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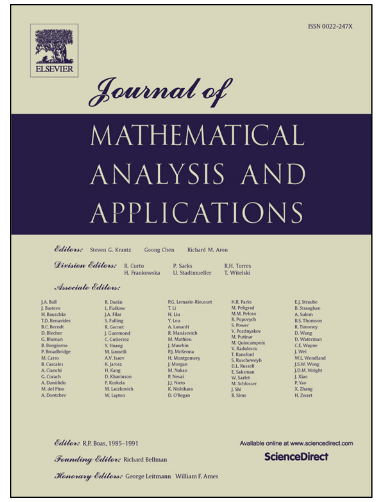


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# BEST CONSTANTS FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR ON FINITE GRAPHS

JAVIER SORIA AND PEDRO TRADACETE

Abstract. We study the behavior of averages for functions defined on finite graphs $G$, in terms of the Hardy-Littlewood maximal operator $M_G$. We explore the relationship between the geometry of a graph and its maximal operator and prove that $M_G$ completely determines $G$ (even though embedding properties for the graphs do not imply pointwise inequalities for the maximal operators). Optimal bounds for the $p$-(quasi)norm of a general graph $G$ in the range $0 < p \leq 1$ are given, and it is shown that the complete graph $K_n$ and the star graph $S_n$ are the extremal graphs attaining, respectively, the lower and upper estimates. Finally, we study weak-type estimates and some connections with the dilation and overlapping indices of a graph.

## 1. Introduction

Given a simple, connected, and finite graph $G = (V, E)$ (conditions that we will always assume from now on), where $V$ is a (finite) set of vertices and $E$ the set of edges between them, for a function $f : V \to \mathbb{R}$ we can consider the (centered) Hardy-Littlewood maximal operator

$$M_G f(v) = \sup_{r\geq 0} \frac{1}{|B(v,r)|} \sum_{w\in B(v,r)} |f(w)|.$$

Here $B(v, r)$ denotes the ball of center $v$ and radius $r$ on the graph, equipped with the geodesic distance induced by the edges in $E$.

The study of this and related maximal operators on metric measure spaces has received a considerable amount of attention recently (see for instance [1, 13, 14]). In particular, A. Naor and T. Tao have shown in [14] that for the infinite rooted $k$-regular tree $T$, the maximal operator satisfies

$$\|M_T\|_{\ell^1(T)\to\ell^{1,\infty}(T)} \lesssim 1,$$

with a constant independent of the degree of $T$ [14, Theorem 1.5]. This kind of averaging operators have also been studied in connection with harmonic functions and the Laplace operator on trees [7, 10].

Our main interest in this work focuses on finding the sharpest constants $C_{G,p}$ in inequalities of the form

$$\|M_G f\|_p \leq C_{G,p}\|f\|_p, \tag{1}$$

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for $0 < p \leq \infty$; i.e.,

$$C_{G,p} = \|M_G\|_p = \sup_f \frac{\|M_G f\|_p}{\|f\|_p},$$

where for a function $f : V \to \mathbb{R}$ we denote by $\|f\|_p = \left(\sum_{v \in V} |f(v)|^p\right)^{1/p}$. It is clear that $|f(v)| \leq M_G f(v) \leq \|f\|_\infty$, and hence $\|M_G\|_\infty = 1$, for every graph $G$. Therefore, we only need to consider the range $0 < p < \infty$.

The sharpest constants $C_{G,p}$ are of interest since they could provide improvements in quantitative estimates arising in some of the multiple applications of the Hardy-Littlewood inequality (for instance, in quantitative versions of Rademacher's differentiation theorem for Lipschitz functions [6].)

Another motivation for studying (1) comes from the discretization results proved for the Hardy-Littlewood maximal function in $\mathbb{R}$ in terms of Dirac deltas [9, 11, 12], which is closely related to the case of a linear tree $L_n$ (see Proposition 4.13 and Remark 4.14). We will see that a richer geometric structure on the graph gives us better estimates for the maximal operator which, in turn, characterize the graph in some extremal cases (see Theorem 3.1). In particular, we will prove that the complete graph on $n$ vertices can be characterized in terms of the equality

$$\|M_G\|_1 = 1 + (n-1)/n,$$

while a graph $G$ of $n$ vertices will satisfy

$$\|M_G\|_1 = 1 + (n-1)/2$$

precisely when $G$ is isomorphic to the star graph $S_n$.

We will also consider weak-type estimates of the form $M_G : \ell^p(V) \to \ell^{p,\infty}(V)$. In general, computing exactly the weak-type $(1,1)$ norm of the Hardy-Littlewood operator in a metric space $\|M\|_{L_1(X) \to L_{1,\infty}(X)}$ is a hard problem. In ultrametric spaces, this norm equals one, while for the real line $\mathbb{R}$, Melas showed [11] it equals $(11 + \sqrt{61})/12$. Optimal bounds in $L^p(\mathbb{R})$, for the uncentered maximal function, are proved in [8].

Other results involving maximal operators on infinite graphs can be found in [3]. In [2] boundedness of some Hardy type averaging operators were also considered in the setting of partially ordered measure spaces, which include the case of infinite trees.

In our analysis of weak-type estimates we will introduce two indices associated with coverings of a graph: the dilation and the overlapping indices. These will provide an upper bound for the weak-type $(1,1)$ estimate of $M_G$ (Theorem 4.9).

The paper is organized as follows: In Section 2 we prove in Theorem 2.4 that $M_G$ completely determines $G$. Lemma 2.5 is our main tool to easily calculate the norm of the maximal operator $M_G$ on the range $0 < p \leq 1$ and, in particular, we consider the case of the complete graph $K_n$. We finish by introducing, in Proposition 2.9, some estimates of restricted type. In Section 3 we show in Theorem 3.1 that $K_n$ and the star graph $S_n$ are optimal cases for the boundedness of $M_G$ in $\ell^p(G)$, $0 < p \leq 1$, and get also sharp estimates for the linear graph $L_n$. To complete the information for the strong-type estimates, we calculate the norm for the star graph, on the range $1 < p < \infty$. Finally, in Section 4, we consider the study of weak-type estimates on $0 < p < \infty$, and establish in Theorem 4.9 a relationship, for $p = 1$, with some geometrical indices associated to the graph.

We refer to [4, 5] for standard notations and definitions on graphs.

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## 2. General properties and best constants

Let $G = (V, E)$ be a simple, connected, and finite graph. Here $V$ denotes the set of vertices and $E$ the set of edges between them. We will work with this space as a metric space endowed with the geodesic distance induced by the edges in $E$. That is, given $v, w \in V$ the distance $d_G(v, w)$ is the number of edges in a shortest path connecting $v$ and $w$. $B(v, r)$ denotes the ball of center $v$ and radius $r$ on the graph, i.e.

$$B(v, r) = \{w \in V(G) : d_G(v, w) \leq r\}.$$

For example, $B(v, r) = \{v\}$, if $0 \leq r < 1$ and $B(v, r) = \{v\} \cup N_G(v)$, if $1 \leq r < 2$, where $N_G(v)$ is the set of neighbors of $v$. Also, given a finite set $A$, denote its cardinality by $|A|$.

Given a function $f : V \to \mathbb{R}$ the Hardy-Littlewood maximal operator is defined as

$$M_G f(v) = \sup_{r \geq 0} \frac{1}{|B(v, r)|} \sum_{w \in B(v,r)} |f(w)|.$$

Since the distance $d_G$ introduced above only takes natural numbers as values, the radius $r > 0$ considered in the definition of the Hardy-Littlewood maximal operator can be taken to be a natural number. Moreover, since the diameter of a graph of $n$ vertices is at most $n - 1$, we can compute

$$M_G f(v) = \max_{k=0,\dots,n-1} \frac{1}{|B(v, k)|} \sum_{w \in B(v,k)} |f(w)|.$$

Given a graph $G$, the degree of a vertex $v \in V_G$, denoted by $d_G(v)$, is the number of edges in $E_G$ which have $v$ as one of the endpoints; that is, $d_G(v) = |N_G(v)|$. For $j \in V$ we will consider the Kronecker delta

$$\delta_j(i) = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{2}$$

We will use the notation $A \lesssim B$, whenever there exists $C > 0$ (independent of the main parameters involved, like the dimension $n \in \mathbb{N}$ or $0 < p < \infty$) such that $A \leq CB$. Similarly for $A \gtrsim B$. As usual, $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

Recall that two graphs $G_1$, $G_2$ are said to be isomorphic if there is a permutation of the vertices $\pi : V \to V$ such that $v, w \in V$ are the endpoints of an edge in $E_{G_1}$ if and only if $\pi(v)$ and $\pi(w)$ are the endpoints of an edge in $E_{G_2}$. In this case, we will write $G_1 \sim G_2$. It is clear that if $G_1 \sim G_2$, then $M_{G_2} f(\pi(v)) = M_{G_1} f(v)$ and hence $\|M_{G_1}\|_p = \|M_{G_2}\|_p$, $0 < p \leq \infty$. That the converse is not true can be seen in Example 2.6.

