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Triple Wronskian vector solitons and rogue waves for the coupled nonlinear Schrödinger equations in the inhomogeneous plasma

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ABSTRACT

The coupled inhomogeneous nonlinear Schrödinger (NLS) equations, which describe the propagation of two nonlinear waves in the inhomogeneous plasma, are investigated. By virtue of the triple Wronskian identities, the coupled inhomogeneous NLS equations are proved to possess the triple Wronskian vector solutions based on the non-isospectral Ablowitz–Kaup–Newell–Segur system. Solving the zero potential Lax pair, we give the bright N -soliton solutions from the triple Wronskian solutions. Amplitude and velocity of the soliton are related to the damping in the plasma. Overtaking interaction, head-on interaction and bound state of the two solitons are given. Solving the non-zero potential Lax pair, we construct the multi-parametric vector rogue-wave solutions of the coupled inhomogeneous NLS equations with the Darboux transformation. Influence of the linear and parabolic density profiles on the background density and amplitude of the rogue wave, is discussed. Bright-dark solitons together with a rogue wave are presented.

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1. Introduction

Plasmas are the most abundant form of matter in the Universe [12,20]. Nonlinear systems for the fluids, plasmas, Bose–Einstein condensates (BECs), optical fibers and so on can be described by the nonlinear evolution equations (NLEEs) [1,2,12,13,18,21–23,28]. A number of the NLEEs have the soliton solutions with the Wronskian structures [10,11,14,17,19,26]. Soliton solutions in the form of the Wronskian determinant are easier to be differentiated, since any order derivative of a Wronskian determinant takes a form consisting of a sum of determinants, the number of which depends on the order of the derivative but not on the size of the determinant [10,11,19].

Ocean rogue waves are the rare events [9,15]. The height of a rogue wave is two or more times those of the surrounding waves [9,15]. Being considered initially for the ocean waves, the notion has been shifted to other fields of physics such as physics of plasma [3], BECs [5] and optical fibers [16,29]. In those fields, propagation of the multiple nonlinear waves in the nonlinear media can be described by a set of the coupled nonlinear

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Schrödinger (NLS) equations, which may possess the vector rogue-wave solutions [4,6,7]. Vector rogue-wave solutions of the coupled Gross–Pitaevskii equations, Manakov system and coupled Hirota equations have been investigated [4,6,7].

In this paper, via the triple Wronskian determinant and Darboux transformation (DT), we will concentrate our attention on the vector soliton and rogue-wave solutions of the coupled inhomogeneous NLS equations in the following form [24,25]:

$$iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 - (\alpha x - \beta^2 x^2)q_1 + i\beta q_1 = 0, \quad (1a)$$

$$iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 - (\alpha x - \beta^2 x^2)q_2 + i\beta q_2 = 0, \quad (1b)$$

which describe the propagation of two nonlinear waves in an inhomogeneous plasma with a parabolic density and constant collisional damping [24,25], where x and t are, respectively, the normalized distance and retarded time, the subscripts denote the partial derivatives, $q_1(x, t)$ and $q_2(x, t)$ are the slowly varying envelopes of two plasma waves, α indicates the coefficient of the linear density profile and β is related to the damping, αx and $\beta^2 x^2$ correspond to the linear and parabolic density profiles, respectively, and the last terms $i\beta q_1$ and $i\beta q_2$ correspond to the damping [24,25]. Bright soliton solutions have been obtained with the Husimi's and lens-type transformations [25]. Lax pair of Eqs. (1) has been constructed with the Ablowitz–Kaup–Newell–Segur (AKNS) formalism [24] and one-soliton solutions of Eqs. (1) have been derived with the Bäcklund transformation [24].

Upon review of the published literature, the triple Wronskian determinant hasn't been applied to Eqs. (1) based on the non-isospectral AKNS system. Moreover, the multi-parametric vector rogue-wave solutions and bright-dark rogue solutions of Eqs. (1) haven't been constructed via the DT (DT in the case of non-isospectral evolution has been treated [8]). The structure of this paper will be arranged as follows: In Section 2, we will adopt the DT and non-isospectral AKNS system to generate the triple Wronskian solutions of Eqs. (1), and give a proof by applying some determinantal identities. In Section 3, the bright N -soliton solutions of Eqs. (1) will be expressed in terms of the triple Wronskian via solving the zero-potential Lax pair. For $N = 1$, influence of β on the soliton amplitude and velocity will be discussed. For $N = 2$, overtaking interaction, head-on interaction and bound-state of two solitons will be given. In Section 4, solving the non-zero potential Lax pair, we will derive the multi-parametric vector rogue-wave solutions via the DT, and discuss the influence of α and β on the vector rogue waves. Bright-dark solitons together with a rogue wave in the inhomogeneous plasma will be presented. Section 5 will be our conclusions.

2. Triple Wronskian solutions

With the 3×3 AKNS scheme, the Lax pair associated with Eqs. (1) can be derived as [24]

$$\Psi_x = U\Psi = \begin{pmatrix} -i\lambda(t) & Q_1 & Q_2 \\ -Q_1^* & i\lambda(t) & 0 \\ -Q_2^* & 0 & i\lambda(t) \end{pmatrix} \Psi, \quad (2a)$$

$$\Psi_t = V\Psi = [2i\lambda^2(t)V_0 + 2\lambda(t)V_1 + iV_2]\Psi, \quad (2b)$$

with

$$V_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} i\beta x & Q_1 & Q_2 \\ -Q_1^* & -i\beta x & 0 \\ -Q_2^* & 0 & -i\beta x \end{pmatrix},$$

$$V_2 = \begin{pmatrix} |Q_1|^2 + |Q_2|^2 - \frac{\alpha x}{2} & Q_{1x} + 2i\beta x Q_1 & Q_{2x} + 2i\beta x Q_2 \\ Q_{1x}^* - 2i\beta x Q_1^* & -|Q_1|^2 + \frac{\alpha x}{2} & -Q_2 Q_1^* \\ Q_{2x}^* - 2i\beta x Q_2^* & -Q_1 Q_2^* & -|Q_2|^2 + \frac{\alpha x}{2} \end{pmatrix},$$

where U and V are the 3×3 matrices, $Q_1 = q_1 e^{-\frac{i\beta x^2}{2}}$, $Q_2 = q_2 e^{-\frac{i\beta x^2}{2}}$, $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ (the superscript T signifies the vector transpose) is the vector eigenfunction, Ψ_i 's ($i = 1, 2, 3$) are the functions of x and t , $\lambda(t) = \frac{\alpha}{4\beta} + C_1 e^{-2\beta t}$ with C_1 as a complex constant and $C_1 - C_1^* \neq 0$, “*” denotes the complex conjugate and the compatibility condition $U_t - V_x + UV - VU = 0$ is exactly equivalent to Eqs. (1). Assume that $(f_1, g_1, h_1)^T$ is a solution of Lax pair (2) with $\lambda(t) = \lambda_1(t) = \frac{\alpha}{4\beta} + C_1 e^{-2\beta t}$, and we can present the DT $(\Psi, q_1, q_2) \rightarrow (\widehat{\Psi}, \widehat{q}_1, \widehat{q}_2)$ as

$$\widehat{\Psi} = \left[\begin{pmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{pmatrix} - \begin{pmatrix} f_1 & -g_1^* & -h_1^* \\ g_1 & f_1^* & 0 \\ h_1 & 0 & f_1^* \end{pmatrix} \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_1^*(t) & 0 \\ 0 & 0 & \lambda_1^*(t) \end{pmatrix} \begin{pmatrix} f_1 & -g_1^* & -h_1^* \\ g_1 & f_1^* & 0 \\ h_1 & 0 & f_1^* \end{pmatrix}^{-1} \right] \Psi,$$

$$\widehat{q}_1 = q_1 + e^{(i\beta x^2/2)} \frac{2i[\lambda_1^*(t) - \lambda_1(t)]f_1 g_1^*}{|f_1|^2 + |g_1|^2 + |h_1|^2}, \quad \widehat{q}_2 = q_2 + e^{(i\beta x^2/2)} \frac{2i[\lambda_1^*(t) - \lambda_1(t)]f_1 h_1^*}{|f_1|^2 + |g_1|^2 + |h_1|^2}, \quad (3)$$

where f_1 , g_1 and h_1 are the functions of x and t , $\widehat{\Psi} = (\widehat{\Psi}_1, \widehat{\Psi}_2, \widehat{\Psi}_3)$ satisfies the Lax pair with the new potential vector function $(\widehat{q}_1, \widehat{q}_2)$, the superscript “-1” signifies the inverse of a matrix, $\widehat{\Psi}_i$'s, \widehat{q}_1 and \widehat{q}_2 are the functions of x and t .

