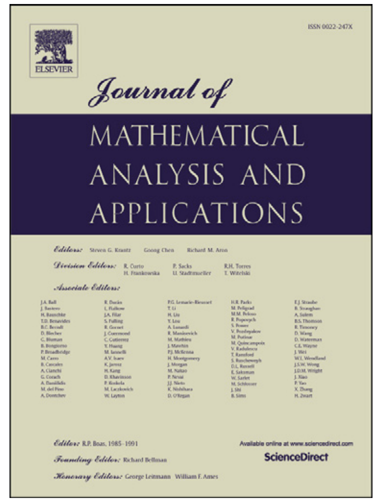

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An estimate of the lower bound of the real parts of the zeros of the partial sums of the Rieman zeta function

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ABSTRACT. Let $\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z}$, $z = x + iy$, be the nth partial sum of the Riemann zeta function and $a_{\zeta_n(z)} := \inf \{\Re z : \zeta_n(z) = 0\}$. In this paper we prove that $a_{\zeta_n(z)} = -\frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)} + \Delta_n$, $n > 2$, with $\limsup_{n\to\infty} |\Delta_n| \leq \log 2$.

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1 Introduction

Let

$$\zeta_n(z) = \sum_{k=1}^{n} \frac{1}{k^z}, \quad n \geq 2, \quad z = x + iy,$$

be the nth partial sum of the Riemann zeta function $\zeta(z)$, and $a_{\zeta_n(z)} := \inf \{\Re z : \zeta_n(z) = 0\}$. By introducing the functions $G_n(z) := \zeta_n(-z) = \sum_{k=1}^{n} k^z$ and defining $b_{G_n(z)} := \sup \{\Re z : G_n(z) = 0\}$, $n \geq 2$, it is immediate that

$$a_{\zeta_n(z)} = -b_{G_n(z)}, \text{ for all } n \geq 2. \tag{1.1}$$

Our objective is to give an estimate of $b_{G_n(z)}$ and, by using (1.1), we will then have that of $a_{\zeta_n(z)}$. It is evident that the numbers $b_{G_n(z)}$ are not easy to calculate. However, the real solutions of the equations $G_{n-1}(x) = n^x$, denoted by $\beta_{G_n(z)}$, which are unique by virtue of Pólya and Szëgo's formula [4, p. 46], are much more easy to determine. Both numbers satisfy

$$b_{G_n(z)} \leq \beta_{G_n(z)}, \quad \text{for all } n \geq 2, \tag{1.2}$$

as an immediate consequence of the fact that the half-plane $\left\{z : \operatorname{Re} z > \beta_{G_n(z)}\right\}$ is a zero-free region of $G_n(z)$, for every $n \geq 2$; for details, see [2, Theorem 3.1] and [3, Lemma 1]. Furthermore, for n prime, it is not hard to prove that $b_{G_n(z)} = \beta_{G_n(z)}$. A proof of this property can be found in [2, Theorem 4.10] and [3, Proposition 5].

Concerning the converse of (1.2), we have the very important contribution of Balazard and Velásquez Castañón [1, Proposition 1, (*ii*)], where it was proved the existence of some n_0 such that

$$\beta_{G_n(z)} \leq b_{G_n(z)}, \quad \text{for all } n \geq n_0.$$

Therefore, the numbers $b_{G_n(z)}$ and $\beta_{G_n(z)}$ are equal from some positive integer n_0. Consequently, to give an estimate of $a_{\zeta_n(z)}$ it is enough to give an estimate of $\beta_{G_n(z)}$. In fact, this process was followed by the aforementioned authors in [1] to prove that

$$\lim_{n\to\infty} \frac{a_{\zeta_n(z)}}{n} = -\log 2 \tag{1.3}$$

or, equivalently, $a_{\zeta_n(z)} = -n\log 2 + o(n)$, by using the property

$$\frac{\beta_{G_n(z)}}{n} \to \log 2, \quad n \to \infty,$$

obtained by Borwein, Fee, Ferguson and van der Waall in [2, p. 25], under the implicit assumption of the existence of limit of the sequence $\left(\frac{\beta_{G_n(z)}}{n}\right)_{n\geq 2}$. There, after the proof of Theorem 3.1, the authors proposed (stated without proof) for $\beta_{G_n(z)}$ the estimate $(n-3/2)\log 2$.

In the present paper we have proved (Theorem 2) that

$$a_{\zeta_n(z)} = -\frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)} + \Delta_n, \quad \text{with} \limsup_{n\to\infty} |\Delta_n| \leq \log 2. \tag{1.4}$$

To do it we have followed the process consisting of, first, to demonstrate (Theorem 1) that

$$\beta_{G_n(z)} = \frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)} + \Delta_n, \quad \text{with} \limsup_{n\to\infty} |\Delta_n| \leq \log 2, \tag{1.5}$$

second, to apply [1, Proposition 1, (ii)] to deduce that the preceding estimate is also true for $b_{G_n(z)}$ and, third, to use (1.1) to obtain (1.4). Furthermore, as we consider relevant the existence of limit of the sequence $\left(\frac{\beta_{G_n(z)}}{n}\right)_{n\geq 2}$, we have given a proof of such a fact. Then, since $\lim_{n\to\infty}(n+a)\log\left(\frac{n-1}{n-2}\right) = 1$, for any $a \in \mathbb{R}$, our estimate (1.5) first confirms the validity of the estimate $(n-3/2)\log 2$, computationally settled by Borwein et al. in [2]. Second, (1.4) implies, in particular, (1.3) and it reveals the secret of the term $o(n)$ in the expression $a_{\zeta_n(z)} = -n\log 2 + o(n)$ of Balazard and Velásquez Castañón in [1].