Let $K_n$ denote the complete graph with $n \geq 2$ vertices, which we are going to label as $V = \{1, \dots, n\}$. As a metric space, this is the simplest among all graphs with $n$ vertices, since given any $j \in V$ we have

$$B(j, r) = \begin{cases} \{j\}, & \text{for } 0 \leq r < 1, \\ V, & \text{for } r \geq 1. \end{cases}$$

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Therefore, the maximal operator takes the form

$$M_{K_n} f(j) = \max \Big\{ |f(j)|, \frac{1}{n} \sum_{k \in V} |f(k)| \Big\}. \tag{3}$$

The operator $M_{K_n}$ is the smallest, in the pointwise ordering, among all $M_G$, with $G$ a graph of $n$ vertices. That is, for every positive function $f : V \to \mathbb{R}$ and every $j \in V$, we have that

$$M_{K_n} f(j) \le M_G f(j). \tag{4}$$

In particular, if $0 < p \le \infty$ and $G$ is a graph with $n$ vertices, then

$$\|M_{K_n}\|_p \le \|M_G\|_p.$$

**Remark 2.1.** Regarding (4), it is worth mentioning that, in general, it is not true that if $G_1 \subset G_2$ (i.e., $V(G_1) = V(G_2)$ and $E(G_1) \subset E(G_2)$), then $M_{G_2} f \le M_{G_1} f$. For example, if $V = \{1, 2, 3, 4\}$, $G_1$ is a linear tree with leaves 1 and 4, $G_2$ is the 4-cycle $C_4$ (with a clockwise orientation of $V$), and $f = \delta_4$ is the Kronecker delta (see (2) for the definition), then $G_1 \subset G_2$, but it is easy to prove that, however, $M_{G_2} \delta_4(1) = 1/3 > 1/4 = M_{G_1} \delta_4(1)$.

Contrary to the minimality property (4) of the complete graph $K_n$, there is no graph $G$ whose maximal operator $M_G$ is the largest in the pointwise ordering among all graphs with $n \ge 3$ vertices ($n = 2$ is trivial since $K_2$ is the only example). That is, there exists no graph $G_{\max}$ such that, for every graph $G$ with $V(G) = V(G_{\max})$, and every function $f : V \to \mathbb{R}$ we have

$$M_G f(j) \le M_{G_{\max}} f(j), \quad \text{for each } j \in V.$$

However, we will prove in Theorem 3.1 that, in terms of the (quasi)norm $\|M_G\|_p$, for $0 < p \le 1$, we do have the existence of a maximal graph (namely, the star $S_n$).

**Proposition 2.2.** *If $G$ is a graph with $n \ge 3$ vertices, then there exists $j \in V(G)$, $f : V \to \mathbb{R}^+$, and another graph $G'$, with $V(G') = V(G)$, so that $M_G f(j) < M_{G'} f(j)$.*

*Proof.* Since $G$ has at least 3 vertices, then there is a vertex $j \in V$ with degree $d_G(j) \ge 2$. Let $k \in V$ be a neighbor of $j$. Let $G' = G_{j,k}$ be a linear tree with $n$ vertices and such that $j$ has degree 1 (it is a leaf in $G'$), and $k$ is the only neighbor of $j$ in $G'$. Let us consider the function $f(j) = 1/3$, $f(k) = 2/3$, and $f(l) = 0$ elsewhere. Then,

$$M_G f(j) = \max\{1/3, 1/(d_G(j) + 1)\} = 1/3 \text{ and } M_{G'} f(j) = \max\{1/3, 1/2\} = 1/2.$$

Hence, $M_G f(j) < M_{G'} f(j)$. □

We can also consider a maximal operator involving the averages for all isomorphic graphs to a given one. That is, given $G$, for $f : V \to \mathbb{R}$ and $j \in V$, we define:

$$M_{[G]} f(j) = \max_{H \sim G} M_H f(j).$$

For this larger operator, we can actually prove the following optimal pointwise estimates (see Proposition 3.2 for further properties):

**Proposition 2.3.** *Let $L_n$ be a linear tree. Then, for every graph with $n$ vertices and any function $f : \{1, \dots, n\} \to \mathbb{R}^+$,*

$$M_{[G]} f(j) \le M_{[L_n]} f(j), \quad j \in \{1, \dots, n\}.$$

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*Proof.* Given $G$ and $j \in \{1, \dots, n\}$, it is easy to find a linear graph $L_n$ so that, for every $0 \le r \le n-1$, there exists $0 \le s(r) \le n-1$ such that

$$B_G(j, r) = B_{L_n}(j, s(r)),$$

and $j$ is a leaf of $L_n$. For example, if $0 \le r < 1$, take $s(r) = r$ and $B_G(j,r) = \{j\} = B_{L_n}(j,r)$. If $1 \le r < 2$, then we order $N_G(j)$ in $L_n$ to obtain that $B_{L_n}(j, d_G(j)) = \{j\} \cup N_G(j) = B_G(j,r)$ (i.e.; $s(r) = d_G(j)$), and so on (see Figure 1). Then, for every $f : \{1, \dots, n\} \to \mathbb{R}^+$,

$$\begin{aligned} M_G f(j) &= \max \left\{ \frac{\sum_{k \in B_G(j,r)} f(k)}{|B_G(j,r)|} : 0 \le r \le n-1 \right\} \\ &\le \max \left\{ \frac{\sum_{k \in B_{L_n}(j,s)} f(k)}{|B_{L_n}(j,s)|} : 0 \le s \le n-1 \right\} \\ &= M_{L_n} f(j) \le M_{[L_n]} f(j), \end{aligned}$$

which gives $M_{[G]} f(j) \le M_{[L_n]} f(j)$, for every $j \in \{1, \dots, n\}$. $\square$

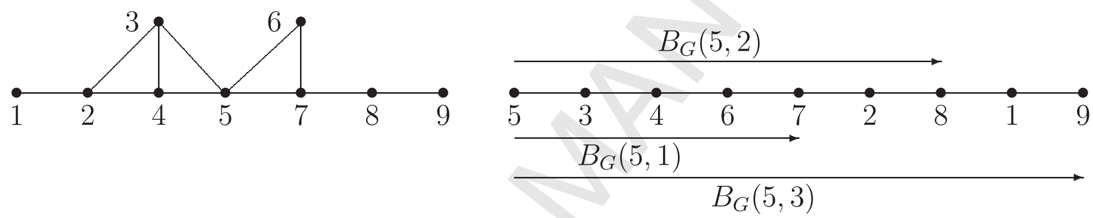


Figure 1: A graph $G$ and its corresponding linear tree for $j = 5$.

We now study the relationship between the geometry of a graph and its maximal operator and prove that $M_G$ completely determines $G$, even though embedding properties for the graphs do not imply pointwise inequalities for the maximal operators (see Remark 2.1).

**Theorem 2.4.** *Let $G_1$ and $G_2$ be two graphs with $V(G_1) = V(G_2) = \{1, \dots, n\}$. The following are equivalent:*

*(i)* $G_1 = G_2$.
*(ii) For every* $f : \{1, \dots, n\} \to \mathbb{R}$, $M_{G_1} f = M_{G_2} f$.
*(iii) For every* $k \in V$, $M_{G_1} \delta_k = M_{G_2} \delta_k$.

*Proof.* The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear. Let us prove $(iii) \Rightarrow (i)$: For each $k \in \{1, \dots, n\}$, we have that

$$M_{G_1} \delta_k(j) = \frac{1}{|B_{G_1}(j, d_{G_1}(j,k))|} = M_{G_2} \delta_k(j) = \frac{1}{|B_{G_2}(j, d_{G_2}(j,k))|}. \tag{5}$$

To prove that $G_1 = G_2$, it suffices to show that $N_{G_1}(j) = N_{G_2}(j)$, for every vertex $j \in \{1, \dots, n\}$. Assume that $|N_{G_1}(j)| = r$, with $r = d_{G_1}(j)$ and choose an ordering of $N_{G_1}(j) = \{v_1, \dots, v_r\}$ in such a way that $d_{G_2}(j, v_1) \le d_{G_2}(j, v_2) \le \dots \le d_{G_2}(j, v_r)$. Since $B_{G_2}(j, d_{G_2}(j, v_1)) \subset B_{G_2}(j, d_{G_2}(j, v_r))$ and, using (5), we also have that, for every $l \in \{1, \dots, r\}$,

$$1 + r = |B_{G_1}(j, 1)| = |B_{G_1}(j, d_{G_1}(j, v_l))| = |B_{G_2}(j, d_{G_2}(j, v_l))|.$$

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Thus, for every $l \in \{1, \dots, r\}$,

$$B_{G_2}(j, d_{G_2}(j, v_1)) = B_{G_2}(j, d_{G_2}(j, v_l)), \tag{6}$$

which implies that $d_{G_2}(j, v_1) = d_{G_2}(j, v_l)$. In fact, if $d_{G_2}(j, v_1) < d_{G_2}(j, v_l)$, for some $v_l$, then $v_l \in B_{G_2}(j, d_{G_2}(j, v_l)) \setminus B_{G_2}(j, d_{G_2}(j, v_1))$, which contradicts (6).

Finally, let us see that $d_{G_2}(j, v_1) = 1$, and hence $d_{G_2}(j, v_l) = 1$, for every $v_l$: If $d_{G_2}(j, v_1) > 1$, then $B_{G_2}(j, d_{G_2}(j, v_1))$ contains a vertex $u \in N_{G_2}(j)$, which necessarily satisfies that $u \notin \{j\} \cup \{v_1, \dots, v_r\}$ (we can always find a geodesic $L$ from $j$ to $v_1$ of the form $L \equiv j, u, \cdots, v_1$). Thus,

$$1 + r = |B_{G_2}(j, d_{G_2}(j, v_1))| \geq 2 + r,$$

which is a contradiction.