With the seed solutions as zero, motivated by the DT, we set

$$q_1 = e^{(i\beta x^2/2)} \frac{v}{s}, \quad q_2 = e^{(i\beta x^2/2)} \frac{w}{s}, \quad (4)$$

where v , w and s can be expressed as [26,27]

$$s = \begin{vmatrix} F_{N \times N} & -G_{N \times N} & -H_{N \times N} \\ G_{N \times N}^* & F_{N \times N}^* & 0 \\ H_{N \times N}^* & 0 & F_{N \times N}^* \end{vmatrix}, \quad v = 2 \begin{vmatrix} F_{N \times (N+1)} & -G_{N \times (N-1)} & -H_{N \times N} \\ G_{N \times (N+1)}^* & F_{N \times (N-1)}^* & 0 \\ H_{N \times (N+1)}^* & 0 & F_{N \times N}^* \end{vmatrix},$$

$$w = 2 \begin{vmatrix} F_{N \times (N+1)} & -G_{N \times N} & -H_{N \times (N-1)} \\ G_{N \times (N+1)}^* & F_{N \times N}^* & 0 \\ H_{N \times (N+1)}^* & 0 & F_{N \times (N-1)}^* \end{vmatrix}, \quad (5)$$

with $F_{N \times M}$, $G_{N \times M}$ and $H_{N \times M}$ ($N = 1, 2, \dots; M = N - 1, N, N + 1$) as

$$F_{N \times M} = \begin{pmatrix} f_1 & f_1^1 & \cdots & f_1^{M-1} \\ f_2 & f_2^1 & \cdots & f_2^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^1 & \cdots & f_N^{M-1} \end{pmatrix}, \quad G_{N \times M} = \begin{pmatrix} g_1 & g_1^1 & \cdots & g_1^{M-1} \\ g_2 & g_2^1 & \cdots & g_2^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_N & g_N^1 & \cdots & g_N^{M-1} \end{pmatrix},$$

$$H_{N \times M} = \begin{pmatrix} h_1 & h_1^1 & \cdots & h_1^{M-1} \\ h_2 & h_2^1 & \cdots & h_2^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_N & h_N^1 & \cdots & h_N^{M-1} \end{pmatrix}, \quad (6)$$

where $f_k^j = \partial^j f_k / \partial x^j$, $g_k^j = \partial^j g_k / \partial x^j$ and $h_k^j = \partial^j h_k / \partial x^j$ ($k = 1, 2, \dots, N$; $j = 1, 2, \dots, M$).

$(f_k, g_k, h_k)^T$ are the linearly independent solutions of the linear system

$$\begin{pmatrix} f_{k,x} \\ g_{k,x} \\ h_{k,x} \end{pmatrix} = \begin{pmatrix} -i\lambda_k(t) & 0 & 0 \\ 0 & i\lambda_k(t) & 0 \\ 0 & 0 & i\lambda_k(t) \end{pmatrix} \begin{pmatrix} f_k \\ g_k \\ h_k \end{pmatrix}, \quad (7a)$$

$$\begin{pmatrix} f_{k,t} \\ g_{k,t} \\ h_{k,t} \end{pmatrix} = \begin{pmatrix} A_k & 0 & 0 \\ 0 & -A_k & 0 \\ 0 & 0 & -A_k \end{pmatrix} \begin{pmatrix} f_k \\ g_k \\ h_k \end{pmatrix}, \quad (7b)$$

where $A_k = -2i\lambda_k^2(t) + 2\lambda_k(t)i\beta x - i\frac{\alpha x}{2}$ and $\lambda_k(t) = \frac{\alpha}{4\beta} + C_k e^{-2\beta t}$ with C_k as the complex constants and $C_k - C_k^* \neq 0$. Besides, the complex conjugates of q_1 and q_2 can be written as

$$q_1^* = e^{(-i\beta x^2/2)} \frac{\bar{v}}{s}, \quad q_2^* = e^{(-i\beta x^2/2)} \frac{\bar{w}}{s}, \quad (8)$$

where \bar{v} and \bar{w} can be expressed as

$$\bar{v} = \begin{vmatrix} F_{N \times (N-1)} & -G_{N \times (N+1)} & -H_{N \times N} \\ G_{N \times (N-1)}^* & F_{N \times (N+1)}^* & 0 \\ H_{N \times (N-1)}^* & 0 & F_{N \times N}^* \end{vmatrix}, \quad \bar{w} = 2 \begin{vmatrix} F_{N \times (N-1)} & -G_{N \times N} & -H_{N \times (N+1)} \\ G_{N \times (N-1)}^* & F_{N \times N}^* & 0 \\ H_{N \times (N-1)}^* & 0 & F_{N \times (N+1)}^* \end{vmatrix}. \quad (9)$$

Based on the triple Wronskian notions [26,27], s , v , w , \bar{v} and \bar{w} can be written as:

$$\begin{aligned} s &= |\widehat{N-1; N-1; N-1}|, & v &= 2|\widehat{N; N-2; N-1}|, \\ \bar{v} &= 2|\widehat{N-2; N; N-1}|, & w &= 2|\widehat{N; N-1; N-2}|, \\ \bar{w} &= 2|\widehat{N-2; N-1; N}|. \end{aligned} \quad (10)$$

According to system (7), we obtain

$$(f_k)_x = -i\lambda_k f_k, \quad (g_k)_x = i\lambda_k g_k, \quad (h_k)_x = i\lambda_k h_k, \quad (11a)$$

$$(f_k)_t = 2i(f_k)_{xx} - 2\beta x(f_k)_x - i\frac{\alpha x}{2} f_k, \quad (g_k)_t = -2i(g_k)_{xx} - 2\beta x(g_k)_x + i\frac{\alpha x}{2} g_k, \quad (11b)$$

$$(h_k)_t = -2i(h_k)_{xx} - 2\beta x(h_k)_x + i\frac{\alpha x}{2} h_k. \quad (11c)$$

Based on the properties of the Wronskian determinant and conditions (11), we can obtain the following various-order derivatives of s , v and w :

$$\begin{aligned} s_x &= |\widehat{N-2, N; N-1; N-1}| + |\widehat{N-1; N-2, N; N-1}| + |\widehat{N-1; N-1; N-2, N}|, \\ s_{xx} &= |\widehat{N-3, N-1, N; N-1; N-1}| + |\widehat{N-2, N+1; N-1; N-1}| \\ &\quad + |\widehat{N-1; N-3, N-1, N; N-1}| + |\widehat{N-1; N-2, N+1; N-1}| \\ &\quad + |\widehat{N-1; N-1; N-3, N-1, N}| + |\widehat{N-1; N-1; N-2, N+1}| \\ &\quad + 2|\widehat{N-2, N; N-2, N; N-1}| + 2|\widehat{N-2, N; N-1; N-2, N}| \\ &\quad + 2|\widehat{N-1; N-2, N; N-2, N}|, \\ v_x &= 2|\widehat{N-1, N+1; N-2; N-1}| + 2|\widehat{N; N-3, N-1; N-1}| + 2|\widehat{N; N-2; N-2, N}|, \\ v_{xx} &= 2(|\widehat{N-2, N, N+1; N-2; N-1}| + |\widehat{N-1, N+2; N-2; N-1}| \\ &\quad + |\widehat{N; N-2; N-2, N+1}| + |\widehat{N; N-3, N; N-1}| \\ &\quad + |\widehat{N; N-2; N-3, N-1, N}| + |\widehat{N; N-4, N-2, N-1; N-1}| \\ &\quad + 2|\widehat{N-1, N+1; N-3, N-1; N-1}| + 2|\widehat{N-1, N+1; N-2; N-2, N}| \\ &\quad + 2|\widehat{N; N-3, N-1; N-2, N}|), \\ w_x &= 2|\widehat{N-1, N+1; N-1; N-2}| + 2|\widehat{N; N-2, N; N-2}| + 2|\widehat{N; N-1; N-3, N-1}|, \\ w_{xx} &= 2(|\widehat{N-2, N, N+1; N-1; N-2}| + |\widehat{N-1, N+2; N-1; N-2}| \\ &\quad + |\widehat{N; N-2, N+1; N-2}| + |\widehat{N; N-1; N-3, N}| \end{aligned}$$