2 The numbers $\beta_{G_n(z)}$

By defining

$$\beta_{G_n(z)} := \sup\{x \in \mathbb{R} : G_{n-1}(x) \geq n^x\}, \quad G_n(z) := \zeta_n(-z), \quad n \geq 2,$$

since the equation $G_{n-1}(x) = n^x$, by virtue of Pólya and Szëgo's formula [4, p. 46], has only one real solution, it follows that $\beta_{G_n(z)}$ is such a solution. Hence, for every $n > 2$, one has

$$\begin{array}{ll} G_{n-1}(x) > n^x, & \text{if } x < \beta_{G_n(z)} \\ G_{n-1}(x) = n^x, & \text{if } x = \beta_{G_n(z)} \\ G_{n-1}(x) < n^x, & \text{if } x > \beta_{G_n(z)} \end{array}. \tag{2.1}$$

Lemma 1 $\left(\beta_{G_n(z)}\right)_{n\geq 2}$ *is an unbounded strictly increasing sequence of positive terms, except* $\beta_{G_2(z)} = 0$. *Furthermore,*

$$b_{G_n(z)} \leq \beta_{G_n(z)} \leq n-2, \text{ for all } n \geq 2,$$

where the first inequality becomes an equality for all prime numbers and in the second the equality is only attained for $n = 2, 3$.

Proof. For $n = 2$, by defining the function $G_1(x)$ as identically equal to 1, it is immediate that $\beta_{G_2(z)} = 0$, so suppose $n > 2$. From (2.1), it trivially follows that $\beta_{G_n(z)} \geq 1$, for all $n > 2$. Now, again by (2.1), let us consider the equalities $G_n(\beta_{G_{n+1}(z)}) = (n+1)^{\beta_{G_{n+1}(z)}}$ and $G_{n-1}(\beta_{G_n(z)}) = n^{\beta_{G_n(z)}}$. By dividing by $(n+1)^{\beta_{G_{n+1}(z)}}$ and $n^{\beta_{G_n(z)}}$, respectively, we have

$$\left(\frac{1}{n+1}\right)^{\beta_{G_{n+1}(z)}} + \left(\frac{2}{n+1}\right)^{\beta_{G_{n+1}(z)}} + \ldots + \left(\frac{n}{n+1}\right)^{\beta_{G_{n+1}(z)}} = 1$$

and

$$\left(\frac{1}{n}\right)^{\beta_{G_n(z)}} + \left(\frac{2}{n}\right)^{\beta_{G_n(z)}} + \ldots + \left(\frac{n-1}{n}\right)^{\beta_{G_n(z)}} = 1.$$

By subtracting we get

$$\left(\frac{1}{n+1}\right)^{\beta_{G_{n+1}(z)}} + \left[\left(\frac{2}{n+1}\right)^{\beta_{G_{n+1}(z)}} - \left(\frac{1}{n}\right)^{\beta_{G_n(z)}}\right] + \ldots +$$

$$\left[\left(\frac{n}{n+1}\right)^{\beta_{G_{n+1}(z)}} - \left(\frac{n-1}{n}\right)^{\beta_{G_n(z)}}\right] = 0,$$

which means that, for some k, with $2 \leq k \leq n$,

$$\left[\left(\frac{k}{n+1}\right)^{\beta_{G_{n+1}(z)}} - \left(\frac{k-1}{n}\right)^{\beta_{G_n(z)}}\right] < 0.$$

That is,

$$\left(\frac{k}{n+1}\right)^{\beta_{G_{n+1}(z)}} < \left(\frac{k-1}{n}\right)^{\beta_{G_n(z)}}, \tag{2.2}$$

which is equivalent to saying that

$$\left(\frac{n+1}{k}\right)^{\beta_{G_{n+1}(z)}} > \left(\frac{n}{k-1}\right)^{\beta_{G_n(z)}}.$$

By taking the logarithm,

$$\beta_{G_{n+1}(z)} \log\left(\frac{n+1}{k}\right) > \beta_{G_n(z)} \log\left(\frac{n}{k-1}\right), \tag{2.3}$$

and, since $\frac{n}{k-1} > \frac{n+1}{k} > 1$, we obtain

$$\frac{\beta_{G_{n+1}(z)}}{\beta_{G_n(z)}} > \frac{\log\left(\frac{n}{k-1}\right)}{\log\left(\frac{n+1}{k}\right)} > 1.$$

This proves that the sequence $\left(\beta_{G_n(z)}\right)_{n\geq 2}$ is strictly increasing.

By supposing that $\left(\beta_{G_n(z)}\right)_{n\geq 2}$ is bounded, there exists a positive integer M such that

$$\beta_{G_n(z)} \leq M, \quad \text{for all } n \geq 2,$$

which means, from (2.1), that $G_{n-1}(M) \leq n^M$ or, equivalently,

$$\frac{G_{n-1}(M)}{n^M} \leq 1, \quad \text{for all } n \geq 2. \tag{2.4}$$

However, it is well known that $G_{n-1}(M) = 1+2^M+\ldots+(n-1)^M$ is a polynomial in n of degree $M+1$ with a positive leader coefficient. Therefore, by taking the limit in (2.4) when n tends to ∞, we are led to a contradiction because the left-hand side of (2.4) tends to ∞ whereas its right-hand side is equal to 1. In consequence, the sequence $\left(\beta_{G_n(z)}\right)_{n\geq 2}$ is unbounded.