Therefore, we obtain that $N_{G_1}(j) \subset N_{G_2}(j)$. Reversing the role of $G_1$ and $G_2$, or using that

$$|N_{G_1}(j)| = |B_{G_1}(j, d_{G_1}(j, v_1))| - 1 = |B_{G_2}(j, d_{G_2}(j, v_1))| - 1 = |N_{G_2}(j)|,$$

we conclude that $N_{G_1}(j) = N_{G_2}(j)$. □

A starting point for our analysis of the norm of $\|M_G\|_p$, for $0 < p \leq 1$, is the following useful result. It is worth mentioning that this estimate is the discrete equivalent version in $\ell^p$ of [12, Theorem 3].

**Lemma 2.5.** *Let $G$ be a graph with $n$ vertices, and $T : \ell^p(G) \to \ell^p(G)$ be a sublinear operator, with $0 < p \leq 1$. Then,*

$$\|T\|_p = \max_{k \in V} \|T\delta_k\|_p.$$

*In particular, $\|M_G\|_p = \max_{k \in V} \|M_G \delta_k\|_p$ and $\|M_{[G]}\|_p = \max_{k \in V} \|M_{[G]}\delta_k\|_p$.*

*Proof.* Since for any $0 < p \leq 1$ and $k \in V$, we have that $\|\delta_k\|_p = 1$, then $\|T\|_p \geq \max_{k \in V} \|T\delta_k\|_p$. For the converse, let $f : V \to \mathbb{R}$, with $\|f\|_p \leq 1$; that is,

$$f = \sum_{k \in V} a_k \delta_k,$$

with $\sum_{k \in V} |a_k|^p \leq 1$. Using Holder's inequality for $0 < p \leq 1$, it follows that

$$\begin{aligned}
\|Tf\|_p^p = \sum_{j \in V} |Tf(j)|^p &= \sum_{j \in V} \Big| T\Big(\sum_{k \in V} a_k \delta_k\Big)(j)\Big|^p \\
&\leq \sum_{j \in V} \Big| \sum_{k \in V} |a_k| T\delta_k(j)\Big|^p \leq \sum_{j \in V} \sum_{k \in V} |a_k T\delta_k(j)|^p \\
&= \sum_{k \in V} |a_k|^p \sum_{j \in V} |T\delta_k(j)|^p = \sum_{k \in V} |a_k|^p \|T\delta_k\|_p^p \\
&\leq \max_{k \in V} \|T\delta_k\|_p^p.
\end{aligned}$$

□

**Example 2.6.** As an application of Lemma 2.5, let us find $\|M_G\|_1$ for all six graphs $G$ with 4 vertices:

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*(i)* $L_4$ (two vertices $\{1,4\}$ of degree 1, two vertices $\{2,3\}$ of degree 2): $\|M_{L_4}\|_1 = 13/6$.

For the vertices 1 and 2 we have

$$M_{L_4}\delta_1(j) = \begin{cases} 1, & j=1, \\ 1/3, & j=2, \\ 1/4, & j=3,4, \end{cases} \quad \text{and} \quad M_{L_4}\delta_2(j) = \begin{cases} 1/2, & j=1, \\ 1, & j=2, \\ 1/3, & j=3,4. \end{cases}$$

Hence, $\|M_{L_4}\delta_1\|_1 = 11/6$ and $\|M_{L_4}\delta_2\|_1 = 13/6$. By symmetry, we also have the estimates for the remaining vertices: $\|M_{L_4}\delta_4\|_1 = 11/6$ and $\|M_{L_4}\delta_3\|_1 = 13/6$. Hence, $\|M_{L_4}\|_1 = 13/6$.

*(ii)* $C_4$ (all four vertices of degree 2): $\|M_{C_4}\|_1 = 23/12$.

Since every vertex has the same degree, we have for $k = 1, \dots, 4$ :

$$M_{C_4}\delta_k(j) = \begin{cases} 1, & j=k, \\ 1/3, & j \equiv k-1, k+1 \pmod 4, \\ 1/4, & j \equiv k+2 \pmod 4). \end{cases}$$

Hence, $\|M_{C_4}\|_1 = \|M_{C_4}\delta_k\|_1 = 23/12$.

*(iii)* $S_4$ (one vertex $\{1\}$ of degree 3, three vertices $\{2,3,4\}$ of degree 1): $\|M_{S_4}\|_1 = 5/2$.

$$M_{S_4}\delta_1(j) = \begin{cases} 1, & j=1, \\ 1/2, & j=2,3,4, \end{cases} \quad \text{and} \quad M_{S_4}\delta_2(j) = \begin{cases} 1/4, & j=1,3,4, \\ 1, & j=2. \end{cases}$$

Hence, $\|M_{S_4}\delta_1\|_1 = 5/2$ and $\|M_{S_4}\delta_2\|_1 = \|M_{S_4}\delta_3\|_1 = \|M_{S_4}\delta_4\|_1 = 7/4$. Hence, $\|M_{S_4}\|_1 = 5/2$ (see Theorem 3.1 for further information).

*(iv)* $K_4$ (all four vertices of degree 3): $\|M_{K_4}\|_1 = 7/4$.

This is a trivial calculation and it also follows from Theorem 3.1, with $n = 4$.

*(v)* $D_4$ (two vertices $\{2,4\}$ of degree 3, two vertices $\{1,3\}$ of degree 2): $\|M_{D_4}\|_1 = 23/12$.

$$M_{D_4}\delta_1(j) = \begin{cases} 1, & j=1, \\ 1/4, & j=2,3,4, \end{cases} \quad \text{and} \quad M_{D_4}\delta_2(j) = \begin{cases} 1/3, & j=1,3, \\ 1, & j=2, \\ 1/4, & j=4. \end{cases}$$

Thus, $\|M_{D_4}\delta_1\|_1 = 7/4, \|M_{D_4}\delta_2\|_1 = 23/12$. As before, by symmetry, we also have the estimates $\|M_{D_4}\delta_3\|_1 = 7/4, \|M_{D_4}\delta_4\|_1 = 23/12$. Hence, we finally obtain that $\|M_{D_4}\|_1 = 23/12$.

*(vi)* $P_4$ (one vertex $\{1\}$ of degree 1, one vertex $\{2\}$ of degree 3, two vertices $\{3,4\}$ of degree 2): $\|M_{P_4}\|_1 = 13/6$.

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$$M_{P_4}\delta_1(j) = \begin{cases} 1, & j = 1, \\ 1/4, & j = 2,3,4, \end{cases} \qquad M_{P_4}\delta_2(j) = \begin{cases} 1/2, & j = 1, \\ 1, & j = 2, \\ 1/3, & j = 3,4, \end{cases}$$

$$M_{P_4}\delta_3(j) = \begin{cases} 1/4, & j = 1,2, \\ 1, & j = 3, \\ 1/3, & j = 4. \end{cases}$$

Hence, $\|M_{P_4}\delta_1\|_1 = 7/4$, $\|M_{P_4}\delta_2\|_1 = 13/6$, $\|M_{P_4}\delta_3\|_1 = \|M_{P_4}\delta_4\|_1 = 11/6$. Thus, $\|M_{P_4}\|_1 = 13/6$.

In the following diagrams we exhibit the different inclusions between all (connected) graphs with 4 vertices and the order relation among the norms of the corresponding maximal operators.

$$\begin{array}{ccccc} & & K_4 & & \\ & & \cup & & \\ P_4 & \subset & D_4 & \supset & C_4 \\ \cup & & & & \cup \\ S_4 & & & & L_4 \end{array} \qquad \begin{array}{ccccc} & & \|M_{K_4}\|_1 & & \\ & & \wedge & & \\ \|M_{P_4}\|_1 & > & \|M_{D_4}\|_1 & = & \|M_{C_4}\|_1 \\ \wedge & & & & \wedge \\ \|M_{S_4}\|_1 & & & & \|M_{L_4}\|_1 \end{array}$$

In particular, these examples show that we may have non-isomorphic graphs with equal norms ($\|M_{C_4}\|_1 = \|M_{D_4}\|_1$ and $\|M_{L_4}\|_1 = \|M_{P_4}\|_1$).

The diagram however motivates the following question: Given two graphs $G_1 \subset G_2$ with $n$ vertices (in the sense that every edge in $G_1$ is an edge in $G_2$), is it always true that $\|M_{G_2}\|_1 \leq \|M_{G_1}\|_1$? Recall that $G_1 \subset G_2$ does not imply, in general, the pointwise inequality $M_{G_2}f \leq M_{G_1}f$ (see Remark 2.1).

We are now going to study some optimal constants, and other estimates, for $\|M_{K_n}\|_p$. In Section 3 we will see that, for $0 < p \leq 1$, they are in fact uniquely determined by $K_n$.

**Proposition 2.7.**

*(i) If* $0 < p \leq 1$, *then*

$$\|M_{K_n}\|_p = \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}.$$

*(ii) If* $1 < p < \infty$, *then*

$$\Big(1 + \frac{n-1}{n^p}\Big)^{1/p} \leq \|M_{K_n}\|_p \leq \Big(1 + \frac{n-1}{n}\Big)^{1/p}. \tag{7}$$

*In particular,* $\|M_{K_n}\|_p \approx 1$.