$$\begin{aligned}
& + |\widehat{N}; \widehat{N-3}, N-1, N; \widehat{N-2}| + |\widehat{N}; \widehat{N-1}; \widehat{N-4}, N-2, N-1| \\
& + 2|\widehat{N-1}, N+1; \widehat{N-1}; \widehat{N-3}, N-1| + 2|\widehat{N-1}, N+1; \widehat{N-2}, N; \widehat{N-2}| \\
& + 2|\widehat{N}; \widehat{N-2}, N; \widehat{N-3}, N-1|), \\
s_t = & -2\beta x(|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|) \\
& - 3\beta N(N-1)|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}| + Ni\frac{\alpha x}{2}|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}| \\
& + 2i(|\widehat{N-3}, N, N-1; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-2}, N+1; \widehat{N-1}; \widehat{N-1}|) \\
& - 2i(|\widehat{N-1}; \widehat{N-3}, N, N-1; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-2}, N+1; \widehat{N-1}|) \\
& - 2i(|\widehat{N-1}; \widehat{N-1}; \widehat{N-3}, N, N-1| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N+1|), \\
v_t = & -4\beta x(|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}| + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N|) \\
& + 4i(|\widehat{N-2}, N+1, N; \widehat{N-2}; \widehat{N-1}| + |\widehat{N-1}, N+2; \widehat{N-2}; \widehat{N-1}|) \\
& - 4i(|\widehat{N}; \widehat{N-4}, N-1, N-2; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N; \widehat{N-1}|) \\
& - 4i(|\widehat{N}; \widehat{N-2}; \widehat{N-3}, N, N-1| + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N+1|) \\
& + (N-2)i\alpha x|\widehat{N}; \widehat{N-2}; \widehat{N-1}| - 4\beta\left(\frac{3}{2}N^2 - \frac{3}{2}N + 1\right)|\widehat{N}; \widehat{N-2}; \widehat{N-1}|, \\
w_t = & -4\beta x(|\widehat{N-1}, N+1; \widehat{N-1}; \widehat{N-2}| + |\widehat{N}; \widehat{N-2}, N; \widehat{N-2}| + |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N-1|) \\
& + 4i(|\widehat{N-2}, N+1, N; \widehat{N-1}; \widehat{N-2}| + |\widehat{N-1}, N+2; \widehat{N-1}; \widehat{N-2}|) \\
& - 4i(|\widehat{N}; \widehat{N-3}, N, N-1; \widehat{N-2}| + |\widehat{N}; \widehat{N-2}, N+1; \widehat{N-2}|) \\
& - 4i(|\widehat{N}; \widehat{N-1}; \widehat{N-4}, N-1, N-2| + |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N|) \\
& + (N-2)i\alpha x|\widehat{N}; \widehat{N-1}; \widehat{N-2}| - 4\beta\left(\frac{3}{2}N^2 - \frac{3}{2}N + 1\right)|\widehat{N}; \widehat{N-1}; \widehat{N-2}|.
\end{aligned}$$

Substituting transformations (4) and (8) into Eqs. (1), we obtain

$$(iD_t + D_x^2 + 2i\beta x D_x + 2i\beta - \alpha x)v \cdot s = 0, \quad (12a)$$

$$(iD_t + D_x^2 + 2i\beta x D_x + 2i\beta - \alpha x)w \cdot s = 0, \quad (12b)$$

$$D_x^2 s \cdot s = 2(v\bar{v} + w\bar{w}), \quad (12c)$$

where D_x and D_t are both the bilinear derivative operators [13] defined by

$$D_x^l D_t^n (\sigma \cdot \varsigma) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^l \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n \sigma(x, t) \varsigma(x', t') \Big|_{\substack{t'=t \\ x'=x}}, \quad l, n = 0, 1, 2, 3, \dots,$$

where n and l are the positive integers, σ is the function of x and t , and ς is the function of the formal variables x' and t' .

Based on the lemmas in Refs. [10,11,19,27], we have given a proof that s , v , \bar{v} , w and \bar{w} satisfy bilinear forms (12) in Appendix A.

3. Properties of the triple Wronskian solitons

Bright N -soliton solutions for Eqs. (1) can be represented in terms of the triple Wronskian,

$$q_1 = 2e^{(i\beta x^2/2)} \frac{|\widehat{N}; \widehat{N-2}; \widehat{N-1}|}{|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|}, \quad (13a)$$

$$q_2 = 2e^{(i\beta x^2/2)} \frac{|\widehat{N}; \widehat{N-1}; \widehat{N-2}|}{|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|}, \quad (13b)$$

where $(f_k, g_k, h_k) = (\gamma_k e^{\theta_k}, \delta_k e^{-\theta_k}, \nu_k e^{-\theta_k})$, $\theta_k = e^{-i(\frac{\alpha}{4\beta} + C_k e^{-2\beta t})x + \frac{-i\alpha^2}{8\beta^2}t + i\frac{\alpha}{2\beta^2}C_k e^{-2\beta t} + \frac{iC_k^2}{2\beta}e^{-4\beta t}}$, γ_k 's, δ_k 's and ν_k 's are the arbitrary complex constants.

In the case of $N = 1$, the one-soliton solutions for Eqs. (1) can be given as

$$\begin{aligned} q_1 &= e^{(i\beta x^2/2)} \frac{v}{s} = 2i\gamma_1 \delta_1^* (C_1 - C_1^*) e^{-2\beta t + i\beta x^2/2} \frac{e^{\theta_1 - \theta_1^*}}{\gamma_1 \gamma_1^* e^{\theta_1 + \theta_1^*} + (\delta_1 \delta_1^* + \nu_1 \nu_1^*) e^{-(\theta_1 + \theta_1^*)}} \\ &= \frac{2i\gamma_1 \delta_1^* (C_1 - C_1^*) e^{-2\beta t}}{\delta_1 \delta_1^* + \nu_1 \nu_1^*} \operatorname{sech} \left[\theta_1 + \theta_1^* + \frac{\ln a}{2} \right] e^{i\beta x^2/2 + i(\theta_1 - \theta_1^*)}, \end{aligned} \quad (14a)$$

$$\begin{aligned} q_2 &= e^{(i\beta x^2/2)} \frac{w}{s} = 2i\gamma_1 \nu_1^* (C_1 - C_1^*) e^{-2\beta t + i\beta x^2/2} \frac{e^{\theta_1 - \theta_1^*}}{\gamma_1 \gamma_1^* e^{\theta_1 + \theta_1^*} + (\delta_1 \delta_1^* + \nu_1 \nu_1^*) e^{-(\theta_1 + \theta_1^*)}} \\ &= \frac{2i\gamma_1 \nu_1^* (C_1 - C_1^*) e^{-2\beta t}}{\delta_1 \delta_1^* + \nu_1 \nu_1^*} \operatorname{sech} \left[\theta_1 + \theta_1^* + \frac{\ln a}{2} \right] e^{i\beta x^2/2 + i(\theta_1 - \theta_1^*)}, \end{aligned} \quad (14b)$$

where $a = \frac{\gamma_1 \gamma_1^*}{\delta_1 \delta_1^* + \nu_1 \nu_1^*}$. Initial phase of the one soliton is determined by $\frac{\alpha}{2C_{1I}\beta^2} + \frac{C_{1R}}{\beta}$ with the subscripts R and I being respectively the real and imaginary parts. Bright one-soliton solutions are characterized by the soliton amplitude and velocity which are, respectively, $\frac{2|C_{1I}||\gamma_1||\delta_1|e^{-2\beta t}}{\delta_1 \delta_1^* + \nu_1 \nu_1^*}$ and $-2C_{1R}e^{-2\beta t}$.