A proof of the fact that $b_{G_n(z)} \leq \beta_{G_n(z)}$, for all $n \geq 2$, and that the equality is attained for n prime can be found in [2, Theorem 3.1 and 4.10] and [3, Lemma 1 and Proposition 5]. Then it only remains to demonstrate that

$$\beta_{G_n(z)} \leq n-2, \quad \text{for all } n \geq 2, \tag{2.5}$$

where the equality it reached only for $n = 2, 3$. To do it, we firstly see, after an easy computation, that the first values of $\beta_{G_n(z)}$ are $\beta_{G_2(z)} = 0$, $\beta_{G_3(z)} = 1$, $\beta_{G_4(z)} \approx 1.7$ and $\beta_{G_5(z)} \approx 2.4$. Then (2.5) follows for $2 \leq n \leq 5$. Hence, assume $n > 5$. Second, we observe, because (2.1), that the inequality (2.5) is equivalent to

$$G_{n-1}(n-2) \leq n^{n-2}, \quad \text{for all } n \geq 2. \tag{2.6}$$

We proceed to give a proof of (2.5) by induction. Hence, we assume (2.5) is true for a fixed $n > 5$ and we must prove

$$\beta_{G_{n+1}(z)} \leq n-1. \tag{2.7}$$

We firstly claim that

$$2\,(n-1)^{n-2} < n^{n-2}, \quad \text{for all } n > 5, \tag{2.8}$$

which is equivalent to saying that $2 < \left(\frac{n}{n-1}\right)^{n-2}$, for all $n > 5$. Indeed, for $n = 6$, by direct computation, one has $2 < \left(\frac{6}{5}\right)^4$, so (2.8) follows for $n = 6$. Now, we are going to prove that the sequence $\left(\left(\frac{n}{n-1}\right)^{n-2}\right)_{n\geq 6}$ is strictly increasing, which means that (2.8) will be proved for any $n > 5$. It is immediate that $\left(\left(\frac{n}{n-1}\right)^{n-2}\right)_{n\geq 6}$ is strictly increasing if and only if

$$\frac{n-2}{n-1} < \frac{\log(n+1) - \log n}{\log n - \log(n-1)}. \tag{2.9}$$

To show (2.9), we define the functions $f(x) := \log(x+1)$ and $g(x) := \log x$, $x > 0$. By applying Cauchy's mean value theorem, there exists some x with $n-1 < x < n$ such that

$$\frac{\log(n+1) - \log n}{\log n - \log(n-1)} = \frac{f(n) - f(n-1)}{g(n) - g(n-1)} = \frac{f'(x)}{g'(x)} = \frac{x}{x+1}.$$

Then

$$\frac{n-2}{n-1} < \frac{\log(n+1) - \log n}{\log n - \log(n-1)} \quad \text{if and only if } \frac{n-2}{n-1} < \frac{x}{x+1},$$

where the last inequality is true if and only if $n-2 < x$. Therefore, (2.9) follows because x is so that $n-1 < x < n$. Consequently, (2.8) is true. Now, from the hypothesis of induction for a fixed $n > 5$, one has $\beta_{G_n(z)} < \beta_{G_n(z)} + 1 \leq n-1$. Then, from (2.1), it follows that $G_{n-1}(n-1) < n^{n-1}$. Since (2.8) is true for all $n > 5$, we have $2n^{n-1} < (n+1)^{n-1}$. Therefore,

$$G_n(n-1) = G_{n-1}(n-1) + n^{n-1} < 2n^{n-1} < (n+1)^{n-1},$$

which means, by (2.1), that $\beta_{G_{n+1}(z)} < n-1$. That is, (2.7) is true. Finally, by using the principle of induction, (2.5) follows and then the proof is completed. ■

Corollary 1 *The sequence* $\left(\frac{\beta_{G_n(z)}}{n}\right)_{n\geq 2}$ *is strictly increasing and upper bounded by* 1*. Then* $\left(\frac{\beta_{G_n(z)}}{n}\right)_{n\geq 2}$ *has limit and it is* $\log 2$*.*

Proof. From Lemma 1, $\beta_{G_n(z)} \leq n-2$ for all $n \geq 2$. Then, $\frac{\beta_{G_n(z)}}{n} \leq \frac{n-2}{n} < 1$, for all $n \geq 2$. That is, $\left(\frac{\beta_{G_n(z)}}{n}\right)_{n\geq 2}$ is upper bounded by 1. On the other hand, since $\beta_{G_2(z)} = 0$, $\beta_{G_3(z)} = 1$, the inequality $\frac{\beta_{G_n(z)}}{n} < \frac{\beta_{G_{n+1}(z)}}{n+1}$ trivially follows for $n = 2$, so assume that $n > 2$. The above inequality is equivalent to $\frac{\beta_{G_{n+1}(z)}}{\beta_{G_n(z)}} > \frac{n+1}{n}$. Then, in order to prove it, we firstly observe that, from (2.3), for any $n > 2$ there exists some k, with $2 \leq k \leq n$, such that

$$\frac{\beta_{G_{n+1}(z)}}{\beta_{G_n(z)}} > \frac{\log\left(\frac{n}{k-1}\right)}{\log\left(\frac{n+1}{k}\right)} = \frac{\log n - \log(k-1)}{\log(n+1) - \log k}.$$

