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Polynomial and rational inequalities on analytic Jordan arcs and domains

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ABSTRACT

In this paper we prove an asymptotically sharp Bernstein-type inequality for polynomials on analytic Jordan arcs. Also a general statement on mapping of a domain bounded by finitely many Jordan curves onto a complement to a system of the same number of arcs with rational function is presented here. This fact, as well as, Borwein–Erdélyi inequality for derivative of rational functions on the unit circle, Gonchar–Grigorjan estimate of the norm of holomorphic part of meromorphic functions and Totik's construction of fast decreasing polynomials play key roles in the proof of the main result.

0. Introduction

Let $\mathbb{T} := \{z \in \mathbf{C} : |z| = 1\}$ denote the unit circle, $\mathbb{D} := \{z \in \mathbf{C} : |z| < 1\}$ denote the unit disk and $\mathbf{C}_\infty := \mathbf{C} \cup \{\infty\}$ denote the extended complex plane. We also use $\mathbb{D}^* := \{z \in \mathbf{C} : \ |z| > 1\} \cup \{\infty\}$ for the exterior of the unit disk and $\|.\|_K$ for the sup norm over the set K.

First, we recall a Bernstein-type inequality proved by Borwein and Erdélyi in [2] (and in a special case, by Li, Mohapatra and Rodriguez in [7]). We rephrase their inequality using potential theory (namely, normal derivatives of Green's functions) and for the necessary concepts, we refer to [12] and [11]. Then we present one of our main tools, the "open-up" step in Proposition 5, similar step was also discussed by Widom, see [17], pp. 205–206 and Lemma 11.1. This way we switch from polynomials and Jordan arcs to rational functions and Jordan curves. Then we use two conformal mappings, Φ_1 and Φ_2 to map the interior of the Jordan domain onto the unit disk and to map the exterior of the domain onto the exterior

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of the unit disk respectively. We transform our rational function with Φ_1 and "construct" a similar rational function (approximate with another, suitable rational function) so that the Borwein–Erdélyi inequality can be applied.

Our main theorem is the following.

Theorem 1. *Let K be an analytic Jordan arc, $z_0 \in K$ not an endpoint. Denote the two normals to K at z_0 by $n_1(z_0)$ and $n_2(z_0)$. Then for any polynomial P_n of degree n we have*

$$|P_n'(z_0)| \le (1+o(1))\, n\, \|P_n\|_K \cdot \max\left(\frac{\partial}{\partial n_1(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty),\ \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty)\right)$$

where $o(1)$ depends on z_0 and K only and tends to 0 as $n \to \infty$.

Remark. This theorem was formulated as a conjecture in [9] on p. 225.

Theorem 1 is asymptotically sharp as the following theorem shows.

Theorem 2. *Let K be a finite union of disjoint, C^2 smooth Jordan arcs and $z_0 \in K$ is a fixed point which is not an endpoint. We denote the two normals to K at z_0 by $n_1(z_0)$ and $n_2(z_0)$. Then there exists a sequence of polynomials P_n with $\deg P_n = n \to \infty$ such that*

$$|P_n'(z_0)| \ge n(1-o(1))\, \|P_n\|_K \cdot \max\left(\frac{\partial}{\partial n_1(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty), \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty)\right).$$

1. A rational inequality on the unit circle

The following theorem was proved in [2] (see also [1], p. 324, Theorem 7.1.7), with slightly different notations.

If f is a rational function, then $\deg(f)$ denotes the maximum of the degrees of the numerator and denominator of f (where we assume that the numerator and the denominator have no common factors).

Theorem *(Borwein–Erdélyi). Let $a_1, \ldots, a_m \in \mathbf{C} \setminus \{|u| = 1\}$ and let*

$$B_m^+(u) := \sum_{j:\ |a_j|>1} \frac{|a_j|^2-1}{|a_j-u|^2}, \qquad B_m^-(u) := \sum_{j:\ |a_j|<1} \frac{1-|a_j|^2}{|a_j-u|^2},$$

and $B_m(u) := \max(B_m^+(u), B_m^-(u))$. If R is a polynomial with $\deg(R) \le m$ and $f(u) = R(u) / \prod_{j=1}^m (u-a_j)$ is a rational function, then

$$|f'(u)| \le B_m(u)\, ||f||_{\mathbb{T}}, \qquad u \in \mathbb{T}.$$

If all the poles of f are inside or outside of $\mathbb{D}$, then this result was improved in [7], Theorem 2 and Corollary 2 on p. 525 using different approach.

We need to relax the condition on the degree of the numerator and the denominator.

If we could allow poles at infinity, then the degree of the numerator can be larger than that of the denominator. More precisely, we can easily obtain the following

Theorem 3. *Using the notations from Borwein–Erdélyi Theorem, if R is a polynomial with $\deg(R) > m$ and $f(u) = R(u) / \prod_{j=1}^{m}(u - a_j)$ is a rational function, then*

$$|f'(u)| \le \max\left(B_m^+(u) + \deg(R) - m,\ B_m^-(u)\right) ||f||_{\mathbb{T}}, \qquad w \in \mathbb{T}. \tag{1}$$

Proof. Let $d := \deg(R) - m > 0$, and let $f_1(\tau; u) = f_1(u) := \frac{f(u)}{(u-\tau)^d}$, where $\tau \in \mathbf{R}$, $\tau > 1$. Then $(\tau - 1)^d |f_1(u)| \le |f(u)| \le (\tau + 1)^d |f_1(u)|$ for $|u| = 1$, so

$$||f_1||_{\mathbb{T}} \le \frac{1}{(\tau - 1)^d} ||f||_{\mathbb{T}}.$$

Since $f_1'(u) = f'(u)\frac{1}{(u-\tau)^d} - d\, f(u)\frac{1}{(u-\tau)^{d+1}}$, therefore

$$|f_1'(u)| \ge |f'(u)| \frac{1}{(\tau + 1)^d} - d||f||_{\mathbb{T}} \frac{1}{(\tau - 1)^{d+1}}.$$

Using Borwein–Erdélyi Theorem for f_1, $|u| = 1$,

$$|f_1'(u)| \le \max\left(B_m^+(u) + d\frac{\tau^2 - 1}{|u - \tau|^2},\ B_m^-(u)\right) ||f_1||_{\mathbb{T}}.$$

Letting $\tau \to \infty$ and combining the last three displayed estimates, we obtain the Theorem. □

Note that if we let all the poles tend to infinity, then we get back the original Bernstein (Riesz) inequality for polynomials on the unit disk. Let us also remark that the original proof of Borwein and Erdélyi also proves (1), with little modifications.

The relation with Green's functions is as follows. It is well known (see e.g. [12], p. 109) that Green's function of the unit disk $\mathbb{D}$ with pole at $a \in \mathbb{D}$ is

$$g_{\mathbb{D}}(u, a) = \log\left|\frac{1 - \bar{a}u}{u - a}\right|$$

and Green's functions of the complement of the unit disk $\mathbb{D}^* = \{|u| > 1\} \cup \{\infty\}$ with pole at $a \in \mathbf{C}$, $|a| > 1$ and with pole at infinity are

$$g_{\mathbb{D}^*}(u, a) = \log\left|\frac{1 - \bar{a}u}{u - a}\right| \text{ and } g_{\mathbb{D}^*}(u, \infty) = \log|u|.$$

For the normal derivatives elementary calculations give ($|u| = 1$, $n_1(u) = -u$ is the inner normal, $n_2(u) = u$ is the outer normal)

$$\frac{\partial}{\partial n_1(u)} g_{\mathbb{D}}(u, a) = \lim_{t \to 0+} \frac{\log\left|\frac{1-\bar{a}(1-t)u}{(1-t)u-a}\right|}{t} = \frac{1 - |a|^2}{|u - a|^2}, \tag{2}$$

$$\frac{\partial}{\partial n_2(u)} g_{\mathbb{D}^*}(u, a) = \lim_{t \to 0+} \frac{\log\left|\frac{1-\bar{a}(1+t)u}{(1+t)u-a}\right|}{t} = \frac{|a|^2 - 1}{|u - a|^2}, \tag{3}$$

$$\frac{\partial}{\partial n_2(u)} g_{\mathbb{D}^*}(u, \infty) = \lim_{t \to 0+} \frac{\log|(1 + t)u|}{t} = 1. \tag{4}$$

They are also mentioned in [4], p. 1739.

Using this notation, we can reformulate these last two theorems as follows. This is actually the result of Borwein and Erdélyi with slightly different wording.

Theorem 4. *Let $f(u) = R(u)/Q(u)$ be an arbitrary rational function with no poles on the unit circle where R and Q are polynomials. Denote the poles of f on $\mathbf{C}_\infty$ by $a_1, \ldots, a_m \in \mathbf{C}_\infty \setminus \{|u| = 1\}$ where each pole is repeated as many times as its order. Then, for $u \in \mathbb{T}$,*

$$|f'(u)| \leq ||f||_{\mathbb{T}} \max\left(\sum_{j:|a_j|<1} \frac{\partial}{\partial n_1(u)} g_{\mathbb{D}}(u, a_j), \sum_{j:|a_j|>1} \frac{\partial}{\partial n_2(u)} g_{\mathbb{D}^*}(u, a_j)\right). \tag{5}$$

Note that if $\deg(R) > \deg(Q)$, then f has a pole at ∞, therefore it is repeated $\deg(R) - \deg(Q)$ times and this pole at ∞ is taken into account in the second term of maximum. Inequality (5) is sharp, the factor on the right hand side cannot be replaced for smaller constant, see, e.g., [1], p. 324.

