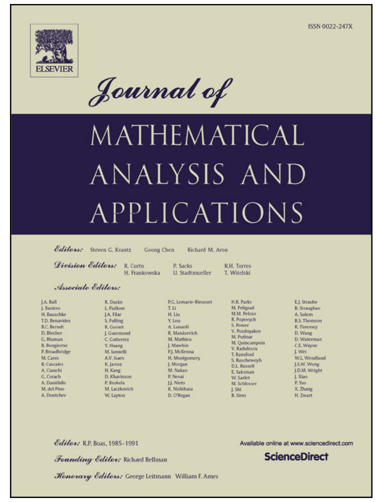
Accepted Manuscript



Decay of the solution to a reduced gravity two and a half layer equations

Ruiying Wei, Yin Li, Zheng-an Yao

PII: S0022-247X(16)00128-1
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.01.081
Reference: YJMAA 20177

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 27 April 2015

Decay of the solution to a reduced gravity two and a half layer equations

Ruiying Wei[1,2], Yin Li[1,2], Zheng-an Yao[2]

1. School of Mathematics and Statistics, Shaoguan University,

512005, Shaoguan, P. R. China.

2. School of Mathematics and Computational Science, Sun Yat-sen University,

510275, Guangzhou, P. R. China

Abstract

In this paper, we study a reduced gravity two and a half layer model in three-dimensional whole space $\mathbb{R}^3$. Under assumption of small initial data, we establish the unique global solution by energy method, furthermore, we obtain the time decay rates of the higher-order spatial derivatives of the solution by the method of spectral analysis and energy estimates if the initial data belongs to $L^1(\mathbb{R}^3)$ additionally.

2000 Mathematics Subject Classification: 35Q30, 76N15, 76P05, 82C40.

Keywords: Reduced gravity model; Decay rates; Fourier splitting method.

1 Introduction

In this paper, we consider the reduced gravity two and a half layer model in the following form:

$$\begin{cases} \partial_t h_1 + div(h_1 u_1) = 0, \\ \partial_t(h_1 u_1) + div(h_1 u_1 \otimes u_1) + (g_1 + g_2) h_1 \nabla h_1 + g_2 h_1 \nabla h_2 = 2\nu_1 div(h_1 \mathbf{D} u_1), \\ \partial_t h_2 + div(h_2 u_2) = 0, \\ \partial_t(h_2 u_2) + div(h_2 u_2 \otimes u_2) + g_2 h_2 \nabla h_1 + g_2 h_2 \nabla h_2 = 2\nu_2 div(h_2 \mathbf{D} u_2), \\ h_1(x,t) \to \tilde{h}_1 > 0, h_2(x,t) \to \tilde{h}_2 > 0 \text{ as } \ |x| \to \infty, \end{cases} \tag{1.1}$$

[1]Corresponding author. Email: *liyin2009521@163.com*

[1]Email: *weiruiying521@163.com, liyin2009521@163.com, mcsyao@mail.sysu.edu.cn*

where $h_i(x,t), u_i(x,t)$ and $\nu_i(i=1,2)$ are the upper and middle layer thickness, horizontal velocity, and the coefficient of lateral friction, $\mathbf{D}u_i := \frac{\nabla u_i + (\nabla u_i)'}{2}(i=1,2)$ denote the stress tensor in the upper and middle layers, respectively. g_1 and g_2 are positive constants, $\tilde{h}_1, \tilde{h}_2$denote the background doping profile which are known positive constants. Without loss of generality, we set $\nu_1 = \nu_2 = \tilde{h}_1 = \tilde{h}_2 = 1$ in the sequel.

We point out here that if we take $h_1 \equiv 0$ or $h_2 \equiv 0$, then the model (1.1) is the following viscous compressible Navier-Stokes equations, the viscosity coefficient is taken as νh, and the pressure is $P(h) = g'h^2$,

$$\begin{cases} \partial_t h + div(hu) = 0, \\ \partial_t(hu) + div(hu \otimes u) + g'h\nabla h = 2\nu div(h\mathbf{D}u). \end{cases}$$

As stated in [2], (1.1) is a useful model of the stratified upper ocean overlying a nearly stationary and nearly unstratified abyss. The simplest way to simulate the ocean circulation is to assume the ocean is homogeneous in density, and this model has no vertical structure. For some reasons, here the stratification in the ocean is simplified as a two-layer fluid. Fluid below the main thermocline moves much slower than that above the main thermocline. As a good approximate, one can assume that the fluid in the lower layer is near stagnant. For more information about this model, see, for instance, [2], [9], and references cited therein. In the present paper, we mainly consider the global existence and large time behavior of global classical solution to the Cauchy problem of the reduced gravity two and a half layer model (1.1) with the following initial condition:

$$(h_1, u_1, h_2, u_2)|_{t=0} = (h_{10}, u_{10}, h_{20}, u_{20}), \quad x \in \mathbb{R}^3. \tag{1.2}$$

The the existence and optimal time-decay rates of the solutions have been the important problem in the PDE theory. For the compressible Navier-Stokes system, there has been much important progress on classical solutions; refer to [1],[11], [12], [13], [14], [15], [16], [18], [19], [21] and references therein. For the compressible Nematic Liquid Crystal Flows, there has been some progress on long-time behavior of solution; refer to [22],[23] and references therein. Considering the reduced gravity two and a half layer model (1.1), Duan and Zhou [2] obtained the stability of weak solutions in two dimension periodic domain $\mathbb{T}^3$. Guo, Li and Yao [4] investigated the existence of global weak solution in one-dimensional bounded spatial domain or periodic domain. Recently, Cui, Yao and Yao [3] proved the global existence by classical methods and got the optimal time decay rates of global smooth solutions by the detailed analysis of the Green's function to the linearized system and elaborate energy estimates to the nonlinear system in three-dimensional under the condition that the $H^N \cap L^1 (N > 4)$ norm of the initial data is small.

$$\|(h_1 - 1, u_1, h_2 - 1, u_2)(t)\|_{H^N} \le C(1+t)^{-\frac{3}{4}}. \tag{1.3}$$

However, to our knowledge there are few results about the large time behavior of the reduced gravity two and a half layer model up to now.

