $q = q_1$ is a real number, the module $|q|$ reduces to the ordinary Euclidean module, i.e. $|q| = \sqrt{q_1 \cdot q_1}$. Suppose that d is a positive integer. A function $f : \mathbb{R}^{2d} \to \mathbb{H}$ can be expressed as

$$f(x,y) := f_1(x,y) + \mathbf{i}\, f_2(x,y) + \mathbf{j}\, f_3(x,y) + \mathbf{k}\, f_4(x,y), \tag{5}$$

where $\mathbf{x} = (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. We may represent $f : \mathbb{R}^{2d} \to \mathbb{H}$ into a symmetric form[35] as

$$f(x,y) = f_1(x,y) + \mathbf{i}\, f_2(x,y) + f_3(x,y)\,\mathbf{j} + \mathbf{i}\, f_4(x,y)\,\mathbf{j}. \tag{6}$$

If $1 \le p < \infty$, the L^p-norm of f is defined by

$$\|f\|_p := \left(\int_{\mathbb{R}^{2d}} |f(x,y)|^p \, dxdy \right)^{1/p}, \tag{7}$$

thus $L^p(\mathbb{R}^{2d}, \mathbb{H})$ is the Banach space of all measurable functions f that have finite L^p-norm. For $p = \infty$, $L^\infty(\mathbb{R}^{2d}, \mathbb{H})$ is the collection of essentially bounded measurable functions with the norm $\|f\|_\infty = ess\ \sup_{x \in \mathbb{R}^{2d}} |f(\mathbf{x})|$. If $f \in L^\infty(\mathbb{R}^{2d}, \mathbb{H})$ is continuous, then $\|f\|_\infty = \sup_{x \in \mathbb{R}^{2d}} |f(\mathbf{x})|$. For $p = 2$, we can define the inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^{2d}} f(x,y)\overline{g(x,y)}\, dxdy \tag{8}$$

here $g(x,y) = g_1(x,y) + \mathbf{i}\, g_2(x,y) + \mathbf{j}\, g_3(x,y) + \mathbf{k}\, g_4(x,y)$, turns $L^2(\mathbb{R}^{2d}, \mathbb{H})$ a Hilbert space.

The two-sided quaternion Fourier transform $\mathcal{F}_Q(f) : \mathbb{R}^{2d} \to \mathbb{H}$ of a quaternion function $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$ is defined as[35]

$$\mathcal{F}_Q(f)(u,v) = \widehat{f}(u,v) := \int_{\mathbb{R}^{2d}} e^{-2\pi \mathbf{i}\, x\cdot u} f(x,y) e^{-2\pi \mathbf{j}\, y \cdot v}\, dxdy. \tag{9}$$

From (6) and the above definition, we have the following lemma:

Lemma 2.1. *If $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$, then $\mathcal{F}_Q(f)$ has a symmetric representation:*

$$\widehat{f}(u,v) = \widehat{f}_1(u,v) + \mathbf{i}\, \widehat{f}_2(u,v) + \widehat{f}_3(u,v)\, \mathbf{j} + \mathbf{i} \widehat{f}_4(u,v)\, \mathbf{j}, \tag{10}$$

where

$$\widehat{f}_l(u,v) = \mathcal{F}_Q(f_l)(u,v),\ l = 1,2,3,4.$$

Now we define a new module of $\widehat{f}$ as follows:

$$|\widehat{f}(u,v)|_Q := (|\widehat{f}_1(u,v)|^2 + |\widehat{f}_2(u,v)|^2 + |\widehat{f}_3(u,v)|^2 + |\widehat{f}_4(u,v)|^2)^{1/2}. \tag{11}$$

Furthermore, we define a new L^p-norm of $\widehat{f}$ as follows:

$$\|\widehat{f}\|_{Q,\,p} := \left(\int_{\mathbb{R}^{2d}} |\widehat{f}(u,v)|_Q^p \, dudv \right)^{1/p}. \tag{12}$$

It is worthwhile to observe that $|\widehat{f}|_Q$ is not equivalent to $|\widehat{f}|$, and $\|\widehat{f}\|_{Q,\,p}$ is not equivalent to $\|\widehat{f}\|_p$. The concept $\|\widehat{f}\|_{Q,\,p}$ do depend on the expression of the quaternion function f. In fact, it requires that every component function f_i of the quaternion function f to be real function, $i = 1,2,3,4$. However, it is useful in some environment, with the L^p-norm of $\widehat{f}$, $\widehat{f}$ behaves like a quaternion function f in the L^p-space.

3. Properties of quaternion Fourier transform

Firstly, let $S(\mathbb{R}^{2d}, \mathbb{H})$ denote the Schwartz space from $\mathbb{R}^{2d}$ into $\mathbb{H}$, we list the elementary properties of two-sided quaternion Fourier transform as follows:

Proposition 3.1. *If $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$, then* *(i) The map $f \longrightarrow \widehat{f}$ is real linear. That is, for $a, b \in \mathbb{R}$, we have $af + bg \longrightarrow a\widehat{f} + b\widehat{g}$.*

(ii) $f(x+h, y+l) \longrightarrow e^{2\pi \mathbf{i}\, h\cdot u}\widehat{f}(u,v)e^{2\pi \mathbf{j}\, l\cdot v}$, *where* $(h,l) \in \mathbb{R}^d \times \mathbb{R}^d$.

(iii) $e^{2\pi \mathbf{i}\, x\cdot h}f(x,y)e^{2\pi \mathbf{j}\, y\cdot l} \longrightarrow \widehat{f}(u-h, v-l)$, *where* $(h,l) \in \mathbb{R}^d \times \mathbb{R}^d$.

(iv) $f(x/\lambda, y/\lambda) \longrightarrow \lambda^{2d}\widehat{f}(\lambda u, \lambda v)$, *where*$\lambda > 0$.

(v) $\dfrac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} f(x,y) \longrightarrow (2\pi \mathbf{i}\, u)^\alpha \widehat{f}(u,v)(2\pi \mathbf{j}\, v)^\beta$, *whenever* $f \in S(\mathbb{R}^{2d}, \mathbb{H})$.

(vi) $(-2\pi \mathbf{i}\, x)^\alpha \widehat{f}(x,y)(-2\pi \mathbf{j}\, y)^\beta \longrightarrow \dfrac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}\widehat{f}(u,v)$, *whenever* $f \in S(\mathbb{R}^{2d}, \mathbb{H})$.

(vii) $f(R_1 x, R_2 y) \longrightarrow \widehat{f}(R_1 u, R_2 v)$, *where* R_1, R_2 *is a rotation.*

(viii) $\widehat{f} \in L^\infty(\mathbb{R}^{2d}, \mathbb{H})$, *and* $\|\widehat{f}\|_{Q,\infty} \leq \|f\|_1$.

(ix) $\widehat{f}$ *is continuous (and hence measurable) function.*

Here the arrow, which we have taken, indicates the quaternion Fourier transform.

Proof. The first four properties are obvious by the definition of the quaternion Fourier transform. The properties (v) and (vi) can be verified by applying integration by parts. Property (vii) can verified by a simple change variables $x_0 = R_1 \cdot x, y_0 = R_2 \cdot y$ in the integral. Then, recall that $|det(R_i)| = 1$ with $i = 1, 2$, and $R_1^{-1}x_0 \cdot u = x_0 \cdot R_1 u, R_2^{-1}y_0 \cdot v = y_0 \cdot R_2 v$, because R is a rotation.

Properties (iv) and (v) in the proposition 3.1 show that, up to factors, the quaternion Fourier transform interchanges differentiation and multiplication by monomials. Indeed, all properties but (iv) and (v) still hold for the $L^1(\mathbb{R}^{2d}, \mathbb{H})$. However, this generalization is useful in many circumstances.

Properties (viii) is just a consequence of absolute inequality. Properties (ix) is easy to prove. By the definition and Properties (iii),

$$\widehat{f}(u,v) - \widehat{f}(u-h, v-l) = \int_{\mathbb{R}^{2d}} e^{-2\pi \mathbf{i}\, x\cdot u}(f(x,y) - e^{2\pi \mathbf{i}\, x\cdot h}f(x,y)e^{2\pi \mathbf{j}\, y\cdot l})e^{-2\pi \mathbf{j}\, y\cdot v}\, dxdy.$$

By Properties (viii) we deduce that

$$\Big|\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\,x\cdot u}(f(x,y)-e^{2\pi\mathbf{i}\,x\cdot h}f(x,y)e^{2\pi\mathbf{j}\,y\cdot l})e^{-2\pi\mathbf{j}\,y\cdot v}\,dxdy\Big| \leq 2\|f\|_1.$$

Since f is integrable, by Lebesgue dominated theorem, we can see that $|\widehat{f}(u,v)-\widehat{f}(u-h,v-l)|\to 0$ when $(h,l)\to 0$. This proves Properties (ix). $\square$

Theorem 3.1 (Riemann-Lebesgue lemma). *If $f\in L^1(\mathbb{R}^{2d},\mathbb{H})$, then $\widehat{f}(\mathbf{u})\to 0$ as $\mathbf{u}=(u,v)\to\infty$.*