2. Mapping complement of a system of arcs onto domains bounded by Jordan curves with rational functions

Let K be a finite union of C^2 smooth, disjoint Jordan arcs on the complex plane, that is,

$$K = \cup_{j=1}^{k_0} \gamma_j, \text{ where } \gamma_j \cap \gamma_k = \emptyset, \ j \neq k.$$

Denote the endpoints of γ_j by ζ_{2j-1}, ζ_{2j}, $j = 1, \ldots, k_0$.

We need the following Proposition to transfer our setting. Although we will use it for one analytic Jordan arc, it can be useful for further researches.

After we worked out the proof, we learned that Widom developed very similar open-up Lemma in his work, see [17], pp. 205–207. The difference is that he considers C^k smooth arcs with Hölder continuous k-th derivative (see also p. 145) while we need this open-up technique for analytic arcs. Furthermore, there is a difference regarding the number of poles. This is discussed after the proof.

Proposition 5. *There exists a rational function F and a domain $G \subset \mathbf{C}_\infty$ such that $\mathbf{C} \setminus G$ is a compact set with k_0 components, $\partial(\mathbf{C}_\infty \setminus G) = \partial G$ is union of finitely many smooth Jordan curves and F is a conformal bijection from G onto $\mathbf{C}_\infty \setminus K$ with $F(\infty) = \infty$.*

Furthermore, if K is analytic, then ∂G is analytic too.

Proof. First, we show that there are polynomials R, Q such that $\deg(R) = k_0 + 1$, $\deg(Q) = k_0$,

$$F(u) := \frac{R(u)}{Q(u)}$$

and

$$F'(u) = 0 \Leftrightarrow F(u) \in \{\zeta_1, \ldots, \zeta_{2k_0}\}. \tag{6}$$

Obviously, $F'(u) = (R'(u) Q(u) - R(u) Q'(u)) / Q^2(u)$ and the numerator is a polynomial of degree $2k_0$. Let $A(u) := \prod_{j=1}^{2k_0} (u - \zeta_j)$. Taking reciprocal, $1/F' = Q^2/A$, that is, the location of the poles are known. Our goal is to find $\beta_0, \beta_1, \beta_2, \ldots, \beta_{2k_0} \in \mathbf{C}$ such that

$$\int \frac{1}{\beta_0 + \sum_{j=1}^{2k_0} \frac{\beta_j}{u - \zeta_j}} du \text{ is a rational function.}$$

Or equivalently, $F_1(u) := \frac{\prod_k (u-\zeta_k)}{\beta_0 \prod_k (u-\zeta_k) + \sum_{j>0} \beta_j \prod_{k\neq j}(u-\zeta_k)}$ must have 0 residue everywhere, $\mathrm{Res}(F_1, u) = 0$ for all $u \in \mathbf{C}$. Since ζ_k's are pairwise different, $\prod_{k\neq j}(u-\zeta_k)$, $j = 1, 2, \ldots, 2k_0$ and $\prod_k (u-\zeta_k)$ are linearly independent, so we can choose β_j's so that

$$\beta_0 \prod_k (u-\zeta_k) + \sum_{j>0} \beta_j \prod_{k\neq j} (u-\zeta_k) = (u-u^*)^{2k_0}$$

where u^* will be specified later. Write $A(u) = \prod_k (u-\zeta_k)$ in the form $A(u) = \sum_{j=0}^{2k_0} c_j (u-u^*)^j$ with suitable c_j's. It is easy to see that $\mathrm{Res}(F_1, u) = 0$ for all $u \neq u^*$, furthermore $\mathrm{Res}(F_1, u^*) = c_{2k_0-1}$. Comparing the coefficients of $A(u)$, we obtain $c_{2k_0} = 1$, $c_{2k_0-1} = -\left(\sum_{j=1}^{2k_0} \zeta_j\right) + 2k_0 u^*$. Rearranging the expression for c_{2k_0-1}, u^* must satisfy the following equation

$$u^* = \frac{\sum_{j=1}^{2k_0} \zeta_j}{2k_0}.$$

With this choice, there exists $F = \int F_1$ with the desired properties.

The domain G is constructed as follows. Denote the unbounded component of $F^{-1}[\mathbf{C}_\infty \setminus K]$ by G. We prove that G is a domain and its boundary consists of finitely many Jordan curves and those curves are smooth. Locally, if $z \in \gamma_j$ for some γ_j and z is not endpoint of γ_j, then, by the construction, z is not a critical value. In other words, for any u such that $F(u) = z$, we know $F'(u) \neq 0$ (u is not a critical place). If $z \in \gamma_j$ is an endpoint and u_1 is any of its inverse image, then $F'(u_1) = 0$ by (6) and since the degree of R and Q are minimal, $F''(u_1) \neq 0$. Therefore $F(u) \approx c(u-u_1)^2 + z$, and the inverse image $F^{-1}[\gamma_j]$ of γ_j near u_1 is a smooth, simple arc. So each bounded component of $\mathbf{C} \setminus G$ is such a compact set that it is a closure of a Jordan domain.

Using continuity and connectedness, $\mathbf{C}_\infty \setminus F^{-1}[\mathbf{C}_\infty \setminus K]$ has at least k_0 bounded components. If there were more than k_0 components, then we obtain contradiction as follows. The boundary of each component is mapped into K, so there should be more than $2k_0$ critical points, but this contradicts the minimality of F. Denote the boundary of the components by κ_j, $j = 1, \ldots, k_0$. These κ_j's are smooth Jordan curves and assume $\kappa_j = \kappa_j(t)$, $t \in [0, 2\pi]$.

It is clear that each component has nonempty interior and contains at least one pole of F, otherwise F maps that component onto some open, bounded, nonempty set and this set would intersect $\mathbf{C}_\infty \setminus K$. Therefore each component contains exactly one pole which is simple by the minimality assumption.

Now, $F = R/Q$ is univalent on G because of the followings. Take smooth Jordan curves $\kappa_{j,\delta}(t)$, $t \in [0, 2\pi]$ satisfying the next properties: $\kappa_{j,\delta} \subset G$, $\kappa_{j,\delta}(t) \to \kappa_j(t)$ as $\delta \to 0$ and $\kappa'_{j,\delta}(t) \to \kappa'_j(t)$ as $\delta \to 0$ and $\kappa_{0,\delta}(t) := 1/\delta \exp(it)$. Since $\deg(R) = \deg(Q)+1$, $F(u) = c_1 u + c_0 + o(1)$ as $u \to \infty$ therefore $F(\kappa_{0,\delta}(t)) \rightrightarrows \infty$ as $\delta \to 0$ and, by continuity, $\mathrm{dist}(F(\kappa_{j,\delta}), \gamma_j) \to 0$. Since F has no critical values outside K, the $F(\kappa_{j,\delta})$'s are smooth Jordan curves. Fix $b \in \mathbf{C} \setminus K$, then there is (at least one) $b' \in G$ with $F(b') = b$, because $F(G)$ is open, $F(G) \subset \mathbf{C} \setminus K$ and $F(\partial G) = F(\kappa_1 \cup \ldots \cup \kappa_{k_0}) \subset K$. If $\delta > 0$ is small enough, then $b \in \mathrm{Int}\, F(\kappa_{0,\delta})$ and $b \in \mathbf{C} \setminus \mathrm{Int}\, F(\kappa_{j,\delta})$ ($j = 1, \ldots, k_0$), so $\mathrm{index}(b, F(\kappa_{0,\delta}) \cup F(\kappa_{1,\delta}) \cup \ldots \cup F(\kappa_{k_0,\delta})) = 1$. Therefore $\mathrm{index}(b', \kappa_{0,\delta} \cup \kappa_{1,\delta} \cup \ldots \cup \kappa_{k_0,\delta}) = 1$, so there is exactly one inverse image, this shows the univalence of F.

We can give another proof for the univalence as follows. There is a (local) branch of F^{-1} such that $F^{-1}[z] = z/c_1 + ..$ as $z \to \infty$, in other words, ∞ is not a branch point of F^{-1}. Furthermore, the function F has branch points only at ζ_j's, $j = 1, \ldots, 2k_0$ and it behaves as a square root there. Therefore every analytic continuations along any curve in $\mathbf{C} \setminus K$ give the same function element. Now we use Lemma 2, p. 175 in [13] with this (local) branch. Therefore we can choose a (global) regular branch of F^{-1} such that $F^{-1}[\infty] = \infty$. Since this branch is regular and F is a rational function, there is no other inverse image of ∞ by F^{-1} in G. By the construction of G and applying the maximum principle, we have $g_{\mathbf{C}_\infty \setminus K}(F(u), \infty) \equiv g_G(u, \infty)$, $u \in G$. Using the majorization principle (see [6], Theorem 1 on p. 624) or Theorem 4.4.1 on p. 112 from [11], we obtain that F is conformal bijection from G onto $\mathbf{C}_\infty \setminus K$.

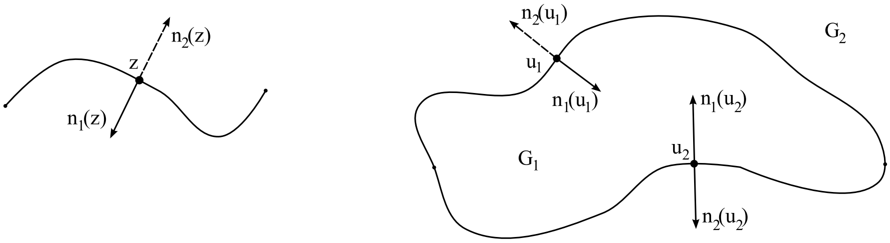

Fig. 1. The γ, z, G_1 and G_2 with the normal vectors.