In this paper, we first establish the global solution by the energy method following the idea of Guo, Wang [14] and Wang [20] under the assumption that the H^3 norm of the initial date is small, but the higher order derivatives can be arbitrarily large. Then we establish the time decay rates by the method of spectral analysis and energy estimates by assuming that the initial data belongs to $L^1(\mathbb{R}^3)$ additionally.

Notation Throughout this paper, ∇^l with an integer $l \geq 0$ stands for the usual any spatial derivatives of order l. When $l < 0$ or l is not a positive integer, ∇^l stands for Λ^l defined by $\Lambda^s u = \mathscr{F}^{-1}(|\xi|^s \hat{u}(\xi))$, where $\hat{u}$ is the Fourier transform of u and $\mathscr{F}^{-1}$ its inverse. We will employ the notation $A \lesssim B$ to mean that $A \leq CB$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem. For the sake of conciseness, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$.

In this subsection, we first reformulate the system (1.1). We denote $n_1 = h_1 - 1, n_2 = h_2 - 1$, then the system (1.1) can be rewritten as

$$\begin{cases} \partial_t n_1 + divu_1 = f_1, \\ \partial_t u_1 - \triangle u_1 - \nabla divu_1 + (g_1 + g_2)\nabla n_1 = f_2, \\ \partial_t n_2 + divu_2 = f_3, \\ \partial_t u_2 - \triangle u_2 - \nabla divu_2 + g_2 \nabla n_2 = f_4, \end{cases} \tag{1.4}$$

with

$$(n_1, u_1, n_2, u_2)|_{t=0} = (h_{10} - 1, u_{10}, h_{20} - 1, u_{20}) =: (n_{10}, u_{10}, n_{20}, u_{20}), \quad x \in \mathbb{R}^3. \tag{1.5}$$

where

$$\begin{aligned} f_1 &:= -div(n_1 u_1), \\ f_2 &:= -u_1 \cdot \nabla u_1 + \frac{1}{n_1 + 1} div[(n_1 + 1)\nabla u_1] - \triangle u_1 + \frac{1}{n_1 + 1} div[(n_1 + 1)(\nabla u_1)'] - \nabla divu_1 - g_2 \nabla n_2, \\ f_3 &:= -div(n_2 u_2), \\ f_4 &:= -u_2 \cdot \nabla u_2 + \frac{1}{n_2 + 1} div[(n_2 + 1)\nabla u_2] - \triangle u_2 + \frac{1}{n_2 + 1} div[(n_2 + 1)(\nabla u_2)'] - \nabla divu_2 - g_2 \nabla n_1. \end{aligned}$$

Our main results are stated in the following theorem.

Theorem 1.1. *Let $N \geq 3$, assume that $(n_{10}, u_{10}, n_{20}, u_{20}) \in H^N(\mathbb{R}^3)$. Then there exists a constant $\delta_0 > 0$ such that if*

$$\|(n_{10}, u_{10}, n_{20}, u_{20})\|_{H^3} \leq \delta_0, \tag{1.6}$$

then the problem (1.1) has a unique global solution satisfying that for all $t \geq 0$,

$$\|(n_1, u_1, n_2, u_2)(t)\|_{H^N}^2 + C\int_0^t (\|\nabla(n_1, n_2)(\tau)\|_{H^{N-1}}^2 + \|\nabla(u_1, u_2)(\tau)\|_{H^N}^2) d\tau \tag{1.7}$$

$$\leq C\|(n_{10},u_{10},n_{20},u_{20})\|_{H^N}^2.$$

If further, $(n_{10},u_{10},n_{20},u_{20})\in L^1(\mathbb{R}^3)$, *then*

$$\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{H^{N-l}}\leq C(1+t)^{-\frac{3+2l}{4}},\quad for\ \ l=0,1,\cdots,N-1. \tag{1.8}$$

and for $2\leq p\leq\infty$, *there holds*

$$\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{L^p}\leq C(1+t)^{-\frac{l}{2}-\frac{3}{2}(1-\frac{1}{p})}, \tag{1.9}$$

especially,

$$\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{L^\infty}\leq C(1+t)^{-\frac{3+l}{2}}. \tag{1.10}$$

2 Energy estimates

In this subsection, we will derive the a priori nonlinear energy estimates for the system (1.7). Hence we assume a priori that for sufficiently small $\delta>0$,

$$\sqrt{\mathcal{E}_0^3(t)}=\|(n_1,u_1,n_2,u_2)(t)\|_{H^3}\leq\delta. \tag{2.1}$$

First of all, by (2.1) and Sobolev's inequality, we obtain

$$\frac{1}{2}\leq n_1+1,n_2+1\leq\frac{3}{2}. \tag{2.2}$$

Now, we list some inequalities, which will be used in the later.

Lemma 2.1. *Let* $0\leq m,\alpha\leq l$, *then we have*

$$\|\nabla^\alpha f\|_{L^p}\lesssim\|\nabla^m f\|_{L^q}^{1-\theta}\|\nabla^l f\|_{L^r}^\theta. \tag{2.3}$$

where $0\leq\theta\leq1$ *and* α *satisfies* $\frac{\alpha}{3}-\frac{1}{p}=(\frac{m}{3}-\frac{1}{q})(1-\theta)+(\frac{l}{3}-\frac{1}{r})\theta$.

Here when $p=\infty$ we require that $0<\theta<1$.