*Proof.* Using (3) we have that the norm of $M_{K_n}$ can be computed as

$$\|M_{K_n}\|_p = \sup\Big\{\Big(\sum_{i=1}^n \max\Big\{x_i, \frac{1}{n}\sum_{j=1}^n x_j\Big\}^p\Big)^{1/p} : x_i \geq 0, \sum_{i=1}^n x_i^p = 1\Big\}. \tag{8}$$

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We start by proving the lower bound, for a general $0 < p < \infty$. For $k \in V$, we consider $\delta_k$, and for every $0 < p < \infty$, we have

$$\|M_{K_n}\delta_k\|_p = \Big(\sum_{i=1}^{n} \max\Big\{\delta_k(i), \frac{1}{n}\sum_{j=1}^{n}\delta_k(j)\Big\}^p\Big)^{1/p} = \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}.$$

Since $\|\delta_k\|_p = 1$, we get that for every $0 < p < \infty$

$$\|M_{K_n}\|_p \geq \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}.$$

Now, by Lemma 2.5, for $0 < p \leq 1$, we get that

$$\|M_{K_n}\|_p = \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}.$$

Finally, to prove the upper bound for the case $1 < p < \infty$, we use Jensen's inequality in (8):

$$\|M_{K_n}\|_p \leq \sup\Big\{\Big(\sum_{i=1}^{n}\max\Big\{x_i^p, \frac{1}{n}\Big\}\Big)^{1/p} : x_i \geq 0, \sum_{i=1}^{n} x_i^p = 1\Big\}.$$

Now, if $x_i^p \leq 1/n$, for every $1 \leq i \leq n$, then

$$\|M_{K_n}\|_p \leq \sup\Big\{\Big(\sum_{i=1}^{n}\frac{1}{n}\Big)^{1/p} : x_i \geq 0, \sum_{i=1}^{n} x_i^p = 1\Big\} = 1.$$

On the other hand, if $x_{i_0}^p > 1/n$, for some index $i_0 \in \{1, \dots, n\}$, then

$$\begin{aligned}\|M_{K_n}\|_p &\leq \sup\Big\{\Big(\sum_{\{x_i^p > 1/n\}} x_i^p + \sum_{\{x_i^p \leq 1/n\}} \frac{1}{n}\Big)^{1/p} : x_i \geq 0, \sum_{i=1}^{n} x_i^p = 1\Big\}\\ &\leq \Big(1 + \frac{n-1}{n}\Big)^{1/p}.\end{aligned}$$

□

It is not an easy task to compute the exact value of $\|M_{K_n}\|_p$ for $p > 1$. At least, from Proposition 2.7, we know that $1 \leq \|M_{K_n}\|_p \leq 2$, for every $n \in \mathbb{N}$ and $p > 1$.

**Remark 2.8.** The estimates we have obtained in Proposition 2.7 *(ii)* are not optimal in general. For example, if we consider the case $n = 2$, then for every function $f : \{1, 2\} \to \mathbb{R}^+$, we have that $M_{K_2}f(j) = (f(j) + \|f\|_\infty)/2$. Thus, if we assume that $\|f\|_\infty = f(2)$ and set $\alpha = f(1)/f(2)$, then for every $0 < p < \infty$,

$$\frac{\|M_{K_2}f\|_p^p}{\|f\|_p^p} = \frac{1}{2^p}\frac{(f(1) + f(2))^p + 2^p f(2)^p}{f(1)^p + f(2)^p} = \frac{1}{2^p}\frac{(1+\alpha)^p + 2^p}{1 + \alpha^p},$$

and hence,

$$\|M_{K_2}\|_p = \frac{1}{2}\Big(\sup_{0 \leq \alpha \leq 1}\frac{(1+\alpha)^p + 2^p}{1 + \alpha^p}\Big)^{1/p}.$$

It is easy to see that, for $1 < p < \infty$, this supremum is attained at the unique root $\alpha_p \in (0, 1)$ of the equation

$$(1+\alpha)^{p-1} = \frac{2^p\alpha^{p-1}}{1 - \alpha^{p-1}}.$$

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In particular, if $p=2$, then $\alpha_2=\sqrt{5}-2$ and $\|M_{K_2}\|_2=(3+\sqrt{5})^{1/2}/2$. However, from (7) we only obtain that $\sqrt{5}/2<\|M_{K_2}\|_2<(3/2)^{1/2}$.

As we have seen, and contrary to what happens for the case $0<p\leq 1$, the lower estimate given in (7) is not optimal when $1<p<\infty$. A closer look to the proof of this result shows that this bound is obtained by evaluating the maximal operator on characteristic functions supported at a singleton (a Kronecker delta). This can be improved by considering arbitrary characteristic functions (what is usually called a restricted type estimate):

$$\|M_{K_n}\|_{p,\mathrm{rest}}=\max\left\{\frac{\|M_{K_n}\chi_A\|_p}{\|\chi_A\|_p}:A\subset V\right\}.$$

Clearly, $\|M_{K_n}\|_{p,\mathrm{rest}}\leq\|M_{K_n}\|_p$. The following result shows that, for some particular values of $n\geq 2$ and $p>1$, we can get a better estimate. Recall that $p'$ denotes the conjugate index to $p$, defined as $1/p+1/p'=1$, and $[x]$ is the integer part of $x$.

**Proposition 2.9.** *Let $n\geq 2$ and $p>1$.*

*(i) If $n\leq p'$, then*

$$\|M_{K_n}\|_{p,\mathrm{rest}}=\left(1+\frac{n-1}{n^p}\right)^{1/p}.$$

*(ii) If $n\leq p$, then*

$$\|M_{K_n}\|_{p,\mathrm{rest}}=\left(1+\frac{(n-1)^{p-1}}{n^p}\right)^{1/p}.$$

*(iii) If $n>\max\{p,p'\}$, $p\in\mathbb{Q}$ with $p=p_1/p_2$ and $p_1$ divides $n$, then*

$$\|M_{K_n}\|_{p,\mathrm{rest}}=\left(1+\frac{(p-1)^{p-1}}{p^p}\right)^{1/p}.$$

*(iv) If $n>\max\{p,p'\}$, but $p$ is not of the previous form, and $[n]_p=[n/p']$, then*

$$\|M_{K_n}\|_{p,\mathrm{rest}}=\left(1+\frac{1}{n^p}\max\left\{\Big(n-[n]_p\Big)[n]_p^{p-1},\Big(n-1-[n]_p\Big)\Big([n]_p+1\Big)^{p-1}\right\}\right)^{1/p}.$$

*In particular, if $n>p'$ we have that*

$$\|M_{K_n}\|_p\geq\|M_{K_n}\|_{p,\mathrm{rest}}>\left(1+\frac{n-1}{n^p}\right)^{1/p}.$$

*Proof.* For $A\subset V$, with $|A|=k\leq n$, we have

$$M_{K_n}\chi_A(j)=\begin{cases}1, & \text{if } j\in A,\\ k/n, & \text{if } j\notin A.\end{cases}$$

Therefore,

$$\|M_{K_n}\chi_A\|_p=\Big(\sum_{j=1}^n M_{K_n}\chi_A(j)^p\Big)^{1/p}=\Big(k+\frac{(n-k)k^p}{n^p}\Big)^{1/p}.$$

Since $\|\chi_A\|_p=k^{1/p}$, we get

$$\|M_{K_n}\|_{p,\mathrm{rest}}=\Big(1+\frac{1}{n^p}\max_{1\leq k\leq n-1}(n-k)k^{p-1}\Big)^{1/p}. \tag{9}$$

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To compute this supremum, let us consider the function $\varphi(x) = (n-x)x^{p-1}$, for $x > 0$. It is easy to see that $x = n/p'$ is the critical point of $\varphi$; that is, $\varphi'(n/p') = 0$.

*(i)* If $n \le p'$, then $\varphi$ is a monotone function on $[1, n-1]$ and the above supremum is attained at the endpoints. This means that

$$\|M_{K_n}\|_{p,\text{rest}} = \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}.$$

*(ii)* If $n \le p$, then $n/p' \ge n-1$ and, as in the previous case, we get that

$$\|M_{K_n}\|_{p,\text{rest}} = \Big(1 + \frac{(n-1)^{p-1}}{n^p}\Big)^{1/p}.$$

*(iii)* If $n > \max\{p, p'\}$, $p \in \mathbb{Q}$, with $p = p_1/p_2$ and $p_1$ divides $n$, then the critical point $n/p'$ is an integer between 1 and $n-1$, so the supremum in (9) is attained at this point. Thus,

$$\|M_{K_n}\|_{p,\text{rest}} = \Big(1 + \frac{(p-1)^{p-1}}{p^p}\Big)^{1/p}.$$

*(iv)* If $n > \max\{p, p'\}$, but $p$ is not of the previous form, then the critical point $n/p' \in [1, n-1]$, but it is not an integer, so the above supremum is

$$\|M_{K_n}\|^p_{p,\text{rest}} = 1 + \frac{1}{n^p} \max\Big\{\Big(n - [n]_p\Big)[n]_p^{p-1}, \Big(n - 1 - [n]_p\Big)\Big([n]_p + 1\Big)^{p-1}\Big\},$$

which corresponds to the evaluation at the closest integer.