Proof. By Lemma 2.1, we can restrict ourself to the case where f is a function in $L^1(\mathbb{R}^{2d},\mathbb{R})$. By the Proposition 1(ii), we can rewrite

$$\begin{aligned}\widehat{f}(\mathbf{u}) &= \int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\,x\cdot u}f(x,y)e^{-2\pi\mathbf{j}\,y\cdot v}\,dxdy\\ &= \frac{1}{2}\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\,x\cdot u}\left[f(x,y)-f(x-h,y-l)\right]e^{-2\pi\mathbf{j}\,y\cdot v}\,dxdy,\end{aligned}$$

where $h=(1/u_1,1/u_2,\cdots,1/2u_n), l=(1/v_1,1/v_2,\cdots,1/v_n)$. Thus,

$$|\widehat{f}(\mathbf{u})|\leq\frac{1}{2}\int_{\mathbb{R}^{2d}}|f(x,y)-f(x-h,y-l)|\,dxdy.$$

Since continuous functions of compact support are dense in the integrable functions space, let $C_c(\mathbb{R}^{2d},\mathbb{R})$ be the space of continuous functions of compact support from $\mathbb{R}^{2d}$ into $\mathbb{R}$, then for any $\varepsilon>0$, we can choose a function $g\in C_c(\mathbb{R}^{2d},\mathbb{R})$ so that $\|f-g\|_1<\varepsilon$. Now

$$f-f_{\mathbf{h}}=(f-g)+(g-g_{\mathbf{h}})+(g_{\mathbf{h}}-f_{\mathbf{h}}),$$

where $f_{\mathbf{h}}(\mathbf{x})=f(\mathbf{x}-\mathbf{h})$, and $\mathbf{h}=(h,l)$.

However, when $\mathbf{u}$ is sufficiently large, while $\mathbf{h}$ becomes very small, $\|f_{\mathbf{h}}-g_{\mathbf{h}}\|_1=\|f-g\|_1<\epsilon$, since g is continuous and has compact support. Clearly we have

$$\|g-g_h\|_1=\int_{\mathbb{R}^{2d}}|g(\mathbf{x})-g(\mathbf{x}-\mathbf{h})|\,d\mathbf{x}\to 0$$

as $\mathbf{h}\to 0$. This proves the theorem. $\square$

Before giving the Plancherel's Theorem, we give the quaternion Fourier transform of the Gaussian function $e^{-\pi\lambda(|x|^2+|u|^2)}$, which is crucial in the proof of Plancherel's Theorem.

Example 3.1 (Quaternion Fourier transform of a Gaussian function)**.** *For $\lambda > 0$, the Gaussian function on $\mathbb{R}^{2d}$ is given by*

$$g_\lambda := \exp\left[-\pi\lambda(|x|^2 + |y|^2)\right] \tag{13}$$

for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Then

$$\widehat{g}_\lambda := \lambda^{-d} \exp\left[-\pi\lambda^{-1}(|u|^2 + |v|^2)\right]. \tag{14}$$

Proof. By Proposition 3.1(iv), it suffices to consider $\lambda = 1$. We deduce

$$\begin{aligned}
\widehat{g}(\mathbf{u}) &= \int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\, x\cdot u} e^{-\pi|x|^2} e^{-\pi|y|^2} e^{-2\pi\mathbf{j}\, y\cdot v}\, dxdy \\
&= \prod_{m=1}^{d} \int_{-\infty}^{\infty} e^{-2\pi\mathbf{i}\, x_m\cdot u_m} e^{-\pi x_m^2}\, dx \prod_{n=1}^{d} \int_{-\infty}^{\infty} e^{-2\pi\mathbf{i}\, y_n\cdot v_n} e^{-\pi y_n^2}\, dy \\
&= \prod_{m=1}^{d} e^{-\pi u_m^2} \prod_{n=1}^{d} e^{-\pi v_n^2} \\
&= e^{-\pi|u|^2} e^{-\pi|v|^2}.
\end{aligned}$$

□

Now we give the Plancherel's Theorem.

Theorem 3.2 (Plancherel)**.** *If $f \in L^1(\mathbb{R}^{2d}, \mathbb{R}) \bigcap L^2(\mathbb{R}^{2d}, \mathbb{R})$, then $\|\widehat{f}\|_2 = \|f\|_2$. Furthermore, if $f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$, then $\|\widehat{f}\|_{Q,2} = \|f\|_2$.*

Proof. We first assume $f \in L^1(\mathbb{R}^{2d}, \mathbb{R}) \bigcap L^2(\mathbb{R}^{2d}, \mathbb{R})$, since the function $\widehat{f}$ is bounded,

$$\int_{\mathbb{R}^{2d}} |\widehat{f}(u, v)|^2 \exp\left[-\varepsilon\pi(|u|^2 + |v|^2)\right] dudv \tag{15}$$

is well defined.

Notice that $f(x, y)f(z, w)\exp[-\varepsilon\pi(|u|^2 + |v|^2)]$ is in $L^1(\mathbb{R}^{3\times 2d})$, we could rewrite (15) as

$$\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\, x\cdot u} f(x, y) e^{-2\pi\mathbf{j}\, y\cdot v}\, dxdy\right)\left(\int_{\mathbb{R}^{2d}} e^{2\pi\mathbf{j}\, w\cdot v} f(z, w) e^{2\pi\mathbf{i}\, z\cdot u}\, dzdw\right)$$
$$\exp\left[-\varepsilon\pi(|u|^2 + |v|^2)\right] dudv.$$

Then, by applying Fubini's theorem and Example 3.1, we will obtain

$$\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \varepsilon^{-d} \exp[-\pi\varepsilon^{-1}(|x-z|^2+|y-w|^2)] f(x,y) f(z,w)\, dxdydzdw. \tag{16}$$

Since the family $\widehat{g}_\varepsilon$ is an approximation to the identity, let $\varepsilon \to 0$, then (16) reduces to $\int_{\mathbb{R}^{2d}} |f(x,y)|^2\, dxdy$. This show that (15) is uniformly in ε.

On the other hand, by the monotone convergence theorem, (15) is equal to $\int_{\mathbb{R}^{2d}} |f(u,v)|^2\, dudv$. Therefore we have showed that $\widehat{f} \in L^2(\mathbb{R}^{2d},\mathbb{R})$ and $\|\widehat{f}\|_2 = \|f\|_2$.

Now we have to prove that the quaternion Fourier transform is a bounded linear operator defined on the dense subset $L^1 \bigcap L^2$ of L^2. In fact, it is isometry. Furthermore, there exists a unique bounded extension, $\mathcal{F}_Q$, of this operator to all of L^2.

In general, if $f \in L^2(\mathbb{R}^{2d},\mathbb{R})$, there exists sequences f^l in $L^1(\mathbb{R}^{2d},\mathbb{R}) \bigcap L^2(\mathbb{R}^{2d},\mathbb{R})$ of $L^2(\mathbb{R}^{2d},\mathbb{R})$ converging to f in the L^2-norm. We may choose the sequence f^l, where $f^l(\mathbf{x})$ equals $f(\mathbf{x})$ when $|\mathbf{x}| \leq l$ and is zero elsewhere. Hence, $\widehat{f^l}$ is a Cauchy sequence in $L^2(\mathbb{R}^{2d})$ that converges to some function in $L^2(\mathbb{R}^{2d},\mathbb{R})$, which we denote it by $\mathcal{F}_Q(f)$, or $\widehat{f}$. Moreover,

$$\|\widehat{f}\|_2 = \lim_{l\to\infty} \|\widehat{f^l}\|_2 = \lim_{l\to\infty} \|f^l\|_2 = \|f\|_2.$$

Finally, let $f \in L^2(\mathbb{R}^{2d},\mathbb{H})$, then we want to prove $\|\widehat{f}\|_{Q,2} = \|f\|_2$. In fact, since previous argument for $f \in L^2(\mathbb{R}^{2d},\mathbb{H})$ gives the desired result for f_i with $i = 1,2,3,4$. The general case, $\|\widehat{f}\|_{Q,2} = \|f\|_2$ follows by (7) and (12). $\square$

Next, let's establish the inversion formula of QFT.

Theorem 3.3 (Inversion formula)**.** *If $f \in L^2(\mathbb{R}^{2d},\mathbb{R})$, then the inversion formula of the quaternion Fourier transform, $\mathcal{F}_Q^{-1}$, can be obtained by letting $(\mathcal{F}_Q^{-1})(x) = \mathcal{F}_Q(-x)$, or simply denoted by $\check{f} := \widehat{f}(-x)$. Moreover, by (6) and (10), the statement that $\check{f} = \widehat{f}(-x)$ still holds for all $f \in L^2(\mathbb{R}^{2d},\mathbb{H})$.*

Proof. For $f \in L^2(\mathbb{R}^{2d},\mathbb{R})$, by Hölder's inequality, the expression

$$\int_{\mathbb{R}^{2d}} f(x,y)\widehat{g}_\varepsilon(u-x,v-y)\, dxdy \tag{17}$$

is well defined. Here $\widehat{g}_\varepsilon$ is defined as Example 3.1, i.e. $\widehat{g}_\varepsilon(u,v) = \varepsilon^{-d}\, \exp[-\pi\varepsilon^{-1}(|u|^2 + |v|^2)]$. By applying Fubini's theorem and Example 3.1, we could express (17) as

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} f(x,y) \left(\int_{\mathbb{R}^{2d}} e^{2\pi \mathbf{i}\,(u-x)\cdot z} g_\varepsilon(z,w) e^{2\pi \mathbf{j}\,(v-y)\cdot w}\, dzdw \right) dxdy \\ = \; &\int_{\mathbb{R}^{2d}} e^{2\pi \mathbf{i}\, u\cdot z} g_\varepsilon(z,w) \widehat{f}(z,w) e^{2\pi \mathbf{j}\, v\cdot w}\, dzdw. \end{aligned}$$

Since the family $\widehat{g}_\varepsilon$ is an approximation to the identity, let $\varepsilon \to 0$, then (17) tends to $f(u,v)$ in $L^2(\mathbb{R}^{2d})$. Further, $g_\varepsilon \widehat{f} \to \widehat{f}$ in $L^2(\mathbb{R}^{2d})$ as $\varepsilon \to 0$ by dominated convergence. Thus, we will get

$$f(u,v) = \int_{\mathbb{R}^{2d}} e^{2\pi \mathbf{i}\, u\cdot z} \widehat{f}(z,w) e^{2\pi \mathbf{j}\, v\cdot w}\, dzdw. \quad a.e. \qquad \square$$

Now let's compute the quaternion Fourier transform of $|x|^{-\alpha}$.

