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The diffusive competition problem with a free boundary in heterogeneous time-periodic environment

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ABSTRACT

In this paper, we consider the diffusive competition problem with a free boundary and sign-changing intrinsic growth rate in heterogeneous time-periodic environment, consisting of an invasive species with density u and a native species with density v. We assume that v undergoes diffusion and growth in $\mathbb{R}^N$, and u exists initially in a ball $B_{h_0}(0)$, but invades into the environment with spreading front $\{r = h(t)\}$. The effect of the dispersal rate d_1, the initial occupying habitat h_0, the initial density u_0 of invasive species u, and the parameter μ (see (1.3)) on the dynamics of this free boundary problem are studied. A spreading–vanishing dichotomy is obtained and some sufficient conditions for the invasive species spreading and vanishing are provided. Moreover, when spreading of u happens, some rough estimates of the spreading speed are also given.

1. Introduction

In this paper, we study the dynamical behavior of the solution $(u(t,r), v(t,r), h(t))$ $(r = |x|,\ x \in \mathbb{R}^N,\ N \geq 2)$ to the following reaction–diffusion problem with a free boundary in heterogeneous time-periodic environment

$$\begin{cases} u_t - d_1\Delta u = u\left(m_1(t,r) - b_1(t,r)u - c_1(t,r)v\right), & t > 0, \quad 0 < r < h(t), \\ v_t - d_2\Delta v = v\left(m_2(t,r) - c_2(t,r)u - b_2(t,r)v\right), & t > 0, \quad 0 < r < \infty, \\ u_r(t,0) = v_r(t,0) = 0,\ u(t,r) = 0, & t > 0, \quad h(t) \leq r < \infty, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,r) = u_0(r), & 0 \leq r \leq h_0, \\ v(0,r) = v_0(r), & 0 \leq r < \infty, \end{cases} \tag{1.1}$$

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where $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, $u(t,r)$ and $v(t,r)$ represent the population densities of two competing species; the positive constants d_1 and d_2 are dispersal rates of u and v, respectively; the initial functions u_0 and v_0 satisfy

$$\begin{cases} u_0 \in C^2([0,h_0]), \quad u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \quad \text{in } [0,h_0), \\ v_0 \in C^2([0,\infty)) \cap L^\infty([0,\infty)), \quad v_0'(0) = 0, \quad v_0 \geq 0 \quad \text{in } [0,\infty), \quad \text{and} \quad v_0 \not\equiv 0; \end{cases} \tag{1.2}$$

$m_i(t,r)$, $b_i(t,r)$, $c_i(t,r)$ represent the intrinsic growth rates of species, self-limitation of species and competition between species, respectively, and $b_i(t,r)$, $c_i(t,r)$ satisfy the following conditions

$$\text{(H1)} \quad \begin{cases} \text{(i)} & m_i \in (C^{\frac{\alpha}{2},1} \cap L^\infty)([0,\infty)\times[0,\infty)),\ b_i, c_i \in (C^{\frac{\alpha}{2},\alpha} \cap L^\infty)([0,\infty)\times[0,\infty)) \text{ for some } \alpha \in (0,1) \\ & \text{and are T-periodic in } t \text{ for some } T > 0; \\ \text{(ii)} & \text{there are positive Hölder continuous and T-periodic functions } b_{i,*}, b_i^*, c_{i,*}, c_i^* \ (i = 1,2) \\ & \text{such that } b_{i,*}(t) \leq b_i(t,r) \leq b_i^*(t),\ c_{i,*}(t) \leq c_i(t,r) \leq c_i^*(t),\ \forall t \in [0,T],\ r \in [0,\infty). \end{cases}$$

Ecologically, this problem describes the dynamical process of a new competitor invading into the habitat of a native species in heterogeneous time-periodic environment. The first species u, which exists initially on a region $B_{h_0}(0)$, stands for the species in the very early stage of its introduction, and disperses through random diffusion over an expanding front $h(t)$, evolving according to the free boundary condition

$$h'(t) = -\mu u_r(t,h(t)), \tag{1.3}$$

where μ is a given positive constant. The second species (v) is native, which undergoes diffusion and growth in the entire available habitat. The equation (1.3) is a special case of the well-known Stefan condition, which has been used in the modeling of a number of applied problems [4,6,38]. We remark that similar free boundary conditions to (1.3) have been used in ecological models over bounded spatial domains in several earlier papers, for example, [30,32–34].

In the absence of a native species, namely $v \equiv 0$, (1.1) reduces to the following diffusive logistic problem with a free boundary in the heterogeneous time-periodic environment

$$\begin{cases} u_t - d_1\Delta u = u(m_1(t,r) - b_1(t,r)u), \quad t > 0, \quad 0 < r < h(t), \\ u_r(t,0) = 0,\ u(t,r) = 0, \quad t > 0, \quad h(t) \leq r < \infty, \\ h'(t) = -\mu u_r(t,h(t)), \quad t > 0, \\ h(0) = h_0,\ u(0,r) = u_0(r), \quad 0 \leq r \leq h_0, \end{cases} \tag{1.4}$$

which has been studied in [10], where the authors showed the spreading–vanishing dichotomy in time-periodic environment, and also determined the spreading speed. The diffusive logistic problem with a free boundary in the heterogeneous time-periodic environment was also studied in [5,42]. In the special case that m_1 and b_1 are independent of time t, problem (1.4) was studied in [9,12,28,41,51] etc. In particular, Du and Lin [12] are the first ones to study the spreading–vanishing dichotomy of species in the homogeneous environment of dimension one, which has been extended in [9] to the situation of higher dimensional space in a radially symmetric case. They showed that, if the diffusion is slow or the occupying habitat is large, the invasive species can establish itself successfully in the underlying habitat, while the species will die out if the initial value of the species is small. Other theoretical advances can also be seen in [2,11,14,16,25–27,37,39] and the references therein.

Recently, Du and Lin [13] considered the following two-species model in higher dimensional domain with radically symmetry

$$\begin{cases} u_t - d_1\Delta u = u(a_1 - b_1 u - c_1 v), \quad t > 0, \quad 0 < r < h(t), \\ v_t - d_2\Delta v = v(a_2 - b_2 u - c_2 v), \quad t > 0, \quad 0 < r < \infty, \\ u_r(t,0) = v_r(t,0) = 0, \ u(t,r) = 0, \quad t > 0, \quad h(t) \le r < \infty, \\ h'(t) = -\mu u_r(t,h(t)), \quad t > 0, \\ h(0) = h_0, \ u(0,r) = u_0(r), \quad 0 \le r \le h_0, \\ v(0,r) = v_0(r), \quad 0 \le r < \infty, \end{cases} \tag{1.5}$$

where u and v represent the invasive and native species, respectively, and a_i, b_i, c_i $(i = 1, 2)$ are positive constants. They showed that a spreading–vanishing dichotomy holds when u is a superior competitor, and the dynamical behavior of (1.5) is similar to that of (1.5) in a fixed domain when u is an inferior competitor. Moreover, when spreading of the invasive species u happens, some estimates of the spreading speed were also given. We remark that similar Lotka–Volterra competitive type problems with a free boundary were introduced in [19,20,44,46,47]. Other studies of Lotka–Volterra prey–predator problems with a free boundary can be found in [30,40,45,50].

The problem (1.1) is a variation of the following diffusive Lotka–Volterra competition problem

$$\begin{cases} u_t - d_1\Delta u = u(m_1(t,x) - b_1(t,x)u - c_1(t,x)v), \quad t > 0, \quad x \in S, \\ v_t - d_2\Delta v = v(m_2(t,x) - c_2(t,x)u - b_2(t,x)v), \quad t > 0, \quad x \in S, \end{cases} \tag{1.6}$$

which is often considered over a bounded spatial domain $S = \Omega \subset \mathbb{R}^N$ with suitable boundary conditions or considered over the entire space $S = \mathbb{R}^N$. For example, the long-time behavior of the bounded domain problem (1.6) with no-flux boundary conditions is well known. The reader may refer to [3,8,17,21–23,31,35] and the references therein for further details. For the entire space problem (1.6), when the functions m_i, b_i, c_i $(i = 1, 2)$ are positive constants, to describe the invasion and spreading phenomenon, there have been many interesting studies on positive traveling waves and asymptotic spreading speed (see [18,29,48]); when the functions m_i, b_i, c_i $(i = 1, 2)$ are independent of x, the authors in [1,49] studied the existence and stability of time periodic traveling waves. As said in [13], the long-time behavior of the Cauchy problem of (1.6) is still poorly understood. To investigate the dynamics of a new competitive species that invades the habitat of a resident species in heterogeneous time-periodic environment (seasonal or daily variations, for example), it is necessary to study the free boundary model (1.1).

Motivated by the works [28,41,51], we will divide the environment into two different circumstances: strong heterogeneous environment and weak heterogeneous environment, where if $m_i(t,r)$ satisfies the following assumptions

$$\begin{aligned} (H_s) \quad & m_i(t,r) \in C^1([0,T] \times [0,\infty)) \cap L^\infty([0,T] \times [0,\infty)), \ i = 1,2, \\ & \text{and} \quad m_1(t,\cdot) \text{ changes sign in } (0,h_0), \quad m_2(t,\cdot) \text{ changes sign in } (0,\infty), \end{aligned}$$

then it is called strong heterogeneous environment for population, and if $m_i(t,r)$ satisfies

$$(H_w) \quad m_i(t,r) \in C^1([0,T] \times [0,\infty)), \text{ and } 0 < \underline{m}_i \le m_i(t,r) \le \bar{m}_i < \infty \text{ for } (t,r) \in [0,T] \times [0,\infty), \ i = 1,2,$$

with $\underline{m}_i$ and $\bar{m}_i$ being positive constants, then it is called weak heterogeneous environment for population.