Adopting the parameter $N = 2$ in solutions (13) yields the two-soliton solutions for Eqs. (1),

$$q_1 = e^{(i\beta x^2/2)} \frac{v}{s}, \quad (15a)$$

$$q_2 = e^{(i\beta x^2/2)} \frac{w}{s}, \quad (15b)$$

where

$$s = \begin{vmatrix} f_1 & f_{1,x} & -g_1 & -g_{1,x} & -h_1 & -h_{1,x} \\ f_2 & f_{2,x} & -g_2 & -g_{2,x} & -h_2 & -h_{2,x} \\ g_1^* & g_{1,x}^* & f_1^* & f_{1,x}^* & 0 & 0 \\ g_2^* & g_{2,x}^* & f_2^* & f_{2,x}^* & 0 & 0 \\ h_1^* & h_{1,x}^* & 0 & 0 & f_1^* & f_{1,x}^* \\ h_2^* & h_{2,x}^* & 0 & 0 & f_2^* & f_{2,x}^* \end{vmatrix}, \quad v = 2 \begin{vmatrix} f_1 & f_{1,x} & f_{1,xx} & -g_1 & -h_1 & -h_{1,x} \\ f_2 & f_{2,x} & f_{2,xx} & -g_2 & -h_2 & -h_{2,x} \\ g_1^* & g_{1,x}^* & g_{1,xx}^* & f_1^* & 0 & 0 \\ g_2^* & g_{2,x}^* & g_{2,xx}^* & f_2^* & 0 & 0 \\ h_1^* & h_{1,x}^* & h_{1,xx}^* & 0 & f_1^* & f_{1,x}^* \\ h_2^* & h_{2,x}^* & h_{2,xx}^* & 0 & f_2^* & f_{2,x}^* \end{vmatrix},$$

$$w = 2 \begin{vmatrix} f_1 & f_{1,x} & f_{1,xx} & -g_1 & -g_{1,x} & -h_1 \\ f_2 & f_{2,x} & f_{2,xx} & -g_2 & -g_{2,x} & -h_2 \\ g_1^* & g_{1,x}^* & g_{1,xx}^* & f_1^* & f_{1,x}^* & 0 \\ g_2^* & g_{2,x}^* & g_{2,xx}^* & f_2^* & f_{2,x}^* & 0 \\ h_1^* & h_{1,x}^* & h_{1,xx}^* & 0 & 0 & f_1^* \\ h_2^* & h_{2,x}^* & h_{2,xx}^* & 0 & 0 & f_2^* \end{vmatrix}.$$

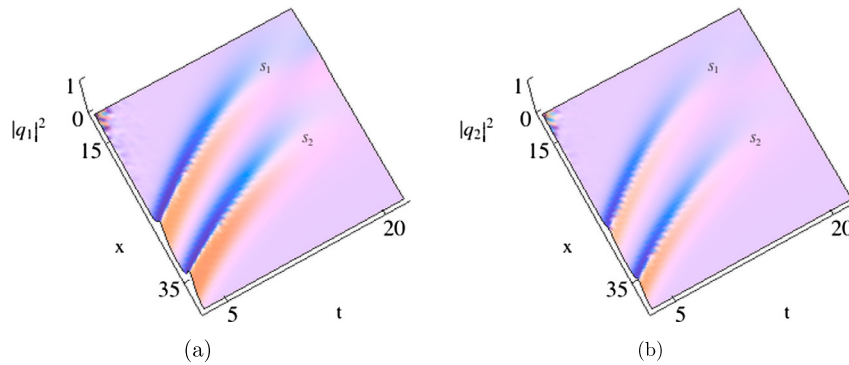


Fig. 1. Overtaking interactions of vector two solitons via solutions (15) with $\gamma_1 = 1$, $\delta_1 = 1.2$, $\nu_1 = 0.8$, $\gamma_2 = 0.9$, $\delta_2 = 1.1$, $\nu_2 = 0.7$, $C_1 = 0.7 + 0.5i$, $C_2 = 0.8 + 0.6i$, $\alpha = 0.02$ and $\beta = 0.03$.

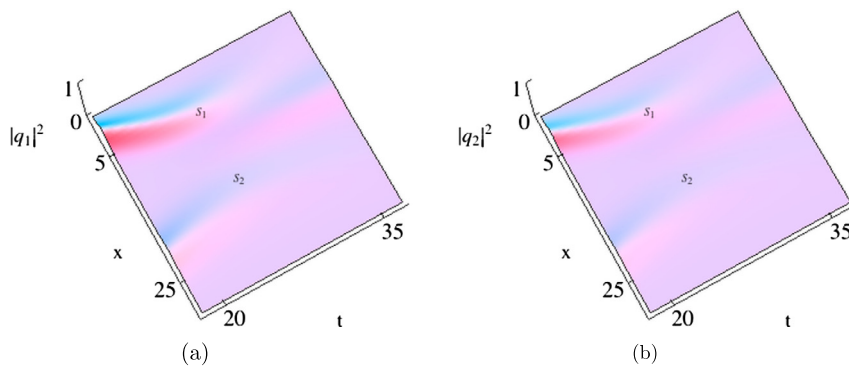


Fig. 2. Head-on interactions of vector two solitons via solutions (15) with $\gamma_1 = 1$, $\delta_1 = 1.2$, $\nu_1 = 0.8$, $\gamma_2 = 0.9$, $\delta_2 = 1.1$, $\nu_2 = 0.7$, $C_1 = 0.7 + 0.6i$, $C_2 = -0.8 + 0.9i$, $\alpha = 0.02$ and $\beta = 0.03$.

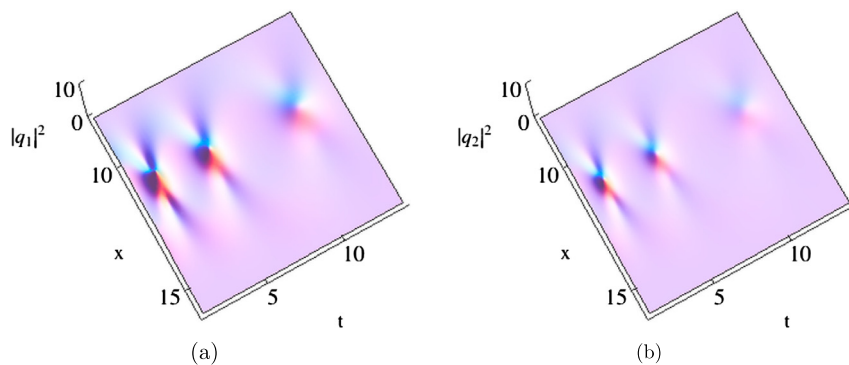


Fig. 3. Bound-state solitons via solutions (15) with $\gamma_1 = 1$, $\delta_1 = 1.2$, $\nu_1 = 0.8$, $\gamma_2 = 0.9$, $\delta_2 = 1.1$, $\nu_2 = 0.7$, $C_1 = 0.5i$, $C_2 = i$, $\alpha = 0.02$ and $\beta = 0.03$.