As for the smoothness assertion (∂G analytic), this follows from standard considerations as follows. Without loss of generality, we may assume that $z = \kappa(t) = t + c_1 t + c_2 t^2 + \dots$, is a convergent power series for $0 \le t \le t_0$ and $z = F(u)$ is such that $F(0) = 0$, $F'(0) = 0$ and $F''(0) \neq 0$. It is known, see e.g. [15], p. 286, that the two branches of the inverse of F near $z = 0$ can be written as $G_0(z) \pm \sqrt{z} G_1(z)$ where G_0, G_1 are holomorphic functions. Denote them by F_1^{-1} and F_2^{-1}. This way $\gamma_1(t) := F_1^{-1}\left[\kappa\left(t^2\right)\right] = G_0\left(\kappa\left(t^2\right)\right) + t\sqrt{1+\kappa_1(t^2)} G_1\left(\kappa\left(t^2\right)\right)$ is a convergent power series in $t \in [0, t_1]$ and similarly for $\gamma_2(t) := F_2^{-1}\left[\kappa\left(t^2\right)\right]$ and $\gamma_1'(0) \neq 0$. Considering $\gamma_1(-t)$ for $t \in [0, t_1]$, we see that $\gamma_2(t) = \gamma_1(-t)$, so γ_1 is actually a convergent power series and it parametrizes the two joining arc. $\square$

As for the number of poles, Widom's open-up mapping is constructed as iterating the Joukowskii mapping (composed with a suitable linear mapping in each step) for each arc and that open-up mapping has 2^{k_0} different, simple poles and the location of poles also depends on the order of arcs. In contrast, our open-up rational function has k_0 simple poles.

With this Proposition, we switch from polynomials on Jordan arcs to rational functions on Jordan curves as follows. We use the following notations, assumptions.

Fix one, C^2 smooth Jordan arc γ with endpoints ζ_1 and ζ_2 and let $z \in \gamma$, $z \neq \zeta_1$, $z \neq \zeta_2$. Denote the two normal vectors of unit length at z to γ by $n_1(z)$, $n_2(z)$, where $n_1(z) = -n_2(z)$. We may assume that n_1 and n_2 depend continuously on z. We use the same letter for normals in different planes and from the context, it is always clear which arc we refer to. We use the rational mapping F and the domain $G_2 := G$ from the previous Proposition for γ. Denote the inward normal vector to ∂G at $u \in \partial G$ by $n_2(u)$ and the outward normal vector to ∂G at u by $n_1(u)$, $n_2(u) = -n_1(u)$. It is easy to see that there are two inverse images of z: $u_1 = u_1(z)$, $u_2 = u_2(z) \in \partial G$ (such that $F(u_1) = F(u_2) = z$) and we can assume that u_1, u_2 are continuous functions of z.

By reindexing u_1 and u_2, we may assume that the normal vector $n_2(u_1)$ is mapped by F to the normal vector $n_2(z)$. This immediately implies that $n_1(u_1)$, $n_2(u_2)$, $n_1(u_2)$ are mapped by F to $n_1(z)$, $n_1(z)$, $n_2(z)$ respectively.

Let us denote the domain $\mathbf{C} \setminus (G \cup \partial G)$ by G_1. Since $\deg F = 2$ and F is a conformal bijection from G_2 onto $\mathbf{C}_\infty \setminus \gamma$, F is a conformal bijection from G_1 onto $\mathbf{C}_\infty \setminus \gamma$. For simplicity, let us denote the inverse of F onto G_1 by F_1^{-1} and onto G_2 by F_2^{-1}.

These geometrical objects are depicted in Fig. 1 where we indicated the normal vectors $n_2(z)$ and $n_2(u_1)$ with dashed arrows (we fix the notations with their help) and we indicated the other normal vectors with simple (not dashed) arrows (their indexings are consequence of the earlier two vectors).

Proposition 6. *Using the notations above, for the Green's functions of $G = G_2$ and G_1 and for $b \in \mathbf{C}_\infty \setminus K$ we have*

$$\frac{\partial}{\partial n_1(z)} g_{\mathbf{C}_\infty \setminus K}(z, b) = \frac{\partial}{\partial n_1(u_1)} g_{G_1}\left(u_1, F_1^{-1}(b)\right) / \left|F'(u_1)\right|$$
$$= \frac{\partial}{\partial n_2(u_2)} g_{G_2}\left(u_2, F_2^{-1}(b)\right) / \left|F'(u_2)\right|$$

and, similarly for the other side,

$$\frac{\partial}{\partial n_2(z)} g_{\mathbf{C}_\infty \setminus K}(z,b) = \frac{\partial}{\partial n_1(u_2)} g_{G_1}\left(u_2, F_1^{-1}(b)\right) / \left|F'(u_2)\right| = \frac{\partial}{\partial n_2(u_1)} g_{G_2}\left(u_1, F_2^{-1}(b)\right) / \left|F'(u_1)\right|.$$

For arbitrary polynomial P, let $f_P(u) = f(u) := P(F(u))$. Then $\|P\|_\gamma = \|f\|_{\partial G}$.

Proof. This immediately follows from the conformal invariance of Green's functions

$$g_{\mathbf{C}_\infty \setminus K}(F(u), b) = g_{G_1}\left(u, F_1^{-1}(b)\right)$$

and

$$g_{\mathbf{C}_\infty \setminus K}(F(u), b) = g_{G_2}\left(u, F_2^{-1}(b)\right).$$

See e.g. [11], p. 107, Theorem 4.4.4. □

This Proposition implies that it is enough to take into account the normal derivatives at, say, u_1 only, i.e. $\frac{\partial}{\partial n_2(u_1)} g_{G_2}\left(u_1, F_2^{-1}(b)\right)$ and $\frac{\partial}{\partial n_1(u_1)} g_{G_1}\left(u_1, F_1^{-1}(b)\right)$ only.

3. Conformal mappings on simply connected domains

Here G_1 is the bounded domain from the previous section and G_2 is the unbounded domain from the previous section. Actually, $G_2 = \mathbf{C}_\infty \setminus \left(G_1^-\right)$. As earlier, $\mathbb{D} = \{v : |v| < 1\}$ and $\mathbb{D}^* = \{v : |v| > 1\} \cup \{\infty\}$. With these notations, $\partial G_1 = \partial G_2$. Using Kellogg–Warschawski theorem (see e.g. [10], p. 49, Theorem 3.6), if the boundary is $C^{1,\alpha}$ smooth, then the Riemann mappings of $\mathbb{D}$, $\mathbb{D}^*$ onto G_1, G_2 respectively and their derivatives can be extended continuously to the boundary.

Under analyticity assumption, we can compare the Riemann mappings as follows.

Proposition 7. *Let $u_0 \in \partial G_1 = \partial G_2$ be fixed. Then there exist two Riemann mappings $\Phi_1 : \mathbb{D} \to G_1$, $\Phi_2 : \mathbb{D}^* \to G_2$ such that $\Phi_j(1) = u_0$ and $\left|\Phi_j'(1)\right| = 1$, $j = 1, 2$.*

If $\partial G_1 = \partial G_2$ is analytic, then there exist $0 \le r_1 < 1 < r_2 \le \infty$ such that Φ_1 extends to $D_1 := \{v : |v| < r_2\}$, $G_1^+ := \Phi_1(D_1)$ and $\Phi_1 : D_1 \to G_1^+$ is a conformal bijection, and similarly, Φ_2 extends to $D_2 := \{v : |v| > r_1\} \cup \{\infty\}$, $G_2^+ := \Phi_2(D_2)$ and $\Phi_2 : D_2 \to G_2^+$ is a conformal bijection.

Proof. The existence of Φ_1 follows immediately from the Riemann mapping theorem by considering arbitrary Riemann mapping and composing this mapping with a suitable rotation and hyperbolic translation toward 1 (that is, $\chi_t(z) = (z-t)/(1-tz)$ with $t \in (-1,1)$ and $t \to -1$, $\chi_t'(1) \to 0$, and $t \to 1$, $\chi_t'(1) \to +\infty$).

The existence of Φ_2 follows the same way, using the same family of hyperbolic translations.

The extension follows from the reflection principle for analytic curves (see e.g. [3], pp. 16–21). □

From now on, we fix such two conformal mappings and let $a_1 := \Phi_1^{-1}\left[F_1^{-1}[\infty]\right]$ and $a_2 := \Phi_2^{-1}[\infty] = \Phi_2^{-1}\left[F_2^{-1}[\infty]\right]$.

The domains of these analytic extensions are depicted in Fig. 2 where D_1 is the grey region on the right and is mapped onto G_1^+ by Φ_1 which is the grey region on the left.

Using these mappings, we have the following relations between the normal derivatives of Green's functions and Blaschke factors.

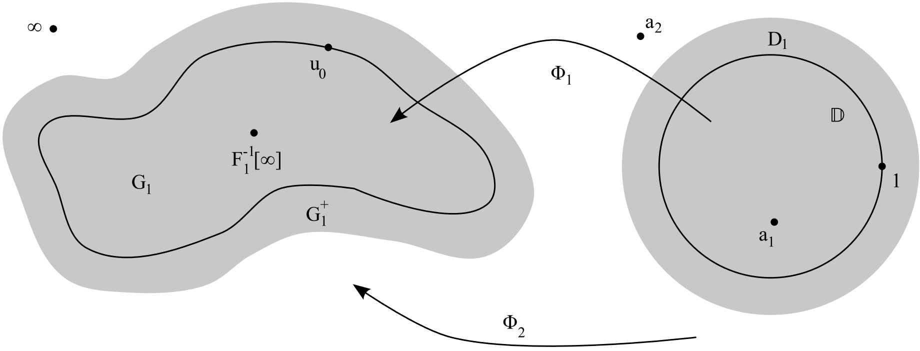

Fig. 2. The two Riemann mappings and the points.

Proposition 8. *The followings hold*

$$\frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right) = \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, a_1) = \frac{1-|a_1|^2}{|1-a_1|^2},$$

$$\frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right) = \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, a_2) = \frac{|a_2|^2-1}{|1-a_2|^2},$$

and if $a_2 = \infty$, *then*

$$\frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right) = \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \infty) = 1.$$

Proof. The second equalities in all three lines follow from (2), (3) and (4).