Theorem 3.4 (Quaternion Fourier transform of $|\mathbf{x}|^{-\alpha}$)**.** *Suppose that f be a function in $C_c(\mathbb{R}^{2d}, \mathbb{H})$, and let $0 < \alpha < 2d$, and*

$$c_\alpha := \pi^{-\alpha/2}\Gamma(\alpha/2). \tag{18}$$

Then we have

$$c_\alpha (|\mathbf{w}|^{-\alpha} \widehat{f}(\mathbf{w}))^\vee(\mathbf{x}) = c_{2d-\alpha} \int_{\mathbb{R}^{2d}} |\mathbf{x}-\mathbf{u}|^{\alpha-2d} f(\mathbf{u})\, d\mathbf{u}. \tag{19}$$

Proof. By Theorem 3.3 together with the representations (6) and (10), it is suffice to prove (19) for all $f \in C_c(\mathbb{R}^{2d}, \mathbb{R})$.

In order to show (19), we will apply the elementary formula

$$c_\alpha |\mathbf{w}|^{-\alpha} = \int_0^\infty \exp[-\pi|\mathbf{w}|^2\lambda]\lambda^{\alpha/2-1}\, d\lambda. \tag{20}$$

To verify this by using the definition of Gamma function and a simple change of variable.

Since $|\mathbf{w}|^{-\alpha}\widehat{f}(\mathbf{w})$ is integrable, by Fubini's theorem and Example 3.1, and (20), we have

$$\begin{aligned} &c_\alpha (|\mathbf{w}|^{-\alpha} \widehat{f}(\mathbf{w}))^\vee(\mathbf{x}) \\ = \; &\int_{\mathbb{R}^{2d}} e^{2\pi \mathbf{i}\, x_1\cdot w_1} \left(\int_0^\infty \exp[-\pi|\mathbf{w}|^2\lambda]\lambda^{\alpha/2-1}\, d\lambda \right) \widehat{f}(\mathbf{w})) e^{2\pi \mathbf{j}\, x_2\cdot w_2}\, d\mathbf{w} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{2d}} e^{2\pi\mathbf{i}\,x_1\cdot w_1}\left(\int_0^\infty e^{-\pi|\mathbf{w}|^2\lambda}\lambda^{\alpha/2-1}\,d\lambda\right)\left(\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\,w_1\cdot y_1} f(\mathbf{y}) e^{-2\pi\mathbf{j}\,w_2\cdot y_2}\,d\mathbf{y}\right) e^{2\pi\mathbf{j}\,x_2\cdot w_2}\,d\mathbf{w} \\
&= \int_0^\infty \lambda^{\alpha/2-1}\left\{\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\,w_1\cdot(y_1-x_1)} e^{-\pi|\mathbf{w}|^2\lambda} e^{-2\pi\mathbf{j}\,w_2\cdot(y_2-x_2)}\,d\mathbf{w}\right) f(\mathbf{y})\,dy\right\} d\lambda \\
&= \int_0^\infty \lambda^{\alpha/2-1}\lambda^{-d}\left\{\int_{\mathbb{R}^{2d}} e^{-\pi\lambda^{-1}|\mathbf{x}-\mathbf{y}|^2} f(\mathbf{y})\,dy\right\} d\lambda \\
&= c_{2d-\alpha}\int_{\mathbb{R}^{2d}} |\mathbf{x}-\mathbf{y}|^{-(2d-\alpha)} f(\mathbf{y})\,d\mathbf{y},
\end{aligned}
$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\mathbf{w} = (w_1, w_2)$ are in $\mathbb{R}^d \times \mathbb{R}^d$. □

In [31], W.Bechner stated the following lemma, which is useful in the calculation of the convolution.

Lemma 3.1 (Beckner)**.** *For $0 < \alpha < n$, $0 < \beta < n$, and $0 < \alpha + \beta < n$,*

$$\left(|x|^{\alpha-n} * |x|^{\beta-n}\right)(y) = \frac{c_{n-\alpha-\beta}\,c_\alpha\,c_\beta}{c_{\alpha+\beta}\,c_{n-\alpha}\,c_{n-\beta}}|y|^{\alpha+\beta-n}, \tag{21}$$

where $x, y \in \mathbb{R}^n$ and $c_\alpha := \pi^{-\alpha/2}\,\Gamma(\alpha/2)$.

Lastly, we extend Theorem 3.4 to the case $L^p(\mathbb{R}^{2d}, \mathbb{H})$, which is key to prove the logarithmic uncertainty principle.

Corollary 3.1. *If $0 < \alpha < d$ and $f \in L^p(\mathbb{R}^{2d}, \mathbb{R})$ with $p = 2d/(d+\alpha)$, then $\widehat{f}$ exists. Moreover, let*

$$g(\mathbf{x}) := c_{2d-\alpha}|\mathbf{x}|^{\alpha-2d} * f(\mathbf{x}), \tag{22}$$

where c_α is defined as in Theorem 3.4, that is $c_\alpha := \pi^{-\alpha/2}\Gamma(\alpha/2)$. Then g is an $L^2(\mathbb{R}^{2d}, \mathbb{R})$ function, and hence has a quaternion Fourier transform $\widehat{g}$. In fact,

$$\widehat{g}(\mathbf{w}) = c_\alpha|\mathbf{w}|^{-\alpha}\widehat{f}(\mathbf{w}), \tag{23}$$

and an application of Plancherel's Theorem will lead to

$$c_{2\alpha}\int_{\mathbb{R}^{2d}} |\mathbf{w}|^{-2\alpha}|\widehat{f}(\mathbf{w})|^2\,d\mathbf{w} = c_{2d-2\alpha}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} f(\mathbf{x})f(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{2\alpha-2d}\,d\mathbf{x}d\mathbf{y}. \tag{24}$$

Furthermore, for $f \in L^p(\mathbb{R}^{2d}, \mathbb{H})$,

$$c_{2\alpha}\int_{\mathbb{R}^{2d}} |\mathbf{w}|^{-2\alpha}|\widehat{f}_Q(\mathbf{w})|^2\,d\mathbf{w} = c_{2d-2\alpha}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \sum_{i=1}^4 f_i(\mathbf{x})f_i(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{2\alpha-2d}\,d\mathbf{x}d\mathbf{y}. \tag{25}$$

Remark 3.1. *Let us recall the properties that we have been proved so far,*

$$f \in L^1(\mathbb{R}^{2d}, \mathbb{R}) \implies \mathcal{F}_Q(f) \in L^\infty(\mathbb{R}^{2d}, \mathbb{H}) \text{ with } \|\mathcal{F}_Q(f)\|_{Q,\infty} \le \|f\|_1.$$
$$f \in L^2(\mathbb{R}^{2d}, \mathbb{R}) \implies \mathcal{F}_Q(f) \in L^2(\mathbb{R}^{2d}, \mathbb{H}) \text{ with } \|\mathcal{F}_Q(f)\|_{Q,2} = \|f\|_2.$$

By the Riesz's interpolation theorem, we obtain

$$\|\widehat{f}\|_{Q,p'} \le \|f\|_p. \tag{26}$$

holds for $1 \le p \le 2$ with $1/p + 1/p' = 1$. That means, $\widehat{f}$ does exist in the L^p-norm sense, and $\widehat{f} \in L^{p'}$ whenever $f \in L^p$.