Proof. This can be found in [[8], p. 125, Theorem]. □

Lemma 2.2. *Assume that* $\|n_j\|_{L^\infty}\leq1$ *and* $p>1$. *Let* $g(n_j)$ *be a smooth function of* n_j *with bounded derivatives of any order, then for any integer* $m\geq1$, *we have*

$$\|\nabla^m g(n_j)\|_{L^p}\lesssim\|\nabla^m n_j\|_{L^p},\quad j=1,2. \tag{2.4}$$

Proof. The proof is similar to the proof of Lemma A.2 in [20] and is omitted here. □

Lemma 2.3. *Let $m \geq 1$ be an integer and define the commutator*

$$[\nabla^m, f]g = \nabla^m(fg) - f\nabla^m g.$$

Then we have

$$\|[\nabla^m, f]g\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}}\|\nabla^{m-1}g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}}\|g\|_{L^{p_4}}. \tag{2.5}$$

and for $m \geq 0$

$$\|\nabla^m(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}}\|\nabla^m g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}}\|g\|_{L^{p_4}}. \tag{2.6}$$

where $p, p_2, p_3 \in (1, \infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Proof. For $p = p_2 = p_3 = 2$, it can be proved by using Lemma 2.1. For the general cases, one may refer to [[7], Lemma 3.1]. □

Next, we will establish the global existence of solution for the reduced gravity two and a half layer model. For this purpose, we begin with the energy estimates including n_1, u_1, n_2, u_2 themselves.

Lemma 2.4. *If $\sqrt{\mathcal{E}_0^3(t)} \leq \delta$, then for $k = 0, \cdots, N-1$, we have*

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla^k(n_1, u_1, n_2, u_2)\|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla^k n_1 \cdot \nabla^k n_2 dx) + C\|\nabla^{k+1}(u_1, u_2)\|_{L^2}^2 \\
&\lesssim \delta\|\nabla^{k+1}(n_1, u_1, n_2, u_2)\|_{L^2}^2.
\end{aligned} \tag{2.7}$$

Proof. Applying ∇^k to (1.4), and multiplying the resulting identities by $(g_1+g_2)\nabla^k n_1, \nabla^k u_1, g_2\nabla^k n_2, \nabla^k u_2$ respectively, summing up them and then integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[(g_1+g_2)\|\nabla^k n_1\|_{L^2}^2 + \|\nabla^k u_1\|_{L^2}^2 + g_2\|\nabla^k n_2\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2] + \|\nabla^{k+1}(u_1, u_2)\|_{L^2}^2 + \|\nabla^k div(u_1, u_2)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} (g_1+g_2)\nabla^k f_1 \cdot \nabla^k n_1 + \nabla^k f_2 \cdot \nabla^k u_1 + g_2\nabla^k f_3 \cdot \nabla^k n_2 + \nabla^k f_4 \cdot \nabla^k u_2 dx \\
&:= \sum_{i=1}^4 I_i.
\end{aligned} \tag{2.8}$$

We shall estimate the terms in the right hand side of (2.8). First, for the term I_1, employing Hölder's inequality and Lemma 2.3, we obtain

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3} (g_1+g_2)\nabla^k div(n_1 u_1) \cdot \nabla^k n_1 dx \\
&\lesssim (\|n_1\|_{L^3}\|\nabla^{k+1}u_1\|_{L^2} + \|\nabla^{k+1}n_1\|_{L^2}\|u_1\|_{L^3})\|\nabla^k n_1\|_{L^6} \\
&\lesssim \delta(\|\nabla^{k+1}n_1\|_{L^2}^2 + \|\nabla^{k+1}u_1\|_{L^2}^2).
\end{aligned} \tag{2.9}$$

Similarly, we can bound

$$I_3 \lesssim \delta(\|\nabla^{k+1}n_2\|_{L^2}^2 + \|\nabla^{k+1}u_2\|_{L^2}^2). \tag{2.10}$$

For the second and fourth terms, we have

$$\begin{aligned} I_2 + I_4 = &-g_2 \int_{\mathbb{R}^3} \nabla^k \nabla n_2 \cdot \nabla^k u_1 + \nabla^k \nabla n_1 \cdot \nabla^k u_2 dx + \sum_{j=1}^{2} \int_{\mathbb{R}^3} \nabla^k(-u_j \cdot \nabla u_j) \cdot \nabla^k u_j \\ &+ \nabla^k(\frac{1}{n_j+1}\partial_l n_j \partial_l u_j^i) \cdot \nabla^k u_j^i + \nabla^k(\frac{1}{n_j+1}\partial_l n_j \partial_i u_j^l) \cdot \nabla^k u_j^i dx. \end{aligned} \tag{2.11}$$

By $(1.4)_{1,3}$ and integrating by parts, we deduce from (2.9), (2.10) that

$$\begin{aligned} &- \int_{\mathbb{R}^3} \nabla^k \nabla n_2 \cdot \nabla^k u_1 + \nabla^k \nabla n_1 \cdot \nabla^k u_2 dx \\ =& \int_{\mathbb{R}^3} \nabla^k n_2 \cdot \nabla^k divu_1 + \nabla^k n_1 \cdot \nabla^k divu_2 dx \\ =& \int_{\mathbb{R}^3} \nabla^k n_2 \cdot \nabla^k(f_1 - \partial_t n_1) + \nabla^k n_1 \cdot \nabla^k(f_3 - \partial_t n_2) dx \\ =& \int_{\mathbb{R}^3} \nabla^k n_2 \cdot \nabla^k f_1 + \nabla^k n_1 \cdot \nabla^k f_3 dx - \frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k n_1 \cdot \nabla^k n_2 dx \\ \lesssim& \delta(\|\nabla^{k+1}n_2\|_{L^2}^2 + \|\nabla^{k+1}u_2\|_{L^2}^2) - \frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k n_1 \cdot \nabla^k n_2 dx. \end{aligned} \tag{2.12}$$

Now, we estimate the term $\sum_{j=1}^2 \int_{\mathbb{R}^3} \nabla^k(-u_j \cdot \nabla u_j) \cdot \nabla^k u_j dx$. If $k = 0$, by Hölder's inequality, we have

$$\int_{\mathbb{R}^3} (u_j \cdot \nabla u_j) \cdot u_j dx \lesssim \|u_j\|_{L^3} \|\nabla u_j\|_{L^2} \|u_j\|_{L^6} \lesssim \delta \|\nabla u_j\|_{L^2}^2. \tag{2.13}$$