The aim of this paper is to study the dynamics of problem (1.1) in the strong and weak heterogeneous time-periodic environment. We employ d_1, h_0, μ and $u_0(r)$ as variable parameters to study problem (1.1) when $m_i(t,r)$ $(i = 1, 2)$ satisfy conditions (H1)–(H3). We derive a spreading–vanishing dichotomy and some sufficient conditions to ensure that spreading and vanishing occur, which yield sharp criteria governing spreading and vanishing both in the strong and weak heterogeneous time-periodic environment. These results give the following biological explanations: slow diffusion, large occupying habitat and big initial density of

invasive species u are benefit for the invasive species to survive in the new environment. Moreover, the estimate of the asymptotic spreading speed is given. Unlike the non-periodic case, the principle eigenvalue of time-periodic eigenvalue problem (see (3.1) later) is not monotone with respect to dispersal rate (see Theorem 2.2 in [24]), so we only consider two particular cases for d_1: slow diffusion and fast diffusion (see Corollary 3.1 later).

After this work was completed, we found the preprint by Wang [43]. There, two types of diffusive competition model with a free boundary in heterogeneous time-periodic environment of dimension one are considered (see (1.2) and (1.3) in [43]). The author obtained the spreading–vanishing dichotomy and sharp criteria for spreading and vanishing by selecting h_0 and μ as variable parameters. When $N = 1$, $b_i(t, r) = 1$ and $c_i(t, r) = const.$ $(i = 1, 2)$, our model (1.1) becomes into (1.3) in [43] with $\alpha_i = 0$ and $\beta_i = -1$. Except the difference between equations we dealing with, we also consider the effect of the dispersal rate d_1 and the initial data u_0 on spreading and vanishing under suitable assumptions.

The rest of our paper is arranged as follows. In Section 2, we exhibit some fundamental results, including the global existence and uniqueness of the solution of problem (1.1) and the comparison principle in the moving domain; An eigenvalue problem under some suitable assumptions is given in Section 3; In Section 4, we investigate the dynamics of problem (1.1) in heterogeneous environment. Section 5 is devoted to studying the asymptotic spreading speed of the free boundary when spreading of the invasive species u occurs.

2. Preliminaries

In this section, we give some fundamental results on solutions of problem (1.1) under (H1).

Lemma 2.1. *For any given (u_0, v_0) satisfying (1.2), problem (1.1) admits a unique solution (u, v, h) defined for all $t > 0$ and*

$$(u, v, h) \in C^{(1+\alpha)/2,1+\alpha}(D) \times C^{(1+\alpha)/2,1+\alpha}(D^\infty) \times C^{1+\alpha/2}([0, \infty)).$$

Moreover,

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(D)} + \|v\|_{C^{(1+\alpha)/2,1+\alpha}(D^\infty)} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C,$$

where $D = \{(t, r) \in \mathbb{R}^2 : t \in [0, \infty),\ r \in [0, h(t))\}$, $D^\infty = \{(t, r) \in \mathbb{R}^2 : t \in [0, \infty),\ r \in [0, \infty)\}$, C only depend on h_0, α, $\|u_0\|_{C^2([0,h_0])}$, $\|v_0\|_{C^2([0,\infty))}$.

Proof. The proof is a simple modification of those of Theorems 2.1, 2.4 and 2.5 in [13]. So we briefly describe the main steps.

Step 1. The local existence and uniqueness of positive solution of (1.1).

The essential idea of this proof is to construct a contraction mapping, and the desired result would then follow from the contraction mapping theorem. This step can be obtained by exactly the same argument used in the proof of Theorem 2.1 in [13].

Step 2. The local solution can be extended to all $t > 0$.

To show this conclusion, we need the following estimates: if (u, v, h) is a solution of (1.1) defined for $t \in (0, T_0)$ for some T_0, then there exist constants C_1, C_2 and C_3 independent of T_0, such that

$$\begin{aligned} 0 < u(t, r) \le C_1, &\quad \text{for } t \in (0, T_0), \quad 0 < r < h(t), \\ 0 < v(t, r) \le C_2, &\quad \text{for } t \in (0, T_0), \quad 0 < r < \infty, \\ 0 < h'(t) \le C_3, &\quad \text{for } t \in (0, T_0). \end{aligned} \tag{2.1}$$

Now we prove (2.1). Applying the strong maximum principle, we immediately obtain that $u(t,r) > 0$, $u_r(t,h(t)) < 0$ for $(t,r) \in (0,T_0) \times [0,h(t))$ and $v(t,r) > 0$ for $(t,r) \in (0,T_0) \times [0,\infty)$. Using the Stefan condition (1.3), we have $h'(t) > 0$ for $t \in (0,T_0)$. Since $b_i(t,r)$ $(i = 1,2)$ satisfy (H1), then $\min_{[0,T]\times[0,\infty)} b_i(t,r) > 0$. Using the maximum principle again, we can obtain C_1 and C_2, more precisely,

$$u(t,r) \le C_1 := \max\left\{\frac{\|m_1\|_{L^\infty([0,T]\times[0,\infty))}}{\min_{[0,T]\times[0,\infty)} b_1(t,r)}, \|u_0\|_{L^\infty([0,h_0])}\right\}, \quad \text{for } t \in (0,T_0) \quad \text{and} \quad r \in [0,h(t))$$

and

$$v(t,x) \le C_2 := \max\left\{\frac{\|m_2\|_{L^\infty([0,T]\times[0,\infty))}}{\min_{[0,T]\times[0,\infty)} b_2(t,r)}, \|v_0\|_{L^\infty([0,\infty))}\right\}, \quad \text{for } t \in (0,T_0) \quad \text{and} \quad r \in [0,\infty).$$

To get C_3. We define

$$\Omega_M = \{(t,r) : 0 < t < T_0, h(t) - M^{-1} < r < h(t)\},$$

and construct an auxiliary function

$$\bar{u}(t,r) = C_1[2M(h(t)-r) - M^2(h(t)-r)^2].$$

We will choose $M > \frac{1}{h_0}$ so that $\bar{u}(t,r) \ge u(t,r)$ holds over Ω_M.

Direct calculations yield that, for $(t,r) \in \Omega_M$,

$$\begin{cases} \bar{u}_t - d_1\Delta\bar{u} = (2MC_1h'(t) + \frac{2(n-1)d_1C_1M}{r})[1 - M(h(t)-r)] + 2d_1C_1M^2 \\ \qquad\qquad \ge 2d_1C_1M^2 \ge u(m_1 - b_1u - c_1v), \\ \bar{u}(t,h(t)-M^{-1}) = C_1 \ge u(t,h(t)-M^{-1}), \\ \bar{u}(t,h(t)) = 0 = u(t,h(t)), \end{cases}$$

provided $M^2 \ge \frac{\|m_1\|_{L^\infty([0,T]\times[0,\infty))}}{2d_1}$. On the other hand, we calculate

$$\bar{u}_r(0,r) = -2C_1M[1 - M(h_0 - r)] \le -C_1M, \quad \text{for } r \in [h_0 - (2M)^{-1}, h_0].$$

Therefore, by choosing

$$M := \max\left\{\frac{1}{h_0}, \sqrt{\frac{\|m_1\|_{L^\infty([0,T]\times[0,\infty))}}{2d_1}}, \frac{4\|u_0\|_{C^1([0,h_0])}}{3C_1}\right\},$$

we will have $\bar{u}_r(0,r) \le u_r(0,r)$ for $r \in [h_0 - (2M)^{-1}, h_0]$. Since $\bar{u}(0,h_0) = u_0(h_0) = 0$, the above inequality implies

$$\bar{u}(0,r) \ge u_0(r), \quad \text{for } r \in [h_0 - (2M)^{-1}, h_0].$$

Moreover, for $r \in [h_0 - M^{-1}, h_0 - (2M)^{-1}]$, we have

$$\bar{u}(0,r) \ge \frac{3}{4}C_1, u_0(r) \le \|u_0\|_{C^1([0,h_0])}M^{-1} \le \frac{3}{4}C_1.$$

Therefore, $u_0(r) \le \bar{u}(0,r)$ for $r \in [h_0 - (2M)^{-1}, h_0]$.

Applying the maximum principle to $\bar{u} - u$ over Ω_M gives that $u(t,r) \le \bar{u}(t,r)$ for $(t,r) \in \Omega_M$, which indicates that

$$-2MC_1 = \bar{u}_r(t,h(t)) \le u_r(t,h(t)), \quad h'(t) = -\mu u_r(t,h(t)) \le C_3 := 2MC_1\mu \quad \text{for } t \in (0,T_0).$$

The rest of the proof is the same as in [13].

Step 3. The solution of (1.1) exists and is unique for all $t > 0$.

This conclusion can be proved by exactly the same argument used in the proof of Theorem 2.4 in [13]. □

In what follows, we discuss the comparison principle for (1.1). The proof is similar to that of Lemma 2.6 in [13], so we omit the details.

Lemma 2.2 *(The comparison principle). Suppose that* $T_0 \in (0,\infty)$, $\underline{h}, \bar{h} \in C^1([0,T_0])$, $\underline{u} \in C(\overline{D^*_{T_0}}) \cap C^{1,2}(D^*_{T_0})$ *with* $D^*_{T_0} := \{(t,r) \in \mathbb{R}^2 : t \in (0,T_0],\ r \in (0,h(t))\}$, $\bar{u} \in C(\overline{D^{**}_{T_0}}) \cap C^{1,2}(D^{**}_{T_0})$ *with* $D^{**}_{T_0} := \{(t,r) \in \mathbb{R}^2 : t \in (0,T_0],\ r \in (0,\bar{h}(t))\}$, $\underline{v}, \bar{v} \in (L^\infty \cap C)((0,T_0] \times [0,\infty)) \cap C^{1,2}((0,T_0) \times [0,\infty))$ *and*

$$\begin{cases}
\bar{u}_t - d_1\Delta\bar{u} \ge \bar{u}(m_1(t,r) - b_1(t,r)\bar{u} - c_1(t,r)\underline{v}), & 0 < t \le T_0, \quad 0 < r < \bar{h}(t),\\
\underline{u}_t - d_1\Delta\underline{u} \le \underline{u}(m_1(x) - b_1(t,r)\underline{u} - c_1(t,r)\bar{v}), & 0 < t \le T_0, \quad 0 < r < \underline{h}(t),\\
\bar{v}_t - d_2\Delta\bar{v} \ge \bar{v}(m_2(t,r) - c_2(t,r)\underline{u} - b_2(t,r)\bar{v}), & 0 < t \le T_0, \quad 0 < r < \infty,\\
\underline{v}_t - d_2\Delta\underline{v} \le \underline{v}(m_2(t,r) - c_2(t,r)\bar{u} - b_2(t,r)\underline{v}), & 0 < t \le T_0, \quad 0 < r < \infty,\\
\bar{u}_r(t,0) = \bar{v}_r(t,0) = 0, \quad \bar{u}(t,r) = 0, & 0 < t \le T_0, \quad \bar{h}(t) \le r < \infty,\\
\underline{u}_r(t,0) = \underline{v}_r(t,0) = 0, \quad \underline{u}(t,r) = 0, & 0 < t \le T_0, \quad \underline{h}(t) \le r < \infty,\\
\underline{h}'(t) \le -\mu\underline{u}_r(t,h(t)), \quad \bar{h}'(t) \ge -\mu\bar{u}_r(t,h(t)), & 0 < t \le T_0,\\
\underline{h}(0) \le h_0 \le \bar{h}(0),\\
\underline{u}(0,r) \le u_0(r) \le \bar{u}(0,r), & 0 \le r \le h_0,\\
\underline{v}(0,r) \le v_0(r) \le \bar{v}(0,r), & 0 \le r \le \infty.
\end{cases}$$