Proof of Corollary 3.1. As in Theorem 3.4, we assume that f is a real function. Since $C_c(\mathbb{R}^{2d}, \mathbb{R})$ is dense in $L^p(\mathbb{R}^{2d}, \mathbb{R})$, we may find a sequence f^l of functions in $C_c(\mathbb{R}^{2d}, \mathbb{R})$ such that $\|f^l - f\|_p \to 0$ as $l \to \infty$. Setting $g := c_{2d-\alpha}|\mathbf{x}|^{\alpha-2d} * f$ and $g^l := c_{2d-\alpha}|\mathbf{x}|^{\alpha-2d} * f^l$. By Fubini's theorem and Lemma 3.1, we have

$$\begin{aligned}
& \int_{\mathbb{R}^{2d}} |g(\mathbf{x})|^2 \, d\mathbf{x} \\
= & \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} c_{2d-\alpha} |\mathbf{x} - \mathbf{y}|^{\alpha-2d} f(\mathbf{y}) \, d\mathbf{y} \right) \left(\int_{\mathbb{R}^{2d}} c_{2d-\alpha} |\mathbf{x} - \mathbf{z}|^{\alpha-2d} f(\mathbf{z}) \, d\mathbf{z} \right) d\mathbf{x} \\
= & \; c_{2d-\alpha}^2 \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} f(\mathbf{y}) f(\mathbf{z}) \left(\int_{\mathbb{R}^{2d}} |\mathbf{x} - \mathbf{y}|^{\alpha-2d} |\mathbf{x} - \mathbf{z}|^{\alpha-2d} \, d\mathbf{x} \right) d\mathbf{y} d\mathbf{z} \\
= & \; \frac{c_{2d-2\alpha}}{c_{2\alpha}} c_\alpha^2 \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} f(\mathbf{y}) |\mathbf{y} - \mathbf{z}|^{2\alpha-2d} f(\mathbf{z}) \, d\mathbf{y} d\mathbf{z}.
\end{aligned}$$

By Hardy-Littlewood-Sobelev inequality (see [34]), g and g^l are in $L^2(\mathbb{R}^{2d}, \mathbb{R})$.

Since $f^l \to f$ in $L^p(\mathbb{R}^{2d}, \mathbb{R})$, we have that $\widehat{f^l} \to \widehat{f}$ in $L^{p'}(\mathbb{R}^{2d}, \mathbb{R})$ by the Remark in Corollary 3.1. An application of Hardy-Littlewood-Sobelev inequality, we have that $g^l \to g$ in $L^2(\mathbb{R}^{2d}, \mathbb{R})$, and hence $\widehat{g^l} \to \widehat{g}$ in $L^2(\mathbb{R}^{2d}, \mathbb{R})$ by Plancherel's theorem.

From Theorem 3.4, we can see

$$\widehat{g^l}(\mathbf{w}) = c_\alpha |\mathbf{w}|^{-\alpha} \widehat{f^l}(\mathbf{w}).$$

To show that

$$\widehat{g}(\mathbf{w}) = c_\alpha |\mathbf{w}|^{-\alpha} \widehat{f^l}(\mathbf{w}).$$

We pass to a subsequence so that $\widehat{g}^l \to \widehat{g}$ and $\widehat{f}^l \to \widehat{f}$ pointwise a.e. By the completeness of L^p space. This implies

$$\widehat{g}(\mathbf{w}) = \lim_{l\to\infty} c_\alpha |\mathbf{w}|^{-\alpha} \widehat{f}^l(\mathbf{w}) = c_\alpha |\mathbf{w}|^{-\alpha} \widehat{f}(\mathbf{w}). \quad a.e. \qquad \square$$

4. Pitt's inequality and the uncertainty principle

The uncertainty principle is a description of feature of a function and its quaternion Fourier transform. Beckner [30] has showed that sharp Pitt's inequality yields a short proof of logarithmic uncertainty estimate, and Heisenberg-Weyl's inequality follows by using logarithmic uncertainty inequality in the real number field. In [30], with aid of rearrangement and symmetrization, Beckner have proved a sharp Pitt's inequality by applying the sharp L^1 Young's inequality for convolution on $\mathbb{R}_+$. Here, as in [31], base on Young's inequality for convolution on $\mathbb{R}_+ \times S^{2d-1}$, we will give a different proof of the Pitt's inequality about quaternion Fourier transform.

Theorem 4.1 (Pitt's inequality)**.** *For $f \in S(\mathbb{R}^{2d}, \mathbb{H})$,*

$$\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi}|^{-\alpha} |\widehat{f}(\boldsymbol{\xi})|_Q^2 \, d\boldsymbol{\xi} \le C_\alpha \int_{\mathbb{R}^{2d}} |\mathbf{x}|^{\alpha} |f(\mathbf{x})|^2 \, d\mathbf{x}, \tag{27}$$
$$C_\alpha := \pi^\alpha \left[\Gamma\left(\frac{2d-\alpha}{4}\right) \Gamma\left(\frac{2d+\alpha}{4}\right)\right].$$

In particular, for $f \in S(\mathbb{R}^{2d}, \mathbb{R})$, and $0 \le \alpha < 2d$,

$$\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi}|^{-\alpha} |\widehat{f}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le C_\alpha \int_{\mathbb{R}^{2d}} |\mathbf{x}|^{\alpha} |f(\mathbf{x})|^2 \, d\mathbf{x}. \tag{28}$$

Proof. We first prove the inequality (28). To this end, we assume that $f \in S(\mathbb{R}^{2d}, \mathbb{R})$, by consider the function

$$F(\mathbf{x}) = |\mathbf{x}|^{\alpha/2} f(\mathbf{x}),$$

then by Corollary 3.1, we can see the left side of (28) is

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi}|^{-\alpha} |\widehat{f}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} &= \frac{c_{2d-\alpha}}{c_\alpha} \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} f(\mathbf{x}) |\mathbf{x}-\mathbf{y}|^{\alpha-2d} f(\mathbf{y}) \, d\mathbf{x} d\mathbf{y} \\
&= \frac{c_{2d-\alpha}}{c_\alpha} \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \frac{F(\mathbf{x})}{|\mathbf{x}|^{\alpha/2}} |\mathbf{x}-\mathbf{y}|^{\alpha-2d} \frac{F(\mathbf{y})}{|\mathbf{y}|^{\alpha/2}} \, d\mathbf{x} d\mathbf{y},
\end{aligned}$$

and the right side of (28) becomes

$$\int_{\mathbb{R}^{2d}} |\mathbf{x}|^\alpha |f(\mathbf{x})|^2 \, d\mathbf{x} = \int_{\mathbb{R}^{2d}} |F(\mathbf{x})|^2 \, d\mathbf{x}.$$

Then (28) is equivalent to prove that

$$\left| \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \frac{f(\mathbf{x})}{|\mathbf{x}|^{\alpha/2}} |\mathbf{x}-\mathbf{y}|^{\alpha-2d} \frac{f(\mathbf{y})}{|\mathbf{y}|^{\alpha/2}} \, d\mathbf{x}d\mathbf{y} \right| \le C_\alpha \frac{c_\alpha}{c_{2d-\alpha}} \int_{\mathbb{R}^{2d}} |f(\mathbf{x})|^2 \, d\mathbf{x}. \qquad (29)$$

Here, for the sake of convenience we shall use f instead of F. Without loss of generality, we may assume that f is nonnegative. By setting

$$\begin{aligned} t = |\mathbf{x}|, &\quad \mathbf{x} = t\,\boldsymbol{\xi}, \\ s = |\mathbf{y}|, &\quad \mathbf{y} = s\,\boldsymbol{\eta}, \\ h(t\,\boldsymbol{\xi}) &= t^d f(t\,\boldsymbol{\xi}), \\ \psi(t, \boldsymbol{\xi}\cdot\boldsymbol{\eta}) &= t^{\alpha/4} \left(t + \frac{1}{t} - 2\boldsymbol{\xi}\cdot\boldsymbol{\eta} \right)^{-(d-\alpha/4)}, \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in S^{2d-1}. \end{aligned}$$

Then (29) has a equivalent formulation as a convolution estimate on the product manifold $\mathbb{R}_+ \times S^{2d-1}$:

$$\|\psi * h\|_{L^2(\mathbb{R}_+\times S^{2d-1})} \le \|\psi\|_{L^1(\mathbb{R}_+\times S^{2d-1})} \|h\|_{L^2(\mathbb{R}_+\times S^{2d-1})}. \qquad (30)$$

Now we observe

$$\|f\|_2^2 = \int_{\mathbb{R}^{2d}} |f(\mathbf{x})|^2 \, d\mathbf{x} = \int_{\mathbb{R}_+\times S^{2d-1}} |t^d f(t\,\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \frac{dt}{t} = \|h\|^2_{L^2(\mathbb{R}_+\times S^{2d-1})}, \qquad (31)$$

and

$$\begin{aligned} \|\psi * h\|^2_{L^2(\mathbb{R}_+\times S^{2d-1})} &= \int_{\mathbb{R}_+\times S^{2d-1}} |\psi * h|^2 \, d\boldsymbol{\xi} \frac{dr}{r} \\ &= \int_{\mathbb{R}_+\times S^{2d-1}} \left(\int_{\mathbb{R}_+\times S^{2d-1}} h(t\,\boldsymbol{\zeta}) \psi(\frac{r}{t}, \boldsymbol{\zeta}\cdot\boldsymbol{\xi}) \, d\boldsymbol{\zeta} \frac{dt}{t} \right) \times \qquad (32) \\ &\quad \times \left(\int_{\mathbb{R}_+\times S^{2d-1}} h(s\,\boldsymbol{\eta}) \psi(\frac{r}{s}, \boldsymbol{\eta}\cdot\boldsymbol{\xi}) \, d\boldsymbol{\eta} \frac{ds}{s} \right) d\boldsymbol{\xi} \frac{dr}{r} \\ &= \int_{\mathbb{R}_+\times S^{2d-1}} \int_{\mathbb{R}_+\times S^{2d-1}} h(t\,\boldsymbol{\zeta}) h(s\,\boldsymbol{\eta}) K(t, s, \boldsymbol{\zeta}\cdot\boldsymbol{\xi}, \boldsymbol{\eta}\cdot\boldsymbol{\xi}) \, d\boldsymbol{\zeta} \frac{dt}{t} \, d\boldsymbol{\eta} \frac{ds}{s}, \end{aligned}$$

where the kernel is

$$K(t, s, \boldsymbol{\zeta}\cdot\boldsymbol{\xi}, \boldsymbol{\eta}\cdot\boldsymbol{\xi}) := \int_{\mathbb{R}_+\times S^{2d-1}} \psi(\frac{r}{t}, \boldsymbol{\zeta}\cdot\boldsymbol{\xi}) \psi(\frac{r}{s}, \boldsymbol{\eta}\cdot\boldsymbol{\xi}) \, d\boldsymbol{\xi} \frac{dr}{r}.$$