Overtaking interactions of two bright solitons are displayed in Fig. 1. Head-on interactions of two bright solitons are presented in Fig. 2. Fig. 3 shows that the two parallel solitons propagate with a periodical mergence, resulting in the temporary enhancement of the excitations at certain positions and bound-state solitons can be formed due to the interaction between the two solitons.

4. Vector rogue waves and bright-dark solitons together with a rogue wave

In Section 2, choosing $q_1 = q_2 = 0$ as the seed solutions, we have constructed the triple Wronskian soliton solutions of Eqs. (1). In this section, to obtain the vector rogue-wave solutions of Eqs. (1), we consider the following cases:

Starting with the seed solutions

$$q_1 = \frac{j_1}{\sqrt{j_1^2 + j_2^2}} e^{-2\beta t + i[a(t)x^2 + b(t)x + c(t)]} \quad \text{and} \quad q_2 = \frac{j_2}{\sqrt{j_1^2 + j_2^2}} e^{-2\beta t + i[a(t)x^2 + b(t)x + c(t)]}$$

with

$$\begin{aligned} a(t) &= \frac{\beta}{2}, \quad b(t) = -\frac{\alpha}{2\beta} + \frac{\alpha}{2\beta} e^{-2\beta t}, \\ c(t) &= \frac{1 - e^{-4t\beta}}{2\beta} + \frac{\alpha^2(3 + e^{-4t\beta} - 4e^{-2t\beta} - 4t\beta)}{16\beta^3}, \end{aligned}$$

we get the corresponding solutions for the linear spectral problem at $\lambda(t) = \frac{\alpha}{4\beta} + ihe^{-2\beta t}$ as

$$\Psi(\epsilon) = \begin{pmatrix} (-l_1 e^{-A} + l_2 e^A) e^{\frac{i}{2}[b(t)x + c(t)]} \\ \left(\frac{j_1}{\sqrt{j_1^2 + j_2^2}} l_2 e^{-A} - \frac{j_1}{\sqrt{j_1^2 + j_2^2}} l_1 e^A \right) e^{-\frac{i}{2}[b(t)x + c(t)]} \\ \left(\frac{j_2}{\sqrt{j_1^2 + j_2^2}} l_2 e^{-A} - \frac{j_2}{\sqrt{j_1^2 + j_2^2}} l_1 e^A \right) e^{-\frac{i}{2}[b(t)x + c(t)]} \end{pmatrix}, \quad (16)$$

where

$$\begin{aligned} l_1 &= \frac{\sqrt{h - \sqrt{h^2 - 1}}}{\sqrt{h^2 - 1}}, \quad l_2 = \frac{\sqrt{h + \sqrt{h^2 - 1}}}{\sqrt{h^2 - 1}}, \quad h = 1 + \epsilon^2, \\ A &= \sqrt{h^2 - 1} \left(x e^{-2\beta t} - \frac{\alpha}{2\beta^2} e^{-2\beta t} + \frac{\alpha}{4\beta^2} e^{-4\beta t} + \frac{\alpha}{4\beta^2} \right) + \frac{ih\sqrt{h^2 - 1}(1 - e^{-4t\beta})}{2\beta}, \end{aligned}$$

with ϵ as the formal parameter. Expanding the vector functions $\Psi(\epsilon)$ at $\epsilon = 0$, we have

$$\Psi(\epsilon) = \Psi^{[0]} + \Psi^{[1]}\epsilon^2 + \dots, \quad (17)$$

where $\Psi^{[k]} = \frac{1}{(2k)!} \frac{\partial^{2k}}{\partial \epsilon^{2k}} \Psi(\epsilon) \Big|_{\epsilon=0}$ ($k = 0, 1, 2, \dots$). Then, we can derive

$$\Psi^{[0]} = \begin{pmatrix} \frac{\alpha - 2e^{2t\beta}\alpha + e^{4t\beta}\alpha - 2i\beta + 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 + 4e^{2t\beta}x\beta^2}{2\beta^2} e^{\Lambda - 4t\beta} \\ \frac{-\alpha + 2e^{2t\beta}\alpha - e^{4t\beta}\alpha + 2i\beta - 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 - 4e^{2t\beta}x\beta^2}{2\sqrt{j_1^2 + j_2^2}\beta^2} j_1 e^{-\Lambda - 4t\beta} \\ \frac{-\alpha + 2e^{2t\beta}\alpha - e^{4t\beta}\alpha + 2i\beta - 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 - 4e^{2t\beta}x\beta^2}{2\sqrt{j_1^2 + j_2^2}\beta^2} j_2 e^{-\Lambda - 4t\beta} \end{pmatrix}, \quad (18)$$

with

$$\Lambda = \frac{3i\alpha^2}{32\beta^3} + \frac{ie^{-4t\beta}\alpha^2}{32\beta^3} - \frac{ie^{-2t\beta}\alpha^2}{8\beta^3} - \frac{it\alpha^2}{8\beta^2} + \frac{i}{4\beta} - \frac{ie^{-4t\beta}}{4\beta} - \frac{ix\alpha}{4\beta} + \frac{ix\alpha e^{-2t\beta}}{4\beta} - 4t\beta. \quad (19)$$

We note that $\Psi^{[0]}$ is a solution of Lax pair (7). By means of expression (3), we obtain the vector rogue-wave solutions of Eqs. (1),

$$q_1 = \frac{j_1}{\sqrt{j_1^2 + j_2^2}} e^{-2\beta t + i[a(t)x^2 + b(t)x + c(t)]} \frac{G_1}{F_1}, \quad (20a)$$

$$q_2 = \frac{j_2}{\sqrt{j_1^2 + j_2^2}} e^{-2\beta t + i[a(t)x^2 + b(t)x + c(t)]} \frac{G_1}{F_1}, \quad (20b)$$

where

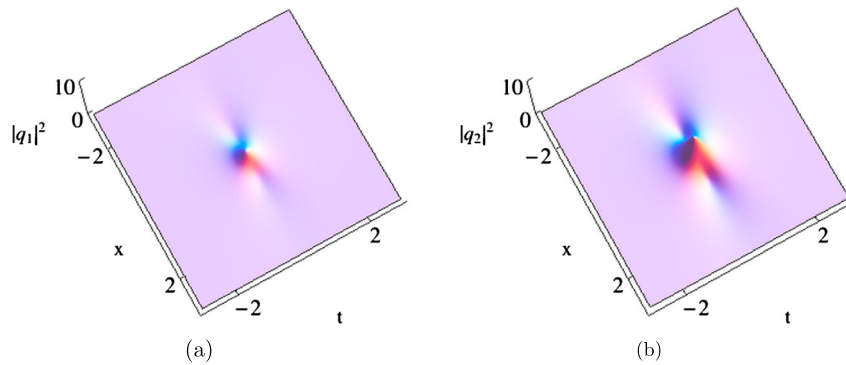


Fig. 4. Vector rogue waves via solutions (20) with $j_1 = 1$, $j_2 = \sqrt{3}$, $\alpha = 0.01$ and $\beta = 0.02$.