By considering the functions $f(x) := \log(x+1)$ and $g(x) := \log x$, $x > 0$, used to prove (2.9), and applying Cauchy's mean value theorem, there exists some x, with $k-1<x<n$, such that

$$\frac{\log n-\log(k-1)}{\log(n+1)-\log k}=\frac{g'(x)}{f'(x)}=\frac{x+1}{x}.$$

Then, since $x<n$, one has $\frac{x+1}{x}>\frac{n+1}{n}$, so from the two previous relations it follows that $\frac{\beta_{G_{n+1}(z)}}{\beta_{G_n(z)}}>\frac{n+1}{n}$. Therefore, the sequence $\left(\frac{\beta_{G_n(z)}}{n}\right)_{n\geq 2}$ is strictly increasing and, in consequence, as we have just proved that it is bounded, there exists $a:=\lim_{n\to\infty}\frac{\beta_{G_n(z)}}{n}$, with $0<a\leq 1$. By (2.1), $G_{n-1}(\beta_{G_n(z)})=n^{\beta_{G_n(z)}}$. Then, dividing by $n^{\beta_{G_n(z)}}$, we have

$$\left(\frac{1}{n}\right)^{\beta_{G_n(z)}}+\left(\frac{2}{n}\right)^{\beta_{G_n(z)}}+\ldots+\left(\frac{n-1}{n}\right)^{\beta_{G_n(z)}}=1, \text{ for every } n\geq 2,$$

or equivalently

$$1=\left[\left(\frac{n-1}{n}\right)^n\right]^{\frac{\beta_{G_n(z)}}{n}}+\left[\left(\frac{n-2}{n}\right)^n\right]^{\frac{\beta_{G_n(z)}}{n}}+\ldots+\left[\left(\frac{n-(n-1)}{n}\right)^n\right]^{\frac{\beta_{G_n(z)}}{n}}.$$

Since $\lim_{n\to\infty}\left(\frac{n-j}{n}\right)^n=e^{-j}$, by taking the limit in the above expression and noting that $a:=\lim_{n\to\infty}\frac{\beta_{G_n(z)}}{n}$, we obtain

$$1=e^{-a}+e^{-2a}+\ldots,$$

where the series is convergent because $0<e^{-a}<1$. As its sum is $\frac{1}{e^a-1}$, it must satisfy $1=\frac{1}{e^a-1}$, which implies that $a=\log 2$. This proves the corollary. ■

Lemma 2 *For every $n\geq 2$, define $m_n:=\left[\beta_{G_n(z)}\right]+1$, where $\left[\beta_{G_n(z)}\right]$ denotes the integer part of $\beta_{G_n(z)}$. Then $m_n+1>\beta_{G_{n+1}(z)}$.*

Proof. We firstly note that in spite of the sequence $\left(\beta_{G_n(z)}\right)_{n\geq 2}$ is strictly increasing, by virtue of Lemma 1, it could be $\left[\beta_{G_n(z)}\right]=\left[\beta_{G_{n+1}(z)}\right]$ for some n. In this case the lemma trivially follows, so from now on we assume that $\left[\beta_{G_n(z)}\right]<\left[\beta_{G_{n+1}(z)}\right]$. As we have just seen in the proof of the preceding lemma, the first few values of $\beta_{G_n(z)}$ are $\beta_{G_2(z)}=0$, $\beta_{G_3(z)}=1$, $\beta_{G_4(z)}\approx 1.7$ and $\beta_{G_5(z)}\approx 2.4$. Then the lemma is true for $2\leq n\leq 5$. Hence, assume $n>5$. Since by the definition of integer part is $\beta_{G_n(z)}<m_n$, in particular, one has $\beta_{G_n(z)}<m_n+1$. Then, by (2.1), $G_{n-1}(m_n+1)<n^{m_n+1}$. The proof will be completed if we can prove that

$$2n^{m_n+1}<(n+1)^{m_n+1}, \quad \text{for all } n>5. \tag{2.10}$$

If this were true, then we will have

$$G_n(m_n+1)=G_{n-1}(m_n+1)+n^{m_n+1}<(n+1)^{m_n+1},$$

and, by (2.1), it implies that $m_n + 1 > \beta_{G_{n+1}(z)}$. To prove inequality (2.10), we observe that it is of type (2.8), so we will show it of a similar form. Indeed, for $n = 6$, $\beta_{G_6(z)} \approx 3.1$, then $m_6 = 4$. Now, a simple computation shows that $2 < \left(\frac{7}{6}\right)^5$, so (2.10) follows for $n = 6$. Then, by proving that the sequence

$$\left(\left(\frac{n+1}{n}\right)^{m_n+1}\right)_{m\geq 6}$$

is strictly increasing, it will eventually demonstrate (2.10). Hence, we are going to show

$$\left(\frac{n+1}{n}\right)^{m_n+1} < \left(\frac{n+2}{n+1}\right)^{m_{n+1}+1}, \quad \text{for all } n > 5. \tag{2.11}$$

Indeed, as it has been done in (2.9), by taking the logarithm and applying Cauchy's mean value theorem to the functions $f(x) := \log(x+1)$ and $g(x) := \log x$, $x > 0$, on the interval $[n, n+1]$, there exists x with $n < x < n+1$ such that (2.11) is equivalent to prove that

$$\frac{m_n+1}{m_{n+1}+1} < \frac{x}{x+1}.$$

But this inequality is true by noting that $n < x$ and $m_n \leq n-2$ for all $n > 5$, by virtue of Lemma 1. Now the proof is completed. ■