We know that $\Phi_1(1) = u_0$ and $\Phi_2(1) = u_0$, moreover $|\Phi_1'(1)| = 1$, $|\Phi_2'(1)| = 1$ imply that $n_j(1)$ is mapped to $n_j(u_0)$ by Φ_j, $j = 1, 2$ and the mappings Φ_j, $j = 1, 2$ also preserve the length at 1 (there is no magnifying factor $\left|\Phi_j'(1)\right|^{-1}$ unlike at Proposition 6). Using the conformal mappings Φ_1 and Φ_2, and the conformal invariance of Green's functions, we obtain the first equalities in all three lines. □

4. Proof of Theorem 1 with rational functions

4.1. Auxiliary results, some notations

Before we start the proof, let us recall three results. The first one is Gonchar–Grigorjan estimate when we have one pole only. See [5], Theorem 2 on p. 572 (in the English translation).

Theorem. *Let* $D_G \subset \mathbf{C}$ *be a simply connected domain and its boundary is* C^1 *smooth. Let* $f_G : D_G \to \mathbf{C}_\infty$ *be a meromorphic function on* D_G *such that it has only one pole. Assume that* f_G *can be extended continuously to the boundary* ∂D_G *of* D_G. *Denote* $f_{G,r}$ *the principal part of* f_G *in* D_G *(with* $f_{G,r}(\infty) = 0$*) and let* $f_{G,h}$ *denote the holomorphic part of* f_G *in* D_G. *Denote the order of the pole of* f_G *by* n_G. *Then* $f_G = f_{G,r} + f_{G,h}$ *and there exists* $C_1(D_G) > 0$ *depending on* D_G *only such that*

$$\|f_{G,h}\|_{\partial D_G} \le C_1(D_G)(\log n_G + 1)\|f_G\|_{\partial D_G} \tag{7}$$

where $\|.\|_{\partial D_G}$ *denotes the sup norm over the boundary of* D_G.

In the main result of this paper we are interested in asymptotics as $n \to \infty$. In particular, if $n_G \ge 2$, then $\log n_G + 1 \le 3\log(n_G)$, so we may write $\log n_G + 1 = O(\log n_G)$.

The second result is a special case of the Bernstein–Walsh estimate, see [11], p. 156, Theorem 5.5.7 a) or [12], p. 153.

Theorem. *Let $\tilde{G} \subset \mathbf{C}_\infty$ be a domain, $\infty \in \tilde{G}$ and denote its Green's function by $g_{\tilde{G}}(u,\infty)$ with pole at infinity. Let $\tilde{f} : \tilde{G} \to \mathbf{C}_\infty$ be a meromorphic function which has only one pole at infinity and we denote the order of the pole by $\tilde{n}$. Assume that $\tilde{f}$ can be extended continuously to the boundary $\partial\tilde{G}$ of $\tilde{G}$. Then*

$$\left|\tilde{f}(u)\right| \le \left\|\tilde{f}\right\|_{\partial\tilde{G}} \exp\left(\tilde{n}\, g_{\tilde{G}}(u,\infty)\right) \tag{8}$$

where $\|.\|_{\partial\tilde{G}}$ denotes the sup norm over $\partial\tilde{G}$.

The third result is a special case of a general construction of fast decreasing polynomials by Totik, see [16], Corollary 4.2 and Theorem 4.1 too on p. 2065.

Theorem. *Let $\tilde{K} \subset \mathbf{C}$ be a compact set, $\tilde{u} \in \partial\tilde{K}$ be a boundary point. Assume that $\tilde{K}$ satisfies the touching outer-disk-condition, that is, there exists a closed disk (with positive radius) such that its intersection with $\tilde{K}$ is $\{\tilde{u}\}$. Then there exist $C_2, C_3 > 0$ such that for all $\tilde{n}$ there exists a polynomial $\tilde{Q}$ with the following properties: $\deg\left(\tilde{Q}\right) \le \tilde{n}^{109/110}$, $\tilde{Q}(\tilde{u}) = 1$, $\left\|\tilde{Q}\right\|_{\tilde{K}} \le 1$ and if $u \in \tilde{K}$, $|u-\tilde{u}| \ge \tilde{n}^{-9/10}$, then $\left|\tilde{Q}(u)\right| \le C_2 \exp\left(-C_3 \tilde{n}^{1/110}\right)$.*

To apply this third theorem, we introduce several notations.

We need $\psi(v) := \frac{1-\overline{a_2}v}{v-a_2} = w$ and its inverse $\psi^{-1}(w) = \frac{1+a_2 w}{w+\overline{a_2}}$. Note that $\psi(a_2) = \infty$, $\psi(1) = \frac{1-\overline{a_2}}{1-a_2}$ and let $b_1 := \frac{1-\overline{a_2}}{1-a_2}$. Obviously, $\psi(\partial\mathbb{D}) = \partial\mathbb{D}$.

Let $\Gamma_1 = \{w : |w| = 1 + \delta_1\}$ and $\delta_1 > 0$ is chosen so that $\Gamma_1 \subset \psi(D_1)$. This δ_1 depends on G_2 only.

Let $D_3 := \{w : |w - 2b_1| < 1\}$, this disk touches the unit disk at b_1. Fix $\delta_{2,3}^{(0)} > 0$, $\delta_{2,3}^{(0)} < 1$, such that $\left\{w : |w| \le 1 + \delta_{2,3}^{(0)}\right\} \subset \psi(D_1)$. Then for every $\delta_{2,3} \in \left(0, \delta_{2,3}^{(0)}\right]$, $\{w : \ |w| = 1 + \delta_{2,3}\} \cap \partial D_3$ consists of exactly two points, $w_1^* = w_1^*(\delta_{2,3})$ and $w_2^* = w_2^*(\delta_{2,3})$. It is easy to see that the length of the two arcs of $\left\{w : |w| = 1 + \delta_{2,3}^{(0)}\right\}$ lying in between w_1^* and w_2^* are different, therefore, by reindexing them, we can assume that the shorter arc is going from w_1^* to w_2^* counterclockwise. Elementary geometric considerations show that for all w, $1 \le |w| \le 1 + \delta_{2,3}$ with $\arg w \in \left\{\arg w_j^*(\delta_{2,3}) : \ j = 1,2\right\}$, we have (since $\delta_{2,3} < 1$)

$$\frac{1}{2}\sqrt{\delta_{2,3}} \le |w - b_1| \le 2\sqrt{\delta_{2,3}}. \tag{9}$$

Let

$$K_w^* := \left\{w : \ |w| \le 1 + \delta_{2,3}^{(0)}\right\} \setminus D_3.$$

Obviously, this K_w^* is a compact set and satisfies the touching-outer-disk condition at $b_1 = \frac{1-\overline{a_2}}{1-a_2}$ of Totik's theorem. See Fig. 3 later.

Consider

$$K_u^* := \Phi_2 \circ \psi^{-1}\left[K_w^* \cap \mathbb{D}^*\right] \cup \Phi_1 \circ \psi^{-1}\left[K_w^* \cap \mathbb{D}^*\right] \cup G_1.$$

This is a compact set and also satisfies the touching-outer-disk condition at $u_0 = \Phi_2(1)$ of Totik's theorem. Obviously, $\partial G_2 \subset K_u^*$, $G_1 \subset K_u^*$, $u_0 \in K_u^*$ and if $w \in K_w^*$, then $\Phi_1 \circ \psi^{-1}(w) \in K_u^*$ and $\Phi_2 \circ \psi^{-1}(w) \in K_u^*$ too. Now applying Totik's theorem, there exists a fast decreasing polynomial for K_u^* at u_0 of degree at most n_1 which we denote by $Q = Q(n_1; u)$. More precisely, Q has the following properties: $Q(u_0) = 1$, $|Q(u)| \le 1$ on $u \in K_u^*$, $\deg Q \le n_1^{109/110} \le n_1$ and if $|u - u_0| > n_1^{-9/10}$, $u \in K_u^*$, then

$$|Q(u)| \le C_2 \exp\left(-C_3 n_1^{1/110}\right). \tag{10}$$

Let $n_1 := \lfloor \sqrt{n} \rfloor$, $n_2 := \lfloor n^{3/4} \rfloor$, $\delta_{2,1} := 1/n$ and $\delta_{2,3} := n^{-2/3}$.

4.2. Proof

In this subsection, we let $f(u) := P_n(F(u))$ where P_n is a fixed polynomial of degree n and F is the open-up rational function (see Proposition 5) for K (from Theorem 1).

Actually, we use only the following facts. f is a rational function such that it has one pole in G_1 and one in G_2. We know that the poles of f are $\infty = F_2^{-1}[\infty]$ and $F_1^{-1}[\infty]$, and the order of the pole in G_1 is n.

It is easy to decompose f into sum of rational functions, that is,

$$f = f_1 + f_2$$

where f_1 is a rational function with pole in G_1, $f_1(\infty) = 0$ and f_2 is a polynomial (rational function with pole at ∞). This decomposition is unique. We use the Gonchar–Grigorjan estimate (7) for f_2 on G_1^+, so we have

$$\|f_2\|_{\partial G_2} \le C_1\left(G_1^+\right)(\log n + 1)\|f\|_{\partial G_2}. \tag{11}$$

Obviously, we have

$$\|f_1\|_{\partial G_2} \le \left(1 + C_1\left(G_1^+\right)(\log n + 1)\right)\|f\|_{\partial G_2}. \tag{12}$$

Consider

$$\varphi_1(v) := f_1(\Phi_1(v)).$$

This is a meromorphic function in D_1. We may assume that φ_1 has only one pole in D_1 otherwise we can decrease $r_2 > 1$ so that the pole in G_2 is not in $\Phi_1(D_1) = G_1^+$. We know that

$$\|\varphi_1\|_{\partial\mathbb{D}} = \|f_1\|_{\partial G_2} \tag{13}$$

and $|\varphi_1'(1)| = |f_1'(u_0)|$.