By Lemma 3.1, we can calculate $K(t, s, \boldsymbol{\zeta}\cdot\boldsymbol{\xi}, \boldsymbol{\eta}\cdot\boldsymbol{\xi})$.

$$
\begin{aligned}
& \int_{\mathbb{R}_+\times S^{2d-1}} \psi(\frac{r}{t}, \boldsymbol{\zeta}\cdot\boldsymbol{\xi})\psi(\frac{r}{s}, \boldsymbol{\eta}\cdot\boldsymbol{\xi})\, d\boldsymbol{\xi}\frac{dr}{r} \\
= & \int_{\mathbb{R}_+\times S^{2d-1}} \left(\frac{r}{t}\right)^{\alpha/4}\left(\frac{r}{t}+\frac{t}{r}-2\boldsymbol{\zeta}\cdot\boldsymbol{\xi}\right)^{-(2d-\alpha/2)/2}\left(\frac{r}{s}\right)^{\alpha/4}\left(\frac{r}{s}+\frac{s}{r}-2\boldsymbol{\eta}\cdot\boldsymbol{\xi}\right)^{-(2d-\alpha/2)/2} d\boldsymbol{\xi}\frac{dr}{r} \\
= & \; s^{d-\alpha/2}t^{d-\alpha/2}\int_{\mathbb{R}_+\times S^{2d-1}} |r\,\boldsymbol{\xi}-t\,\boldsymbol{\xi}|^{-(2d-\alpha/2)}\,|r\,\boldsymbol{\xi}-s\,\boldsymbol{\eta}|^{-(2d-\alpha/2)}\, r^{2d-1}\, d\boldsymbol{\xi}dr \\
= & \; |\mathbf{x}|^{d-\alpha/2}|\mathbf{y}|^{d-\alpha/2}\int_{\mathbb{R}^{2d}} |\mathbf{z}-\mathbf{x}|^{-(2d-\alpha/2)}|\mathbf{z}-\mathbf{y}|^{-(2d-\alpha/2)}\, d\mathbf{z} \\
= & \; |\mathbf{x}|^{d-\alpha/2}|\mathbf{y}|^{d-\alpha/2}\,\frac{c_{2d-\alpha}}{c_\alpha}\left[\frac{c_{\alpha/2}}{c_{2d-\alpha/2}}\right]^2 |\mathbf{x}-\mathbf{y}|^{\alpha-2d}.
\end{aligned} \tag{33}
$$

Thus, from (32) and (33), we could know

$$
\|\psi * h\|^2_{L^2(\mathbb{R}_+\times S^{2d-1})} = \frac{c_{2d-\alpha}}{c_\alpha}\left[\frac{c_{\alpha/2}}{c_{2d-\alpha/2}}\right]^2 \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \frac{f(\mathbf{x})}{|\mathbf{x}|^{\alpha/2}}|\mathbf{x}-\mathbf{y}|^{\alpha-2d}\frac{f(\mathbf{y})}{|\mathbf{y}|^{\alpha/2}}\, d\mathbf{x}d\mathbf{y}. \tag{34}
$$

Now Young's inequality (30) and (31) implies the best constant C_α in (29) is

$$
\left[\frac{c_{2d-\alpha/2}}{c_{\alpha/2}}\right]^2 \|\psi\|^2_{L^1(\mathbb{R}_+\times S^{2d-1})}.
$$

To compute the integration of ψ, we observe from Lemma 7 that

$$
\begin{aligned}
\|\psi\|_{L^1(\mathbb{R}_+\times S^{2d-1})} &= \int_{\mathbb{R}_+\times S^{2d-1}} t^{\alpha/4}\left(t+\frac{1}{t}-2\boldsymbol{\xi}\cdot\boldsymbol{\eta}\right)^{-(d-\alpha/4)} d\boldsymbol{\xi}\frac{dt}{t} \\
&= \int_{\mathbb{R}_+\times S^{2d-1}} t^{-d}(t^2+1-2t\boldsymbol{\xi}\cdot\boldsymbol{\eta})^{-(d-\alpha/4)}t^{2d-1}\, d\boldsymbol{\xi}dt \\
&= \int_{\mathbb{R}^{2d}} |\mathbf{x}-\boldsymbol{\xi}|^{-(2d-\alpha/2)}|\mathbf{x}|^{-d}\, d\mathbf{x} \\
&= \frac{c_{d-\alpha/2}\, c_{\alpha/2}}{c_{d+\alpha/2}\, c_{2d-\alpha/2}},
\end{aligned}
$$

where $|\boldsymbol{\xi}| = 1$. Evidently,

$$
C_\alpha = \left[\frac{c_{d-\alpha/2}}{c_{d+\alpha/2}}\right]^2 = \pi^\alpha\left[\Gamma\left(\frac{2d-\alpha}{4}\right)\Gamma\left(\frac{2d+\alpha}{4}\right)\right].
$$

Because of (7), (12) and (28), we can easily deduce (27). □

Since inequalities (28) and (27) are equations for $\alpha = 0$, by differentiating the sharp Pitt's inequalities at $\alpha = 0$, we will obtain logarithmic estimate of uncertainty.

Corollary 4.1 (Logarithmic uncertainty principle). *For $f \in S(\mathbb{R}^{2d}, \mathbb{H})$,*

$$\int_{\mathbb{R}^{2d}} \ln|\mathbf{x}|\,|f(\mathbf{x})|^2\,d\mathbf{x} + \int_{\mathbb{R}^{2d}} \ln|\boldsymbol{\xi}|\,|\widehat{f}(\boldsymbol{\xi})|_Q^2\,d\boldsymbol{\xi} \geq D \int_{\mathbb{R}^{2d}} |f(\mathbf{x})|^2\,d\mathbf{x} \tag{35}$$

$$D := \psi(2d/4) - \ln(\pi), \quad \psi := \frac{d}{dt}\left[\ln \Gamma(t)\right].$$

In particular, for $f \in S(\mathbb{R}^{2d}, \mathbb{R})$,

$$\int_{\mathbb{R}^{2d}} \ln|\mathbf{x}|\,|f(\mathbf{x})|^2\,d\mathbf{x} + \int_{\mathbb{R}^{2d}} \ln|\boldsymbol{\xi}|\,|\widehat{f}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \geq D \int_{\mathbb{R}^{2d}} |f(\mathbf{x})|^2\,d\mathbf{x}. \tag{36}$$

Remark In fact, the logarithmic uncertainty (36) implies the Heisenberg-Weyl's inequality. Since the logarithm is a concave function, by Jensen's inequality,

$$\ln\left[\int_{\mathbb{R}^{2d}} \ln|\mathbf{x}|^2\,|f(\mathbf{x})|^2\,d\mathbf{x}\right]^{1/2} \geq \int_{\mathbb{R}^{2d}} \ln|\mathbf{x}|\,|f(\mathbf{x})|^2\,d\mathbf{x},$$

$$\ln\left[\int_{\mathbb{R}^{2d}} \ln|\boldsymbol{\xi}|^2\,|\widehat{f}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\right]^{1/2} \geq \int_{\mathbb{R}^{2d}} \ln|\boldsymbol{\xi}|\,|\widehat{f}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}.$$

With the aid of the logarithmic uncertainty principle (36), we arrive at

$$\ln\left[\int_{\mathbb{R}^{2d}} \ln|\mathbf{x}|^2\,|f(\mathbf{x})|^2\,d\mathbf{x} \int_{\mathbb{R}^{2d}} \ln|\boldsymbol{\xi}|^2\,|\widehat{f}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\right]^{1/2} \geq D = \psi(2d/4) - \ln(\pi),$$

which holds for all $d \in \mathbb{N}$. By utilizing the product structure, $\prod f(x_k)$ with $\|f\|_2 = 1$, note that $\psi(2d/4) - \ln(2d/4) \geq -1/(2d/4)$, and let $d \to \infty$. This implies the Heisenberg-Weyl's uncertainty principle in one dimension:

$$\left[\int_{\mathbb{R}} |\mathbf{x}|^2\,|f(\mathbf{x})|^2\,d\mathbf{x} \int_{\mathbb{R}} |\boldsymbol{\xi}|^2\,|\widehat{f}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\right]^{1/2} \geq \frac{1}{4\pi}.$$

By utilizing the product structure, $\prod f(x_k)$ with $\|f\|_2 = 1$ again, one obtains the n-dimensional form for $f \in S(\mathbb{R}^{2d}, \mathbb{R})$:

$$\left(\int_{\mathbb{R}^{2d}} |\mathbf{x}|^2\,|f(\mathbf{x})|^2\,d\mathbf{x}\right)\left(\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi}|^2\,|\widehat{f}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\right) \geq \left(\frac{2d}{4\pi}\right)^2.$$