Corollary 2 *For all $n \geq 2$, one has $\beta_{G_{n+1}(z)} - \beta_{G_n(z)} < 2$.*

Proof. By the definition of integer part we have

$$\left[\beta_{G_n(z)}\right] \leq \beta_{G_n(z)} < \left[\beta_{G_n(z)}\right] + 1 := m_n.$$

By applying the above lemma, $m_n + 1 > \beta_{G_{n+1}(z)}$. Then we get

$$\beta_{G_{n+1}(z)} - \beta_{G_n(z)} < m_n + 1 - \left[\beta_{G_n(z)}\right] = 2,$$

so the corollary follows. ■

Lemma 3 $\liminf_{n\to\infty}\left(\beta_{G_{n+1}(z)} - \beta_{G_n(z)}\right) \geq \log 2$.

Proof. In Lemma 1 we have proved, see (2.2) and (2.3), that given $n > 2$, there exists some k, with $2 \leq k \leq n$, such that

$$\beta_{G_{n+1}(z)} \log\left(\frac{n+1}{k}\right) > \beta_{G_n(z)} \log\left(\frac{n}{k-1}\right).$$

By writing $\frac{n}{k-1} = \frac{n+1}{k} . \frac{nk}{(n+1)(k-1)}$ and substituting in the preceding inequality one has

$$\beta_{G_{n+1}(z)} \log\left(\frac{n+1}{k}\right) > \beta_{G_n(z)}\left[\log\left(\frac{n+1}{k}\right) + \log\left(\frac{nk}{(n+1)(k-1)}\right)\right].$$

That is, for every integer $n > 2$, there exists $k = k_n$ (k depends on n), with $2 \le k_n \le n$, such that

$$\begin{aligned}\beta_{G_{n+1}(z)} - \beta_{G_n(z)} &> \beta_{G_n(z)} \frac{\log\left(\frac{nk_n}{(n+1)(k_n-1)}\right)}{\log\left(\frac{n+1}{k_n}\right)} \\ &= \frac{\beta_{G_n(z)}}{n} n \frac{\log\left(\frac{nk_n}{(n+1)(k_n-1)}\right)}{\log\left(\frac{n+1}{k_n}\right)}. \end{aligned} \tag{2.12}$$

Now, we claim that the sequence $(k_n)_{n>2}$ is unbounded. Indeed, if $(k_n)_{n>2}$ were bounded, then clearly

$$\lim_{n\to\infty} n \frac{\log\left(\frac{nk_n}{(n+1)(k_n-1)}\right)}{\log\left(\frac{n+1}{k_n}\right)} = \infty,$$

which is impossible by noting that, on one hand, $\beta_{G_{n+1}(z)} - \beta_{G_n(z)}$ is upper bounded, by virtue of Corollary 2, and, on the other hand, $\lim_{n\to\infty} \frac{\beta_{G_n(z)}}{n} = \log 2$, from Corollary 1. Thus, the claim follows. We write

$$\begin{aligned} & n \frac{\log\left(\frac{nk_n}{(n+1)(k_n-1)}\right)}{\log\left(\frac{n+1}{k_n}\right)} \\ &= \frac{\frac{n}{n+1} \frac{n+1-k_n}{k_n-1}}{\log\left(1 + \frac{n+1-k_n}{k_n}\right)} \log\left(1 + \frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}} \\ &= \frac{\frac{n}{n+1}}{\frac{k_n-1}{k_n} \log\left(1 + \frac{1}{\frac{k_n}{n+1-k_n}}\right)^{\frac{k_n}{n+1-k_n}}} \log\left(1 + \frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}}. \end{aligned} \tag{2.13}$$

Now, since for $x > 0$ is always $x > \log(1 + x)$, we have

$$\frac{1}{\log\left(1 + \frac{1}{x}\right)^x} = \frac{\frac{1}{x}}{\log\left(1 + \frac{1}{x}\right)} > 1,$$

so, by using this property for $x = \frac{k_n}{n+1-k_n}$ in the last equality of (2.13), we get

$$n\frac{\log\left(\frac{nk_n}{(n+1)(k_n-1)}\right)}{\log\left(\frac{n+1}{k_n}\right)} > \frac{\frac{n}{n+1}}{\frac{k_n-1}{k_n}}\log\left(1+\frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}}$$

$$= \frac{nk_n}{(n+1)(k_n-1)}\log\left(1+\frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}}$$

$$> \log\left(1+\frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}}, \tag{2.14}$$

because $\frac{nk_n}{(n+1)(k_n-1)} > 1$ by taking into account that $2 \leq k_n \leq n$. Then, from (2.12), (2.13) and (2.14), we obtain

$$\beta_{G_{n+1}(z)} - \beta_{G_n(z)} > \frac{\beta_{G_n(z)}}{n}\log\left(1+\frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}}. \tag{2.15}$$

Since we have just proved that $(k_n)_{n>2}$ is unbounded, it implies that $\frac{(k_n-1)(n+1)}{n+1-k_n} \to \infty$ and then

$$\lim_{n\to\infty}\log\left(1+\frac{1}{\frac{(k_n-1)(n+1)}{n+1-k_n}}\right)^{\frac{(k_n-1)(n+1)}{n+1-k_n}} = 1.$$

Therefore, by applying Corollary 1, the limit of the right-hand side of (2.15) is $\log 2$. Consequently,

$$\liminf_{n\to\to\infty}\left(\beta_{G_{n+1}(z)} - \beta_{G_n(z)}\right) \geq \log 2.$$