We decompose “the essential part of” φ_1 as follows

$$Q \circ \Phi_1 \cdot \varphi_1 = \varphi_{1r} + \varphi_{1e} \tag{14}$$

where φ_{1r} is a rational function, $\varphi_{1r}(\infty) = 0$ and φ_{1e} is holomorphic in $\mathbb{D}$. We use the Gonchar–Grigorjan estimate (7) again for φ_1 on $\mathbb{D}$, this way the following sup norm estimate holds

$$\|\varphi_{1e}\|_{\partial\mathbb{D}} \le C_1(\mathbb{D})(\log n + 1)\|Q \circ \Phi_1 \cdot \varphi_1\|_{\partial\mathbb{D}} \le C_1(\mathbb{D})(\log n + 1)\|\varphi_1\|_{\partial\mathbb{D}} \tag{15}$$

where $C_1(\mathbb{D})$ is a constant independent of φ_1.

As a remark, let us note that we may write $\log n + 1 \le O(\log n)$ for simplicity since we are interested in asymptotics as $n \to \infty$ in the main theorem. Otherwise, if $n = 0$ or $n = 1$, then P_n is a constant or linear polynomial and the error term $o(1)$ in the main theorem (Theorem 1) can be sufficiently large (depending on K and z_0) for these two particular values of n. In this manner, we write $(\log n + 1)$ in general, but we simplify it to $O(\log n)$ frequently.

Furthermore, we can estimate $\varphi_{1e}(v)$ on $v \in D_1 \setminus \mathbb{D}$ as follows

$$|\varphi_{1e}(v)| = |(Q \cdot f_1) \circ \Phi_1(v) - \varphi_{1r}(v)| \le |(Q \cdot f_1) \circ \Phi_1(v)| + |\varphi_{1r}(v)|. \tag{16}$$

We also need to estimate Q outside $\mathbb{D}$ (and K_w^*) as follows. Using $\deg Q \le n_1^{109/110} \le n_1$ and Bernstein–Walsh estimate (8), we can write for $v \in D_1 \setminus \mathbb{D}$

$$|Q(\Phi_1(v))| \le 1 \cdot \exp\left(n_1 g_{G_2}(\Phi_1(v), \infty)\right).$$

Since the set $\Phi_1(D_1 \setminus \mathbb{D})$ is bounded,

$$C_6 := \sup\{g_{G_2}(\Phi_1(v), \infty) : \ v \in D_1 \setminus \mathbb{D}\} < \infty.$$

Therefore, for all $v \in D_1 \setminus \mathbb{D}$,

$$|(Q \cdot f_1) \circ \Phi_1(v)| \le e^{C_6 n_1} \|f_1\|_{\partial G_2}.$$

This way we can continue (16) and we use $u = \Phi_1(v)$ here and that φ_{1r} is a rational function with no poles outside $\mathbb{D}$ and the maximum principle for φ_{1r}

$$\le e^{C_6 n_1} |f_1(u)| + \|\varphi_{1r}\|_{\partial\mathbb{D}} \le e^{C_6 n_1} \|f_1\|_{\partial G_2} + \|\varphi_1\|_{\partial\mathbb{D}} + \|\varphi_{1e}\|_{\partial\mathbb{D}}$$

and here we used that f_1 has no pole in G_2 and the maximum principle. We can estimate these three sup norms with the help of (12) and (13), (12) and (15), (13), (12). Hence we have for $v \in D_1 \setminus \mathbb{D}$

$$\begin{aligned} |\varphi_{1e}(v)| &\le \left(e^{C_6 n_1} + 1 + C_1(\mathbb{D})(\log n + 1)\right)\left(1 + C_1\left(G_1^+\right)(\log n + 1)\right)\|f\|_{\partial G_2} \\ &= O\left(\log(n)\, e^{C_6 n_1}\right)\|f\|_{\partial G_2}. \end{aligned} \tag{17}$$

Approximate and interpolate φ_{1e} as follows with rational function which has only one pole, namely at $a_2 = \Phi_2^{-1}[\infty]$. Consider $\varphi_{1e} \circ \psi^{-1}(w)$ on $\psi(D_1)$. Using the properties of ψ, we have

$$\|\varphi_{1e}\|_{\partial\mathbb{D}} = \left\|\varphi_{1e} \circ \psi^{-1}\right\|_{\partial\mathbb{D}}$$

and $\varphi_{1e} \circ \psi^{-1}$ is a holomorphic function in $\psi(D_1)$. We interpolate and use integral estimates for the error, see e.g. [11], p. 170, proof of Theorem 6.3.1 or [14], p. 11. Therefore, let

$$q_N(w) := w^N (w - b_1)^2$$

where $N = n + \lfloor\sqrt{n}\rfloor + \lfloor n^{3/4}\rfloor = n(1 + o(1))$. We define the approximating polynomial

$$p_{1,N}(w) := \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\varphi_{1e} \circ \psi^{-1}(\omega)}{q_N(\omega)} \frac{q_N(w) - q_N(\omega)}{w - \omega} d\omega.$$

It is well known that $p_{1,N}$ does not depend on Γ_1. Since b_1 is a double pole of q_N, therefore $p_{1,N}$ and $p'_{1,N}$ coincide there with $\varphi_{1e} \circ \psi^{-1}$ and $\left(\varphi_{1e} \circ \psi^{-1}\right)'$ respectively.

The error of the approximating polynomial $p_{1,N}$ to $\varphi_{1e} \circ \psi^{-1}$ is

$$\begin{aligned}\varphi_{1e} \circ \psi^{-1}(w) - p_{1,N}(w) &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\varphi_{1e} \circ \psi^{-1}(\omega)}{\omega - w} \frac{q_N(w)}{q_N(\omega)} d\omega \\ &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\omega - w}\, q_N(w)\, \frac{\varphi_{1e} \circ \psi^{-1}(\omega)}{q_N(\omega)}\, d\omega, \end{aligned} \tag{18}$$

here $w \in \mathbb{D}$ can be arbitrary. It is easy to see that for $w \in \mathbb{D}$, $|q_N(w)| \le 4$ and

$$\frac{1}{2\pi} \int_{\Gamma_1} \left| \frac{1}{\omega - w} \right| |d\omega| \le \frac{1 + \delta_1}{\delta_1}.$$

Therefore, using (17), we can estimate the error (of approximation of $p_{1,N}$ to $\varphi_{1e} \circ \psi^{-1}$) as follows

$$\begin{aligned}\left|\varphi_{1e} \circ \psi^{-1}(w) - p_{1,N}(w)\right| &\le \frac{4(1+\delta_1)}{\delta_1} O\left(\log(n) e^{C_6 n_1}\right) \|f\|_{\partial G_2} \frac{1}{\delta_1^2 (1+\delta_1)^N} \\ &= \frac{4(1+\delta_1)}{\delta_1^3} \frac{O\left(\log(n) e^{C_6 n_1}\right)}{(1+\delta_1)^N} \|f\|_{\partial G_2}\end{aligned}$$

which tends to 0 as $n \to \infty$, because $n_1 = \lfloor \sqrt{n} \rfloor$ and

$$\frac{e^{C_6 n_1}}{(1+\delta_1)^N} = \exp\left(C_6 \sqrt{n} - \log(1+\delta_1)\, n\,(1 + o(1))\right) \to 0.$$

Considering $p_{1,N} \circ \psi$, it is a rational function with pole at a_2 only, the order of its pole at a_2 is at most N and we know that

$$\|\varphi_{1e} - p_{1,N} \circ \psi\|_{\partial \mathbb{D}} = o(1) \|f\|_{\partial G_2} \tag{19}$$

where $o(1)$ is independent of P_n and f and depends only on G_2 and tends to 0 as $n \to \infty$, furthermore

$$\varphi_{1e}'(1) = (p_{1,N} \circ \psi)'(1). \tag{20}$$

Now we interpolate and approximate $f_2 \circ \Phi_1$. As earlier, we do not need the full information of this function, it is enough to deal with $f_2 \circ \Phi_1$ locally around 1 and preserve the sup norm. Therefore we "chop off" "the unnecessary parts of $f_2 \circ \Phi_1$" with the fast decreasing polynomial Q.

We have the following description about the growth of Green's function.

Lemma 9. *There exists $C_4 > 0$ depending on $\delta_{2,3}^{(0)}$, that is, depending on G_2 only and is independent of P_n, n and f such that for all $1 \le |w| \le 1 + \delta_{2,3}^{(0)}$ we have*

$$\left| \frac{\left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)'(w)}{\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}(w)} \right| \le C_4$$

and

$$g_{G_2}\left(\Phi_1 \circ \psi^{-1}(w), \infty\right) \le C_4 (|w| - 1). \tag{21}$$

Furthermore, there exists $C_5 > 0$ *which depends on* G_2 *and independent of* P_n, n *and* f *such that for all* $1 \le |\zeta| \le 1 + \delta_{2,3}^{(0)}$ *we have*

$$\left| \frac{\left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)' (\zeta)}{\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1} (\zeta)} \right| \le 1 + C_5 |\zeta - b_1|$$

and

$$g_{G_2} \left(\Phi_1 \circ \psi^{-1} (\zeta), \infty \right) \le (|\zeta| - 1) (1 + C_5 |\zeta - b_1|) . \tag{22}$$

Proof. For simplicity, let $\zeta^* := \arg \zeta$ where $\arg \zeta = \zeta / |\zeta|$, if $\zeta \neq 0$ and $\arg 0 = 0$.