If $k \geq 1$, by using Hölder's inequality, Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla^k(u_j \cdot \nabla u_j) \cdot \nabla^k u_j dx \\ \lesssim& (\|u_j\|_{L^3} \|\nabla^{k+1}u_j\|_{L^2} + \|\nabla^k u_j\|_{L^2} \|\nabla u_j\|_{L^3}) \|\nabla^k u_j\|_{L^6} \\ \lesssim& \delta \|\nabla^{k+1}u_j\|_{L^2}^2 + \|u_j\|_{L^2}^{1-\frac{k}{k+1}} \|\nabla^{k+1}u_j\|_{L^2}^{\frac{k}{k+1}} \|\nabla^\alpha u_j\|_{L^2}^{\frac{k}{k+1}} \|\nabla^{k+1}u_j\|_{L^2}^{1-\frac{k}{k+1}} \|\nabla^k u_j\|_{L^6} \\ \lesssim& \delta \|\nabla^{k+1}u_j\|_{L^2}^2. \end{aligned} \tag{2.14}$$

where α is defined by

$$\frac{1}{3} - \frac{1}{3} = (\frac{\alpha}{3} - \frac{1}{2}) \times \frac{k}{k+1} + (\frac{k+1}{3} - \frac{1}{2}) \times (1 - \frac{k}{k+1}).$$

Since $k \geq 1$, we have $\alpha = \frac{k+1}{2k} \in (\frac{1}{2}, 1]$.

For the fourth term, by using Hölder's inequality, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla^k(\frac{1}{n_j+1}\partial_l n_j \partial_l u_j^i) \cdot \nabla^k u_j^i dx \\ \lesssim& \Big(\|\frac{1}{n_j+1}\|_{L^3} \|\nabla^k(\partial_l n_j \partial_l u_j^i)\|_{L^2} + \|\nabla^k(\frac{1}{n_j+1})\|_{L^6} \|\nabla n_j\|_{L^3} \|\nabla u_j\|_{L^3}\Big) \|\nabla^k u_j\|_{L^6} \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
&\lesssim \|n_j\|_{L^3}\Big(\|\nabla n_j\|_{L^\infty}\|\nabla^{k+1}u_j\|_{L^2}+\|\nabla^{k+1}n_j\|_{L^2}\|\nabla u_j\|_{L^\infty}\Big)\|\nabla^{k+1}u_j\|_{L^2}\\
&\quad+\|\nabla n_j\|_{L^3}\|\nabla u_j\|_{L^3}\|\nabla^{k+1}n_j\|_{L^2}\|\nabla^{k+1}u_j\|_{L^2}\\
&\lesssim \delta(\|\nabla^{k+1}n_j\|_{L^2}^2+\|\nabla^{k+1}u_j\|_{L^2}^2).
\end{aligned}
$$

Similarly, we have

$$
\int_{\mathbb{R}^3}\nabla^k(\frac{1}{n_j+1}\partial_l n_j\partial_i u_j^l)\cdot\nabla^k u_j^i dx \lesssim \delta(\|\nabla^{k+1}n_j\|_{L^2}^2+\|\nabla^{k+1}u_j\|_{L^2}^2). \tag{2.16}
$$

Combining (2.8) – (2.16), we deduce (2.7) for $0\le k\le N-1$, this yields the desired result. □

Lemma 2.5. *If $\sqrt{\mathcal{E}_0^3(t)}\le\delta$, then for $k=0,\cdots,N-1$, we have*

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla^{k+1}(n_1,u_1,n_2,u_2)\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^{k+1}n_1\cdot\nabla^{k+1}n_2dx)+C\|\nabla^{k+2}(u_1,u_2)\|_{L^2}^2\\
&\lesssim \delta\|\nabla^{k+1}(n_1,u_1,n_2,u_2)\|_{L^2}^2.
\end{aligned} \tag{2.17}
$$

Proof. Applying ∇^{k+1} to (1.4), and multiplying the resulting identities by $(g_1+g_2)\nabla^{k+1}n_1, \nabla^{k+1}u_1, g_2\nabla^{k+1}n_2$, $\nabla^{k+1}u_2$ respectively, summing up them and then integrating over $\mathbb{R}^3$ by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[(g_1+g_2)\|\nabla^{k+1}n_1\|_{L^2}^2+g_2\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^{k+1}(u_1,u_2)\|_{L^2}^2]+\|\nabla^{k+2}(u_1,u_2)\|_{L^2}^2+\|\nabla^{k+1}div(u_1,u_2)\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(g_1+g_2)\nabla^{k+1}f_1\cdot\nabla^{k+1}n_1+\nabla^{k+1}f_2\cdot\nabla^{k+1}u_1dx+g_2\nabla^k f_3\cdot\nabla^{k+1}n_2+\nabla^{k+1}f_4\cdot\nabla^k u_2dx \qquad (2.18)\\
&:=\sum_{i=1}^4 J_i.
\end{aligned}
$$

We shall estimate the terms in the right hand side of (2.18). First, for the term J_1, employing the Leibniz formula and by Hölder's inequality, we obtain

$$
\begin{aligned}
J_1&=-\int_{\mathbb{R}^3}(g_1+g_2)\nabla^{k+1}div(n_1u_1)\cdot\nabla^{k+1}n_1dx \qquad (2.19)\\
&=-\int_{\mathbb{R}^3}(g_1+g_2)([\nabla^{k+1},divu_1]n_1+divu_1\nabla^{k+1}n_1+[\nabla^{k+1},u_1]\nabla n_1+u_1\nabla^{k+1}\nabla n_1)\cdot\nabla^{k+1}n_1dx\\
&\lesssim(\|\nabla^2u_1\|_{L^3}\|\nabla^k n_1\|_{L^6}+\|\nabla^{k+2}u_1\|_{L^2}\|n_1\|_{L^\infty}+\|\nabla u_1\|_{L^\infty}\|\nabla^{k+1}n_1\|_{L^2})\|\nabla^{k+1}n_1\|_{L^2}\\
&\qquad+(\|\nabla u_1\|_{L^\infty}\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^{k+1}u_1\|_{L^6}\|\nabla n_1\|_{L^3}+\|\nabla u_1\|_{L^\infty}\|\nabla^{k+1}n_1\|_{L^2})\|\nabla^{k+1}n_1\|_{L^2}\\
&\lesssim\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^{k+2}u_1\|_{L^2}^2).
\end{aligned}
$$