Let (u,v,h) *be the unique solution of* (1.1)*, then*

$$h(t) \le \bar{h}(t) \quad \text{in } (0,T_0], \quad u(t,r) \le \bar{u}(t,r), \quad \underline{v}(t,r) \le v(t,r) \quad \text{for } (t,r) \in (0,T_0] \times [0,\infty),$$
$$h(t) \ge \underline{h}(t) \quad \text{in } (0,T_0], \quad \underline{u}(t,r) \le u(t,r), \quad v(t,r) \le \bar{v}(t,r) \quad \text{for } (t,r) \in (0,T_0] \times [0,\infty).$$

3. Some eigenvalue problems

In this section, we mainly study an eigenvalue problem and analyze the property of its principle eigenvalue. These results play an important role in later sections.

Consider the following eigenvalue problem

$$\begin{cases}
\varphi_t - d\Delta\varphi = m(t,|x|)\varphi + \lambda\varphi, & \text{in } [0,T] \times B_R,\\
\varphi = 0, & \text{on } [0,T] \times \partial B_R,\\
\varphi(0,x) = \varphi(T,x) & \text{in } B_R.
\end{cases} \tag{3.1}$$

It is well known [3,23] that (3.1) possesses a unique principal eigenvalue $\lambda_1 = \lambda_1(d,m,R,T)$, which corresponds to a positive eigenfunction $\varphi \in C^{1,2}([0,T] \times B_R)$. Moreover, $\varphi(t,x)$ is radially symmetric in x for all t.

In what follows, we present some further properties of $\lambda_1 = \lambda_1(d,m,R,T)$. We now discuss the dependence of λ_1 on d for fixed R.

Lemma 3.1. *(See [5].) Let $m(t,|x|)$ be functions satisfying* (H1). *Then*

(i) $\lambda_1(\cdot,m,R,T)\to-\max_{\bar{B}_R}\frac{1}{T}\int_0^T m(t,|x|)dt$ *as* $d\to 0$;
(ii) $\lambda_1(\cdot,m,R,T)\to+\infty$ *as* $d\to+\infty$.

Corollary 3.1. *(See [5].)* (i) *If* $\max_{\bar{B}_R}\frac{1}{T}\int_0^T m(t,|x|)dt>0$, *then there exists a constant* $d_*=d_*(m,R,T)\in(0,+\infty)$ *such that* $\lambda_1(d,m,R,T)\le 0$ *for* $0<d\le d_*$; (ii) *There exists a constant* $d^*=d^*(m,R,T)\in(0,+\infty)$ *such that* $\lambda_1(d,m,R,T)>0$ *for* $d>d^*$.

We assume

$$\text{(H2)}\quad 0<m_*(t):=\liminf_{|x|\to\infty}m(t,|x|)\le m^*(t):=\limsup_{|x|\to\infty}m(t,|x|)<\infty,$$

where $m_*(t),m^*(t)\in C^{\frac{\alpha}{2}}([0,T])$ are positive T-periodic functions. Clearly, this condition allows $m(t,|x|)$ to change sign in a bounded domain with respect to x.

Lemma 3.2. *(See [5].) Let $m(t,|x|)$ be functions satisfying* (H1). *Then*

(i) $\lambda_1(d,m,\cdot,T)$ *is a strictly decreasing continuous function in* $(0,+\infty)$ *for fixed* d,m,T, *and* $\lambda_1(d,\cdot,R,T)$ *is a strictly decreasing continuous function in the sense that* $\lambda_1(d,k_1(t,r),R,T)<\lambda_1(d,k_2(t,r),R,T)$ *if the two* T*-periodic continuous functions* $k_1(t,r)$ *and* $k_2(t,r)$ *satisfy* $k_1(t,r)\ge,\not\equiv k_2(t,r)$ *on* $[0,T]\times B_R$;
(ii) $\lambda_1(d,m,R,T)\to+\infty$ *as* $R\to 0$;
(iii) $\lim_{R\to\infty}\lambda_1(d,m,R,T)<0$ *under the assumption* (H2).

Corollary 3.2. *(See [5].) There exists a threshold* $h^*=h^*(d,m,T)\in(0,\infty]$ *such that* $\lambda_1(d,m,R,T)\le 0$ *for* $R\ge h^*$ *and* $\lambda_1(d,m,R,T)>0$ *for* $0<R<h^*$. *Moreover,* $h^*\in(0,\infty)$ *if the assumption* (H2) *holds. If we replace* R *in* (3.1) *by* $h(t)$, *then it follows from the strict increasing monotony of* $h(t)$ *and Lemma 3.2 that* $\lambda_1(d,m,h(t),T)$ *is a strictly monotone decreasing function of* t.

4. Strong and weak heterogeneous time-periodic environments

In this section, we will give the dynamics of problem (1.1) with $m_i(t,|x|)$ $(i=1,2)$ satisfying (H1) and (H2). The condition (H2) means that we only consider (1.1) in some cases of strong and weak heterogeneous time-periodic environments, where the growth rates of species satisfy some positivity conditions at infinity. To get an entire analysis, we need to add the following assumption:

$$\text{(H3)}\quad m_{1,*}(t)-c_1^*(t)V^*(t)>0,$$

where $V_*(t)$, $V^*(t)$ are the unique positive solutions of the T-periodic ordinary differential problems

$$\begin{cases}V'(t)=V(m_{2,*}(t)-b_2^*(t)V),\\ V(0)=V(T),\end{cases}\tag{4.1}$$

and

$$\begin{cases}V'(t)=V(m_2^*(t)-b_{2,*}(t)V),\\ V(0)=V(T),\end{cases}\tag{4.2}$$

respectively.

Throughout this section, (H1)–(H3) are assumed to hold even if they are not explicitly mentioned.

4.1. Spreading–vanishing dichotomy

In this subsection, we prove the spreading–vanishing dichotomy. In view of (2.1), we see that the free boundary $h(t)$ is a strictly increasing function with respect to time t. Thus, $h_\infty := \lim_{t\to\infty} h(t)$ is well defined. We first prove that if the habitat of the invasive species is limited in the long run, then the invasive species u vanishes.

Lemma 4.1. *If $h_\infty < \infty$, then $\limsup_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$ and $\lim_{k\to\infty} v(t+kT,r) = V(t,r)$ uniformly in any bounded subset of $[0,T]\times[0,\infty)$, where $V(t,|x|)$ is the unique positive solution of*

$$\begin{cases} V_t - d_2\Delta V = V(m_2(t,|x|) - b_2(t,|x|)V), & (t,x)\in[0,T]\times\mathbb{R}^N,\\ V(0,|x|) = V(T,|x|). \end{cases} \tag{4.3}$$

Proof. Since m_2 satisfies the assumption (H2), Theorem 1.3 in [36] is available, and then the existence and uniqueness of $V(t,|x|)$ can be established.

We now argue indirectly, that is, we assume that $\limsup_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = \delta > 0$. Then there exists a sequence $(t_n, r_n) \in [0,\infty)\times[0,h(t_n)]$ with $t_n \to \infty$ as $n\to\infty$ such that $u(t_n,r_n) \geq \frac{\delta}{2}$ for all $n\in N$. Since $0 \leq r_n < h_\infty < \infty$, there exists a subsequence of $\{r_n\}$, denoted by itself, and $r_0 \in [0,h_\infty]$, such that $r_n \to r_0$ as $n\to\infty$. We claim that $r_0 < h_\infty$. If this is not true, then $r_n - h(t_n) \to 0$ as $n\to\infty$. According to Lemma 2.1 and the above assumption, we have

$$\left|\frac{\delta}{2(r_n - h(t_n))}\right| \leq \left|\frac{u(t_n,r_n)}{r_n - h(t_n)}\right| = \left|\frac{u(t_n,r_n) - u(t_n,h(t_n))}{r_n - h(t_n)}\right| = |u_r(t_n,\bar{r}_n)| \leq C,$$

where $\bar{r}_n \in (r_n, h(t_n))$. It is a contradiction since $r_n - h(t_n) \to 0$. Without loss of generality, we assume $r_n \to r_0 \in [0, h_\infty - \sigma]$ as $n\to\infty$ for some $\sigma > 0$.

Define

$$u_n(t,r) = u(t+t_n,r) \quad \text{and } v_n(t,r) = v(t+t_n,r) \quad \text{for } (t,r)\in D_n,$$

with $D_n := \{(t,r)\in\mathbb{R}^2 : t\in(-t_n,\infty),\ r\in[0,h(t+t_n)]\}$.