The fact that $\|M_{K_n}\|_{p,\text{rest}} > \big(1 + \frac{n-1}{n^p}\big)^{1/p}$, if $n > p'$, is an easy computation. For example, if $p' < n \le p$, then $p > 2$, which is equivalent to the inequality

$$\Big(1 + \frac{(n-1)^{p-1}}{n^p}\Big)^{1/p} > \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}.$$

□

## 3. Optimal estimates for $\|M_G\|_p$

In this Section we are going to prove our main result, namely that if $0 < p \le 1$, the norm of $M_G$ is bounded below and above by some optimal constants, and that equality at the endpoints is only obtained for some specific graphs. Throughout we fix $n \in \mathbb{N}$ and $V = \{1, \ldots, n\}$. Let $S_n$ denote the star graph of $n$ vertices; i.e., a graph with one vertex of degree $n-1$ and $n-1$ leaves (vertices of degree 1). It is clear that, on $V$, there are $n$ different (but isomorphic) $n$-star graphs.

**Theorem 3.1.** *Let $G$ be a graph with $n$ vertices and $0 < p \le 1$. Then, the following optimal estimates hold:*

$$\Big(1 + \frac{n-1}{n^p}\Big)^{1/p} \le \|M_G\|_p \le \Big(1 + \frac{n-1}{2^p}\Big)^{1/p}.$$

*Moreover,*

*(i)* $\|M_G\|_p = \Big(1 + \frac{n-1}{n^p}\Big)^{1/p}$ *if and only if* $G = K_n$;

*(ii)* $\|M_G\|_p = \Big(1 + \frac{n-1}{2^p}\Big)^{1/p}$ *if and only if* $G \sim S_n$.

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*Proof.* This theorem contains several claims. We will prove each of them separately.

*Claim 1: For every graph $G$ we have $\left(1+\frac{n-1}{n^p}\right)^{1/p} \leq \|M_G\|_p \leq \left(1+\frac{n-1}{2^p}\right)^{1/p}$.*

Using (4) we have that $M_{K_n} f \leq M_G f$. Hence, Proposition 2.7 gives us the lower estimate

$$\left(1+\frac{n-1}{n^p}\right)^{1/p} = \|M_{K_n}\|_p \leq \|M_G\|_p.$$

For the upper bound, given $i \in V$ we have that

$$\|M_G\delta_i\|_p^p = M_G\delta_i(i)^p + \sum_{k\in V\setminus\{i\}} M_G\delta_i(k)^p = 1 + \sum_{k\in V\setminus\{i\}} \left(\frac{1}{|B_k|}\sum_{j\in B_k}\delta_i(j)\right)^p,$$

where $B_k$ is a ball in $G$ with center $k$ and a certain radius grater than or equal to 1. Note that for $k \in V\setminus\{i\}$ necessarily $|B_k| \geq 2$. Thus, we get

$$\|M_G f\|_p^p \leq 1 + \frac{n-1}{2^p}.$$

Since this holds for each $i \in V$, by Lemma 2.5 we obtain the upper estimate

$$\|M_G\|_p \leq \left(1+\frac{n-1}{2^p}\right)^{1/p}.$$

*Claim 2: $G = K_n$ if and only if $\|M_G\|_p = \left(1+\frac{n-1}{n^p}\right)^{1/p}$.*

By Proposition 2.7, we have that $\|M_{K_n}\|_p = \left(1+\frac{n-1}{n^p}\right)^{1/p}$ and hence it remains to show that any graph $G$ with $n$ vertices, which is not $K_n$, must necessarily satisfy $\|M_G\|_p > \left(1+\frac{n-1}{n^p}\right)^{1/p}$. To see this, suppose $G \neq K_n$. Then, there exist $i \neq j$ in $V = \{1,\dots,n\}$ such that $d_G(i,j) > 1$. Let us consider the sets

$$A = B(i,1) = \{k \in V : d_G(i,k) \leq 1\} \quad \text{and} \quad B = B(j,1) = \{k \in V : d_G(j,k) \leq 1\}.$$

Clearly $|A|, |B| \geq 2$. We will analyze two cases:

(a) $\min\{|A|,|B|\} \leq n/2$.
(b) $\min\{|A|,|B|\} > n/2$.

In case (a), we may suppose without loss of generality that $|A| \leq n/2$. We pick any $k \in A$ such that $k \neq i$ (i.e., $d_G(i,k) = 1$) and define $\delta_k$ as in (2). Then, since $M_G\delta_k(l) \geq 1/n$, for every $l \in V$,

$$\begin{aligned}\|M_G\delta_k\|_p^p &= \sum_{l=1}^n M_G\delta_k(l)^p \\ &= M_G\delta_k(k)^p + M_G\delta_k(i)^p + \sum_{l\neq i,k} M_G\delta_k(l)^p \\ &\geq 1 + \left(\frac{1}{|A|}\sum_{m\in A}\delta_k(m)\right)^p + \frac{n-2}{n^p}.\end{aligned}$$

Using the hypotheses ($k \in A$ and $|A| \leq n/2$), we now get

$$\|M_G\|_p^p \geq \|M_G\delta_k\|_p^p \geq 1 + \left(\frac{2}{n}\right)^p + \frac{n-2}{n^p} > 1 + \frac{n-1}{n^p}.$$

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This finishes the proof for (a).

We now consider case (b), in which both $A$ and $B$ have cardinality strictly larger than $n/2$. In particular, we have that $A \cap B \neq \emptyset$. If we pick $k \in A \cap B$ and consider the function $\delta_k$ as above, then

$$\begin{aligned}\|M_G\delta_k\|_p^p &= M_G\delta_k(i)^p + M_G\delta_k(j)^p + M_G\delta_k(k)^p + \sum_{l \neq i,j,k} M_G\delta_k(l)^p \\ &\geq \left(\frac{1}{|A|}\sum_{m\in A}\delta_k(m)\right)^p + \left(\frac{1}{|B|}\sum_{m\in B}\delta_k(m)\right)^p + 1 + \frac{n-3}{n^p}.\end{aligned}$$

Hence, using that $k \in A \cap B$ and $|A|, |B| \leq n-1$, we get

$$\|M_G\|_p^p \geq \|M_G\delta_k\|_p^p \geq \frac{2}{(n-1)^p} + 1 + \frac{n-3}{n^p} > 1 + \frac{n-1}{n^p}.$$

This proves the claim.

*Claim 3: $G \sim S_n$ if and only if $\|M_G\|_p = \left(1 + \frac{n-1}{2^p}\right)^{1/p}$.*

We first compute $\|M_{S_n}\|_p$. Let $k \in V$ be the vertex of degree $n-1$ in $S_n$. We have that, for any $f : V \to \mathbb{R}^+$, with $\|f\|_1 = 1$,

$$M_{S_n}f(j) = \begin{cases} \max\left\{f(j), \dfrac{1}{n}\right\}, & \text{if } j = k, \\ \max\left\{f(j), \dfrac{f(j)+f(k)}{2}, \dfrac{1}{n}\right\}, & \text{if } j \neq k. \end{cases} \tag{10}$$

In particular, for $\delta_k$ we get

$$\|M_{S_n}\|_p^p \geq \|M_{S_n}\delta_k\|_p^p = \sum_{j=1}^{n} M_{S_n}\delta_k(j)^p = 1 + \frac{n-1}{2^p}. \tag{11}$$

Since the converse inequality always holds we get that $\|M_{S_n}\|_p = \left(1 + \frac{n-1}{2^p}\right)^{1/p}$.

Now, suppose that $G$ is not isomorphic to $S_n$, and hence $n \geq 3$. Then there exist two different vertices $i, j \in V$ whose degrees satisfy that $d_G(i), d_G(j) > 1$. Note that for every function $f : V \to \mathbb{R}^+$, with $\|f\|_1 \leq 1$, either $M_Gf(k) = f(k)$ or $M_Gf(k) \leq 1/(d_k + 1)$. Given such $f : V \to \mathbb{R}^+$, let

$$A = \{k \in V : M_Gf(k) = f(k)\}.$$

Then we have

$$\|M_Gf\|_p^p = \sum_{k\in A} M_Gf(k)^p + \sum_{k\in V\setminus A} M_Gf(k)^p \leq \sum_{k\in A} f(k)^p + \sum_{k\in V\setminus A}\frac{1}{(d_k+1)^p}.$$

Now, if both $i, j \in A$, then

$$\|M_Gf\|_p^p \leq 1 + \frac{n-2}{2^p} < 1 + \frac{n-1}{2^p}.$$

Otherwise, if $i \notin A$, then since $A \neq \emptyset$, we have

$$\|M_Gf\|_p^p \leq 1 + \frac{1}{(d_i+1)^p} + \frac{n-2}{2^p} \leq 1 + \frac{1}{3^p} + \frac{n-2}{2^p} < 1 + \frac{n-1}{2^p}.$$

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Similarly, the same holds if $j \notin A$. Hence,

$$\|M_G\|_p^p \leq \max\left\{1 + \frac{n-2}{2^p}, 1 + \frac{1}{3^p} + \frac{n-2}{2^p}\right\} < 1 + \frac{n-1}{2^p}.$$

This proves the claim and finishes the proof. □

In view of the optimality of $\|M_{S_n}\|_1$ and Proposition 2.3, it is natural to compare the norms for the corresponding maximal operators for $[L_n]$ and $[S_n]$. The following results show the different behavior of these two graphs:

**Proposition 3.2.** *For $n \geq 3$, we have*

$$\|M_{[L_n]}\|_1 = \|M_{[S_n]}\|_1 = \|M_{S_n}\|_1 = \frac{n+1}{2}.$$

*Proof.* We use Lemma 2.5 to estimate $\|M_{[L_n]}\|_1$. Given $j, k \in V$, $j \neq k$, we take any linear tree $L$ for which $k$ is a leaf and $j$ is a neighbor of $k$, to get that

$$\frac{1}{2} \geq M_{[L_n]}\delta_j(k) \geq M_L\delta_j(k) = \frac{1}{2}.$$

Since $M_{[L_n]}\delta_j(j) = 1$, then $\|M_{[L_n]}\delta_j\|_1 = 1 + \frac{n-1}{2}$, which proves that $\|M_{[L_n]}\|_1 = \frac{n+1}{2}$. Let us now calculate $\|M_{[S_n]}\|_1$: If $f \geq 0$,

$$\begin{aligned} M_{[S_n]}f(j) &= \max\left\{\max_{\{1\leq k\neq j\leq n\}}\left\{f(j), \frac{f(k)+f(j)}{2}, \frac{1}{n}\right\}, \max\left\{f(j), \frac{1}{n}\right\}\right\} \\ &= \max\left\{\frac{f(j)+\|f\|_\infty}{2}, \frac{1}{n}\right\}. \end{aligned}$$

Let $A = \left\{k : \frac{f(k)+\|f\|_\infty}{2} \geq \frac{1}{n}\right\}$. Then

$$\begin{aligned} \|M_{[S_n]}f\|_1 &= \sum_{k\in A}\frac{f(k)+\|f\|_\infty}{2} + \sum_{k\notin A}\frac{1}{n} = \sum_{k\in A}\frac{f(k)}{2} + \frac{\|f\|_\infty}{2}|A| + \frac{1}{n}(n-|A|) \\ &\leq \frac{1}{2} + |A|\Big(\frac{\|f\|_\infty}{2} - \frac{1}{n}\Big) + 1 \leq \frac{1}{2} + n\Big(\frac{1}{2} - \frac{1}{n}\Big) + 1 = \frac{n+1}{2}. \end{aligned}$$

Thus, $\|M_{[S_n]}\|_1 \leq \frac{n+1}{2}$. On the other hand, Theorem 3.1 gives us the converse inequality, since $\|M_{[S_n]}\|_1 \geq \|M_{S_n}\|_1 = \frac{n+1}{2}$. Therefore,

$$\frac{n+1}{2} \geq \|M_{[S_n]}\|_1 \geq \|M_{S_n}\|_1 = \frac{n+1}{2},$$

which finishes the proof. □

**Proposition 3.3.** *For $n \geq 2$ we have*

$$\|M_{L_n}\|_p \approx \begin{cases} \Big(\dfrac{n^{1-p}-1}{1-p}\Big)^{1/p}, & 0 < p < 1, \\ \log n, & p = 1. \end{cases} \tag{12}$$

*Proof.* Let us enumerate $L_n = \{1, 2, \ldots, n\}$, where 1 and $n$ are its leaves. We have

$$M_{L_n}\delta_1(j) = \frac{1}{2j-1}, \quad 1 \leq j \leq [n/2].$$

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Hence,

$$\|M_{L_n}\|_1 \geq \|M_{L_n}\delta_1\|_1 \geq \sum_{j=1}^{[n/2]} \frac{1}{2j-1} \gtrsim \log n.$$

Conversely, since $\|M_{L_n}\delta_k\|_1 = \|M_{L_n}\delta_{n-k+1}\|_1$, then using Lemma 2.5,

$$\begin{aligned}\|M_{L_n}\|_1 &= \max_{1\leq k\leq[n/2]} \|M_{L_n}\delta_k\|_1 \\ &\leq \max_{1\leq k\leq[n/2]} \Bigg( \sum_{j=1}^{[k/2]} \frac{1}{k} + \sum_{j=[k/2]+1}^{[(k+n)/2]} \frac{1}{2|k-j|+1} + \sum_{j=[(k+n)/2]+1}^{n} \frac{1}{n-k+1} \Bigg) \\ &\lesssim \max_{1\leq k\leq[n/2]} (1+\log n) \lesssim \log n.\end{aligned}$$

Similarly, for $0<p<1$ we have

$$\|M_{L_n}\|_p \geq \|M_{L_n}\delta_1\|_p \geq \Big( \sum_{j=1}^{[n/2]} \frac{1}{(2j-1)^p} \Big)^{1/p} \gtrsim \Big( \frac{n^{1-p}-1}{1-p} \Big)^{1/p}.$$

And, as before,

$$\begin{aligned}\|M_{L_n}\|_p &= \max_{1\leq k\leq[n/2]} \|M_{L_n}\delta_k\|_p \\ &\leq \max_{1\leq k\leq[n/2]} \Bigg( \sum_{j=1}^{[k/2]} \frac{1}{k^p} + \sum_{j=[k/2]+1}^{[(k+n)/2]} \frac{1}{(2|k-j|+1)^p} + \sum_{j=[(k+n)/2]+1}^{n} \frac{1}{(n-k+1)^p} \Bigg)^{1/p} \\ &\lesssim \max_{1\leq k\leq[n/2]} \Bigg( \frac{k^{1-p}}{2} + \frac{2\big((k-1)^{1-p}-(k-n-1)^{1-p}\big)}{1-p} + \frac{(n-k+1)^{1-p}}{2} \Bigg)^{1/p} \\ &\lesssim \Big( \frac{n^{1-p}-1}{1-p} \Big)^{1/p}.\end{aligned}$$

□

Note that if $n \geq 4$, then $\|M_{[L_n]}\|_1 > \|M_{L_n}\|_1$. Indeed, by Theorem 3.1 and the fact that $L_n \not\sim S_n$, then we have that

$$\|M_{L_n}\|_1 < \|M_{S_n}\|_1 = \|M_{[L_n]}\|_1.$$

Observe also that in (12), $\lim_{p\to 1^-} \Big(\frac{n^{1-p}-1}{1-p}\Big)^{1/p} = \log n$.

Similar computations yield exactly the same estimate for the cyclic graph $C_n$. Note that this graph could be considered as a discretized version of the one-dimensional torus (although the metric has to be normalized with $n$).

To finish this Section, we complete the information about the strong-type estimates, on the range $1<p<\infty$, for the star graph.

**Proposition 3.4.** *If $1<p<\infty$, then*

$$\Big(1+\frac{n-1}{2^p}\Big)^{1/p} \leq \|M_{S_n}\|_p \leq \Big(\frac{n+5}{2}\Big)^{1/p},$$

*i.e.,* $\|M_{S_n}\|_p \approx n^{1/p}$.

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*Proof.* Using an easy modification of (10), it follows that if $f \geq 0$,

$$\begin{aligned}\|M_{S_n} f\|_p^p &\leq \|f\|_p^p + \frac{n}{n^p}\|f\|_1^p + \sum_{j=1}^n \left(\frac{f(j)+f(1)}{2}\right)^p \\ &\leq \|f\|_p^p + n^{1-p}n^{p-1}\|f\|_p^p + \frac{1}{2^p}\sum_{j=1}^n 2^{p/p'}\Big(f^p(j)+f^p(1)\Big) \\ &\leq 2\|f\|_p^p + \frac{1}{2}\Big(\|f\|_p^p + n\|f\|_p^p\Big) = \frac{n+5}{2}\|f\|_p^p.\end{aligned}$$

Conversely, using (11) we can prove that $\|M_{S_n}\|_p \geq \big(1+\frac{n-1}{2^p}\big)^{1/p}$. □

## 4. Weak-type estimates

Let $G$ be a connected graph with $n$ vertices, $V = \{1,\dots,n\}$, $f : V \to \mathbb{R}^+$, and let $\{f_j^*\}_{j=1,\dots,n}$ be the decreasing rearrangement of the sequence $\{f_j\}_{j=1,\dots,n}$. We now consider weak-type estimates of the form $M_G : \ell^p(V) \to \ell^{p,\infty}(V)$, $0 < p < \infty$, where

$$\|f\|_{p,\infty} := \sup_{t>0} t\Big|\{j \in V : f_j > t\}\Big|^{1/p}.$$

It is easily seen that also $\|f\|_{p,\infty} = \max_{j\in V} j^{1/p} f_j^*$. For this purpose we define

$$\|M_G\|_{p,\infty} = \sup_f \frac{\|M_G f\|_{p,\infty}}{\|f\|_p}.$$

It is clear that if $0 < p < \infty$, then $\|M_G\|_{p,\infty} \leq \|M_G\|_p$ and also

$$\|M_G\|_{p,\infty} \leq n^{1/p}, \quad \text{if } |G| = n. \tag{13}$$

**Theorem 4.1.** *If $0 < p < \infty$, then*

$$\|M_{K_n}\|_{p,\infty} = \begin{cases} n^{1/p-1}, & \text{if } 0 < p \leq 1,\\ 1, & \text{if } p \geq 1.\end{cases} \tag{14}$$

*In particular, for every connected graph $G$ with $n$ vertices,*

$$\|M_G\|_{p,\infty} \geq \begin{cases} n^{1/p-1}, & \text{if } 0 < p \leq 1,\\ 1, & \text{if } p \geq 1.\end{cases}$$