Similarly, we can bound

$$
J_3\lesssim\delta(\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^{k+2}u_2\|_{L^2}^2). \tag{2.20}
$$

For the second and fourth terms, we have

$$
J_2+J_4=-g_2\int_{\mathbb{R}^3}\nabla^{k+1}\nabla n_2\cdot\nabla^{k+1}u_1+\nabla^k\nabla n_1\cdot\nabla^{k+1}u_2dx+\sum_{j=1}^2\int_{\mathbb{R}^3}\nabla^{k+1}(-u_j\cdot\nabla u_j)\cdot\nabla^{k+1}u_j \tag{2.21}
$$

$$+\nabla^{k+1}(\frac{1}{n_j+1}\partial_l n_j\partial_l u_j^i)\cdot\nabla^{k+1}u_j^i+\nabla^{k+1}(\frac{1}{n_j+1}\partial_l n_j\partial_i u_j^l)\cdot\nabla^{k+1}u_j^i dx.$$

By $(1.4)_{1,3}$ and integrating by parts, we deduce from (2.19), (2.20) that

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\nabla^{k+1}\nabla n_2\cdot\nabla^{k+1}u_1+\nabla^{k+1}\nabla n_1\cdot\nabla^{k+1}u_2dx \\
=&\int_{\mathbb{R}^3}\nabla^{k+1}n_2\cdot\nabla^{k+1}divu_1+\nabla^{k+1}n_1\cdot\nabla^{k+1}divu_2dx \\
=&\int_{\mathbb{R}^3}\nabla^{k+1}n_2\cdot\nabla^{k+1}(-f_1-\partial_t n_1)+\nabla^{k+1}n_1\cdot\nabla^{k+1}(-f_3-\partial_t n_2)dx \\
=&-\int_{\mathbb{R}^3}\nabla^{k+1}n_2\cdot\nabla^{k+1}f_1+\nabla^{k+1}n_1\cdot\nabla^{k+1}f_3dx-\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{k+1}n_1\cdot\nabla^{k+1}n_2dx \\
\lesssim&\,\delta(\|\nabla^{k+1}(n_1,n_2)\|_{L^2}^2+\|\nabla^{k+2}(u_1,u_2)\|_{L^2}^2)-\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{k+1}n_1\cdot\nabla^{k+1}n_2dx.
\end{aligned}\tag{2.22}$$

By using Hölder's inequality, Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^{k+1}(u_j\cdot\nabla u_j)\cdot\nabla^{k+1}u_jdx \\
\lesssim&\,(\|u_j\|_{L^3}\|\nabla^{k+2}u_j\|_{L^2}+\|\nabla^{k+1}u_j\|_{L^2}\|\nabla u_j\|_{L^3})\|\nabla^{k+1}u_j\|_{L^6} \\
\lesssim&\,\delta(\|\nabla^{k+1}u_j\|_{L^2}^2+\|\nabla^{k+2}u_j\|_{L^2}^2).
\end{aligned}\tag{2.23}$$

For the fourth term, by using Hölder's inequality, Lemma 2.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^{k+1}(\frac{1}{n_j+1}\partial_l n_j\partial_l u_j^i)\cdot\nabla^{k+1}u_j^idx \\
\lesssim&\Big(\|\frac{1}{n_j+1}\|_{L^\infty}\|\nabla^k(\partial_l n_j\partial_l u_j^i)\|_{L^2}+\|\nabla^k(\frac{1}{n_j+1})\|_{L^6}\|\nabla n_j\|_{L^6}\|\nabla u_j\|_{L^6}\Big)\|\nabla^{k+2}u_j\|_{L^2} \\
\lesssim&\,\|n_j\|_{L^\infty}\Big(\|\nabla n_j\|_{L^\infty}\|\nabla^{k+1}u_j\|_{L^2}+\|\nabla^{k+1}n_j\|_{L^2}\|\nabla u_j\|_{L^\infty}\Big)\|\nabla^{k+2}u_j\|_{L^2} \\
&+\|\nabla^{k+1}n_j\|_{L^2}\|\nabla^2 n_j\|_{L^2}\|\nabla^2 u_j\|_{L^2}\|\nabla^{k+2}u_j\|_{L^2} \\
\lesssim&\,\delta(\|\nabla^{k+1}n_j\|_{L^2}^2+\|\nabla^{k+1}u_j\|_{L^2}^2+\|\nabla^{k+2}u_j\|_{L^2}^2).
\end{aligned}\tag{2.24}$$

Similarly, we have

$$\int_{\mathbb{R}^3}\nabla^{k+1}(\frac{1}{n_j+1}\partial_l n_j\partial_i u_j^l)\cdot\nabla^{k+1}u_j^idx\lesssim\delta(\|\nabla^{k+1}n_j\|_{L^2}^2+\|\nabla^{k+1}u_j\|_{L^2}^2+\|\nabla^{k+2}u_j\|_{L^2}^2).\tag{2.25}$$

Combining (2.18) – (2.25), we deduce (2.17) for $0\le k\le N-1$, this yields the desired result.

□

The following lemma provides the dissipation estimate for n_1, n_2.