Letting $t_n = \bar{t}_n + \bar{n}T$ with $\bar{t}_n \in [0,T)$ and $\bar{n}\in\mathbb{N}$, passing to a subsequence if necessary, we may assume that $\bar{t}_n \to t_0$ as $n\to\infty$. Since $h_\infty < \infty$, we have $\lim_{t\to\infty} h'(t_n) = 0$. It follows from Lemma 2.1 that $\{(u_n,v_n)\}$ is bounded, by the parabolic regularity, we have, up to a subsequence if necessary, $(u_n,v_n) \to (\bar{u},\bar{v})$ as $n\to\infty$, with $(\bar{u},\bar{v})$ satisfying

$$\begin{cases} \bar{u}_t - d_1\Delta\bar{u} = \bar{u}(m_1(t+t_0,r) - b_1(t+t_0,r)\bar{u} - c_1(t+t_0,r)\bar{v}), & t\in(-\infty,\infty), \quad 0 < r < h_\infty,\\ \bar{v}_t - d_2\Delta\bar{v} = \bar{v}(m_2(t+t_0,r) - c_2(t+t_0,r)\bar{u} - b_2(t+t_0,r)\bar{v}), & t\in(-\infty,\infty), \quad 0 < r < \infty,\\ \bar{u}(t,h_\infty) = 0, \quad t\in(-\infty,\infty). \end{cases}$$

Since $\bar{u}(0,r_0) = \lim_{n\to\infty} u_n(0,r_n) = \lim_{n\to\infty} u(t_n,r_n) \geq \frac{\delta}{2}$, by the maximum principle, we have $\bar{u} > 0$ in $(-\infty,\infty)\times(0,h_\infty)$. Thus, we can apply the Hopf boundary lemma to conclude that $\sigma_0 := \bar{u}_r(0,h_\infty) < 0$, which implies that $u_r(t_{n_i}, h(t_{n_i})) = \partial_r u_{n_i}(0,h(t_{n_i})) \leq \frac{\sigma_0}{2} < 0$ for all large i, and hence $h'(t_{n_i}) \geq -\mu\frac{\sigma_0}{2} > 0$ for all large i. Since $h'(t)\to 0$ as $t\to\infty$ under the condition $h_\infty < \infty$, this is a contradiction.

Next we prove that $\lim_{n\to\infty} v(t+nT,r) = V(t,r)$ uniformly in any bounded subset of $[0,T]\times[0,\infty)$. In what follows, we use a squeezing argument developed in [15] to prove our result. The proof can be done by modifying the arguments of [9,15,28]. Due to both time-periodic and sign-changing are considered here, we provide the details of proof for the reader's convenience.

Since $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([0,h(t)])}=0$ for $t\geq 0$ and $u(t,r)=0$ for $r\geq h(t)$, then for any small $\varepsilon>0$, there exists $T_0>0$ such that $0<c_2(t,r)u(t,r)\leq\|c_2\|_{L^\infty([0,T]\times[0,\infty))}u(t,r)\leq\varepsilon$ for any $t\geq T_0$ and $r\in[0,\infty)$. For any $L>0$, we consider the following problem

$$\begin{cases} z_t-d_2\Delta z=z(m_2(t,r)-\varepsilon-b_2(t,r)z), & (t,r)\in[0,T]\times[0,L],\\ z_r(t,0)=0,\ z(t,L)=0,\\ z(0,r)=z(T,r). \end{cases} \tag{4.4}$$

Since $m_2(t,r)$ satisfies the condition (H2), we have $\Sigma_{d_2}=\{R>0:\lambda_1(d_2,m_2,R,T)=0\}\neq\emptyset$ by Corollary 3.2. Thus, we may assume $L_0\in\Sigma_{d_2}$, and then $\lambda_1(d_2,m_2,L,T)<0$ for any $L>L_0$. Since $\lambda_1(d_2,k(t,r),R,T)$ is a strictly decreasing continuous function in $k(t,r)$, then $\lambda_1(d_2,m_2-\varepsilon,L,T)<0$ for small ε. Therefore, for any $L>L_0$, (4.4) has a unique positive solution (see [3,23]), denoted by z_L^ε.

We next consider the following boundary blow-up problem

$$\begin{cases} w_t-d_2\Delta w=w(m_2(t+t^*,|x|+r^*)-b_2(t+t^*,|x|+r^*)w), & (t,x)\in[0,T]\times B_L,\\ w(t+t^*,L+r^*)=\infty,\\ w(t^*,r+r^*)=w(t^*+T,r+r^*), \end{cases} \tag{4.5}$$

where r^* is a constant satisfying $r^*>L_0$. It follows from Lemma 3.1 in [36] that (4.5) has a unique positive solution $w_L(t+t^*,r+r^*):=w_L^*(t,r)$ for any $L\gg 1$.

Now we choose a decreasing sequence $\{\varepsilon_n\}$ and an increasing sequence $\{L_n\}$ such that $\varepsilon_n>0$, $L_n>L_0$ for all n and $\varepsilon_n\to 0$, $L_n\to\infty$ as $n\to\infty$. Clearly, both $z_{L_n}^{\varepsilon_n}$ and w_{L_n} converge to $V(t,r)$ as $n\to\infty$, and for each n, there exists $T_n>T_0$ such that $h(t)\geq L_n$ for $t\geq T_n$. Since $L_n>L_0$, from [3,23] we know that the following problem

$$\begin{cases} Z_t-d_2\Delta Z=Z(m_2(t,r)-\varepsilon_n-b_2(t,r)Z), & t\geq T_n,\quad 0<r<L_n,\\ Z_r(t,0)=Z(t,L_n)=0, & t\geq T_n,\\ Z(T_n,r)=v(T_n,r), & 0<r<L_n, \end{cases}$$

admits a unique positive solution $Z_n(t,r)$ satisfying

$$Z_n(t+kT,r)\to z_{L_n}^{\varepsilon_n}(t,r)\quad\text{uniformly for } (t,r)\in[0,T]\times[0,L_n]\quad\text{as } k\to\infty.$$

Moreover, it follows from the comparison principle that

$$Z_n(t,r)\leq v(t,r)\quad\text{for } t\geq T_n\quad\text{and } r\in[0,L_n].$$

Hence

$$\liminf_{k\to\infty}v(t+kT,r)\geq z_{L_n}^{\varepsilon_n}(t,r)\quad\text{uniformly for } (t,r)\in[0,T]\times[0,L_n].$$

By letting $n\to\infty$ in the above inequality, we attain

$$\liminf_{k\to\infty}v(t+kT,r)\geq V(t,r)\quad\text{locally uniformly for } (t,r)\in[0,T]\times[0,\infty). \tag{4.6}$$

Similarly one can prove

$$\limsup_{k\to\infty}v(t+kT,r)\leq w_{L_n}\quad\text{uniformly for } (t,r)\in[0,T]\times[0,L_n],$$

which implies (by sending $n \to \infty$)

$$\limsup_{k\to\infty} v(t+kT,r) \le V(t,r) \quad \text{locally uniformly for } (t,r) \in [0,T]\times[0,\infty). \tag{4.7}$$

The desired result would then follow directly (4.6) and (4.7). □

Lemma 4.2. *If $h_\infty = \infty$, then $U(t,r) \le \liminf_{k\to\infty} u(t+kT,r) \le \limsup_{k\to\infty} u(t+kT,r) \le \hat{U}(t,r)$ uniformly in any compact subset of $[0,T]\times[0,\infty)$, where $U(t,|x|)$ is the unique positive solution of*

$$\begin{cases} U_t - d_1\Delta U = U(m_1(t,|x|) - b_1(t,|x|)U - c_1(t,|x|)V(t,|x|)), & (t,x) \in [0,T]\times\mathbb{R}^N, \\ U(0,|x|) = U(T,|x|), \end{cases} \tag{4.8}$$

and $\hat{U}(t,|x|)$ is the unique positive solution of

$$\begin{cases} \hat{U}_t - d_1\Delta \hat{U} = \hat{U}(m_1(t,|x|) - b_1(t,|x|)\hat{U}), & (t,x) \in [0,T]\times\mathbb{R}^N, \\ \hat{U}(0,|x|) = \hat{U}(T,|x|), \end{cases} \tag{4.9}$$

where $V(t,|x|)$ satisfies (4.3).

Proof. By Theorem 1.4 in [36], we have

$$0 < V_*(t) \le \liminf_{r\to\infty} V(t,r) \le \limsup_{r\to\infty} V(t,r) \le V^*(t), \tag{4.10}$$

where $V_*(t)$ and $V^*(t)$ are defined in (4.1) and (4.2).

Moreover, since (H3) holds, then we know that

$$\begin{aligned} 0 < m_{1,*}(t) - c_1^*(t)V^*(t) &\le \liminf_{|x|\to\infty}(m_1(t,|x|) - c_1(t,|x|)V(t,|x|)) \\ &\le \limsup_{|x|\to\infty}(m_1(t,|x|) - c_1(t,|x|)V(t,|x|)) \le m_1^*(t) - c_{1,*}(t)V_*(t). \end{aligned} \tag{4.11}$$

Therefore, Theorem 1.3 in [36] is available, and then the existence and uniqueness of $U(t,|x|)$ can be established.

Define

$$\bar{v}(t,r) = (1 + He^{-Kt})V(t,r), \tag{4.12}$$

where V satisfies (4.3) and K, H are positive constants to be determined later. Direct calculations yield

$$\begin{aligned} \bar{v}_t - d_2\Delta\bar{v} - \bar{v}(m_2(t,r) - b_2(t,r)\bar{v}) &= He^{-Kt}V(t,r)[-K + (1+He^{-Kt})b_2(t,r)V(t,r)] \\ &\ge He^{-Kt}V(t,r)[-K + b_2(t,r)V(t,r)] \end{aligned}$$

and $\bar{v}(0,r) = (1+H)V(0,r)$. Since the positive time-periodic functions $b_2(t,r)$ and $V(t,r)$ satisfy (H1) and (4.10) for any $t \in [0,T]$, then we have $\min_{[0,T]\times[0,\infty)} b_2(t,r) > 0$ and $\min_{[0,T]\times[0,\infty)} V(t,r) > 0$, and thus we can choose

$$K = \frac{1}{2}\min_{[0,T]\times[0,\infty)} b_2(t,r) \min_{[0,T]\times[0,\infty)} V(t,r), \quad 1 + H = \max\left\{\frac{\|v_0\|_{L^\infty([0,\infty))}}{\min_{[0,\infty)} V(0,r)}, 1\right\} \tag{4.13}$$

such that

$$\bar{v}_t - d_2\Delta\bar{v} - \bar{v}(m_2(t,r) - b_2(t,r)\bar{v}) \geq He^{-Kt}V(t,r)[-K + b_2(t,r)V(t,r)] \geq 0$$

and $\bar{v}(0,r) = (1+H)V(0,r) \geq \|v_0\|_{L^\infty([0,\infty))} \geq v_0(r)$. By the comparison principle, we have $v(t,r) \leq \bar{v}(t,r)$.