We can express Green's function in the following ways for $u \in G_2$,

$$g_{G_2} (u, \infty) = \log \left| \psi \circ \Phi_2^{-1} (u) \right|$$

and for $w \in \mathbb{D}^*$

$$g_{G_2} \left(\Phi_1 \circ \psi^{-1} (w), \infty \right) = \log \left| \psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1} (w) \right| .$$

The first displayed inequality in the Lemma comes from continuity considerations and the conformal bijection properties. Integrating this inequality along radial rays, we obtain (21). If we are close to 1, then more is true:

$$\left| \left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)' (b_1) \right| = 1.$$

Using continuity, we see that there exists $C_5 > 0$ such that for all ζ, $1 \le |\zeta| \le 1 + \delta_{2,3}^{(0)}$, we have

$$\left| \frac{\left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)' (\zeta)}{\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1} (\zeta)} \right| \le 1 + C_5 |\zeta - b_1| .$$

In particular, for all η from the segment $[\zeta^*, \zeta]$, $\eta \in [\zeta^*, \zeta]$,

$$\left| \frac{\left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)' (\eta)}{\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1} (\eta)} \right| \le 1 + C_5 |\eta - b_1|$$

and $|\eta - b_1| \le |\zeta - b_1|$. Therefore, integrating with respect to η along $[\zeta^*, \zeta]$, we obtain

$$\begin{aligned} g_{G_2} \left(\Phi_1 \circ \psi^{-1} (\zeta), \infty \right) &= \Re \int_{\zeta^*}^{\zeta} \frac{\left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)' (\eta)}{\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1} (\eta)} d\eta \\ &\le \int_{\zeta^*}^{\zeta} \left| \frac{\left(\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1}\right)' (\eta)}{\psi \circ \Phi_2^{-1} \circ \Phi_1 \circ \psi^{-1} (\eta)} \right| |d\eta| \le \int_{\zeta^*}^{\zeta} 1 + C_5 |\zeta - b_1| \, |d\eta| \\ &= (|\zeta| - 1) (1 + C_5 |\zeta - b_1|) . \qquad \square \end{aligned}$$

Now we give the approximating polynomial as follows

$$p_{2,N} (w) := \frac{1}{2\pi i} \int_{\Gamma} \frac{(Q \cdot f_2) \circ \Phi_1 \circ \psi^{-1} (\omega)}{q_N (\omega)} \frac{q_N (w) - q_N (\omega)}{w - \omega} d\omega$$

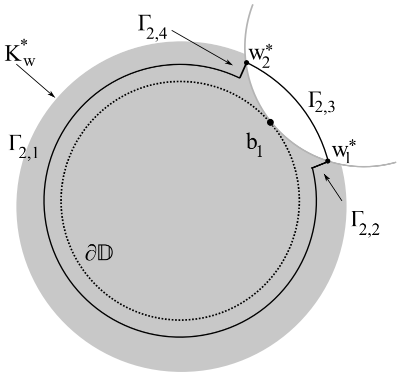

Fig. 3. K_w^* and the arcs that make up Γ_2.

where Γ can be arbitrary with $\mathbb{D} \subset \operatorname{Int}\Gamma$ and $\Gamma \subset \psi(D_1)$. We remark that we use the same interpolating points, but we need a different Γ for the error estimate.

Now we construct $\Gamma = \Gamma_2$ for the estimate and investigate the error. We use $\delta_{2,1} = 1/n$, $\delta_{2,3} = n^{-2/3}$ and $n_2 = \lfloor n^{3/4} \rfloor$. We give four Jordan arcs that will make up Γ_2. Let $\Gamma_{2,3}$ be the (shorter, circular) arc between $w_1^*(\delta_{2,3})$ and $w_2^*(\delta_{2,3})$, $\Gamma_{2,1}$ be the longer circular arc between $w_1^*(\delta_{2,3})\frac{1+\delta_{2,1}}{1+\delta_{2,3}}$ and $w_2^*(\delta_{2,3})\frac{1+\delta_{2,1}}{1+\delta_{2,3}}$, $\Gamma_{2,2} := \{w : 1+\delta_{2,1} \le |w| \le 1+\delta_{2,3},\ \arg w = \arg(w_1^*(\delta_{2,3}))\}$ and similarly $\Gamma_{2,4} := \{w : 1+\delta_{2,1} \le |w| \le 1+\delta_{2,3},\ \arg w = \arg(w_2^*(\delta_{2,3}))\}$ be the two segments connecting $\Gamma_{2,1}$ and $\Gamma_{2,3}$. Finally let Γ_2 be the union of $\Gamma_{2,1}$, $\Gamma_{2,2}$, $\Gamma_{2,3}$ and $\Gamma_{2,4}$. Fig. 3 depicts these arcs and K_w^* defined above.

We estimate the error of $p_{2,N}$ to $(Q \cdot f_2) \circ \Phi_1 \circ \psi^{-1}$ on each integral separately:

$$(Q \cdot f_2) \circ \Phi_1 \circ \psi^{-1}(w) - p_{2,N}(w) = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(Q \cdot f_2) \circ \Phi_1 \circ \psi^{-1}(\omega)}{\omega - w} \frac{q_N(w)}{q_N(\omega)} d\omega$$

$$= \frac{1}{2\pi i} \left(\int_{\Gamma_{2,1}} + \int_{\Gamma_{2,2}} + \int_{\Gamma_{2,3}} + \int_{\Gamma_{2,4}} \right).$$

For the first term, we use Bernstein–Walsh estimate (8) for the polynomial f_2 on G_2 and the fast decreasing polynomial Q as follows. If $w \in \Gamma_{2,1}$, then with (21), $g_{G_2}\left(\Phi_1 \circ \psi^{-1}(w), \infty\right) \le C_4 \delta_{2,1} = C_4/n$, therefore

$$\left|f_2\left(\Phi_1 \circ \psi^{-1}(w)\right)\right| \le \|f_2\|_{\partial G_2} \exp\left(n \frac{C_4}{n}\right) \le \|f\|_{\partial G_2} C_1\left(G_1^+\right)(\log n + 1)\, e^{C_4}$$

$$= O(\log(n)) \|f\|_{\partial G_2}$$

where we used (11). Now we use the fast decreasing property of Q as follows. We know that $\Gamma_{2,1} \subset K_w^*$ (if $n \ge 1/\delta_{2,3}^{(0)}$) and with the elementary geometric considerations (9) we have $\sqrt{\delta_{2,3}}/2 \ge n_1^{-9/10}$ which is equivalent to $n^{-1/3}/2 \ge n^{-9/20}$ (this is true if n is large). It is also important that $\sup\left\{\left|\left(\Phi_1 \circ \psi^{-1}\right)'(w)\right| :\ w \in \psi(D_1)\right\} < \infty$ and $K_w^* \subset \psi(D_1)$ therefore the growth order of the distances is preserved by $\Phi_1 \circ \psi^{-1}$. Hence the fast decreasing polynomial Q is small, see (10), and we can write

$$\left|(Q \cdot f_2)\left(\Phi_1 \circ \psi^{-1}(w)\right)\right| \le O\left(\frac{\log(n)}{\exp\left(C_3 n^{1/220}\right)}\right) \|f\|_{\partial G_2}$$

and integrating along $\Gamma_{2,1}$, we can write for $w \in \mathbb{D}$

$$\left| \frac{1}{2\pi i} \int\limits_{\Gamma_{2,1}} \frac{(Q \cdot f_2) \circ \Phi_1 \circ \psi^{-1}(\omega)}{\omega - w} \frac{q_N(w)}{q_N(\omega)} d\omega \right|$$

$$\leq \frac{1}{2\pi} \int\limits_{\Gamma_{2,1}} \frac{1}{|\omega - w|} O\left(\frac{\log(n)}{\exp\left(C_3 n^{1/220}\right)} \right) \|f\|_{\partial G_2} 4 \frac{1}{(1+\delta_{2,1})^N \delta_{2,1}^2} |d\omega|$$

$$\leq \frac{2}{\pi} \frac{2\pi(1+\delta_{2,1})}{(1+\delta_{2,1})^N \delta_{2,1}^3} O\left(\frac{\log(n)}{\exp\left(C_3 n^{1/220}\right)} \right) \|f\|_{\partial G_2} = O\left(\frac{n^3 \log(n)}{\exp\left(C_3 n^{1/220}\right)} \right) \|f\|_{\partial G_2}$$

here we used $\delta_{2,1} = 1/n$.

We estimate the third term, the integral on $\Gamma_{2,3}$, as follows for $w \in \mathbb{D}$

$$\left| \frac{1}{2\pi i} \int\limits_{\Gamma_{2,3}} \frac{(Q \cdot f_2) \circ \Phi_1 \circ \psi^{-1}(\omega)}{\omega - w} \frac{q_N(w)}{q_N(\omega)} d\omega \right|$$

$$\leq \frac{1}{2\pi} \int\limits_{\Gamma_{2,3}} 4 \frac{1}{|\omega - w|} \left| (Q \cdot f_2)\left(\Phi_1 \circ \psi^{-1}(\omega)\right) \right| \frac{1}{|q_N(\omega)|} |d\omega| . \tag{23}$$

Here, $|\omega| = 1 + \delta_{2,3}$, $|w - \omega| \geq \delta_{2,3}$, $|q_N(\omega)| \geq \delta_{2,3}^2 (1+\delta_{2,3})^N$. Roughly speaking, f_2 grows and this time Q grows too (the bad guys) and only $|q_N(\omega)|^{-1}$ decreases (the good guy). We estimate their growth using Bernstein–Walsh estimate (8) for f_2 on G_2 and Lemma 9 (and estimate (11) as well) in the following way. Here, as earlier, $\omega \in \Gamma_{2,3}$

$$\left| f_2\left(\Phi_1 \circ \psi^{-1}(\omega)\right) \right| \leq \|f_2\|_{\partial G_2} \exp\left(n g_{G_2}\left(\Phi_1 \circ \psi^{-1}(\omega), \infty\right) \right)$$

$$\leq C_1\left(G_1^+\right) (\log n + 1) \|f\|_{\partial G_2} \exp\left(n(|\omega| - 1)(1 + C_5|\omega - b_1|)\right)$$

$$\leq C_1\left(G_1^+\right) (\log n + 1) \|f\|_{\partial G_2} \exp\left(n\delta_{2,3} + C_5 n \delta_{2,3} 2\sqrt{\delta_{2,3}} \right)$$

$$= C_1\left(G_1^+\right) (\log n + 1) \|f\|_{\partial G_2} \exp\left(n\delta_{2,3}\right) e^{2C_5}$$

where in the last two steps we used $|\omega - b_1| \leq 2\sqrt{\delta_{2,3}}$ from (9) and $\delta_{2,3} = n^{-2/3}$.