Lemma 2.6. *If $\sqrt{\mathcal{E}_0^3(t)}\le\delta$, then for $k=0,\cdots,N-1$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^k u_1\cdot\nabla^{k+1}n_1+\nabla^k u_2\cdot\nabla^{k+1}n_2dx+C\|\nabla^{k+1}(n_1,n_2)\|_{L^2}^2 \\
\lesssim&\,\|\nabla^{k+1}(u_1,u_2)\|_{L^2}^2+\|\nabla^{k+2}(u_1,u_2)\|_{L^2}^2.
\end{aligned}\tag{2.26}$$

Proof. Applying ∇^k to $(1.4)_2$ and $(1.4)_4$, multiplying $\nabla^{k+1}n_1, \nabla^{k+1}n_2$ respectively, summing up and integrating by part, we get

$$\begin{aligned}(g_1+g_2)\|\nabla^{k+1}n_1\|_{L^2}^2 + g_2\|\nabla^{k+1}n_2\|_{L^2}^2 \le & -\int_{\mathbb{R}^3} \nabla^k\partial_t u_1\cdot\nabla^{k+1}n_1 + \nabla^k\partial_t u_2\cdot\nabla^{k+1}n_2 dx \\ & + C\|\nabla^{k+2}u_1\|_{L^2}\|\nabla^{k+1}n_1\|_{L^2} + C\|\nabla^{k+2}u_2\|_{L^2}\|\nabla^{k+1}n_2\|_{L^2} \\ & + \int_{\mathbb{R}^3} \nabla^k f_2\cdot\nabla^{k+1}n_1 + \nabla^k f_4\cdot\nabla^{k+1}n_2 dx.\end{aligned} \tag{2.27}$$

For the first integration in the right-hand side of (2.27), by $(1.4)_{1,2}$ and integrating by parts, we deduce from (2.9), (2.10) that

$$\begin{aligned}& -\int_{\mathbb{R}^3} \nabla^k\partial_t u_1\cdot\nabla^{k+1}n_1 + \nabla^k\partial_t u_2\cdot\nabla^{k+1}n_2 dx \\ & = -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u_1\cdot\nabla^{k+1}n_1 + \nabla^k u_2\cdot\nabla^{k+1}n_2 dx + \int_{\mathbb{R}^3} \nabla^k u_1\cdot\nabla^{k+1}\partial_t n_1 + \nabla^k u_2\cdot\nabla^{k+1}\partial_t n_2 dx \\ & = -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u_1\cdot\nabla^{k+1}n_1 + \nabla^k u_2\cdot\nabla^{k+1}n_2 dx + \int_{\mathbb{R}^3} \nabla^k divu_1\cdot\nabla^k(f_1 - divu_1) + \nabla^k divu_2\cdot\nabla^k(f_3 - divu_2) dx \\ & = -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u_1\cdot\nabla^{k+1}n_1 + \nabla^k u_2\cdot\nabla^{k+1}n_2 dx + \|\nabla^{k+1}(u_1,u_2)\|_{L^2}^2 + C\|\nabla^{k+1}(u_1,u_2)\|_{L^2}\|\nabla^k(f_1,f_3)\|_{L^2} \\ & = -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u_1\cdot\nabla^{k+1}n_1 + \nabla^k u_2\cdot\nabla^{k+1}n_2 dx + \|\nabla^{k+1}(u_1,u_2)\|_{L^2}^2 + C\delta\|\nabla^{k+1}(n_1,n_2)\|_{L^2}^2.\end{aligned} \tag{2.28}$$

For the last integration in the right-hand side of (2.27), by using Hölder's inequality , Lemma 2.3 and together with (2.24), (2.25), we obtain

$$\begin{aligned}& \int_{\mathbb{R}^3} \nabla^k f_2\cdot\nabla^{k+1}n_1 + \nabla^k f_4\cdot\nabla^{k+1}n_2 dx \\ & = -g_2\int_{\mathbb{R}^3} \nabla^k\nabla n_2\cdot\nabla^{k+1}n_1 + \nabla^k\nabla n_1\cdot\nabla^{k+1}n_2 dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3} \nabla^k(-u_j\cdot\nabla u_j)\cdot\nabla^{k+1}n_j \\ & \quad + \nabla^k(\frac{1}{n_j+1}\partial_l n_j\partial_l u_j^i)\cdot\nabla^{k+1}n_j^i + \nabla^k(\frac{1}{n_j+1}\partial_l n_j\partial_i u_j^l)\cdot\nabla^{k+1}n_j^i dx \\ & \le \frac{g_2}{2}(\|\nabla^{k+1}n_1\|_{L^2}^2 + \|\nabla^{k+1}n_2\|_{L^2}^2) + \delta(\|\nabla^{k+1}n_j\|_{L^2}^2 + \|\nabla^{k+1}u_j\|_{L^2}^2) \\ & \quad + C(\|u_j\|_{L^\infty}\|\nabla^{k+1}u_j\|_{L^2} + \|\nabla^k u_j\|_{L^6}\|\nabla u_j\|_{L^3})\|\nabla^{k+1}n_j\|_{L^2} \\ & \le \frac{g_2}{2}(\|\nabla^{k+1}n_1\|_{L^2}^2 + \|\nabla^{k+1}n_2\|_{L^2}^2) + C\delta(\|\nabla^{k+1}n_j\|_{L^2}^2 + \|\nabla^{k+1}u_j\|_{L^2}^2).\end{aligned} \tag{2.29}$$

Combining (2.27) – (2.29), by Cauchy's inequality, since δ is small, we then obtain (2.26).

□

Next, we will combine all the energy estimates that we have derived to prove (1.7) of Theorem 1.1.

Proof. We first close the energy estimates at each l-th level in our weaker sense. Let $N \ge 3$ and $0 \le l \le m-1$ with $1 \le m \le N$. Summing up the estimates (2.7) of Lemma 2.4 for from $k = l$ to $m-1$, since

$\sqrt{\mathcal{E}_0^3} \leq \delta$ is small, we obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{l\leq k\leq m-1}\Big(\|\nabla^k(n_1,u_1,n_2,u_2)\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^k n_1\cdot\nabla^k n_2 dx\Big)+C\sum_{l+1\leq k\leq m}\|\nabla^k(u_1,u_2)\|_{L^2}^2\\&\lesssim\delta\sum_{l+1\leq k\leq m}\|\nabla^k(n_1,u_1,n_2,u_2)\|_{L^2}^2.\end{aligned}\tag{2.30}$$

Let $k=m-1$ in the estimates (2.17) of Lemma 2.5 , we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\nabla^m(n_1,u_1,n_2,u_2)\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^m n_1\cdot\nabla^m n_2 dx\Big)+C\|\nabla^{m+1}(u_1,u_2)\|_{L^2}^2\\&\lesssim\delta\|\nabla^m(n_1,u_1,n_2,u_2)\|_{L^2}^2.\end{aligned}\tag{2.31}$$