Since $h_\infty = \infty$ and $\lim_{k\to\infty}\bar{v}(t+kT,r) = \lim_{k\to\infty}(1+He^{-K(t+kT)})V(t+kT,r) = \lim_{k\to\infty}(1+He^{-K(t+kT)})V(t,r) = V(t,r)$ uniformly in $[0,T]\times[0,\infty)$, then for any given $0 < \varepsilon \ll 1$ and $L \gg 1$, there exists $k_\varepsilon \in \mathbb{N}$ such that $h(t+kT) > L$ and $v(t+kT,r) \leq \bar{v}(t+kT,r) \leq V(t,r)+\varepsilon$ for any $k \geq k_\varepsilon$ and $(t,r) \in [0,T]\times[0,L]$.

Let $\underline{u}^\varepsilon_L(t,r)$ be the unique solution of

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} = \underline{u}(m_1(t,r) - c_1(t,r)(V(t,r)+\varepsilon) - b_1(t,r)\underline{u}), & t \geq k_\varepsilon T, \quad 0 < r < L, \\ \underline{u}_r(t,0) = 0 = \underline{u}(t,L), & t \geq k_\varepsilon T, \\ \underline{u}(k_\varepsilon T,r) = u(k_\varepsilon T,r), & 0 < r < L. \end{cases}$$

The comparison principle implies $u(t+kT,r) \geq \underline{u}^\varepsilon_L(t+kT,r)$ for $k \geq k_\varepsilon$ and $(t,r) \in [0,T]\times[0,L]$. Since $L \gg 1$, we can deduce that $\underline{u}^\varepsilon_L(t+kT,r) \to U^\varepsilon_L(t,r)$ as $k \to \infty$, where $U^\varepsilon_L(t,r)$ is the unique positive periodic solution of

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} = \underline{u}(m_1(t,r) - c_1(t,r)(V(t,r)+\varepsilon) - b_1(t,r)\underline{u}), & t \in [0,T], \quad 0 < r < L, \\ \underline{u}_r(t,0) = 0 = \underline{u}(t,L), & t \in [0,T], \\ \underline{u}(0,r) = \underline{u}(T,r), & 0 < r < L. \end{cases}$$

Hence, $\liminf_{k\to\infty} u(t+kT,r) \geq U^\varepsilon_L(t,r)$ uniformly in $[0,T]\times[0,L]$. Similar as before, we know that $\lim_{L\to\infty} U^\varepsilon_L(t,r) = U^\varepsilon(t,r)$ uniformly in any compact subset of $[0,T]\times[0,\infty)$, where $U^\varepsilon(t,r)$ is the unique positive solution of

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} = \underline{u}(m_1(t,r) - c_1(t,r)(V(t,r)+\varepsilon) - b_1(t,r)\underline{u}), & (t,r) \in [0,T]\times(0,\infty), \\ \underline{u}(0,r) = \underline{u}(T,r). \end{cases}$$

Letting $\varepsilon \to 0$, it follows that $\liminf_{k\to\infty} u(t+kT,r) \geq U(t,r)$ uniformly in any compact subset of $[0,T]\times[0,\infty)$, where $U(t,r)$ satisfies (4.8).

On the other hand, since $v(t,r)$ is positive by (2.1), we know that $u(t,r)$ satisfies

$$\begin{cases} u_t - d_1\Delta u \leq u\left(m_1(t,r) - b_1(t,r)u\right), & t > 0, \quad 0 < r < h(t), \\ u_r(t,0) = 0,\ u(t,r) = 0, & t > 0, \quad h(t) \leq r < \infty, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,r) = u_0(r), & 0 \leq r \leq h_0. \end{cases}$$

Now we consider the following problem

$$\begin{cases} \bar{u}_t - d_1\Delta\bar{u} = \bar{u}\left(m_1(t,r) - b_1(t,r)\bar{u}\right), & t > 0, \quad 0 < r < \bar{h}(t), \\ \bar{u}_r(t,0) = 0,\ \bar{u}(t,r) = 0, & t > 0, \quad \bar{h}(t) \leq r < \infty, \\ \bar{h}'(t) = -\mu u_r(t,\bar{h}(t)), & t > 0, \\ \bar{h}(0) = h_0,\ \bar{u}(0,r) = u_0(r), & 0 \leq r \leq h_0. \end{cases} \tag{4.14}$$

It follows from the comparison principle that

$$0 \leq u(t,r) \leq \bar{u}(t,r) \quad \text{and} \quad h(t) \leq \bar{h}(t) \quad \text{for} \quad t \geq 0,\ 0 \leq r < h(t). \tag{4.15}$$

Since $h_\infty = \infty$, then we have $\bar{h}_\infty = \infty$. By Lemma 4.2 in [5], we have $\lim_{k\to\infty} \bar{u}(t+kT, r) = \hat{U}(t, r)$ uniformly in any compact subset of $[0, T] \times [0, \infty)$, where $\hat{U}(t, r)$ is defined in (4.9). Thus, $\liminf_{k\to\infty} v(t + kT, x) \le \hat{U}(t, x)$ uniformly in any compact subset of $[0, T] \times [0, \infty)$, which completes the proof of Lemma 4.2. □

The following result gives a sufficient condition for spreading and an estimate of h_∞ when $h_\infty < \infty$.

Lemma 4.3. *If $h_\infty < \infty$, then $h_\infty \le h^*(d_1, m_1 - c_1 V, T)$, where $V(t, |x|)$ is the unique positive solution of (4.3).*

Proof. By Corollary 3.2, we know that under the assumption (H3) there exists $h^* = h^*(d_1, m_1 - c_1 V, T) > 0$ such that $\lambda_1(d_1, h^*, m_1 - c_1 V, T) = 0$.

We assume $h_\infty > h^*(d_1, m_1 - c_1 V, T)$ to get a contradiction. Note that $h^*(d_1, k, T)$ is a strictly decreasing continuous function in $k(t, |x|)$, and due to Lemma 4.1, it is easy to see that for any given $0 < \varepsilon \ll 1$ there exists $k_\varepsilon \in \mathbb{N}$ such that for $k \ge k_\varepsilon$

$$\begin{aligned} &h(t + kT) > \max\{h_0, h^*(d_1, m_1 - c_1(V + \varepsilon), T)\} \\ &\text{and} \quad v(t + kT, r) \le V(t, r) + \varepsilon, \quad (t, r) \in [0, T] \times [0, h_\infty]. \end{aligned}$$

Set $L = h(t + kT)$, then $L > h^*(d_1, m_1 - c_1(V + \varepsilon), T)$. Let $\underline{u}(t, r)$ be the unique positive solution of the following initial boundary value problem with fixed boundary

$$\begin{cases} \underline{u}_t - d_1 \Delta \underline{u} = \underline{u}(m_1(t, r) - c_1(t, r)(V(t, r) + \varepsilon) - b_1(t, r)\underline{u}), & t \ge k_\varepsilon T, \quad 0 < r < L, \\ \underline{u}_r(t, 0) = 0 = \underline{u}(t, L), & t \ge k_\varepsilon T, \\ \underline{u}(k_\varepsilon T, r) = u(k_\varepsilon T, r), & 0 < r < L. \end{cases}$$

By the comparison principle

$$u(t + kT, r) \ge \underline{u}(t + kT, r), \quad \text{for any } k \ge k_\varepsilon,\ (t, r) \in [0, T] \times [0, L].$$

Since $\lambda_1(d_1, L, m_1 - c_1(V + \varepsilon), T) < \lambda_1(d_1, h^*(d_1, m_1 - c_1(V + \varepsilon), T), m_1 - c_1(V + \varepsilon), T) = 0$, we know that $\underline{u}(t + kT, r) \to \underline{u}^*(t, r)$ as $k \to \infty$ uniformly for $r \in [0, L]$, where $\underline{u}^*(t, r)$ is the unique positive solution of

$$\begin{cases} \underline{u}_t - d_1 \Delta \underline{u} = \underline{u}(m_1(t, r) - c_1(t, r)(V(t, r) + \varepsilon) - b_1(t, r)\underline{u}), & (t, r) \in [0, T] \times [0, L], \\ \underline{u}(0, r) = \underline{u}(T, r). \end{cases}$$

Hence, $\liminf_{k\to\infty} u(t + kT, r) \ge \lim_{k\to\infty} \underline{u}(t + kT, r) = \underline{u}^*(t, r) > 0$ in $[0, T] \times [0, L]$. This contradicts to Lemma 4.1. □

According to Lemma 4.3, we directly have the following Corollary 4.1.

Corollary 4.1. *If $h_0 > h^*(d_1, m_1 - c_1 V, T)$, then $h_\infty = \infty$.*

Combining Lemmas 4.1–4.3, we have the following spreading–vanishing dichotomy theorem.