*Proof.* Let $f : V \to \mathbb{R}^+$, with $\|f\|_p = 1$ (we may assume that $f$ is not a constant function), and let $A(f) = \|f\|_1/n$. Since $M_{K_n} f(j) = \max\{f_j, A(f)\}$, if we define

$$j(f) = \min\{1 \leq j \leq n-1 : f_{j+1}^* < A(f) \leq f_j^*\},$$

then

$$\big(M_{K_n} f\big)^*(j) = \begin{cases} f_j^*, & \text{if } 1 \leq j \leq j(f),\\ A(f), & \text{if } j(f) < j \leq n,\end{cases}$$

and

$$\begin{aligned}\|M_{K_n} f\|_{p,\infty} &= \max\Big\{\max_{1\leq j\leq j(f)} j^{1/p} f_j^*, \max_{j(f)<j\leq n} j^{1/p} A(f)\Big\} \\ &= \max\Big\{\max_{1\leq j\leq j(f)} j^{1/p} f_j^*, n^{1/p-1}\|f\|_1\Big\}. \qquad (15)\end{aligned}$$

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If we now take $f = \delta_1$, then $j(f) = 1$ and

$$\|M_{K_n}\delta_1\|_{p,\infty} = \max\left\{1, n^{1/p-1}\right\} = \begin{cases} n^{1/p-1}, & \text{if } 0 < p \le 1, \\ 1, & \text{if } p \ge 1, \end{cases}$$

and, hence

$$\|M_{K_n}\|_{p,\infty} \ge \begin{cases} n^{1/p-1}, & \text{if } 0 < p \le 1, \\ 1, & \text{if } p \ge 1. \end{cases}$$

Conversely, if $0 < p \le 1$ let us see that $f_j^* \le 1/j$, for every $1 \le j \le n$. In fact, if there exists $1 \le j_0 \le n$ for which $f_{j_0}^* > 1/j_0$, then

$$1 \ge \sum_{j=1}^{j_0} (f_j^*)^p > \sum_{j=1}^{j_0} \frac{1}{{j_0}^p} = j_0^{1-p} \ge 1,$$

which is a contradiction. Thus, $j^{1/p} f_j^* \le j^{1/p-1} \le n^{1/p-1}$, and hence $\|M_{K_n}\|_{p,\infty} \le n^{1/p-1}$.

Finally, if $1 < p < \infty$, we have that $j^{1/p} f_j^* \le \|f\|_{p,\infty} \le \|f\|_p = 1$ and also, using Hölder's inequality, $n^{1/p-1}\|f\|_1 \le n^{1/p-1}\|f\|_p n^{1/p'} = \|f\|_p = 1$, and the result follows from (15).

The last part is a consequence of the trivial estimate $M_{K_n} f(j) \le M_G f(j)$, for every $j \in V$, and (14). □

**Remark 4.2.** The fact that $\|M_{K_n}\|_{1,\infty} = 1$ also follows from the general theory for ultrametric spaces. Recall that an ultrametric space is a metric space with the stronger inequality

$$d(x, y) \le \max\{d(x, z), d(z, y)\}$$

instead of the triangle inequality. It is clear that $K_n$ is an ultrametric space. In fact, it is the only graph with this property: Indeed, if $G \ne K_n$, there exist two vertices $x, y$ with $d_G(x, y) = r \ge 2$. Pick a geodesic path joining $x$ and $y$, and let $z$ be a neighbor of $x$ in that path. It follows that $d_G(x, z) = 1$, $d_G(z, y) = r - 1$, and so

$$d_G(x, y) = r > \max\{d(x, z), d(z, y)\}.$$

**Proposition 4.3.** *Let $0 < p < \infty$. Then,*

$$\max\{n^{1/p}/2, 1\} \le \|M_{S_n}\|_{p,\infty} \le n^{1/p}. \tag{16}$$

*In particular, $\|M_{S_n}\|_{p,\infty} \approx n^{1/p}$, for every $n \ge 1$ and $0 < p < \infty$, and also, for every connected graph $G$ with $n$ vertices, $\|M_G\|_{p,\infty} \le 2\|M_{S_n}\|_{p,\infty}$, $0 < p < \infty$.*

*Proof.* Assuming that $j = 1$ is the vertex of degree $n - 1$ in $S_n$, and taking $f = \delta_1$, using (10) we get that

$$M_{S_n} f(j) = \begin{cases} 1, & \text{if } j = 1, \\ 1/2, & \text{if } 2 \le j \le n, \end{cases}$$

and hence,

$$\|M_{S_n} f\|_{p,\infty} = \max\left\{1, \max\{j^{1/p}/2 : j = 2, \dots, n\}\right\} = \begin{cases} 1, & \text{if } n < 2^p, \\ n^{1/p}/2, & \text{if } n \ge 2^p, \end{cases}$$

which, together with the trivial inequality (13), proves (16). To finish, both estimates $\|M_{S_n}\|_{p,\infty} \approx n^{1/p}$ and $\|M_G\|_{p,\infty} \le 2\|M_{S_n}\|_{p,\infty}$ are just a simple remark. □

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The case $p = 1$ in Proposition 4.3 was previously studied in [14, Proposition 1.5, Remark 1.2].

Motivated by the classical weak-type $(1,1)$ bounds for the Hardy-Littlewood operator on $\mathbb{R}^n$ we will introduce two indices associated to a graph $G$: the dilation and overlapping indices. The dilation index of a graph is related to the so called doubling condition, and measures the growth of the number of vertices in a ball when its radius is tripled.

**Definition 4.4.** *Given a graph $G$ we define its dilation index as*

$$\mathcal{D}(G) = \max\left\{\frac{|B(x,3r)|}{|B(x,r)|} : x \in V,\, r \in \mathbb{N},\, r \leq \operatorname{diam}(G)\right\}.$$

**Example 4.5.** The dilation index of the complete graph of $n$ vertices and the star $S_n$ can be easily computed for $n \in \mathbb{N}$:

$$\mathcal{D}(K_n) = 1 \qquad \text{and} \qquad \mathcal{D}(S_n) = \frac{n}{2}.$$

For the linear tree $L_n$ it is easy to check that $\mathcal{D}(L_n) < 3$ for all $n \in \mathbb{N}$, and that $\lim_{n\to\infty} \mathcal{D}(L_n) = 3$. For small number of vertices we have: $\mathcal{D}(L_3) = 3/2$, $\mathcal{D}(L_4) = 2$, $\mathcal{D}(L_5) = 2$, $\mathcal{D}(L_6) = 2$, $\mathcal{D}(L_7) = 7/3\ldots$

The dilation index can be used to give an elementary version of the Vitali covering lemma [9]:

**Lemma 4.6.** *Let $G$ be a graph with $n$ vertices and $A \subset V$ any set of vertices. If $\{B_j\}_{j\in J}$ is a finite collection of balls covering $A$, then there exists $I \subset J$ such that $B_i \cap B_k = \emptyset$, for $i,k \in I$, and*

$$|A| \leq \mathcal{D}(G) \sum_{i\in I} |B_i|. \tag{17}$$

*Proof.* Let $B_{i_1}$ be a ball in $\{B_j\}_{j\in J}$ with the largest radius; let $B_{i_2}$ be a ball in $\{B_j\}_{j\in J\setminus\{i_1\}}$, with the largest radius among those which are disjoint from $B_{i_1}$; let $B_{i_3}$ be a ball in $\{B_j\}_{j\in J\setminus\{i_1,i_2\}}$, with the largest radius among those which are disjoint from $B_{i_1}$ and $B_{i_2}$, and so on. Let $k$ be the index where this process stops, and set $I = \{i_1,\ldots,i_k\}$.

That $\{B_i\}_{i\in I}$ are pairwise disjoint is trivial by construction. To prove (17), given a ball $B_i = B(x_i, r_i)$ let us consider $\widetilde{B}_i = B(x_i, 3r_i)$. We claim that $A \subset \bigcup_{i\in I} \widetilde{B}_i$. Indeed, otherwise there is a vertex $v \in A \setminus \bigcup_{i\in I} \widetilde{B}_i$, and since $A \subset \bigcup_{j\in J} B_j$ we have that $v \in B_{j_0} = B_{j_0}(x_{j_0}, r_{j_0})$, for some $j_0 \in J\setminus I$. Since the ball $B_{j_0}$ has not been chosen, there exists $i \in I$ such that $B_{j_0} \cap B_i \neq \emptyset$ and $r_i \geq r_{j_0}$. Finally, if we take $u \in B_{j_0} \cap B_i$, then

$$d_G(v, x_i) \leq d_G(v, x_{j_0}) + d_G(x_{j_0}, u) + d_G(u, x_i) \leq r_{j_0} + r_{j_0} + r_i \leq 3r_i,$$

and hence $v \in \widetilde{B}_i$, which is a contradiction.