Naturally, with the representations (7) and (12), and replacing $f(x_1, x_2)$ by $e^{-2\pi \mathbf{i}\, x_1 \cdot \xi_1} f(x_1, x_2) e^{-2\pi \mathbf{j}\, x_2 \cdot \xi_2}$ and changing variables, the corresponding n-dimensional form for $f \in S(\mathbb{R}^{2d}, \mathbb{H})$ is given by:

$$\left(\int_{\mathbb{R}^{2d}} |\mathbf{x} - \mathbf{x_0}|^2\,|f(\mathbf{x})|^2\,d\mathbf{x}\right)\left(\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi} - \boldsymbol{\xi_0}|^2\,|\widehat{f}(\boldsymbol{\xi})|_Q^2\,d\boldsymbol{\xi}\right) \geq \left(\frac{d}{2\pi}\right)^2.$$

We introduce the standard derivations

$$\Delta_f \mathbf{x} = \left(\int_{\mathbb{R}^{2d}} |\mathbf{x} - \mathbf{x_0}|^2 \, |f(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2},$$

and

$$\Delta_f \boldsymbol{\xi} = \left(\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi} - \boldsymbol{\xi_0}|^2 \, |\widehat{f}(\boldsymbol{\xi})|_Q^2 \, d\boldsymbol{\xi} \right)^{1/2}.$$

It is easy to observe that the expectation $\overline{\mathbf{x}} = \int_{\mathbb{R}^{2d}} \mathbf{x}|f(\mathbf{x})|^2 \, d\mathbf{x}$ is that choice for for which the uncertainty $\Delta_f \boldsymbol{\xi}$ is the smallest. Similarly, $\Delta_f \boldsymbol{\xi}$ is minimized at $\overline{\boldsymbol{\xi}} = \int_{\mathbb{R}^{2d}} \boldsymbol{\xi}|\widehat{f}(\boldsymbol{\xi})|_Q^2 \, d\boldsymbol{\xi}$.

Now the uncertainty principle of quaternion Fourier transform becomes:

Theorem 4.2 (Heisenberg-Weyl's uncertainty principle)**.** *If $f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$ with $\|f\|_2^2 = 1$, then*

$$\Delta_f \mathbf{x} \cdot \Delta_f \boldsymbol{\xi} \geq \frac{d}{2\pi}. \tag{37}$$

The Heisenberg uncertainty principle of quaternion Fourier transform can be formulated in terms of the Hermite operator $L := -\Delta + |\mathbf{x}|^{\mathbf{2}}$, which acts on Schwartz functions by the formula

$$L(f) := -\Delta f + |\mathbf{x}|^2 f,$$

here $f \in S(\mathbb{R}^{2d}, \mathbb{H})$.

At first, we integrate by parts, then by the arithmetic mean-geometric mean inequality and Theorem 4.2, we have

$$\begin{aligned}
\langle Lf, f \rangle &= \int_{\mathbb{R}^{2d}} (-\Delta f + |\mathbf{x}|^2 f)\overline{f} \, d\mathbf{x} = \int_{\mathbb{R}^{\mathbf{2d}}} (|\nabla \mathbf{f}|^{\mathbf{2}} + |\mathbf{x}|^{\mathbf{2}}|\mathbf{f}|^{\mathbf{2}}) \, \mathbf{dx} \\
&\geq 2 \left(\int_{\mathbb{R}^{2d}} |\nabla f|^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{\mathbb{R}^{2d}} |\mathbf{x}|^2 |f|^2 \, d\mathbf{x} \right)^{1/2} \\
&= 2 \left(\int_{\mathbb{R}^{2d}} |2\pi\boldsymbol{\xi}|^2 |\widehat{f}|^2 \, d\boldsymbol{\xi} \right)^{1/2} \left(\int_{\mathbb{R}^{2d}} |\mathbf{x}|^2 |f|^2 \, d\mathbf{x} \right)^{1/2} \\
&\geq 2 \times \frac{2d}{4\pi} \times 2\pi \times \left(\int_{\mathbb{R}^{2d}} |f|^2 \, d\mathbf{x} \right)^{1/2} \\
&= (2d)\langle f, f \rangle.
\end{aligned}$$

In the penultimate equation we have used Plancherel's theorem.

Now we have showed that the Heisenberg uncertainty principle implies $\langle Lf, f\rangle \geq (2d)\langle f, f\rangle$ for all $f \in S(\mathbb{R}^{2d}, \mathbb{H})$. This usually denoted by $L \geq (2d)I$.

Next, consider the operator A_i and A_i^* defined on $S(\mathbb{R}^{2d}, \mathbb{H})$ by

$$A_i(f) := \frac{\partial f}{\partial x_i} + x_i f \quad \text{and} \quad A_i^*(f) := -\frac{\partial f}{\partial x_i} + x_i f, \quad i = 1, 2, \cdots, 2d.$$

The operator A_i and A_i^* are sometimes called the annihilation and creation operators, respectively. By integrating by parts, for all $f, g \in S(\mathbb{R}^{2d}, \mathbb{H})$, we can find that

(i) $\langle A_i f, g\rangle = \langle f, A_i^* g\rangle$,

(ii) $\langle A_i f, A_i f\rangle = \langle A_i^* A_i f, A_i f\rangle \geq 0$,

(iii) $\sum_{i=1}^{2d} A_i^* A_i = L - (2d)I$,

for $i = 1, 2, \cdots, 2d$. In particular, this again show that $L \geq (2d)I$.

Now for $t \in \mathbb{R}$, let

$$A_{i,t}(f) := \frac{\partial f}{\partial x_i} + t x_i f \quad \text{and} \quad A_{i,t}^*(f) := -\frac{\partial f}{\partial x_i} + t x_i f, \quad i = 1, 2, \cdots, 2d.$$

Use the fact that $\langle \sum_{i=1}^{2d} A_{i,t}^* A_{i,t} f, f\rangle \geq 0$ and $\sum_{i=1}^{2d} A_{i,t}^* A_{i,t} f = -\Delta f + t^2|\mathbf{x}|^2 f - (2d)f$, then we have

$$t^2\langle |\mathbf{x}|^2 f, f\rangle - t(2d)\langle f, f\rangle - \langle \Delta f, f\rangle \geq 0,$$

or

$$t^2\langle |\mathbf{x}|^2 f, f\rangle - t(2d)\langle f, f\rangle + \langle \nabla f, \nabla f\rangle \geq 0.$$

If $f \not\equiv 0$, then $\langle |\mathbf{x}|^2 f, f\rangle \neq 0$. We can choose t to be equal to $\frac{1}{2}(2d)\langle f, f\rangle / \langle |\mathbf{x}|^2 f, f\rangle$, then

$$\langle |\mathbf{x}|^2 f, f\rangle \langle \nabla f, \nabla f\rangle \geq \left(\frac{2d}{2}\right)^2 \langle f, f\rangle.$$

Assume that $\|f\|_2^2 = 1$, and use Plancherel's theorem, we obtain

$$\langle |\mathbf{x}|^2 f, f\rangle \langle |\boldsymbol{\xi}|^2 \widehat{f}, \widehat{f}\rangle \geq \left(\frac{2d}{4\pi}\right)^2,$$

which is the Heisenberg uncertainty principle. Now we have showed that $L \geq (2d)I$ implies the Heisenberg uncertainty principle conversely.

The heuristic assertion stated before the next section can be made precise as follows. If $f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$ with $\|f\|_2^2 = 1$, then we say that its position is ε_p-concentrated on a ball B_1 centered at $\mathbf{x_0}$ if

$$\left(\int_{B_1^c} |\mathbf{x} - \mathbf{x_0}|^2 |f(\mathbf{x})|^2 \, d\mathbf{x}\right)^{1/2} \leq \varepsilon_p \int_{\mathbb{R}^{2d}} |\mathbf{x} - \mathbf{x_0}|^2 |f(\mathbf{x})|^2 \, d\mathbf{x}. \tag{38}$$

If $0 \leq \varepsilon_p \leq 1/2$, then the preponderance of its position is contained in a ball B_1.

Correspondingly, its momentum is ε_m-concentrated on a ball B_2 centered at $\boldsymbol{\xi}_0$ if

$$\left(\int_{B_2^c} |\boldsymbol{\xi} - \boldsymbol{\xi}_0|^2 |\widehat{f}(\boldsymbol{\xi})|_Q^2 \, d\mathbf{x}\right)^{1/2} \leq \varepsilon_m \int_{\mathbb{R}^{2d}} |\boldsymbol{\xi} - \boldsymbol{\xi}_0|^2 |\widehat{f}(\boldsymbol{\xi})|_Q^2 \, d\boldsymbol{\xi}. \tag{39}$$

Similarly, If $0 \leq \varepsilon_m \leq 1/2$, then the preponderance of its momentum is contained in a ball B_2.