Theorem 4.1. *Let $(u(t, r), v(t, r), h(t))$ be any solution of (1.1). Then, the following alternative holds:*

Either (i) *spreading: $h_\infty = \infty$ and $U(t, r) \le \liminf_{k\to\infty} u(t + kT, r) \le \limsup_{k\to\infty} u(t + kT, r) \le \hat{U}(t, r)$ uniformly in any compact subset of $[0, T] \times [0, \infty)$;*
or (ii) *vanishing: $h_\infty \le h^*(d_1, m_1 - c_1 V, T)$, $\limsup_{t\to\infty} \|u(t, \cdot)\|_{C([0,h(t)])} = 0$ and $\lim_{k\to\infty} v(t + kT, r) = V(t, r)$ uniformly in any bounded subset of $[0, T] \times [0, \infty)$.*

Remark 4.1. If we add the assumption $m_{2,*} - c_2^*(t)U^*(t) > 0$ in (H3), where $U^*(t)$ is the unique positive solution of

$$\begin{cases} U'(t) = U(m_1^*(t) - b_{1,*}(t)U), \\ U(0) = U(T), \end{cases}$$

then, using similar arguments as in Theorem 3.2 and Theorem 5.2 of [43], we have, if spreading occurs,

$$U_*(t,r) \le \liminf_{k\to\infty} u(t+kT,r) \le \limsup_{k\to\infty} u(t+kT,r) \le U^*(t,r),$$
$$V_*(t,r) \le \liminf_{k\to\infty} v(t+kT,r) \le \limsup_{k\to\infty} v(t+kT,r) \le V^*(t,r)$$

uniformly in any bounded subset of $[0,T]\times[0,\infty)$, where (U^*,V_*) and (U_*,V^*) are positive T-periodic solutions of

$$\begin{cases} U_t - d_1\Delta U = U(m_1(t,r) - b_1(t,r)U - c_1(t,r)V), & \text{in } [0,T]\times(0,\infty), \\ V_t - d_2\Delta V = V(m_2(t,r) - c_2(t,r)U - b_2(t,r)V), & \text{in } [0,T]\times(0,\infty), \\ U(0,r) = U(T,r),\ V(0,r) = V(T,r), & \text{in } [0,\infty). \end{cases}$$

Here the existence of (U^*,V_*) and (U_*,V^*) can be obtained similarly to the proof of Theorem 2.1 in [43].

4.2. Sharp criteria for spreading and vanishing

In this subsection, we will establish sharp criteria by selecting d_1, h_0, μ and $u_0(r)$ as varying parameters to distinguish the spreading–vanishing dichotomy for the invasive species u. The following Theorem 4.2 shows that the invader cannot establish itself and the native species always survives the invasion if $\lambda_1(d_1,m_1,h_0,T) > 0$ and the initial density $u_0(r)$ is small.

Theorem 4.2. *If $\lambda_1(d_1,m_1,h_0,T) > 0$ and $\|u_0\|_{C([0,h_0))}$ is small, then $h_\infty < \infty$, $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([0,h(t)])} = 0$ and $\lim_{k\to\infty} v(t+kT,r) = V(t,r)$ uniformly in any bounded subset of $[0,T]\times[0,\infty)$, where $V(t,|x|)$ satisfies (4.3).*

Proof. In (4.15), we have known that $u(t,r) \le \bar{u}(t,r)$ and $h(t) \le \bar{h}(t)$ for $t \ge 0$ and $0 \le r < h(t)$. According to Lemma 5.4 in [5], we have that $\lim_{t\to\infty}\|\bar{u}(t,\cdot)\|_{C([0,\bar{h}(t)])} = 0$ and $\bar{h}_\infty < \infty$ for $t \ge 0$, which implies $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([0,h(t)])} = 0$ and $h(t) < \infty$ for $t \ge 0$.

On the other hand, we can use the same way as the proof of Lemma 4.1 to deduce that $\lim_{k\to\infty} v(t+kT,r) = V(t,r)$ uniformly in any bounded subset of $[0,T]\times[0,\infty)$ under the above assumptions. □

Actually, due to Lemma 5.5 in [5], we can prove a more general result by using the same arguments as Theorem 4.2.

Theorem 4.3. *If $\lambda_1(d_1,m_1,h_0,T) > 0$, then there exists $\mu_0 > 0$ depending on u_0 such that when $0 < \mu \le \mu_0$, we have $h_\infty < \infty$, $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([0,h(t)])} = 0$, and $\lim_{k\to\infty} v(t+kT,r) = V(t,r)$ uniformly in any bounded subset of $[0,T]\times[0,\infty)$, where $V(t,|x|)$ is the unique positive solution of (4.3).*

Corollary 4.2. *Assume one of the following assumptions holds:*

(i) *The diffusion d_1 is fast, i.e. $d_1 > d^*(m_1,h_0,T)$;*
(ii) *The initially occupying habitat h_0 satisfies $h_0 < h^*(m_1,h_0,T)$.*

Then $\lambda_1(d_1, m_1, h_0, T) > 0$*, and hence we can establish the corresponding vanishing results for cases* (i) *and* (ii) *from Theorems 4.2 and 4.3.*

Next, we show that the invasive species can spread successfully if $\lambda_1(d_1, m_1 - c_1V, h_0, T) \le 0$.

Theorem 4.4. *If* $\lambda_1(d_1, m_1 - c_1V, h_0, T) \le 0$*, then* $h_\infty = \infty$*, which implies spreading of the invasive species happens, where* $V(t, |x|)$ *is the unique positive solution of* (4.3)*.*

Proof. First, we prove the case $\lambda_1(d_1, m_1 - c_1V, h_0, T) < 0$.

Recall that, in the proof of Lemma 4.2, we have $\lim_{k\to\infty} \bar{v}(t+kT, r) = V(t, r)$ uniformly in $[0, T] \times [0, \infty)$. Then for any given $0 < \varepsilon \ll 1$, there exists $k_\varepsilon \in \mathbb{N}$ with $k_\varepsilon \gg 1$ such that $v(t + kT, r) \le \bar{v}(t + kT, r) \le V(t, r) + \varepsilon$ for any $k \ge k_\varepsilon$ and $(t, r) \in [0, T] \times [0, \infty)$.

Let φ_1 be the corresponding eigenfunction of problem (3.1) with $\lambda_1 = \lambda_1(d_1, m_1 - c_1V, h_0, T)$.

Now we set

$$\underline{u}(t, r) = \begin{cases} \epsilon\varphi_1(t, r), & \text{for} \quad t \ge k_\varepsilon T,\ r \in [0, h(k_\varepsilon T)], \\ 0, & \text{for} \quad t \ge k_\varepsilon T,\ r > h(k_\varepsilon T). \end{cases}$$

Choose $\epsilon > 0$ so small that

$$\lambda_1 + \varepsilon b_1\varphi_1 + \epsilon\varepsilon c_1 \le 0 \quad \text{and} \quad \epsilon\varphi_1(k_\varepsilon T, r) \le u(k_\varepsilon T, r) \quad \text{for } t \ge k_\varepsilon T,\ r \in [0, h(k_\varepsilon T)].$$

Then direct calculation yields

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} - \underline{u}(m_1(t, r) - b_1(t, r)\underline{u} - c_1(t, r)(V(t, r) + \varepsilon)) \\ \quad \le (\lambda_1 + \varepsilon b_1\varphi_1 + \epsilon\varepsilon c_1)\epsilon\varphi_1 \le 0, \quad t > k_\varepsilon T,\ 0 < r < h(k_\varepsilon T), \\ \underline{u}_r(t, 0) = 0 = u_r(t, 0), \quad t > k_\varepsilon T, \\ \underline{u}(t, r) = 0 \le u(t, r), \quad t > k_\varepsilon T,\ r \ge h(k_\varepsilon T), \\ 0 = h_0' \le -\mu\underline{u}_r(t, h_0), \quad t > k_\varepsilon T, \\ \underline{u}(k_\varepsilon T, r) = \epsilon\varphi_1(k_\varepsilon T, r) \le u(k_\varepsilon T, r), \quad 0 \le r \le h(k_\varepsilon T). \end{cases}$$

By the comparison principle, we have

$$u(t, r) \ge \underline{u}(t, r) \quad \text{for } (t, r) \in [k_\varepsilon T, \infty) \times [0, h(k_\varepsilon T)].$$

It follows that

$$\liminf_{t\to\infty} \|u(t, \cdot)\|_{C([0,h(t)])} \ge \inf_{t\in[0,T]} \epsilon\varphi_1(t, 0) > 0.$$

According to Lemma 4.1, we see that $h_\infty = \infty$. Hence, by Lemma 4.2, spreading happens.

While for $\lambda_1(d_1, m_1 - c_1V, h_0, T) = 0$, using the monotonically of $h(t)$ (see Lemma 2.1), we can select $t^* > 0$ such that $h(t^*) > h_0$. It follows from Corollary 3.2 that $\lambda_1(d_1, m_1 - c_1V, h(t^*), T) < \lambda_1(d_1, m_1 - c_1V, h_0, T) = 0$. Therefore, after replacing h_0 with $h(t^*)$, the same method employed above can obtain the desired result again. □

Corollary 4.3. (1) *If* $\max_{B_{h_0}} \int_0^T (m_1 - c_1V)dt > 0$*, then* $d_*(m_1 - c_1V, h_0, T)$ *exists such that* $h_\infty = \infty$ *for* $d_1 \le d_*(m_1 - c_1V, h_0, T)$.

(2) $h_\infty = \infty$ *for* $h_0 \ge h^*(d_1, m_1 - c_1V, T)$*. The existence of* $h^*(d_1, m_1 - c_1V, T)$ *is obtained by combining the fact*

$$0 < m_{1,*}(t) - c_1^*(t)V^*(t) \le \liminf_{|x|\to\infty} (m_1(t,|x|) - c_1(t,|x|)V(t,|x|))$$
$$\le \limsup_{|x|\to\infty} (m_1(t,|x|) - c_1(t,|x|)V(t,|x|)) \le m_1^*(t) - c_{1,*}(t)V_*(t)$$

and (iii) *in Lemma 3.2.*

Remark 4.2. The condition $\max_{B_{h_0}} \int_0^T (m_1 - c_1 V)dt > 0$ in Corollary 4.3(1) means that some $r_0 \in B_{h_0}$ exists such that $\int_0^T m_1(t, r_0)dt$ is large enough. Corollary 4.3(1) suggests that if the mean growth rate of u over $[0, T]$ is large in a site of initial habitat, then spreading occurs, which coincides with the biological phenomenon.

Next, we give a sufficient condition for the spreading of u provided the principle eigenvalue $\lambda_1(d_1, m_1 - c_1 V, h_0, T) > 0$, where $V(t, r)$ is the unique positive solution of (4.3).