■

3 The numbers γ_n

We define the numbers

$$\gamma_n := \frac{\log 2}{\log\left(\frac{n}{n-1}\right)}, \quad n \geq 2.$$

Lemma 4 *For every $n \geq 2$ one has $\beta_{G_n(z)} \leq \gamma_n$, and the sequence $(\gamma_n)_{n\geq 2}$ is strictly increasing. Furthermore, $\lim_{n\to\infty}(\gamma_n - \gamma_{n-1}) = \lim_{n\to\infty}\frac{\gamma_n}{n} = \log 2$.*

Proof. Since $\beta_{G_2(z)} = 0$, $\beta_{G_3(z)} = 1$ and $\gamma_2 = 1$, $\gamma_3 := \frac{\log 2}{\log\left(\frac{3}{2}\right)} > 1$, the first part of the lemma trivially follows for $n = 2, 3$. Then we assume $n > 3$. By (2.1), for any $x \in \left[\beta_{G_{n-1}(z)}, \beta_{G_n(z)}\right]$, one has

$$1 + 2^x + \ldots + (n-2)^x \leq (n-1)^x, \tag{3.1}$$

because $\beta_{G_{n-1}(z)} \leq x$, and

$$1 + 2^x + \ldots + (n-2)^x + (n-1)^x \geq n^x, \tag{3.2}$$

because $x \leq \beta_{G_n(z)}$. By adding $(n-1)^x$ to (3.1), from (3.2), one deduces

$$n^x \leq 1 + 2^x + \ldots + (n-2)^x + (n-1)^x \leq 2\,(n-1)^x\,.$$

That is, $\left(\frac{n}{n-1}\right)^x \leq 2$ or, equivalently, $x \leq \frac{\log 2}{\log\left(\frac{n}{n-1}\right)} = \gamma_n$ for any x of the interval $\left[\beta_{G_{n-1}(z)}, \beta_{G_n(z)}\right]$. Hence, it follows that $\beta_{G_n(z)} \leq \gamma_n$. Now, we define $f(x) := \frac{\log 2}{\log x}$, $x > 1$. This function is strictly decreasing. Then, as $\frac{n}{n-1} < \frac{n-1}{n-2}$, by taking into account the definition of γ_n, one has $\gamma_n = f(\frac{n}{n-1}) > f(\frac{n-1}{n-2}) = \gamma_{n-1}$, which proves the first part of the lemma. To show the second part, by using the mean value theorem applied on the function $f(x)$, we have

$$\frac{f(\frac{n-1}{n-2}) - f(\frac{n}{n-1})}{\frac{n-1}{n-2} - \frac{n}{n-1}} = f'(x), \text{ for some } x \in \left(\frac{n}{n-1}, \frac{n-1}{n-2}\right).$$

Then we obtain

$$\gamma_n - \gamma_{n-1} = -f'(x)\left(\frac{n-1}{n-2} - \frac{n}{n-1}\right) = \frac{\log 2}{x\,(\log x)^2}.\frac{1}{(n-2)\,(n-1)}. \tag{3.3}$$

Now, noticing $\frac{n}{n-1} < x < \frac{n-1}{n-2}$, the limit in (3.3), when $n \to \infty$, exists and it is immediate that

$$\lim_{n\to\infty} \left(\gamma_n - \gamma_{n-1}\right) = \log 2.$$

Finally, according to the definition of γ_n, it is clear that $\lim_{n\to\infty} \frac{\gamma_n}{n}$ exists and it is equal to $\log 2$. Then the lemma follows. ■

Lemma 5 *Let* $\gamma_n := \frac{\log 2}{\log\left(\frac{n}{n-1}\right)}$ *and* $G_n(x) := 1 + 2^x + \ldots + n^x$, $n \geq 2$. *Then,*

$$\lim_{n\to\infty} \inf \frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} \geq 1.$$

Proof. An elementary computation gives us $\frac{G_2(\gamma_2)}{3^{\gamma_2}} = 1$, $\frac{G_3(\gamma_3)}{4^{\gamma_3}} \approx 1.018$ and $\frac{G_4(\gamma_4)}{5^{\gamma_4}} \approx 1.009$, then the lemma follows for $n = 3$, 4 and 5, so assume $n > 5$. We fix an integer k such that $2 < k < n-2$. Then, noticing $\gamma_{n-1} := \frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)}$, we have

$$\begin{aligned}
&\frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} \\
&\quad = \frac{1 + 2^{\gamma_{n-1}} + \ldots + (n-1-k)^{\gamma_{n-1}} + \ldots + (n-2)^{\gamma_{n-1}} + (n-1)^{\gamma_{n-1}}}{n^{\gamma_{n-1}}} \\
&\quad = \frac{\left(\frac{1}{n-2}\right)^{\gamma_{n-1}} + \ldots + \left(\frac{n-1-k}{n-2}\right)^{\gamma_{n-1}} + \left(\frac{n-k}{n-2}\right)^{\gamma_{n-1}} + \ldots + \left(\frac{n-3}{n-2}\right)^{\gamma_{n-1}} + 3}{\left(\frac{n}{n-2}\right)^{\gamma_{n-1}}}.
\end{aligned} \tag{3.4}$$