Therefore, $A \subset \bigcup_{i\in I} \widetilde{B}_i$ and we have

$$|A| \leq \sum_{i\in I} |\widetilde{B}_i| \leq \mathcal{D}(G) \sum_{i\in I} |B_i|.$$

$\square$

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Another useful quantity for weak-type $(1,1)$ estimates of the maximal operator is the overlapping index of a graph, which represents the smallest number of balls that necessarily overlap in a covering of the graph:

**Definition 4.7.** *Given a graph $G$ we define its overlapping index as*

$$\mathcal{O}(G) = \min\Big\{ r \in \mathbb{N} : \forall \{B_j\}_{j\in J},\ B_j \text{ a ball in } G,\ \exists I \subset J,$$
$$\bigcup_{j\in J} B_j = \bigcup_{i\in I} B_i \text{ and } \sum_{i\in I} \chi_{B_i} \le r \Big\}.$$

**Example 4.8.** The overlapping index of the following families of graphs can be computed easily:

$$\mathcal{O}(K_n) = 1,\ \forall n \in \mathbb{N}; \qquad \mathcal{O}(S_n) = n-1,\ \forall n \ge 2;$$
$$\mathcal{O}(L_n) = \begin{cases} 1 & n \le 2, \\ 2 & n \ge 3; \end{cases} \qquad \mathcal{O}(C_n) = \begin{cases} 1 & n \le 3, \\ 2 & n \ge 4. \end{cases}$$

The dilation and overlapping indices provide an upper bound for the weak-type $(1,1)$ norm of the maximal operator of a graph:

**Theorem 4.9.** *Given a graph $G$, we have*

$$\|M_G\|_{1,\infty} \le \min\big\{\mathcal{D}(G), \mathcal{O}(G)\big\}.$$

*Proof.* The proof follows the same kind of arguments used for estimating in $\mathbb{R}^n$ the weak-type boundedness of the classical centered Hardy-Littlewood maximal operator $M$. Given $f : V \to \mathbb{R}$ and $t > 0$, let

$$A_t = \{1 \le j \le n : M_G f(j) > t\}.$$

For each $j \in A_t$, take a ball $B_j \subset G$ centered at $j$, satisfying that

$$\sum_{k\in B_j} |f(k)| > t|B_j|.$$

On the one hand, by Lemma 4.6, there exists $I \subset A_t$ such that $(B_i)_{i\in I}$ are pairwise disjoint and

$$|A_t| \le \mathcal{D}(G) \sum_{i\in I} |B_i|.$$

Therefore, we get

$$|A_t| \le \mathcal{D}(G) \sum_{i\in I} |B_i| \le \mathcal{D}(G) \sum_{i\in I} \frac{1}{t} \sum_{k\in B_j} |f(k)| \le \frac{\mathcal{D}(G)}{t} \|f\|_1.$$

Thus, we have $\|M_G\|_{1,\infty} \le \mathcal{D}(G)$.

On the other hand, using the definition of the overlapping index, we can also select $L \subset A_t$ such that $A_t \subset \cup_{l\in L} B_l$ and $\sum_{l\in L} \chi_{B_l}(k) \le \mathcal{O}(G)$, $k = 1, \dots, n$. Hence,

$$|A_t| \le \sum_{l\in L} |B_l| < \frac{1}{t} \sum_{l\in L} \sum_{k\in B_l} |f(k)| \le \frac{\mathcal{O}(G)}{t} \|f\|_1,$$

which shows that $\|M_G\|_{1,\infty} \le \mathcal{O}(G)$. □

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To illustrate these results, we will consider now the particular case of the linear tree $L_n$. We will see that for this graph, the behavior of the maximal operator is similar to what happens in the euclidean setting $\mathbb{R}$. First we prove an interpolation result in $L^{p,\infty}(\mu)$, for a general measure $\mu$.

**Lemma 4.10.** *If $T$ is a sublinear operator, of weak-type (1,1) with constant $C_1$ and bounded in $L^\infty$ with constant $C_\infty$, then $T : L^p \to L^{p,\infty}$ is bounded, for $1 < p < \infty$, and*

$$\|Tf\|_{p,\infty} \le (p')^{1/p'} p^{1/p} C_1^{1/p} C_\infty^{1/p'} \|f\|_p.$$

*Proof.* Fix $t > 0$, $0 < \lambda < 1$, and set $r = (1-\lambda)t/C_\infty$. For $f \in L^p$, write $f = f\chi_{\{|f|>r\}} + f\chi_{\{|f|\le r\}} = f_1 + f_2$. Then,

$$\begin{aligned}
\mu(\{|Tf| > t\}) &\le \mu(\{|Tf_1| > \lambda t\}) + \mu(\{|Tf_2| > (1-\lambda)t\}) \\
&\le \frac{C_1}{\lambda t}\|f_1\|_1 + \mu(\{C_\infty \|f_2\|_\infty > (1-\lambda)t\}) \\
&= \frac{C_1}{\lambda t}\int_{\{|f|>(1-\lambda)t/C_\infty\}} |f|\,d\mu \\
&\le \frac{C_1}{\lambda t}\left(\frac{(1-\lambda)t}{C_\infty}\right)^{1-p} \int_{\{|f|>(1-\lambda)t/C_\infty\}} |f|^p\,d\mu \\
&\le \frac{C_1}{C_\infty^{1-p}} \frac{(1-\lambda)^{1-p}}{\lambda} t^{-p}\|f\|_p^p.
\end{aligned}$$

Hence,

$$\|Tf\|_{p,\infty} \le \inf_{0<\lambda<1} \frac{1}{(1-\lambda)^{1/p'}\lambda^{1/p}} C_1^{1/p} C_\infty^{1/p'} \|f\|_p = (p')^{1/p'} p^{1/p} C_1^{1/p} C_\infty^{1/p'} \|f\|_p.$$

□

**Proposition 4.11.** *If $1 \le p < \infty$, then*

$$1 \le \|M_{L_n}\|_{p,\infty} \le (p')^{1/p'}(2p)^{1/p} \le 3.$$

*Proof.* Since $\mathcal{O}(L_n) \le 2$, by Theorem 4.9 we have that $\|M_{L_n}\|_{1,\infty} \le 2$. Since $\|M_{L_n}\|_\infty = 1$ and using Lemma 4.10, we finally obtain the result (notice that $(p')^{1/p'}(2p)^{1/p} \le 3$ is a trivial estimate). The fact that $\|M_{L_n}\|_{p,\infty} \ge 1$ follows easily since $(M_{L_n}\delta_1)^*(1) = 1$ and hence $\|M_{L_n}\delta_1\|_{p,\infty} \ge 1$. □

**Proposition 4.12.** *If $\{G_n\}_{n\in\mathbb{N}}$ is a family of graphs such that $G_n$ has $n$ vertices and $\|M_{G_n}\|_{1,\infty} \approx 1$, uniformly in $n$, then, for every $0 < p < 1$ and $n \in \mathbb{N}$, $\|M_{G_n}\|_{p,\infty} \approx n^{1/p-1}$, uniformly in $n$ and $p$. In particular $\|M_{L_n}\|_{p,\infty} \approx n^{1/p-1}$, $0 < p < 1$.*

*Proof.* The inequality $\|M_{G_n}\|_{p,\infty} \ge n^{1/p-1}$ follows, as before, from Theorem 4.1. Now, if $0 < p < 1$ and $\|f\|_p = 1$, then $\|f\|_1 \le 1$ and

$$\begin{aligned}
j^{1/p}(M_{G_n}f)^*(j) &= j^{1/p-1} j (M_{G_n}f)^*(j) \le j^{1/p-1}\|M_{G_n}f\|_{1,\infty} \\
&\lesssim j^{1/p-1}\|f\|_1 \le n^{1/p-1}.
\end{aligned}$$

Thus, $\|M_{G_n}\|_{p,\infty} \lesssim n^{1/p-1}$. □

**Proposition 4.13.** *For the linear graph $L_n$, we have that $\lim_{n\to\infty} \|M_{L_n}\|_{1,\infty} = 2$.*

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*Proof.* As we have already seen, we have that $\|M_{L_n}\|_{1,\infty} \leq 2$. For the converse inequality, let us assume for simplicity that $n = 2k+1$. Then:

$$M_{L_n}\delta_k(j) = \begin{cases} \dfrac{1}{k}, & j \leq \left[\dfrac{k}{2}\right] \text{ or } j > k + \left[\dfrac{k+1}{2}\right], \\ \dfrac{1}{2|k-j|+1}, & \left[\dfrac{k}{2}\right] < j \leq k + \left[\dfrac{k+1}{2}\right]. \end{cases}$$

Thus, we have

$$\|M_{L_n}\|_{1,\infty} \geq \|M_{L_n}\delta_k\|_{1,\infty} \geq \frac{n}{k} = \frac{2n}{n-1},$$

which tends to 2, as $n \to \infty$. □

**Remark 4.14.** Observe that $\lim_{n\to\infty} \|M_{L_n}\|_{1,\infty} = 2 = \|\mathcal{M}\|_{1,\infty}$, where $\mathcal{M}$ is the non-centered maximal function in $\mathbb{R}$. This should be compared to the discretization result proved in [12, Theorem 3], namely if $M$ is centered Hardy-Littlewood maximal operator in $\mathbb{R}$ and we consider the discrete measures

$$\mathcal{D} = \left\{\mu = \sum_{k=1}^{N} \delta_{a_k} : a_k \in \mathbb{R},\ a_{k+1} = a_k + H,\ H \text{ fixed},\ N \in \mathbb{N}\right\},$$

then

$$\sup_{\mu\in\mathcal{D}} \frac{\|M\mu\|_{1,\infty}}{\|\mu\|} = \frac{3}{2}.$$

**Remark 4.15.** The estimates given in Propositions 4.11 and 4.13 for $M_{L_n}$ also hold for the Hardy-Littlewood maximal operator of the cyclic graph $C_n$, with similar proofs.

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