Theorem 4.5. *$h_\infty = \infty$ if $\|u_0\|_{C([0,h_0])}$ is sufficiently large or if $\mu \ge \mu^0$, where μ^0 depends on u_0, v_0 and h_0.*

Proof. Recall that in (4.11) we have

$$0 < m_{1,*}(t) - c_1^*(t)V^*(t) \le \liminf_{|x|\to\infty} (m_1(t,|x|) - c_1(t,|x|)V(t,|x|))$$
$$\le \limsup_{|x|\to\infty} (m_1(t,|x|) - c_1(t,|x|)V(t,|x|)) \le m_1^*(t) - c_{1,*}(t)V_*(t).$$

Thus,

$$\lim_{L\to\infty} \lambda_1(d_1, m_1 - c_1 V, \sqrt{L}, T) < 0$$

by (iii) in Lemma 3.2. Therefore, there exists $L^* > 0$, such that $\lambda_1(d_1, m_1 - c_1 V, \sqrt{L^*}, T) < 0$.

Next, we construct a suitable lower solution to problem (1.1). First, we consider the following eigenvalue problem

$$\begin{cases} \varphi_t - d_1\varphi_{rr} - \frac{1}{2}\varphi_r = \mu\varphi, \quad 0 < t < T,\ 0 < r < 1, \\ \varphi_r(t,0) = \varphi(t,1) = 0, \quad 0 < t < T, \\ \varphi(0,r) = \varphi(T,r), \quad 0 < r < 1. \end{cases}$$

It follows from [3,23] that the above eigenvalue problem admits a unique principal eigenvalue μ_1 with associated T-periodic eigenfunction $\varphi > 0$ in $(t, r) \in [0, T] \times (0, 1)$ with $\|\varphi\|_{L^\infty([0,T]\times[0,1])} = 1$. By the moving-plane argument in [7], we have $\varphi_r(t, r) < 0$ in $(t, r) \in [0, T] \times (0, 1]$. We claim that $\mu_1 > 0$. In fact, multiplying the equation of φ by φ and integrating over $[0, T] \times (0, 1)$, we obtain

$$\mu_1 \int_0^T \int_0^1 \varphi^2 dr dt = \int_0^T \int_0^1 \varphi_t \varphi dr dt + d_1 \int_0^T \int_0^1 |\varphi_r|^2 dr dt - \frac{1}{2} \int_0^T \int_0^1 \varphi_r \varphi dr dt$$
$$= d_1 \int_0^T \int_0^1 |\varphi_r|^2 dr dt - \frac{1}{2} \int_0^T \int_0^1 \varphi_r \varphi > 0.$$

Define

$$\begin{cases} \underline{h}(t) = \sqrt{t+\delta}, \quad t \geq 0, \\ \underline{u}(t,r) = \frac{M}{(t+\delta)^l}\varphi(\xi,\eta), \quad \xi = \int_0^t \underline{h}^{-2}(s)ds, \quad \eta = \frac{r}{\sqrt{t+\delta}}, \quad t \geq 0, \quad 0 \leq r \leq \sqrt{t+\delta}, \end{cases}$$

where δ, l, M are positive constants to be determined later. We are now in a position to show that $(\underline{u}, \bar{v}, \underline{h})$ is a lower solution of problem (1.1), where $\bar{v}(t,x) = (1 + He^{-Kt})V(t,r)$ is defined in (4.12).

From Lemma 2.2, we have $0 \leq u(t,r) \leq C_1$ for $t \geq 0, r \in [0, h(t)]$, which implies that the parameters δ, l and M at least need to be chosen to satisfy $\underline{u}(t,r) \leq C_1$. Since $m_1(t,r)$, $b_1(t,r)$, $c_1(t,r)$ and $V(t,r)$ are bounded, then there exists a positive constant Q such that $m_1(t,r) - b_1(t,r)\underline{u} - c_1(t,r)\bar{v} \geq -Q$. Direct calculations yield

$$\begin{aligned} &\underline{u}_t - d_1\Delta\underline{u} - \underline{u}(m_1(t,r) - b_1(t,r)\underline{u} - c_1(t,r)\bar{v}) \\ &= -\frac{M}{(t+\delta)^{l+1}}\{l\varphi(\xi,\eta) - (t+\delta)[\underline{h}^{-2}(t)\varphi_\xi(\xi,\eta) - r\underline{h}^{-2}(t)\underline{h}'(t)\varphi_\eta(\xi,\eta)] \\ &\quad + d(t+\delta)[\underline{h}^{-2}(t)\varphi_{\eta\eta}(\xi,\eta) + \frac{\underline{h}^{-2}(t)(N-1)}{\eta}\varphi_\eta(\xi,\eta)] \\ &\quad - (t+\delta)\varphi(\xi,\eta)(m_1(x) - b_1(t,r)\underline{u} - c_1(t,r)\bar{v})\} \\ &= -\frac{M}{(t+\delta)^{l+1}}\{l\varphi(\xi,\eta) - (t+\delta)\underline{h}^{-2}(t)[\mu_1\varphi(\xi,\eta) + \frac{1}{2}\varphi_\eta(\xi,\eta)] \\ &\quad + (t+\delta)r\underline{h}^{-2}(t)\underline{h}'(t)\varphi_\eta(\xi,\eta) - Q(t+\delta)\varphi(\xi,\eta)\} \\ &\leq -\frac{M}{(t+\delta)^{l+1}}\{l\varphi(\xi,\eta) - \mu_1\varphi(\xi,\eta) - Q(t+\delta)\varphi(\xi,\eta)\}, \end{aligned}$$

for $0 < r < \underline{h}(t)$, $0 < t \leq L^*$.

Choosing $0 < \delta \leq 1$, $\mu_1 + Q(L^* + 1) < l$, we obtain

$$\underline{u}_t - d_1\Delta\underline{u} - \underline{u}(m_1(t,r) - b_1(t,r)\underline{u} - c_1(t,r)\bar{v}) \leq -\frac{M}{(t+\delta)^{l+1}}(l\varphi(\xi,\eta) - \mu_1\varphi(\xi,\eta) - Q(L^*+1)\varphi(\xi,\eta)) < 0,$$

for $0 < r < \underline{h}(t)$ and $0 < t \leq L^*$.

(i) We may choose $0 < \delta \leq h_0^2$ and select $\mu > 0$ being sufficiently large such that $\mu \geq \mu^0 := -\frac{(L^*+1)^l}{2M\varphi_r(t,1)}$, then we have

$$\underline{h}'(t) + \mu\underline{u}_r(t, h(t)) = \frac{1}{2\sqrt{t+\delta}} + \frac{\mu M\varphi_r(t,1)}{(t+\delta)^{l+1/2}} \leq 0 \quad \text{for } 0 < t \leq L^*. \tag{4.16}$$

Moreover, we select $M > 0$ being sufficiently small such that

$$\underline{u}(0,r) = \frac{M}{\delta^l}\varphi(0, \frac{r}{\sqrt{\delta}}) < u_0(r) \quad \text{in } [0, \sqrt{\delta}\,]. \tag{4.17}$$

(ii) We may select M and $\|u_0\|_{C([0,h_0))}$ being sufficiently large such that (4.16) and (4.17) hold.

Either by (i) or (ii), we have

$$\begin{cases} \underline{u}_t - d_1\Delta\underline{u} \leq \underline{u}(m_1(t,r) - b_1(t,r)\underline{u} - c_1(t,r)\bar{v}), \quad 0 < t \leq L^*, \quad 0 < r < \underline{h}(t), \\ \underline{u}_r(t,0) = 0, \ \underline{u}(t,\underline{h}(t)) = 0, \quad 0 < t \leq L^*, \\ \underline{h}'(t) + \mu\underline{u}_r(t,\underline{h}(t)) \leq 0, \quad 0 < t \leq L^*, \\ \underline{u}(0,r) \leq u_0(r), \quad 0 \leq r \leq \sqrt{\delta}. \end{cases}$$

By the comparison principle, we have $\underline{h}(t) \le h(t)$ in $[0, L^*]$. Specially, we derive $h(L^*) \ge \underline{h}(L^*) = \sqrt{L^* + \delta} \ge \sqrt{L^*}$. Since $\lambda_1(d_1, m_1 - c_1 V, \sqrt{L^*}, T) < 0$, according to the strictly monotone decreasing of $\lambda_1(d_1, m_1 - c_1 V, R, T)$ in R, we have $h(L^*) \ge \sqrt{L^*} > h^*(d_1, m_1 - c_1 V, T)$, which implies $h_\infty > h^*(d_1, m_1 - c_1 V, T)$. From Corollary 4.1, we obtain $h_\infty = \infty$. □

Corollary 4.4. *For any $d_1 > 0$, $h_\infty = \infty$ if $\|u_0\|_{C([0,h_0])}$ is sufficiently large or if $\mu > \mu^0$, where μ^0 depends on u_0, v_0 and h_0.*

Similarly, due to the strict monotone decreasing of $\lambda^*(d_1, m_1 - c_1 V, h(t), T)$ in $h(t)$, and Theorem 4.5, we obtain

Corollary 4.5. *If $0 < h_0 < h^*(d_1, m_1 - c_1 V, T)$, then $h_\infty = \infty$ if $\|u_0\|_{C([0,h_0])}$ is sufficiently large or if $\mu > \mu^0$, where μ^0 depends on u_0, v_0 and h_0.*

If h_0 is fixed, some sufficient conditions for spreading–vanishing of u depending on d_1 and $u_0(r)$ are derived from Corollaries 4.2, 4.3 and 4.4.

Theorem 4.6. *There exist $d^*(m_1, h_0, T)$ and $d_*(h_0, m_1 - c_1 V, T)$ defined in $(0, \infty)$ such that*

(i) *vanishing occurs if $d_1 > d^*(m_1, h_0, T)$ and initial value $u_0(r)$ is small;*
(ii) *spreading happens if one of the following results holds:*
  (a) *if $0 < d_1 \le d_*(m_1 - c_1 V, h_0, T)$ under the assumption $\max_{B_{h_0}} \int_0^T (m_1 - c_1 V) dt > 0$;*
  (b) *if $\|u_0\|_{C([0,h_0])}$ is sufficiently large for any $d_1 > 0$.*

Similarly, if d_1 is fixed, some sufficient conditions for spreading–vanishing of u depending on h_0 and $u_0(r)$ are obtained from Corollaries 4.2, 4.3 and 4.5.