By defining

$$A_{n,k} := \left(\frac{1}{n-2}\right)^{\gamma_{n-1}} + \left(\frac{2}{n-2}\right)^{\gamma_{n-1}} + \ldots + \left(\frac{n-1-k}{n-2}\right)^{\gamma_{n-1}}$$
$$= (n-2) \sum_{j=1}^{n-1-k} \frac{1}{n-2} \left(\frac{j}{n-2}\right)^{\gamma_{n-1}},$$

is immediate that

$$A_{n,k} > (n-2) \int_0^{\frac{n-1-k}{n-2}} x^{\gamma_{n-1}} dx = \frac{n-2}{\gamma_{n-1}+1} \left(\frac{n-1-k}{n-2}\right)^{\gamma_{n-1}+1}. \tag{3.5}$$

Then, from (3.4) and (3.5), we can write

$$\frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} = \frac{A_{n,k} + \left(\frac{n-k}{n-2}\right)^{\gamma_{n-1}} + \ldots + \left(\frac{n-3}{n-2}\right)^{\gamma_{n-1}} + 3}{\left(\frac{n}{n-2}\right)^{\gamma_{n-1}}} > B_{n,k} \tag{3.6}$$

where, $B_{n,k}$, for every $n > 5$ and k satisfying $2 < k < n-2$, is defined as

$$B_{n,k} := \frac{\frac{n-2}{\gamma_{n-1}+1}\left(\frac{n-1-k}{n-2}\right)^{\gamma_{n-1}+1} + \left(\frac{n-k}{n-2}\right)^{\gamma_{n-1}} + \ldots + \left(\frac{n-3}{n-2}\right)^{\gamma_{n-1}} + 3}{\left(\frac{n}{n-2}\right)^{\gamma_{n-1}}}. \tag{3.7}$$

For each fixed integer j, from the second part of Lemma 4, it is immediate that

$$\lim_{n\to\infty} \frac{n-2}{\gamma_{n-1}+1} = \frac{1}{\log 2}$$

and

$$\lim_{n\to\infty} \left(\frac{n-1-j}{n-2}\right)^{\gamma_{n-1}+1} = \lim_{n\to\infty} \left(\frac{n-1-j}{n-2}\right)^{\gamma_{n-1}} = \frac{1}{2^{j-1}}.$$

Then by taking the limit in (3.7) when $n \to \infty$, for any fixed $k > 2$, we have

$$\lim_{n\to\infty} B_{n,k} = \frac{\frac{1}{\log 2}\frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \ldots + \frac{1}{2} + 1 + 2}{4} = \frac{\frac{1}{\log 2}\frac{1}{2^{k-1}} + 4(1-\frac{1}{2^k})}{4} =$$
$$1 + \frac{1}{\log 2}\frac{1}{2^{k+1}} - \frac{1}{2^k}. \tag{3.8}$$

Then, noticing (3.8), from (3.6), we get

$$\lim_{n\to\infty} \inf \frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} \geq \lim_{n\to\infty} \inf B_{n,k} = \lim_{n\to\infty} B_{n,k} = 1 + \frac{1}{\log 2}\frac{1}{2^{k+1}} - \frac{1}{2^k}.$$

Therefore, since k is arbitrary,

$$\lim_{n\to\infty} \inf \frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} \geq 1$$

and consequently the lemma follows. ■

4 The estimate of $a_{\zeta_n(z)} := \inf\{\Re z : \zeta_n(z) = 0\}$

Theorem 1 *Let $\left(\beta_{G_n(z)}\right)_{n>2}$ be the sequence of the real solutions of the equations $G_{n-1}(x) = n^x$, $n > 2$. Then*

$$\beta_{G_n(z)} = \frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)} + \Delta_n,$$

with $\limsup_{n\to\infty} |\Delta_n| \le \log 2$.

Proof. We firstly say that for those n such that $\frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} \ge 1$, the theorem follows. Indeed, by (2.1), $\frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} \ge 1$ is equivalent to saying that $\gamma_{n-1} \le \beta_{G_n(z)}$. From Lemma 4, $\beta_{G_n(z)} \le \gamma_n$. Then we have $\gamma_{n-1} \le \beta_{G_n(z)} \le \gamma_n$, so we can write $\beta_{G_n(z)} = \gamma_{n-1} + \Delta_n$, with $\Delta_n := \beta_{G_n(z)} - \gamma_{n-1}$. Since $0 \le \Delta_n \le \gamma_n - \gamma_{n-1}$, by virtue of the second part of Lemma 4, we get

$$\limsup_{n\to\infty} |\Delta_n| \le \limsup_{n\to\infty} \left(\gamma_n - \gamma_{n-1}\right) = \lim_{n\to\infty} \left(\gamma_n - \gamma_{n-1}\right) = \log 2.$$

Then, noting that $\gamma_{n-1} = \frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)}$, the theorem is proved. Therefore, from now on, we suppose $\frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} < 1$. For every integer $n > 2$, we define