Adding the inequality (2.31) with (2.30), we get

$$\begin{aligned}&\frac{d}{dt}\sum_{l\leq k\leq m}\Big(\|\nabla^k(n_1,u_1,n_2,u_2)\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^k n_1\cdot\nabla^k n_2 dx\Big)+C_1\sum_{l+1\leq k\leq m+1}\|\nabla^k(u_1,u_2)\|_{L^2}^2\\&\leq C_2\delta\sum_{l+1\leq k\leq m}\|\nabla^k(n_1,u_1,n_2,u_2)\|_{L^2}^2.\end{aligned}\tag{2.32}$$

Summing up the estimates (2.26) of Lemma 2.6 for from $k=l$ to $m-1$, we have

$$\begin{aligned}&\frac{d}{dt}\sum_{l\leq k\leq m-1}\int_{\mathbb{R}^3}(\nabla^k u_1\cdot\nabla^{k+1}n_1+\nabla^k u_2\cdot\nabla^{k+1}n_2)dx+C_3\sum_{l+1\leq k\leq m}\|\nabla^k(n_1,n_2)\|_{L^2}^2\\&\leq C_4\sum_{l+1\leq k\leq m+1}\|\nabla^k(u_1,u_2)\|_{L^2}^2.\end{aligned}\tag{2.33}$$

Multiplying (2.33) by $\frac{2C_2\delta}{C_3}$, adding it with (2.32), since $\delta>0$ is small, we deduce that there exists a constant $C_5>0$ such that for $0\leq l\leq m-1$

$$\begin{aligned}&\frac{d}{dt}\Bigg\{\sum_{l\leq k\leq m}\Big(\|\nabla^k(n_1,u_1,n_2,u_2)\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^k n_1\cdot\nabla^k n_2 dx\Big)\\&\quad+\frac{2C_2\delta}{C_3}\sum_{l\leq k\leq m-1}\int_{\mathbb{R}^3}(\nabla^k u_1\cdot\nabla^{k+1}n_1+\nabla^k u_2\cdot\nabla^{k+1}n_2)dx\Bigg\}\\&\quad+C_5\Bigg\{\sum_{l+1\leq k\leq m}\|\nabla^k(n_1,n_2)\|_{L^2}^2+\sum_{l+1\leq k\leq m+1}\|\nabla^k(u_1,u_2)\|_{L^2}^2\Bigg\}\leq 0.\end{aligned}\tag{2.34}$$

Next, we define $\mathcal{E}_l^m(t)$ to be C_5^{-1} times the expression under the time derivative in (2.34). Observe that since δ is small, $\mathcal{E}_l^m(t)$ is equivalent to $\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{H^{m-l}}^2$, that is, there exists a constant $C_6>0$ such that for $0\leq l\leq m-1$

$$C_6^{-1}\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{H^{m-l}}^2\leq\mathcal{E}_l^m(t)\leq C_6\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{H^{m-l}}^2.\tag{2.35}$$

Then we may write (2.34) as that for $0\leq l\leq m-1$

$$\frac{d}{dt}\mathcal{E}_l^m(t)+C(\|\nabla^{l+1}(n_1,n_2)\|_{H^{m-l-1}}^2+\|\nabla^{l+1}(u_1,u_2)\|_{H^{m-l}}^2)\leq 0.\tag{2.36}$$

Taking $l=0$ and $m=3$ in (2.36), and then integrating directly in time, we get

$$\|(n_1,u_1,n_2,u_2)(t)\|_{H^3}^2 \lesssim \mathcal{E}_0^3(t) \lesssim \mathcal{E}_0^3(0) \lesssim \|(n_{10},u_{10},n_{20},u_{20})\|_{H^3}^2. \tag{2.37}$$

By a standard continuity argument, this closes the a priori estimates (2.1) if at the initial time we assume that $\|(n_{10},u_{10},n_{20},u_{20})\|_{H^3}^2 \le \delta_0$ is sufficiently small. This in turn allows us to take $l=0$ and $m=N$ in (2.36), and then integrate it directly in time to obtain

$$\|(n_1,u_1,n_2,u_2)(t)\|_{H^N}^2 + C\int_0^t \|\nabla(n_1,n_2)(\tau)\|_{H^{N-1}}^2 + \|\nabla(u_1,u_2)(\tau)\|_{H^N}^2 d\tau \le C\|(n_{10},u_{10},n_{20},u_{20})\|_{H^N}^2.$$

This proved (1.7). □

3 Convergence rate of the solution

The aim of this section is to establish the decay rates of the solution stated in Theorem 1.1 under additional assumptions that the initial data belong to L^1. Firstly, we derive the decay rates for the linearized reduced gravity two and a half layer equations. Then, we establish the decay rates for the flows (1.1) by the method of spectral analysis and energy estimates. The Cauchy problem to the linearized two and a half layer system is as follows:

$$\begin{cases} \partial_t n_1 + div u_1 = 0, \\ \partial_t u_1 - \triangle u_1 - \nabla div u_1 + (g_1+g_2)\nabla n_1 + g_2 \nabla n_2 = 0, \\ \partial_t n_2 + div u_2 = 0, \\ \partial_t u_2 - \triangle u_2 - \nabla div u_2 + g_2\nabla n_2 + g_2\nabla n_1 = 0. \end{cases} \tag{3.1}$$

Initial data of the system is given as

$$(n_1,u_1,n_2,u_2)|_{t=0} = (n_{10},u_{10},n_{20},u_{20}), \quad x\in\mathbb{R}^3. \tag{3.2}$$

Then the solution operator $S(x,t)$ to the model (1.1) satisfies

$$\begin{cases} \partial_t S - A(\nabla_x) S = 0, \\ S(x,0) = \delta(x) I_{8\times 8}. \end{cases} \tag{3.3}$$

where $\delta(x)$ is the Dirac function. Taking the Fourier transform, we have

$$\begin{cases} \partial_t \hat{S} - A(\xi)\hat{S} = 0, \\ \hat{S}(\xi,0) = I_{8\times 8}. \end{cases} \tag{3.4}$$

where

$$A(\xi) = \begin{pmatrix} 0 & -i\xi' & 0 & 0 \\ -(g_1+g_2)i\xi & -|\xi|^2 I_{3\times3} - \xi\otimes\xi & -g_2 i\xi & 0 \\ 0 & 0 & 0 & -i\xi' \\ -g_2 i\xi & 0 & -g_2 i\xi & -|\xi|^2 I_{3\times3} - \xi\otimes\xi \end{pmatrix}_{8\times 8} \tag{3.5}$$