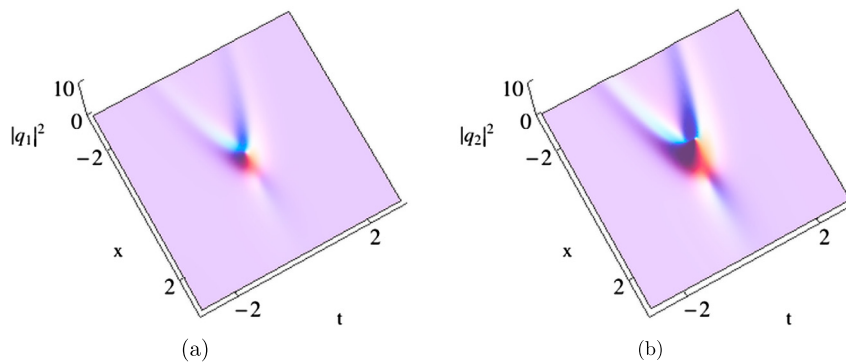


Fig. 5. Vector rogue waves via solutions (20) with $j_1 = 1$, $j_2 = \sqrt{3}$, $\alpha = 2$ and $\beta = 0.02$.

$$\begin{aligned} G_1 &= \alpha^2 - 4e^{2t\beta}\alpha^2 + 6e^{4t\beta}\alpha^2 - 4e^{6t\beta}\alpha^2 + e^{8t\beta}\alpha^2 + 4\beta^2 - 8e^{4t\beta}\beta^2 + 4e^{8t\beta}\beta^2 \\ &\quad + 8e^{2t\beta}x\alpha\beta^2 - 16e^{4t\beta}x\alpha\beta^2 + 8e^{6t\beta}x\alpha\beta^2 + 16ie^{4t\beta}\beta^3 - 16ie^{8t\beta}\beta^3 - 12e^{8t\beta}\beta^4 + 16e^{4t\beta}x^2\beta^4, \\ F_1 &= \alpha^2 - 4e^{2t\beta}\alpha^2 + 6e^{4t\beta}\alpha^2 - 4e^{6t\beta}\alpha^2 + e^{8t\beta}\alpha^2 + 4\beta^2 - 8e^{4t\beta}\beta^2 + 4e^{8t\beta}\beta^2 \\ &\quad + 8e^{2t\beta}x\alpha\beta^2 - 16e^{4t\beta}x\alpha\beta^2 + 8e^{6t\beta}x\alpha\beta^2 + 4e^{8t\beta}\beta^4 + 16e^{4t\beta}x^2\beta^4. \end{aligned} \quad (21)$$

Vector rogue waves with the presence of linear density profile α , parabolic density profile and damping term β are shown in Fig. 4. We note that solutions (20) are simply the solutions of the scalar counterpart of Eqs. (1) with no interaction dynamics. With the increase of the value of linear density profile α , it can be seen that the amplitudes of the vector rogue waves keep unchanged. However, the rogue wave seems to emerge from a small amplitude wave that moves in the direction of lower potential growing in amplitude and decreasing in width. At the turning point, the rogue wave reaches its maximum amplitude, as shown in Fig. 5. Increasing the parabolic density profile β , we find that the vector rogue waves exist on top of a monotonically decreasing background, as shown in Fig. 6.

Ref. [4] has provided the evidence of interactions between the vector dark-bright solitons and the rogue wave. The experimental conditions for the observation of a vector dark-bright soliton together with a rogue wave have also been discussed [4]. Motivated by those, we will construct the bright- and dark-rogue solutions of Eqs. (1).

When $j_1 = 0$ and $j_2 = 1$, we obtain the seed solutions $q_1 = 0$ and $q_2 = e^{-2\beta t + i[a(t)x^2 + b(t)x + c(t)]}$ with

$$\begin{aligned} a(t) &= \frac{\beta}{2}, \quad b(t) = -\frac{\alpha}{2\beta} + \frac{\alpha}{2\beta}e^{-2\beta t}, \\ c(t) &= \frac{1 - e^{-4t\beta}}{2\beta} + \frac{\alpha^2(3 + e^{-4t\beta} - 4e^{-2t\beta} - 4t\beta)}{16\beta^3}. \end{aligned}$$

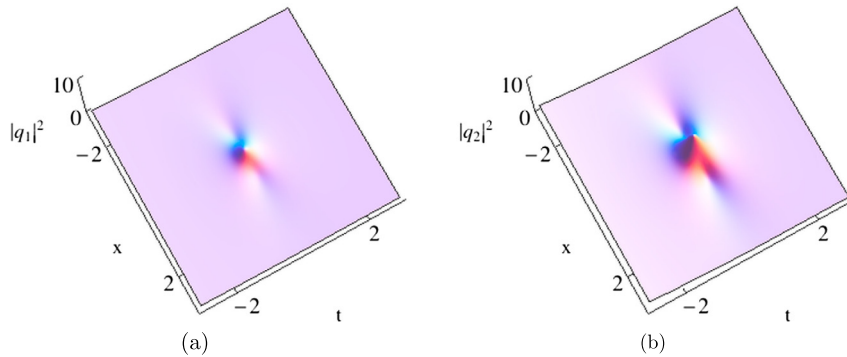


Fig. 6. Vector rogue waves via solutions (20) with $j_1 = 1$, $j_2 = \sqrt{3}$, $\alpha = 0.01$ and $\beta = 0.1$.

We get the corresponding solutions for the linear spectral problem at $\lambda(t) = \frac{\alpha}{4\beta} + ih e^{-2\beta t}$ as

$$\Psi(\epsilon) = \begin{pmatrix} (-l_1 e^{-A} + l_2 e^A) e^{\frac{i}{2}[b(t)x+c(t)]} \\ e^{i(\frac{\alpha}{4\beta} + C_k e^{-2\beta t})x + \frac{i\alpha^2}{8\beta^2}t - i\frac{\alpha}{2\beta^2}C_k e^{-2\beta t} - \frac{iC_k^2}{2\beta} e^{-4\beta t}} \\ (l_2 e^{-A} - l_1 e^A) e^{-\frac{i}{2}[b(t)x+c(t)]} \end{pmatrix}, \quad (22)$$

where

$$l_1 = \frac{\sqrt{h - \sqrt{h^2 - 1}}}{\sqrt{h^2 - 1}}, \quad l_2 = \frac{\sqrt{h + \sqrt{h^2 - 1}}}{\sqrt{h^2 - 1}}, \quad C_k = ih, \quad h = 1 + \epsilon^2, \\ A = \sqrt{h^2 - 1} \left(x e^{-2\beta t} - \frac{\alpha}{2\beta^2} e^{-2\beta t} + \frac{\alpha}{4\beta^2} e^{-4\beta t} + \frac{\alpha}{4\beta^2} \right) + \frac{ih\sqrt{h^2 - 1}(1 - e^{-4t\beta})}{2\beta},$$

with ϵ as the formal parameter. Expanding the vector functions $\Psi(\epsilon)$ at $\epsilon = 0$, we have

$$\Psi(\epsilon) = \Psi^{[0]} + \Psi^{[1]}\epsilon^2 + \dots, \quad (23)$$

where $\Psi^{[k]} = \frac{1}{(2k)!} \frac{\partial^{2k}}{\partial \epsilon^{2k}} \Psi(\epsilon)|_{\epsilon=0}$ ($k = 0, 1, 2, \dots$). Then, we can derive

$$\Psi^{[0]} = \begin{pmatrix} \frac{\alpha - 2e^{2t\beta}\alpha + e^{4t\beta}\alpha - 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 + 4e^{2t\beta}x\beta^2}{2\beta^2} e^{A-4t\beta} \\ e^{-e^{-2t\beta}x + \frac{e^{-2\beta t}\alpha}{2\beta^2} + \frac{it\alpha^2}{8\beta^2} + \frac{ie^{-4t\beta}}{2\beta} + \frac{ix\alpha}{4\beta}} \\ \frac{-\alpha + 2e^{2t\beta}\alpha - e^{4t\beta}\alpha + 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 - 4e^{2t\beta}x\beta^2}{2\beta^2} e^{-A-4t\beta} \end{pmatrix}, \quad (24)$$

with

$$A = \frac{3i\alpha^2}{32\beta^3} + \frac{ie^{-4t\beta}\alpha^2}{32\beta^3} - \frac{ie^{-2t\beta}\alpha^2}{8\beta^3} - \frac{it\alpha^2}{8\beta^2} + \frac{i}{4\beta} - \frac{ie^{-4t\beta}}{4\beta} - \frac{ix\alpha}{4\beta} + \frac{ix\alpha e^{-2t\beta}}{4\beta} - 4t\beta. \quad (25)$$