As for q_N,

$$\frac{1}{|q_N(\omega)|} \leq \frac{1}{\delta_{2,3}^2} \frac{1}{(1+\delta_{2,3})^N} = \frac{1}{\delta_{2,3}^2} \exp\left(-(n + n_1 + n_2)\log(1+\delta_{2,3})\right)$$

$$\leq \frac{1}{\delta_{2,3}^2} \exp\left(-n\delta_{2,3} - n_1\delta_{2,3} - n_2\delta_{2,3} + (n + n_1 + n_2)\frac{\delta_{2,3}^2}{2} \right)$$

$$\leq \frac{1}{\delta_{2,3}^2} \exp\left(-n\delta_{2,3} - n_1\delta_{2,3} - n_2\delta_{2,3}\right) \exp\left(3n\, n^{-4/3}\right)$$

$$\leq \frac{\exp\left(-n\delta_{2,3} - n_1\delta_{2,3} - n_2\delta_{2,3}\right)}{\delta_{2,3}^2} e^3$$

where we used $n_1 = \lfloor n^{1/2} \rfloor$, $n_2 = \lfloor n^{3/4} \rfloor$ and $\delta_{2,3} = n^{-2/3}$.

As for Q (this time it is a bad guy), we use Bernstein–Walsh estimate (8) for Q on $G_1 \cup \partial G_1$ and that $G_1 \cup \partial G_1 \subset K_u^*$. Therefore, $\|Q\|_{\partial G_2} = 1$ and we know that $\deg Q \le n_1^{109/110} \le n^{109/220}$, hence

$$\begin{aligned}\left|Q\left(\Phi_1 \circ \psi^{-1}(\omega)\right)\right| &\le \|Q\|_{\partial G_2} \exp\left(n_1 g_{G_2}\left(\Phi_1 \circ \psi^{-1}(\omega), \infty\right)\right) \\ &\le \exp\left(n_1(|\omega| - 1)(1 + C_5|\omega - b_1|)\right) \le \exp\left(n^{109/220}\delta_{2,3}\left(1 + C_5 2\sqrt{\delta_{2,3}}\right)\right) \\ &= \exp\left(n^{109/220}\delta_{2,3} + 2C_5 n^{109/220} n^{-1}\right) \le \exp\left(n^{109/220}\delta_{2,3}\right)e^{2C_5}.\end{aligned}$$

Here we used again (9) and the definition of $\delta_{2,3}$.

We multiply together all these three last displayed estimates, this way we can continue our main estimate (23). Note that $\exp(n\delta_{2,3})$ cancels, and $\exp(-n_1\delta_{2,3})$ kills the factor $\exp\left(n^{109/220}\delta_{2,3}\right)$, in more detail:

$$\begin{aligned}&\le \frac{2}{\pi}\int\limits_{\Gamma_{2,3}} \frac{1}{\delta_{2,3}} C_1\left(G_1^+\right)(\log n + 1)\|f\|_{\partial G_2}\exp(n\delta_{2,3})e^{2C_5} \\ &\quad\cdot \frac{\exp(-n\delta_{2,3} - n_1\delta_{2,3} - n_2\delta_{2,3})}{\delta_{2,3}^2} e^3 \exp\left(n^{109/220}\delta_{2,3}\right)e^{2C_5}|d\omega| \\ &= \frac{2e^{4C_5+3}C_1\left(G_1^+\right)}{\pi}\|f\|_{\partial G_2}\frac{\log n + 1}{\delta_{2,3}^3}\int\limits_{\Gamma_{2,3}}|d\omega| \\ &\quad\cdot \exp\left(\left(n^{109/220} - n_1\right)\delta_{2,3}\right)\exp(-n_2\delta_{2,3}) \le \|f\|_{\partial G_2} O\left(\frac{n^2\log(n)}{\exp\left(n^{1/12}\right)}\right)\end{aligned}$$

where we used several estimates: length of $\Gamma_{2,3}$ is at most 4π, the definitions of n_1, n_2 and $\delta_{2,3}$ and that $n_1 > n^{109/220}$, therefore $\exp\left(\left(n^{109/220} - n_1\right)\delta_{2,3}\right) \le 1$.

For $\Gamma_{2,2}$ and $\Gamma_{2,4}$, we apply the same estimate which we detail for $\Gamma_{2,2}$ only. We again start with the integral for $w \in \mathbb{D}$

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int\limits_{\Gamma_{2,2}}\frac{(Q\cdot f_2)\circ\Phi_1\circ\psi^{-1}(\omega)}{w - \omega}\frac{q_N(w)}{q_N(\omega)}d\omega\right| \\ &\qquad \le \frac{1}{2\pi}\int\limits_{\Gamma_{2,2}} 4\frac{1}{|w - \omega|}\left|(Q\cdot f_2)\left(\Phi_1\circ\psi^{-1}(\omega)\right)\right|\frac{1}{|q_N(\omega)|}|d\omega|. \end{aligned} \tag{24}$$

Since $\omega \in \Gamma_{2,2}$, we can rewrite it in the form $\omega = (1+\delta)w_1^*/|w_1^*|$ where $\delta_{2,1} \le \delta \le \delta_{2,3}$ (with $w_1^* = w_1^*(\delta_{2,3})$). We use essentially the same steps to estimate f_2 (the only one bad guy this time) and q_N and Q (this time it is a good guy). In estimating f_2, the only difference is that $|\omega| - 1 = \delta$, so

$$\begin{aligned}\left|f_2\left(\Phi_1\circ\psi^{-1}(\omega)\right)\right| &\le \|f_2\|_{\partial G_2}\exp\left(n g_{G_2}\left(\Phi_1\circ\psi^{-1}(\omega), \infty\right)\right) \\ &\le C_1\left(G_1^+\right)(\log n + 1)\|f\|_{\partial G_2}\exp\left(n(|\omega| - 1)(1 + C_5|\omega - b_1|)\right) \\ &\le C_1\left(G_1^+\right)(\log n + 1)\|f\|_{\partial G_2}\exp\left(n\delta + C_5 n\delta_{2,3}2\sqrt{\delta_{2,3}}\right) \\ &= C_1\left(G_1^+\right)(\log n + 1)\|f\|_{\partial G_2}\exp(n\delta)e^{2C_5}.\end{aligned}$$

Similarly for q_N, we can write

$$\begin{aligned}\frac{1}{|q_N(\omega)|} &\le \frac{1}{\delta_{2,1}^2}\frac{1}{(1+\delta)^N} = \frac{1}{\delta_{2,1}^2}\exp\left(-(n+n_1+n_2)\log(1+\delta)\right)\\ &\le \frac{1}{\delta_{2,1}^2}\exp\left(-n\delta - n_1\delta - n_2\delta + (n+n_1+n_2)\frac{\delta_{2,3}^2}{2}\right)\\ &\le \frac{1}{\delta_{2,1}^2}\exp\left(-n\delta - n_1\delta - n_2\delta\right)\exp\left(3n\, n^{-4/3}\right)\\ &\le \frac{\exp\left(-n\delta - n_1\delta - n_2\delta\right)}{\delta_{2,1}^2}e^3 \le \frac{\exp(-n\delta)}{\delta_{2,1}^2}e^3.\end{aligned}$$

As for Q, we know that ω is far from b_1 so Q is small there. More precisely, following the same argument as for $\Gamma_{2,1}$, we know that $\sqrt{\delta_{2,3}}/2 \ge n_1^{-9/10}$, hence (10) holds for Q at ω, that is, we can write

$$\left|Q\left(\Phi_1 \circ \psi^{-1}(\omega)\right)\right| \le C_2 \exp\left(-C_3 n^{1/220}\right).$$

Putting these all together, we see that $\exp(n\delta)$ cancels and actually Q make the integrand small. So we can continue the estimate (24)

$$\begin{aligned}&\le \frac{2}{\pi}\int\limits_{\Gamma_{2,2}} \frac{1}{\delta_{2,1}}C_1\left(G_1^+\right)(\log n+1)\,\|f\|_{\partial G_2}\exp(n\delta)\,e^{2C_5}\frac{\exp(-n\delta)}{\delta_{2,1}^2}e^3\\ &\cdot C_2\exp\left(-C_3 n^{1/220}\right)|d\omega| = \frac{2e^{2C_5+3}C_2C_1\left(G_1^+\right)}{\pi}\|f\|_{\partial G_2}\int\limits_{\Gamma_{2,2}}|d\omega|\\ &\cdot\frac{\log n+1}{\delta_{2,1}^3}\exp\left(-C_3n^{1/220}\right) \le \|f\|_{\partial G_2}\,O\left(\frac{n^3\log(n)}{\exp\left(C_3 n^{1/220}\right)}\right)\end{aligned}$$

where we used that the length of $\Gamma_{2,2}$ is at most 1 (since $\delta_{2,3}^{(0)} < 1$) and $\delta_{2,1} = 1/n$.