Corollary 4.2. *With the assumption* (1) *and* (2)*, let r_j denotes the radius of B_j with $j = 1, 2$. Then we have*

$$r_1 \cdot r_2 \geq \frac{2d}{4\pi} \sqrt{(1 - \varepsilon_p)(1 - \varepsilon_m)} \tag{40}$$

Proof. By assumption (1) and the fact that $|\mathbf{x} - \mathbf{x_0}| \leq r_1$ when $x \in B_1$, and $\|f\|_2^2 = 1$,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |\mathbf{x} - \mathbf{x_0}|^2 |f(\mathbf{x})|^2 \, d\mathbf{x} &\leq (1 - \varepsilon_p)^{-1} \int_{B_1} |\mathbf{x} - \mathbf{x_0}|^2 |f(\mathbf{x})|^2 \, d\mathbf{x} \\ &\leq (1 - \varepsilon_p)^{-1} r_1^2. \end{aligned}$$

Similarly, we deduce that

$$\int_{\mathbb{R}^{2d}} |\boldsymbol{\xi} - \boldsymbol{\xi}_0|^2 |\widehat{f}(\boldsymbol{\xi})|_Q^2 \, d\boldsymbol{\xi} \leq (1 - \varepsilon_m)^{-1} r_2^2.$$

Combining these inequalities and applying Theorem 4.2,

$$(1 - \varepsilon_p)^{-1} (1 - \varepsilon_m)^{-1} r_1^2 \cdot r_2^2 \geq \left(\frac{2d}{4\pi}\right)^2,$$

from which the corollary follows. $\square$

5. The uncertainty principle of Donoho and Stark

The classical uncertainty principle is based on the interpretation of the standard deviation $\Delta_f \mathbf{x}$ as the size of the essential support of f. Other notions of the support lead to different versions of the uncertainty principle. As an example we present a beautiful uncertainty principle on $\mathbb{R}^{2d}$ of Donoho and Stark [32].

Definition 5.1. *A function $f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$ is said to be ε-concentrated on a measurable set $T \subseteq \mathbb{R}^{2d}$, if*

$$\left(\int_{T^c} |f(x_1, x_2)|^2 \, dx_1 dx_2 \right)^{1/2} \leq \varepsilon \|f\|_2. \tag{41}$$

If $0 \leq \varepsilon \leq 1/2$, then the most of energy is concentrated on T, and T is indeed the essential support of f. If $\varepsilon = 0$, then T is the exact support of f.

Theorem 5.1 (Uncertainty principle of Donoho and Stark)**.** *Suppose that $f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$ with $f \neq 0$, is ε_T-concentrated on $T \subseteq \mathbb{R}^{2d}$ and $\widehat{f}$ is ε_Ω-concentrated on $\Omega \subseteq \mathbb{R}^{2d}$. Then*

$$|T| \cdot |\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2. \tag{42}$$

Proof. Without loss of generality, we may assume that T and Ω have finite measure. Let

$$P_T f = \chi_T \cdot f,$$

and

$$Q_\Omega f = \mathcal{F}_Q^{-1} \left[\chi_\Omega (\mathcal{F}_Q f) \right] = \int_\Omega e^{2\pi \mathbf{i} \, x \cdot u} \widehat{f}(u, v) e^{2\pi \mathbf{j} \, y \cdot v} \, du dv.$$

Both operators are orthogonal projections on $L^2(\mathbb{R}^{2d}, \mathbb{H})$. With this notation, f is ε_T-concentrated on T if and only if

$$\|f - P_T f\|_2 \leq \varepsilon \|f\|_2,$$

and $\widehat{f}$ is ε_Ω-concentrated on Ω if and only if

$$\|f - Q_\Omega f\|_2 = \|\chi_{\Omega^c} \cdot \widehat{f}\|_{Q,2} \leq \varepsilon_\Omega \|f\|_2.$$

Since $\|Q_\Omega\|_2 \leq 1$, we obtain that

$$\begin{aligned}\|f - Q_\Omega P_T f\|_2 &\leq \|f - Q_\Omega f\|_2 + \|Q_\Omega(f - P_T f)\|_2 \\ &\leq (\varepsilon_T + \varepsilon_\Omega)\|f\|_2,\end{aligned}$$

and consequently

$$\|Q_\Omega P_T f\|_2 \geq \|f\|_2 - \|f - Q_\Omega P_T f\|_2 \geq (1 - \varepsilon_T - \varepsilon_\Omega)\|f\|_2. \tag{43}$$

Next we compute the integral kernel and the L^2-norm of $Q_\Omega P_T$.

$$\begin{aligned}Q_\Omega P_T f(x,y) &= \mathcal{F}_Q^{-1}\left(\chi_\Omega (P_T f)^\wedge\right)(x,y) \\ &= \int_\Omega e^{2\pi\mathbf{i}\,x\cdot u}\left(\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\,t\cdot u}\chi_T(t,s)f(t,s)e^{-2\pi\mathbf{j}\,s\cdot v}\,dtds\right)e^{2\pi\mathbf{j}\,y\cdot v}\,dudv.\end{aligned}$$

Since both T and Ω have finite measure and since $f \in L^2(T,H) \subseteq L^1(T,H)$, this double integral converges absolutely. By Fubini's theorem,

$$\begin{aligned}Q_\Omega P_T f(x,y) &= Q_\Omega P_T f_1(x,y) + \mathbf{i}\,Q_\Omega P_T f_2(x,y) + Q_\Omega P_T f_3(x,y)\,\mathbf{j} + \mathbf{i}\,Q_\Omega P_T Q_\Omega P_T f_4(x,y)\,\mathbf{j} \\ &= \int_{\mathbb{R}^{2d}} K(x,y;t,s)f_1(t,s)\,dtds + \mathbf{i}\int_{\mathbb{R}^{2d}} K(x,y;t,s)f_2(t,s)\,dtds + \\ &\quad + \int_{\mathbb{R}^{2d}} K(x,y;t,s)f_3(t,3)\,dtds\,\mathbf{j} + \mathbf{i}\int_{\mathbb{R}^{2d}} K(x,y;t,s)f_4(t,s)\,dtds\,\mathbf{j},\end{aligned}$$

where the kernel is

$$\begin{aligned}K(x,y;t,s) &= \chi_T(t,s)\int_\Omega e^{2\pi\mathbf{i}\,(x-t)\cdot u}\chi_\Omega(x,y)e^{2\pi\mathbf{j}\,(y-s)\cdot v}\,dudv \\ &= \chi_T(t,s)T_{(t,s)}\mathcal{F}_Q^{-1}\left(\chi_\Omega(x,y)\right),\end{aligned}$$

here $T_{(t,s)}$ is the translation operator, i.e. $T_{(t,s)}f(\cdot,\cdot) = f(\cdot - t, \cdot - s)$. The L^2-norm of $Q_\Omega P_T$ is given by

$$\|Q_\Omega Q_T\|_2^2 = \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |K(x,y;t,s)|^2\,dxdydsdt.$$

Since the translation operator $T_{(t,s)}$ is unitary, and Inversion Fourier transform $\mathcal{F}_Q^{-1}$ is isometry. We have for fixed (t,s) that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} |K(x,y;t,s)|^2\,dxdy &= \chi_T(t,s)\|T_{(t,s)}\mathcal{F}_Q^{-1}\chi_\Omega\|_2^2 \\ &= \chi_T(t,s)\|\chi_\Omega\|_2^2 \\ &= \chi_T(t,s)|\Omega|,\end{aligned}$$

and therefore

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |K(x,y;t,s)|^2 \, dxdydsdt = |T| \cdot |\Omega|. \tag{44}$$

Finally combining (43), (44), and the fact that the operator norm $\|Q_\Omega P_T\|$ is dominated by the Hilbert Schmidt norm (see [36]), we obtain

$$\begin{aligned} (1-\varepsilon_T-\varepsilon_\Omega)^2\|f\|_2^2 &\leq \|Q_\Omega P_T f\|_2^2 \\ &\leq \|Q_\Omega P_T\|_2^2 \cdot \|f\|_2^2 \\ &= |T| \cdot |\Omega| \|f\|_2^2. \end{aligned}$$

Hence

$$|T| \cdot |\Omega| \geq (1-\varepsilon_T-\varepsilon_\Omega)^2. \qquad \square$$

Choose $\varepsilon_T = \varepsilon_\Omega = 0$ in Theorem 5.1 and observe that f is concentrated on T if and only if $\operatorname{supp} f \subseteq T$, and $\widehat{f}$ is concentrated on Ω if and only if $\operatorname{supp} \widehat{f} \subseteq \Omega$. Thus we have the following result.

Theorem 5.2. *Suppose that* $f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$, $\operatorname{supp} f \subseteq T$ *and* $\operatorname{supp} \widehat{f} \subseteq \Omega$. *Then* $|T| \cdot |\Omega| \geq 1$.

This theorem should be contrasted with the following qualitative uncertainty principle [33].

Theorem 5.3. *Suppose that* $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$, $\operatorname{supp} f \subseteq T$ *and* $\operatorname{supp} \widehat{f} \subseteq \Omega$. *If* $|T| \cdot |\Omega| \geq \infty$, *then* $f = 0$.

Note that $f \in L^p(\mathbb{R}^{2d}, \mathbb{H})$ and $|T| < \infty$, then we can see that $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$ by Hölder's inequality. On the contrary, if $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$, then by inversion formula we have that $|f| \leq \int_\Omega |\widehat{f}|_Q \, d\boldsymbol{\xi} \leq \|f\|_1 \cdot |\Omega|$ and $\|f\|_p \leq \|f\|_1 \cdot |\Omega| \cdot |T|^{1/p}$. From which we can know that $f \in L^p(\mathbb{R}^{2d}, \mathbb{H})$. Hence the theorem applies equally to L^p function.