We note that $\Psi^{[0]}$ is a solution of Lax pair (7). By means of expression (3), we obtain the bright- and dark-rogue solutions of Eqs. (1),

$$q_1 = e^{(i\beta x^2/2)} \frac{2i[\lambda^*(t) - \lambda(t)]f_1 g_1^*}{|f_1|^2 + |g_1|^2 + |h_1|^2}, \quad (26a)$$

$$q_2 = e^{-2\beta t + i[a(t)x^2 + b(t)x + c(t)]} + e^{(i\beta x^2/2)} \frac{2i[\lambda^*(t) - \lambda(t)]f_1 h_1^*}{|f_1|^2 + |g_1|^2 + |h_1|^2}, \quad (26b)$$

where

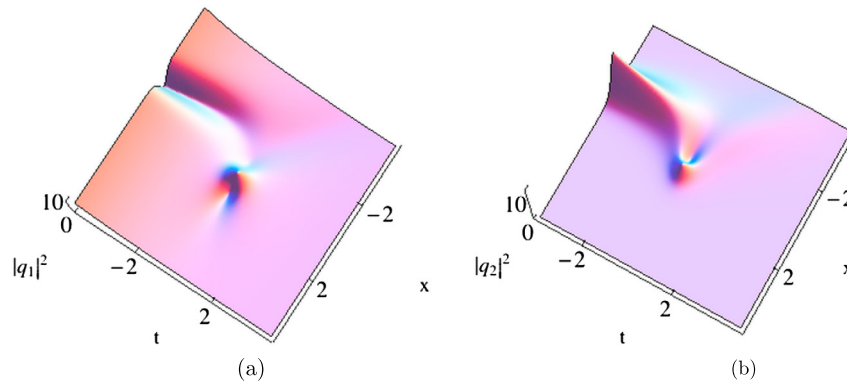


Fig. 7. Bright-dark solitons together with a rogue wave via solutions (26) with $\alpha = 0.01$ and $\beta = 0.1$.

$$g_1 = e^{-e^{-2t\beta}x + \frac{e^{-2\beta t}\alpha}{2\beta^2} + \frac{it\alpha^2}{8\beta^2} + \frac{ie^{-4t\beta}}{2\beta} + \frac{ix\alpha}{4\beta}},$$

$$f_1 = \frac{\alpha - 2e^{2t\beta}\alpha + e^{4t\beta}\alpha - 2i\beta + 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 + 4e^{2t\beta}x\beta^2}{2\beta^2}e^{\Lambda-4t\beta},$$

$$h_1 = \frac{-\alpha + 2e^{2t\beta}\alpha - e^{4t\beta}\alpha + 2i\beta - 2ie^{4t\beta}\beta + 2e^{4t\beta}\beta^2 - 4e^{2t\beta}x\beta^2}{2\beta^2}e^{-\Lambda-4t\beta}.$$

In this case, each wave component q_j is a mixture of a dark and a bright pulse. Fig. 7(a) shows a dark soliton together with a rogue wave, while Fig. 7(b) shows a bright soliton together with a rogue wave. From Fig. 7, solutions (26) describe a dark and a bright pulse, respectively, asymptotically as $t \rightarrow \pm\infty$.

5. Conclusions

The coupled inhomogeneous NLS equations, i.e., Eqs. (1), which describe the propagation of two nonlinear waves in the inhomogeneous plasma, have been investigated. We note that, in the plasma, α indicates the coefficient of the linear density profile and β is related to the damping. Main results of this paper are as follows:

(A) We have constructed triple Wronskian solutions (4) of Eqs. (1). Solving the zero-potential Lax pair, we have given the bright N -soliton solutions (13). Based on Lemmas 1–3, Triple Wronskian identities (29) and (32), we have given a proof that s , v , \bar{v} , w and \bar{w} satisfy bilinear forms (12). For $N = 1$ and 2, vector one- and two-soliton solutions, i.e., solutions (14) and (15), have been derived. Overtaking interactions of the two bright solitons have been depicted in Fig. 1. Head-on interactions of the two bright solitons have been presented in Fig. 2. Bound-state solitons can be formed due to the interaction between the two solitons, as shown in Fig. 3.

(B) Solving the non-zero potential Lax pair and using the generalized DT, we have obtained the multi-parametric vector rogue-wave solutions, i.e., solutions (20) and (26) of Eqs. (1). With the seed solutions as zero, we have constructed the triple Wronskian vector soliton solutions. With the seed solutions $q_1 \neq 0$ and $q_2 \neq 0$, Vector rogue-wave solutions (20) have been obtained, while the seed solutions $q_1 = 0$ and $q_2 \neq 0$ have produced the bright- and dark-rogue solutions (26). Vector rogue waves with the presence of α and β have been shown in Fig. 4. With the increase of α , Fig. 5 has shown the parabolic trajectory of the rogue waves. Increasing β , we have found that the vector rogue waves exist on top of a monotonically decreasing background, as shown in Fig. 6. Fig. 7(a) has shown a dark soliton together with a rogue wave, while Fig. 7(b) has shown a bright soliton together with a rogue wave.

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Appendix A

Lemma 1. Suppose that M is an $n \times (n - 2)$ matrix, n is a positive integer which satisfies $n > 2$ and a, b, c and d are the n -dimensional column vectors. Then we have

$$|M, a, b||M, c, d| - |M, a, c||M, b, d| + |M, a, d||M, b, c| = 0. \quad (27)$$

Lemma 2. For a matrix $\Pi = (\pi_{ij})_{n \times n} = [\Pi_1, \dots, \Pi_n]$ and a column vector $\Upsilon = (\gamma_1, \dots, \gamma_n)^T$, we have the relation

$$\sum_{j=1}^n |\Pi_1, \dots, \Pi_{j-1}, \Upsilon \Pi_j, \Pi_{j+1}, \dots, \Pi_n| = |\Pi| \sum_{j=1}^n \gamma_j, \quad (28)$$

where π_{ij} 's ($i, j = 1, \dots, n$) are the elements of Π in row i and column j , Π_i 's are the n column vectors and γ_i 's are the elements of Υ .

As a direct result of Lemma 2, two triple Wronskian identities are obtained as

$$\begin{aligned} & |\widehat{N_1 - 1}; \widehat{N_2 - 1}; \widehat{N_3 - 1}| \sum_{k=1}^N (-i\lambda_k(t) - 2i\lambda_k^*(t)) \\ &= |\widehat{N_1 - 2}, N_1; \widehat{N_2 - 1}; \widehat{N_3 - 1}| - |\widehat{N_1 - 1}; \widehat{N_2 - 2}, N_2; \widehat{N_3 - 1}| - |\widehat{N_1 - 1}; \widehat{N_2 - 1}; \widehat{N_3 - 2}, N_3|, \\ & |\widehat{N_1 - 1}; \widehat{N_2 - 1}; \widehat{N_3 - 1}| \left[\sum_{k=1}^N (-i\lambda_k(t) - 2i\lambda_k^*(t)) \right]^2 \\ &= |\widehat{N_1 - 3}, N_1 - 1, N_1; \widehat{N_2 - 1}; \widehat{N_3 - 1}| + |\widehat{N_1 - 2}, N_1 + 1; \widehat{N_2 - 1}; \widehat{N_3 - 1}| \\ &+ |\widehat{N_1 - 1}; \widehat{N_2 - 3}, N_2 - 1, N_2; \widehat{N_3 - 1}| - 2|\widehat{N_1 - 2}, N_1; \widehat{N_2 - 2}, N_2; \widehat{N_3 - 1}| \\ &+ |\widehat{N_1 - 1}; \widehat{N_2 - 2}, N_2 + 1; \widehat{N_3 - 1}| - 2|\widehat{N_1 - 2}, N_1; \widehat{N_2 - 1}; \widehat{N_3 - 2}, N_3| \\ &+ |\widehat{N_1 - 1}; \widehat{N_2 - 1}; \widehat{N_3 - 3}, N_3 - 1, N_3| + 2|\widehat{N_1 - 1}; \widehat{N_2 - 2}, N_2; \widehat{N_3 - 2}, N_3| \\ &+ |\widehat{N_1 - 1}; \widehat{N_2 - 1}; \widehat{N_3 - 2}, N_3 + 1|. \end{aligned} \quad (29a)$$