Summarizing these estimates on $\Gamma_{2,1}$, $\Gamma_{2,3}$ and $\Gamma_{2,2}$ (and also on $\Gamma_{2,4}$), we have uniformly for $|w| \le 1$,

$$\left|p_{2,N}(w) - (Q\cdot f_2)\circ\Phi_1\circ\psi^{-1}(w)\right| = o(1)\,\|f\|_{\partial G_2}$$

where $o(1)$ tends to 0 as $n\to\infty$ but it is independent of P_n and f_2. Obviously, $p_{2,N}\circ\psi$ is a rational function with pole at $v = a_2$ only, the order of the pole at a_2 (of $p_{2,N}\circ\psi$) is $\deg p_{2,N} = N = n+n_1+n_2 = (1+o(1))\,n$ and using the properties of $w = \psi(v)$, we uniformly have for $|v| \le 1$

$$|p_{2,N}\circ\psi(v) - (Q\cdot f_2)\circ\Phi_1(v)| = o(1)\,\|f\|_{\partial G_2},$$

that is,

$$\|p_{2,N}\circ\psi - (Q\cdot f_2)\circ\Phi_1\|_{\partial\mathbb{D}} = o(1)\,\|f\|_{\partial G_2}. \tag{25}$$

Since b_1 is double zero of q, $p_{2,N}'(b_1) = \left((Q\cdot f_2)\circ\Phi_1\circ\psi^{-1}\right)'(b_1)$, and dividing both sides with $\left(\psi^{-1}\right)'(b_1)$, we obtain

$$(p_{2,N}\circ\psi)'(1) = ((Q\cdot f_2)\circ\Phi_1)'(1). \tag{26}$$

Consider the "constructed" rational function

$$h(v) := \varphi_{1,r}(v) + p_{1,N} \circ \psi(v) + p_{2,N} \circ \psi(v).$$

This function h has a pole at a_1 (because of $\varphi_{1,r}$) and the order of its pole at a_1 is at most n, and h has a pole at a_2 (because of $p_{1,N} \circ \psi$ and $p_{2,N} \circ \psi$) and the order of its pole at a_2 is at most $N = n(1 + o(1))$. We use the identity

$$f \circ \Phi_1 = (Q \cdot f + (1 - Q) \cdot f) \circ \Phi_1$$

to calculate the derivatives as follows

$$(((1-Q) \cdot f) \circ \Phi_1)'(1) = \left((1-Q)' \cdot f\right)(u_1) \cdot \Phi_1'(1) + ((1-Q) \cdot f')(u_1) \cdot \Phi_1'(1)$$

where the second term is zero because of the fast decreasing polynomial ($Q(u_1) = 1$) and for the first term we can apply Theorem 1.3 from [8] in the following way ($\|1 - Q\|_{\partial G_2} \le 2$):

$$\left|(1-Q)'(u_1)\right| \le (1 + o(1)) \deg(Q)\, 2 \frac{\partial}{\partial n_2(u_1)} g_{G_2}(u_1, \infty)$$

where $o(1)$ depends on G_2 and u_1 only and tends to 0 as $\deg Q \to \infty$ (note: $\deg Q \le n^{109/220} \le \sqrt{n}$). Therefore

$$\begin{aligned}\left|\left((1-Q)' \cdot f\right)(u_1) \cdot \Phi_1'(1)\right| &\le \|f\|_{\partial G_2} \sqrt{n} 2 (1 + o(1)) \frac{\partial}{\partial n_2(u_1)} g_{G_2}(u_1, \infty) \\ &= \|f\|_{\partial G_2} O\left(\sqrt{n}\right) \frac{\partial}{\partial n_2(u_1)} g_{G_2}(u_1, \infty) \\ &\le o(1)\, n \|f\|_{\partial G_2} \max\left(\frac{\partial}{\partial n_2(u_1)} g_{G_2}(u_1, \infty), \frac{\partial}{\partial n_1(u_1)} g_{G_1}(u_1, a_1)\right). \end{aligned} \tag{27}$$

This way we need to consider $(Q \cdot f) \circ \Phi_1$ only. The derivatives at 1 of the original f and h coincide, because of (14), (20) and (26), so

$$h'(1) = \varphi_{1,r}'(1) + (p_{1,N} \circ \psi)'(1) + (p_{2,N} \circ \psi)'(1) = ((Q \cdot f) \circ \Phi_1)'(1). \tag{28}$$

As for the sup norms, we use (14), (19), (25), so we write

$$\|(Q \cdot f) \circ \Phi_1 - h\|_{\partial \mathbb{D}} = o(1) \|f\|_{\partial G_2}. \tag{29}$$

Now we apply the Borwein–Erdélyi inequality (5) for h as follows:

$$|h'(1)| \le \|h\|_{\partial \mathbb{D}} \max\left(\sum_\alpha \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha), \sum_\alpha \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \alpha)\right) \tag{30}$$

where the summation is taken over all poles in $\mathbb{D}$ and in $\mathbb{D}^*$ respectively, counting multiplicities. We will continue this estimate later after simplifying these expressions. Using Propositions 8 and 7, we can write

$$\begin{aligned}\sum_\alpha \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha) \le n \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, a_1) &= n \frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right) \\ &= n \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty) |F'(u_0)| \end{aligned}$$

where in the last step we used Proposition 7 with $z_0 = F(u_0)$ and identifying $u_0 = u_1$. Similarly, we can simplify the second term in the maximum in (30)

$$\sum_{\alpha} \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \alpha) = \deg(p_{1,N} + p_{2,N}) \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, a_2)$$
$$\leq N \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, a_2) = (1 + o(1)) n \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)$$
$$= (1 + o(1)) n \frac{\partial}{\partial n_1(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty) |F'(u_0)|$$

where $o(1)$ here does not depend on anything. Note that we "used a slightly bit more the pole at a_2", but it does not cause problem. So we can continue the main estimate (30)

$$\leq \|h\|_{\partial\mathbb{D}} \max\left(n \frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right), (1 + o(1)) n \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)\right)$$
$$\leq \|h\|_{\partial\mathbb{D}} (1 + o(1)) n$$
$$\cdot \max\left(\frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right), \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)\right).$$

Summarizing these estimates, we have for h

$$|h'(1)| \leq \|h\|_{\partial\mathbb{D}} (1 + o(1)) n$$
$$\cdot \max\left(\frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right), \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)\right).$$

Now we rewrite this inequality for $Q \cdot f$ using (28) and (29), so

$$\left|(Q \cdot f)'(u_1)\right| \leq \|Q \cdot f\|_{\partial G_2} (1 + o(1)) n$$
$$\cdot \max\left(\frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right), \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)\right)$$
$$+ o(1) n \|f\|_{\partial G_2} \cdot \max\left(\frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right), \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)\right).$$

Now, we use the estimate $\|Q \cdot f\|_{\partial G_2} \leq \|f\|_{\partial G_2}$ and (27), so

$$|f'(u_1)| \leq \|f\|_{\partial G_2} (1 + o(1)) n$$
$$\cdot \max\left(\frac{\partial}{\partial n_1(u_0)} g_{G_1}\left(u_0, F_1^{-1}[\infty]\right), \frac{\partial}{\partial n_2(u_0)} g_{G_2}\left(u_0, F_2^{-1}[\infty]\right)\right). \tag{31}$$

In the final step, we use $f = P_n \circ F$ and Proposition 6, so we get the main theorem.

5. Sharpness

In this section we show that the result is asymptotically sharp, that is, we prove Theorem 2. The idea is similar to that of [9]. Note that we assume C^2 smoothness only.

Fig. 4. The sets K and K^*.

Proof. We may assume that

$$\frac{\partial}{\partial n_1(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty) \leq \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty).$$

Furthermore, we assume that $n_1(.)$ and $n_2(.)$ are defined on the component of K containing z_0 and they are continuous there except for the endpoints.

It is easy to see that for every $\varepsilon > 0$ there exists a compact set $K^* = K^*(\varepsilon)$ such that ∂K^* is finite union of disjoint, C^2 smooth Jordan curves, $K \subset K^*$, $z_0 \in \partial K^*$ and the normal vector $n(K^*, z_0)$ to K^* (pointing outward) at z_0 is equal to $n_2(z_0)$ and

$$\begin{aligned}\frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty)(1-\varepsilon) &\leq \frac{\partial}{\partial n(K^*, z_0)} g_{\mathbf{C}_\infty \setminus K^*}(z_0, \infty) \\ &\leq \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty).\end{aligned}$$

These conditions, roughly speaking, require that near z_0, K^* is on the $n_1(z_0)$-side of K and the whole K^* shrinks to K as $\varepsilon \to 0$. Fig. 4 depicts K and the grey area is K^*.

Now we apply the sharpness result of [8] (Theorem 1.4, p. 194). This gives a sequence of polynomials for $K^*(\varepsilon)$, say $P_{\varepsilon,n}$, with $\deg P_{\varepsilon,n} \leq n$ such that

$$\begin{aligned}\left|P'_{\varepsilon,n}(z_0)\right| &\geq n(1 - o_\varepsilon(1)) \|P_{\varepsilon,n}\|_{K^*(\varepsilon)} \frac{\partial}{\partial n(K^*, z_0)} g_{\mathbf{C}_\infty \setminus K^*}(z_0, \infty) \\ &\geq n(1 - o_\varepsilon(1))(1-\varepsilon) \|P_{\varepsilon,n}\|_K \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0, \infty)\end{aligned}$$

where $o_\varepsilon(1)$ depends on $K^*(\varepsilon)$ and z_0 and tends to 0 as $\deg P_{\varepsilon,n} \to \infty$. Since ε was arbitrary, we see that $(1 - o_\varepsilon(1))(1-\varepsilon) = 1 - o(1)$, that is, choosing a suitable subsequence of $\{P_{\varepsilon,n}\}$ we obtain the assertion. □

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