$$g_n(x) := \frac{G_{n-1}(x)}{n^x}, \quad x \ge 0,$$

which is a convex and strictly decreasing one-to-one function which maps the interval $[0,\infty)$ onto $(0, n-1]$, and, from (2.1), satisfies $g_n(\beta_{G_n(z)}) = 1$. Then the inverse function of $g_n(x)$, denoted by $h_n(x)$, maps $(0, n-1]$ onto $[0,\infty)$ and shares the aforementioned properties with $g_n(x)$. By Lemma 1, $\beta_{G_n(z)} < \beta_{G_{n+1}(z)}$ for all $n > 2$. Thus, let β_n be the mean point of each interval $\left[\beta_{G_n(z)}, \beta_{G_{n+1}(z)}\right]$. Since $h_n(1) = \beta_{G_n(z)}$, for all $n > 2$, and $h_n(x)$ is strictly decreasing, there exists an $\epsilon_n > 0$ such that $\beta_{{}_n} = h_n(1-\epsilon_n)$. Now we claim that $\epsilon := \inf\{\epsilon_n : n > 2\}$ is a positive number. Otherwise, there exists a subsequence of $(\epsilon_n)_n$, denoted by the same form, such that $\epsilon_n \to 0$ and then by continuity of every $h_n(x\ )$ at $x = 1$ we would have

$$\lim_{n\to\infty} (\beta_{{}_n} - \beta_{G_n(z)}) = \lim_{n\to\infty} (h_n(1-\epsilon_n) - h_n(1)) = 0.$$

But, this is impossible because $\beta_{{}_n} - \beta_{G_n(z)} = \frac{1}{2}\left(\beta_{G_{n+1}(z)} - \beta_{G_n(z)}\right)$ and, by taking into account Lemma 3, it does not tend to 0. Hence, the claim follows. Now, given the above ϵ, for every $n > 2$ let us define $\beta_{{}_{n,\epsilon}} := h_n(1-\epsilon)$. Then, since $\epsilon \le \epsilon_n$ for all $n > 2$, noting that $h_n(x)$ is strictly decreasing and that β_n is the mean point of the interval $\left[\beta_{G_n(z)}, \beta_{G_{n+1}(z)}\right]$, one has

$$\beta_{G_n(z)} < \beta_{{}_{n,\epsilon}} \le \beta_{{}_n} < \beta_{G_{n+1}(z)}. \tag{4.1}$$

By using Lemma 5, and under the assumption made on $\frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}}$, there exists n_0 such that

$$1-\epsilon \leq \frac{G_{n-1}(\gamma_{n-1})}{n^{\gamma_{n-1}}} < 1, \quad \text{for all } n \geq n_0.$$

From the definition of $g_n(x)$, the above inequalities are equivalent to write

$$1-\epsilon \leq g_n(\gamma_{n-1}) < 1, \quad \text{for all } n \geq n_0.$$

Then, as $h_n(x)$ is strictly decreasing,

$$h_n(1-\epsilon) \geq \gamma_{n-1} > h_n(1), \quad \text{for all } n \geq n_0,$$

and, noticing $h_n(1-\epsilon) = \beta_{n,\epsilon}$ and $h_n(1) = \beta_{G_n(z)}$, we get

$$\beta_{n,\epsilon} \geq \gamma_{n-1} > \beta_{G_n(z)}. \tag{4.2}$$

Finally, from (4.1) and (4.2), we obtain

$$\beta_{G_n(z)} < \gamma_{n-1} \leq \beta_{n,\epsilon} \leq \beta_n < \beta_{G_{n+1}(z)}. \tag{4.3}$$

Now consider two cases:

Case 1: $\gamma_n < \beta_{G_{n+1}(z)}$. Then, from (4.3) and Lemma 4, we have

$$\beta_{G_n(z)} < \gamma_{n-1} < \gamma_n < \beta_{G_{n+1}(z)} \leq \gamma_{n+1}, \tag{4.4}$$

and, by writing

$$\beta_{G_{n+1}(z)} = \gamma_n + (\beta_{G_{n+1}(z)} - \gamma_n) = \gamma_n + \Delta_n,$$

with $\Delta_n := \beta_{G_{n+1}(z)} - \gamma_n$, from (4.4), one has that $0 \leq \Delta_n \leq \gamma_{n+1} - \gamma_n$. Now, by using the second part of Lemma 4, the theorem follows.

Case 2: $\beta_{G_{n+1}(z)} \leq \gamma_n$. Then from (4.3), we get

$$\beta_{G_n(z)} < \gamma_{n-1} < \beta_{G_{n+1}(z)} \leq \gamma_n, \tag{4.5}$$

so we can express

$$\beta_{G_{n+1}(z)} = \gamma_n - \left(\gamma_n - \beta_{G_{n+1}(z)}\right) = \gamma_n + \Delta_n$$

with $\Delta_n := -\left(\gamma_n - \beta_{G_{n+1}(z)}\right)$. By (4.5), $|\Delta_n| = \gamma_n - \beta_{G_{n+1}(z)} < \gamma_n - \gamma_{n-1}$, and then in this case the theorem also follows by applying the second part of Lemma 4. This completes the proof. ■

Theorem 2 *Let* $a_{\zeta_n(z)} := \inf\{\Re z : \zeta_n(z) = 0\}$. *Then*

$$a_{\zeta_n(z)} = -\frac{\log 2}{\log\left(\frac{n-1}{n-2}\right)} + \Delta_n, \quad n > 2, \tag{4.6}$$

with $\limsup_{n\to\infty} |\Delta_n| \leq \log 2$.

Proof. From [1, Proposition 1, (ii)], there exists n_0 such that

$$b_{G_n(z)} := \sup\{\Re z : G_n(z) = 0\} = \beta_{G_n(z)}, \quad n \geq n_0.$$

By using (1.1), we get $a_{\zeta_n(z)} = -b_{G_n(z)}$, for all $n \geq 2$, so $a_{\zeta_n(z)} = -\beta_{G_n(z)}$ for $n \geq n_0$. Now, by applying Theorem 1, the estimate (4.6) follows. ■

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