Lemma 3. Let $W = (\widehat{N - 1}; \widehat{N - 3}; \widehat{N - 3})$, $\Phi_1 = (\phi, \varphi, \chi)$ and $\Phi_2 = (\phi^{(1)}, \dots, \phi^{(N)}; \psi^{(1)}, \dots, \psi^{(N-2)}; \chi^{(1)}, \dots, \chi^{(N-2)})$. The determinants of the two matrices

$$\chi_1 = \begin{pmatrix} W & 0 & 0 & \chi^{(N-2)} & \chi^{(N-1)} & \chi^{(N)} & \varphi^{(N-2)} & \varphi^{(N-1)} \\ 0 & \Phi_1 & \Phi_2 & \chi^{(N-1)} & \chi^{(N)} & \chi^{(N+1)} & \varphi^{(N-1)} & \varphi^{(N)} \end{pmatrix}, \quad (30)$$

$$\chi_2 = \begin{pmatrix} W & 0 & 0 & \varphi^{(N-2)} & \varphi^{(N-1)} & \varphi^{(N)} & \chi^{(N-2)} & \chi^{(N-1)} \\ 0 & \Phi_1 & \Phi_2 & \varphi^{(N-1)} & \varphi^{(N)} & \varphi^{(N+1)} & \chi^{(N-1)} & \chi^{(N)} \end{pmatrix}, \quad (31)$$

are equal to zero.

As a direct result of Lemma 3, two triple Wronskian equations are obtained as

$$\begin{aligned} & |\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-2}; \widehat{N-2}, N+1| + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N||\widehat{N-1}; \widehat{N-2}; \widehat{N}| \\ & + |\widehat{N-1}; \widehat{N-1}; \widehat{N-3}, N-1, N||\widehat{N}; \widehat{N-2}; \widehat{N-1}| - |\widehat{N-1}; \widehat{N-3}, N-1; \widehat{N}||\widehat{N}; \widehat{N-1}; \widehat{N-2}| \\ & - |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N||\widehat{N}; \widehat{N-2}; \widehat{N-2}, N| = 0, \end{aligned} \quad (32a)$$

$$\begin{aligned} & |\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-2}, N+1; \widehat{N-2}| - |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N||\widehat{N-1}; \widehat{N}; \widehat{N-2}| \\ & + |\widehat{N-1}; \widehat{N-3}, N-1, N; \widehat{N-1}||\widehat{N}; \widehat{N-1}; \widehat{N-2}| + |\widehat{N-1}; \widehat{N}; \widehat{N-3}, N-1||\widehat{N}; \widehat{N-2}; \widehat{N-1}| \\ & - |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}||\widehat{N}; \widehat{N-2}, N; \widehat{N-2}| = 0. \end{aligned} \quad (32b)$$

Using the triple Wronskian notations of s_x , s_{xx} , v_x , v_{xx} , w_x , w_{xx} , s_t , v_t , w_t and identities (29), we obtain

$$\begin{aligned} & (iv_t + v_{xx})s + v(s_{xx} - is_t) - 2s_x v_x + 2i\beta x(v_x s - s_x v) + 2i\beta v s - \alpha x v s \\ & = -4i\beta x|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|(|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}| \\ & + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N|) - i4\beta\left(\frac{3}{2}N^2 - \frac{3}{2}N + 1\right)|\widehat{N}; \widehat{N-2}; \widehat{N-1}||\widehat{N-1}; \widehat{N-1}; \widehat{N-1}| \\ & + 8|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|(|\widehat{N-2}, N, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N-1; \widehat{N-2}, N| \\ & + |\widehat{N}; \widehat{N-3}, N; \widehat{N-1}| + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N+1|) \\ & + 4i\beta x|\widehat{N}; \widehat{N-2}; \widehat{N-1}|(|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| \\ & + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|) + 6i\beta N(N-1)|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-2}; \widehat{N-1}| \\ & + 8|\widehat{N}; \widehat{N-2}; \widehat{N-1}|(|\widehat{N-2}, N+1; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-3}, N-1, N; \widehat{N-1}| \\ & + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-2}, N| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-3}, N-1, N|) \\ & + 4i\beta x[|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|(|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}| \\ & + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N|) - |\widehat{N}; \widehat{N-2}; \widehat{N-1}|(|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| \\ & + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|)] \\ & - 8|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-1}||\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| \\ & + 4i\beta|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-2}; \widehat{N-1}| \\ & - 8|\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}||\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| \\ & - 8|\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}||\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N| \\ & - 8|\widehat{N}; \widehat{N-2}; \widehat{N-2}, N||\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| \\ & - 8|\widehat{N}; \widehat{N-2}; \widehat{N-2}, N||\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|, \end{aligned} \quad (33)$$

$$\begin{aligned} & (iw_t + w_{xx})s + w(s_{xx} - is_t) - 2s_x w_x + 2i\beta x(w_x s - s_x w) + 2i\beta w s - \alpha x w s \\ & = -4i\beta x|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|(|\widehat{N-1}, N+1; \widehat{N-1}; \widehat{N-2}| + |\widehat{N}; \widehat{N-2}, N; \widehat{N-2}| \\ & + |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N-1|) - i4\beta\left(\frac{3}{2}N^2 - \frac{3}{2}N + 1\right)|\widehat{N}; \widehat{N-1}; \widehat{N-2}||\widehat{N-1}; \widehat{N-1}; \widehat{N-1}| \\ & + 4i\beta x|\widehat{N}; \widehat{N-1}; \widehat{N-2}|(|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| \\ & + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|) + 6i\beta N(N-1)|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-1}; \widehat{N-2}| \end{aligned}$$

$$\begin{aligned}
& + 8|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|(|\widehat{N-2}, N, N+1; \widehat{N-1}; \widehat{N-2}| + |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N| \\
& + |\widehat{N}; \widehat{N-2}, N; \widehat{N-3}, N-1| + |\widehat{N}; \widehat{N-2}, N+1; \widehat{N-2}|) \\
& + 8|\widehat{N}; \widehat{N-1}; \widehat{N-2}|(|\widehat{N-2}, N+1; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-3}, N-1, N| \\
& + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-2}, N| + |\widehat{N-1}; \widehat{N-3}, N-1, N; \widehat{N-1}|) \\
& + 4i\beta|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-1}; \widehat{N-2}| \\
& + 4i\beta x[|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|(|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}| \\
& + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N|) - |\widehat{N}; \widehat{N-2}; \widehat{N-1}|(|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| \\
& + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|)] \\
& - 8|\widehat{N-1}, N+1; \widehat{N-1}; \widehat{N-2}||\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| \\
& - 8|\widehat{N}; \widehat{N-1}; \widehat{N-3}, N-1||\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| \\
& - 8|\widehat{N}; \widehat{N-1}; \widehat{N-3}, N-1||\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N| \\
& - 8|\widehat{N}; \widehat{N-2}, N; \widehat{N-2}||\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N| \\
& - 8|\widehat{N}; \widehat{N-2}, N; \widehat{N-2}||\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}|,
\end{aligned} \tag{34}$$

$$\begin{aligned}
D_x^2 s \cdot s - 2v\bar{v} - 2w\bar{w} &= 4|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N-2}, N; \widehat{N-2}, N; \widehat{N-1}| \\
& + 4|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N-2}, N; \widehat{N-1}; \widehat{N-2}, N| \\
& - 4|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}||\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| \\
& - 4|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}||\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N| \\
& - 4|\widehat{N}; \widehat{N-2}; \widehat{N-1}||\widehat{N-2}; \widehat{N}; \widehat{N-1}| \\
& - 4|\widehat{N}; \widehat{N-1}; \widehat{N-2}||\widehat{N-2}; \widehat{N-1}; \widehat{N}|.
\end{aligned} \tag{35}$$

Using Lemmas 1–3, identities (29) and (32), one can check that Eqs. (33), (34) and (35) are equivalent to zero.

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