Theorem 4.7. *There exist $h^*(d_1, m_1, T)$ and $h^*(d_1, m_1 - c_1 V, T)$ defined in $(0, \infty)$ such that*

(i) *vanishing occurs if $h_0 < h^*(d_1, m_1, T)$ and the initial value $u_0(r)$ is small;*
(ii) *spreading happens if one of the following holds:*
  (a) *if $h_0 \ge h^*(d_1, m_1 - c_1 V, T)$;*
  (b) *if $0 < h_0 < h^*(d_1, m_1 - c_1 V, T)$ and $\|u_0\|_{C([0,h_0])}$ is sufficiently large.*

Next, if d_1 is fixed, the initial number $u_0(r)$ governs the spreading and vanishing of the invasive species. Then we can derive the sharp criteria for spreading–vanishing of an invasive species u from Corollaries 4.2, 4.5 and Theorem 4.4, by the same arguments as Theorem 5.7 in [28].

Theorem 4.8. *For any $d_1 > 0$ and given v_0, which satisfies (1.2), if $u_0(r) = \varepsilon\theta(r)$ for some $\varepsilon > 0$ and $\theta(r)$ such that u_0 satisfies (1.2), then ε^* exists depending on θ, v_0 and d_1 such that spreading occurs if $\varepsilon > \varepsilon^*$, and vanishing happens if $0 < \varepsilon \le \varepsilon^*$. Moreover, $\varepsilon^* = 0$ if $h_0 \ge h^*(d_1, m_1 - c_1 V, T)$, $\varepsilon^* \ge 0$ if $0 < h_0 < h^*(d_1, m_1 - c_1 V, T)$, and $\varepsilon^* > 0$ if $h_0 < h^*(d_1, m_1, T)$.*

Now we can derive the sharp criteria for spreading–vanishing of an invasive species u from Corollaries 4.2, 4.3, and 4.5 by choosing the expansion capability μ as a parameter. The proof is similar to that of Theorem 3.9 in [13].

Theorem 4.9. *For any $d_1 > 0$ and given (u_0, v_0), which satisfies (1.2), μ^* exists depending on u_0, v_0, h_0 and d_1 such that spreading occurs if $\mu > \mu^*$, and vanishing occurs if $0 < \mu < \mu^*$. Moreover, $\mu^* = 0$ if $h_0 \ge h^*(d_1, m_1 - c_1 V, T)$, $\mu^* \ge 0$ if $0 < h_0 < h^*(d_1, m_1 - c_1 V, T)$, and $\mu^* > 0$ if $h_0 < h^*(d_1, m_1, T)$.*

5. Estimates of the spreading speed

In this section, we give some rough estimates on the spreading speed of $h(t)$ for the case that spreading of u happens. We first consider the following problem

$$\begin{cases} U_t - d\Delta U + K(t)U_r = U(a(t) - b(t)U), \quad (t,r) \in [0,T] \times (0,\infty), \\ U(t,0) = 0, \quad t \in [0,T], \\ U(0,r) = U(T,r), \quad r \in (0,\infty), \end{cases} \tag{5.1}$$

where $d > 0$ is a given constant, and K, a, b are given T-periodic Hölder continuous functions with a, b positive and K nonnegative. From Propositions 2.1, 2.3 and Theorem 2.4 in [10], we have the following Proposition 5.1.

Proposition 5.1. *For any given positive T-periodic functions $a, b \in C^{\frac{\nu_0}{2}}([0,T])$ and any nonnegative continuous T-periodic function $K \in C^{\frac{\nu_0}{2}}([0,T])$, problem (5.1) admits a positive T-periodic solution $U^K \in C^{1,2}([0,T] \times [0,\infty))$ if and only if $\frac{1}{T}\int_0^T a(t)dt > \frac{1}{T^2}(\int_0^T K(t)dt)^2/(4d)$, and such a solution is unique when it exists. Furthermore, $U_r^K(t,r) > 0$ in $[0,T] \times [0,\infty)$ and $U^K(t,r) \to V(t)$ uniformly for $t \in [0,T]$ as $r \to +\infty$, where $V(t)$ is the unique positive solution of the problem*

$$\begin{cases} \frac{dV}{dt} = V(a(t) - b(t)V), \quad t \in [0,T], \\ V(0) = V(T). \end{cases} \tag{5.2}$$

In addition, for any given nonnegative T-periodic function $K_1 \in C^{\frac{\nu_0}{2}}([0,T])$, the assumption $K_1 \leq, \not\equiv K$ implies $U_r^{K_1}(t,0) > U_r^K(t,0)$, $U^{K_1}(t,r) > U^K(t,r)$ for $(t,r) \in [0,T] \times (0,+\infty)$. Besides, for each $\mu > 0$, there exists a positive continuous T-periodic function $K_0(t) = K_0(\mu,a,b)(t) > 0$ such that $\mu U_r^{K_0}(t,0) = K_0(t)$ on $[0,T]$. Moreover, $0 < \frac{1}{T}\int_0^T K_0(\mu,a,b)(t)dt < 2\sqrt{\frac{d}{T}\int_0^T a(t)dt}$ for every $\mu > 0$.

Making use of the function $K_0(\mu,a,b)$, we have the following estimate for the spreading speed of $h(t)$.

Theorem 5.1. *Assume* (H1)–(H3) *hold. If $h_\infty = +\infty$, then*

$$\frac{1}{T}\int_0^T K_0(\mu, m_{1,*} - c_1^* V^*, b_1^*)dt \leq \liminf_{t\to+\infty} \frac{h(t)}{t} \leq \limsup_{t\to+\infty} \frac{h(t)}{t} \leq \frac{1}{T}\int_0^T K_0(\mu, m_1^*, b_{1,*})dt.$$

Proof. Consider the following auxiliary problem as (4.14)

$$\begin{cases} \bar{u}_t - d_1\Delta\bar{u} = \bar{u}(m_1(t,r) - b_1(t,r)\bar{u}), \quad t > 0, \quad 0 < r < \bar{h}(t), \\ \bar{u}_r(t,0) = 0, \ \bar{u}(t,r) = 0, \quad t > 0, \quad \bar{h}(t) \leq r < \infty, \\ \bar{h}'(t) = -\mu\bar{u}_r(t,\bar{h}(t)), \quad t > 0, \\ \bar{u}(0,r) = u_0(r), \quad 0 \leq r \leq h_0. \end{cases}$$

By the comparison principle, it follows that $\bar{h}(t) \geq h(t) \to +\infty$ as $t \to \infty$. By Theorem 4.4 in [10], $\limsup_{t\to+\infty} \frac{\bar{h}(t)}{t} \leq \frac{1}{T}\int_0^T K_0(\mu, m_1^*, b_{1,*})dt$. Thus, we have

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \leq \limsup_{t\to+\infty} \frac{\bar{h}(t)}{t} \leq \frac{1}{T}\int_0^T K_0(\mu, m_1^*, b_{1,*})dt.$$

Next, we prove that $\liminf_{t\to+\infty}\frac{h(t)}{t}\geq\frac{1}{T}\int_0^T K_0(\mu,m_{1,*}-c_1^*V^*,b_1^*)dt$.

As in the proof of Lemma 4.2, we know, for any $0<\varepsilon\ll 1$, there exists $k_\varepsilon>0$ such that $v(t+kT,r)\leq V(t,r)+\varepsilon$ for any $k\geq k_\varepsilon$ and $(t,r)\in[0,T]\times[0,\infty)$. Since $h_\infty=+\infty$, we may assume that $h(k_\varepsilon T)>h^*(d_1,m_1-c_1(V+\varepsilon),T)$. Let $(\underline{u}(t,r),\underline{h}(t))$ be the unique solution of

$$\begin{cases}\underline{u}_t-d_1\Delta\underline{u}=\underline{u}(m_1(t,r)-c_1(t,r)(V(t,r)+\varepsilon)-b_1(t,r)\underline{u}), & t>k_\varepsilon T,\quad 0<r<\underline{h}(t),\\ \underline{u}_r(t,0)=0,\ \underline{u}(t,r)=0, & t>k_\varepsilon T,\quad \underline{h}(t)\leq r<\infty,\\ \underline{h}'(t)=-\mu\underline{u}_r(t,h(t)), & t>k_\varepsilon T,\\ \underline{u}(k_\varepsilon T,r)=u(k_\varepsilon T,r)>0, & \underline{h}(k_\varepsilon T)=h(k_\varepsilon T),\quad 0<r\leq\underline{h}(k_\varepsilon T).\end{cases}$$

The comparison principle implies $\underline{u}(t+kT,r)\leq u(t+kT,r)$ and $\underline{h}(t+kT)\leq h(t+kT)$ for any $k\geq k_\varepsilon$ and $(t,r)\in[0,T]\times[0,\infty)$. By Lemma 4.3, $\underline{h}_\infty=\infty$ since $\underline{h}(k_\varepsilon T)=h(k_\varepsilon T)>h^*(d_1,m_1-c_1(V+\varepsilon),T)$. Moreover, by Theorem 1.4 in [36], we have $\limsup_{r\to\infty}c_1(t,r)V(t,r)\leq c_1^*(t)V^*(t)$. It follows from Theorem 4.4 in [10] that $\liminf_{t\to+\infty}\frac{\underline{h}(t)}{t}\geq\frac{1}{T}\int_0^T K_0(\mu,m_{1,*}-c_1^*(V^*+\varepsilon),b_1^*)dt$, which implies $\frac{1}{T}\int_0^T K_0(\mu,m_{1,*}-c_1^*(V^*+\varepsilon),b_1^*)dt\leq\liminf_{t\to+\infty}\frac{h(t)}{t}$ for any small $\varepsilon>0$. Let $\varepsilon\to 0$ and using the continuity of K_0 with respect to its components, we immediately obtain the desired result. □

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