Theorem 5.3 for the ordinary Fourier transform is due to Benedicks [33], whose elegant proof, we reproduce for the quaternion Fourier transform below. It relies on the following form of the Poissson summation formula.

Lemma 5.1 (Poissson summation formula)**.** *If $f \in L^1(\mathbb{R}^{2d}, \mathbb{H})$, the series*

$$\phi(x_1, x_2) := \sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} f(x_1 + k_1, x_2 + k_2)$$

converges in $L^1([0,1]^{2d}, \mathbb{H})$, then the quaternion Fourier series of ϕ is

$$\sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} e^{2\pi\mathbf{i}\, k_1\cdot x_1} \widehat{f}(k_1, k_2) e^{2\pi\mathbf{j}\, k_2\cdot x_2}.$$

Proof. Since ϕ converges in $L^1([0,1]^{2d}, \mathbb{H})$, we can take the quaternion Fourier series of ϕ,

$$\begin{aligned}
&\int_{[0,1]^{2d}} e^{-2\pi\mathbf{i}\, k_1\cdot x_1} \left(\sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} f(x_1 + k_1, x_2 + k_2) \right) e^{-2\pi\mathbf{j}\, k_2\cdot x_2}\, dx_1 dx_2 \\
= &\sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} \int_{[0,1]^{2d}} e^{-2\pi\mathbf{i}\, k_1\cdot x_1} f(x_1 + k_1, x_2 + k_2) e^{-2\pi\mathbf{j}\, k_2\cdot x_2}\, dx_1 dx_2 \\
= &\sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} \int_{[0,1]^{2d}} e^{-2\pi\mathbf{i}\, k_1\cdot y_1} f(y_1, y_2) e^{-2\pi\mathbf{j}\, k_2\cdot y_2}\, dy_1 dy_2 \\
= &\int_{\mathbb{R}^{2d}} e^{-2\pi\mathbf{i}\, k_1\cdot y_1} f(y_1, y_2) e^{-2\pi\mathbf{j}\, k_2\cdot y_2}\, dy_1 dy_2 \\
= &\widehat{f}(k_1, k_2).
\end{aligned}$$

So the quaternion Fourier series of ϕ is $\sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} e^{2\pi\mathbf{i}\, k_1\cdot x_1} \widehat{f}(k_1, k_2) e^{2\pi\mathbf{j}\, k_2\cdot x_2}$. □

Proof of Theorem 5.3.

Proof. We may assume that $|T| < 1$ by replacing $f(\mathbf{x})$ by $f(c\mathbf{x})$ for some $c > 0$. We have

$$\int_{[0,1]^{2d}} \sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} \chi_\Omega(\xi_1 + k_1, \xi_2 + k_2)\, d\xi_1 d\xi_2 = \int_{\mathbb{R}^{2d}} \chi_\Omega(\xi_1, \xi_2)\, d\xi_1 d\xi_2 = |\Omega| < \infty,$$

$$\int_{[0,1]^{2d}} \sum_{(k_1,k_2)\in\mathbb{Z}^{2d}} \chi_T(x_1 + k_1, x_2 + k_2)\, dx_1 dx_2 = \int_{\mathbb{R}^{2d}} \chi_T(x_1, x_2)\, dx_1 dx_2 = |T| < 1.$$

These inequalities implies, respectively, that

(i) These exists $E \subseteq [0,1]^{2d}$ with $|E| = 1$ such that $\sum \chi_\Omega(\mathbf{a} + \mathbf{k}) < +\infty$ for $\mathbf{a} := (a_1, a_2) \in E$, and hence $\widehat{f}(\mathbf{a} + \mathbf{k}) \neq 0$ for only finitely many $\mathbf{k} := (k_1, k_2)$ if $\mathbf{a} \in E$.

(ii) There exists $F \subseteq [0,1]^{2d}$ with $|F| > 0$ such that $\sum \chi_T(\mathbf{x}+\mathbf{k}) = 0$ for $\mathbf{x} := (x_1, x_2) \in F$, and hence $f(\mathbf{x}+\mathbf{k}) = 0$ for $\mathbf{k}$ if $\mathbf{x} \in F$.

Given $\mathbf{a} \in E$, let

$$\phi_{\mathbf{a}}(\mathbf{x}) = \sum_{(\mathbf{k_1},\mathbf{k_2})\in\mathbb{Z}^{\mathbf{2d}}} \mathbf{e}^{-\mathbf{2}\pi\mathbf{i}\,\mathbf{a_1}\cdot(\mathbf{x_1}+\mathbf{k_1})}\mathbf{f}(\mathbf{x_1}+\mathbf{k_1},\mathbf{x_2}+\mathbf{k_2})\mathbf{e}^{-\mathbf{2}\pi\mathbf{j}\,\mathbf{a_2}\cdot(\mathbf{x_2}+\mathbf{k_2})}.$$

Since $\phi_{\mathbf{a}}(\mathbf{x}) \in \mathbf{L^1}([\mathbf{0},\mathbf{1}]^{\mathbf{2d}}, \mathbb{H})$ and by Lemma 5.1, the quaternion Fourier transform of ϕ_a is $\sum e^{2\pi\mathbf{i}\,k_1\cdot x_1}\widehat{f}(k_1,k_2)e^{2\pi\mathbf{j}\,k_2\cdot x_2}$. Since $a \in E$, $\widehat{\phi}_a$ is a trigonometric polynomial, thus ϕ_a is a trigonometric polynomial by inversion formula. A trigonometric polynomial ϕ_a, however, cannot vanish on a set of positive measure, unless it vanishes identically. We conclude that $\phi_a = 0$ for almost all $a \in E$, whence $\widehat{\phi}_a(\mathbf{a}+\mathbf{k}) = 0$ for $a \in E$ and $k \in \mathbb{Z}^{2d}$. In other word, $\widehat{f} = 0$ *a.e.*, so $f = 0$. □

From Theorem 5.3 we obtain a qualitative statement about the quaternion Fourier transform: either $f \equiv 0$ or $|\mathrm{supp}\, f| \cdot |\mathrm{supp}\, \widehat{f}| = \infty$.

We summarize the main results in Table 1.

Table 1: The uncertainty principle.

Theorem	Condition	Conclusion
Pitt's inequality	$f \in S(\mathbb{R}^{2d}, \mathbb{H})$	$\int_{\mathbb{R}^{2d}} \lvert\boldsymbol{\xi}\rvert^{-\alpha}\lvert\widehat{f}(\boldsymbol{\xi})\rvert_Q^2\, d\boldsymbol{\xi} \le C_\alpha \int_{\mathbb{R}^{2d}} \lvert\mathbf{x}\rvert^{\alpha}\lvert f(\mathbf{x})\rvert^2\, d\mathbf{x}$, $C_\alpha := \pi^\alpha\left[\Gamma\left((2d-\alpha)/4\right)\Gamma\left((2d+\alpha)/4\right)\right]$.
Logarithmic uncertainty principle	$f \in S(\mathbb{R}^{2d}, \mathbb{H})$	$\int_{\mathbb{R}^{2d}} \ln\lvert\mathbf{x}\rvert\, \lvert f(\mathbf{x})\rvert^2\, d\mathbf{x} + \int_{\mathbb{R}^{2d}} \ln\lvert\boldsymbol{\xi}\rvert\, \lvert\widehat{f}(\boldsymbol{\xi})\rvert_Q^2\, d\boldsymbol{\xi} \ge D\int_{\mathbb{R}^{2d}} \lvert f(\mathbf{x})\rvert^2\, d\mathbf{x}$, $D := \psi(2d/4) - \ln(\pi), \quad \psi := \frac{d}{dt}[\ln\Gamma(t)]$.
Heisenberg-Weyl's uncertainty principle	$f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$ with $\lVert f\rVert_2^2 = 1$	$\Delta_f\mathbf{x}\cdot\Delta_f\boldsymbol{\xi} \ge \frac{d}{2\pi}$.
Uncertainty principle of Donoho and Stark	$0 \ne f \in L^2(\mathbb{R}^{2d}, \mathbb{H})$ is ε_T-concentrated, $\widehat{f}$ is ε_Ω-concentrated.	$\lvert T\rvert\cdot\lvert\Omega\rvert \ge (1-\varepsilon_T-\varepsilon_\Omega)^2$.

6. Conclusion

Firstly, we constructed the multiple dimensional quaternion Fourier transform by employing a symmetric rewriting only in terms of **i** and **j**. Secondly, some important properties of the quaternion Fourier transform such as Plancherel's theorem, Inversion formula, quaternion Fourier transform of $|\mathbf{x}|^{-\alpha}$ were demonstrated. Thirdly, the Heisenberg-Weyl's uncertainty principle associated with the quaternion Fourier transform was established by using logarithmic estimate obtained from a sharp form of Pitt's inequality. Finally the uncertainty principle of Donoho and Stark associated with the quaternion Fourier transform was formulated by applying the concept of ε_Ω-concentrated and Hilbert-Schmidt operator.

Acknowledgement

The author acknowledge financial support from the research acknowledges financial support from the research grant of the University of Macau No. MYRG142 (Y3-L2)-FST11-KKI, MRG002/KKI/2013/FST, MYRG099(Y1-L2)-FST13-KKI, MRG 012/KKI/2014/FST and the Macao Science and Technology Development Fund FDCT/094/2011/A, FDCT/099/2012/A3.

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