The characteristic polynomial of $A(\xi)$ is $(\lambda+|\xi|^2)^4[(\lambda^2+2|\xi|^2\lambda)^2+(\lambda^2+2|\xi|^2\lambda)(g_1+2g_2)|\xi|^2+g_1g_2|\xi|^4]$, which implies the eigenvalues are

$$\lambda_0(\xi)=-|\xi|^2(quadruple), \lambda_1(\xi)=-|\xi|^2+ib_1$$
$$\lambda_2(\xi)=-|\xi|^2-ib_1, \lambda_3(\xi)=-|\xi|^2+ib_2, \lambda_4(\xi)=-|\xi|^2-ib_2,$$

where

$$b_1(\xi)=\frac{1}{2}\sqrt{2(g_1+2g_2+\sqrt{g_1^2+4g_2^2})|\xi|^2-4|\xi|^4},$$
$$b_2(\xi)=\frac{1}{2}\sqrt{2(g_1+2g_2-\sqrt{g_1^2+4g_2^2})|\xi|^2-4|\xi|^4}.$$

As in [3], [5], [6], [10], we can compute the exact expression of the Fourier transform $\hat{S}(\xi,t)$ of the system (3.4). Making use of the semigroup theory for evolutional equation, the solutions $U=(n_1,u_1,n_2,u_2)'$ of the linear Cauchy problem $(3.1)-(3.2)$ can be expressed as

$$U(t)=S(t)*U_0. \tag{3.6}$$

Next, we give the following inequality, which can be found in [3].

Lemma 3.1. *Let $k\geq 0$ be an integer and, then for any $t\geq 0$, the solution $U(t)$ of system $(3.1)-(3.2)$ satisfies*

$$\|\nabla^k U(t)\|_{L^2}\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|U_0\|_{L^1}+\|\nabla^k U_0\|_{L^2}). \tag{3.7}$$

Now, we turn to establish the time decay rates for the reduced gravity two and a half layer equations $(1.4)-(1.5)$.

Lemma 3.2. *Under the assumptions of Theorem 1.1, the global solution (n_1,u_1,n_2,u_2) of problem (1.4) satisfies*

$$\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{H^{N-l}}\leq C(1+t)^{-\frac{3+2l}{4}}, \quad for \quad l=0,1. \tag{3.8}$$

Proof. Adding $\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{L^2}^2$ to both sides of (2.36) gives

$$\frac{d}{dt}\mathcal{E}_l^m(t)+C\mathcal{E}_l^m(t)\leq\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{L^2}^2. \tag{3.9}$$

Taking $l=1$ and $m=N$ in (3.9), we get

$$\frac{d}{dt}\mathcal{E}_1^N(t)+C\mathcal{E}_1^N(t)\leq\|\nabla(n_1,u_1,n_2,u_2)(t)\|_{L^2}^2. \tag{3.10}$$

It follows from Gronwall inequality and Lemma 3.1 that

$$\mathcal{E}_1^N(t)\leq\mathcal{E}_1^N(0)e^{-Ct}+\int_0^t e^{-C(t-\tau)}\|\nabla(n_1,u_1,n_2,u_2)(\tau)\|_{L^2}^2d\tau. \tag{3.11}$$

In order to derive the time decay rate for $\mathcal{E}_1^N(t)$, we need to control the term $\|\nabla(n_1,u_1,n_2,u_2)(t)\|_{L^2}^2$. In fact, by Duhamel principle, we know

$$U(t) = S(t) * U_0 + \int_0^t S(t-\tau) * (f_1, f_2 + g_2\nabla n_2, f_3, f_4 + g_2\nabla n_1)' d\tau. \tag{3.12}$$

For the nonlinear terms of the model (1.4), employing the Hölder's inequality and Lemma 2.3, we get

$$\begin{cases} \|f_1\|_{L^1} \lesssim \delta\|\nabla(n_1,u_1)\|_{L^2},\ \|f_2 + g_2\nabla n_2\|_{L^1} \lesssim \delta\|\nabla u_1\|_{L^2}, \\ \|f_3\|_{L^1} \lesssim \delta\|\nabla(n_2,u_2)\|_{L^2},\ \|f_4 + g_2\nabla n_1\|_{L^1} \lesssim \delta\|\nabla u_2\|_{L^2}, \\ \|f_1\|_{L^2} \lesssim \delta\|\nabla(n_1,u_1)\|_{L^2},\ \|f_2 + g_2\nabla n_2\|_{L^2} \lesssim \delta\|\nabla(n_1,u_1)\|_{L^2}, \\ \|f_3\|_{L^2} \lesssim \delta\|\nabla(n_2,u_2)\|_{L^2},\ \|f_4 + g_2\nabla n_1\|_{L^2} \lesssim \delta\|\nabla(n_2,u_2)\|_{L^2} \\ \|\nabla f_1\|_{L^2} \lesssim \delta\|\nabla^2(n_1,u_1)\|_{L^2},\ \|\nabla(f_2 + g_2\nabla n_2)\|_{L^2} \lesssim \delta\|\nabla(n_1,u_1)\|_{H^1}, \\ \|\nabla f_3\|_{L^2} \lesssim \delta\|\nabla^2(n_1,u_1)\|_{L^2},\ \|\nabla(f_4 + g_2\nabla n_1)\|_{L^2} \lesssim \delta\|\nabla(n_2,u_2)\|_{H^1}. \end{cases} \tag{3.13}$$

Together with (2.13) − (2.14) and Lemma 3.1,, we have

$$\begin{aligned} \|\nabla(n_1,u_1,n_2,u_2)\|_{L^2}^2 &\le (1+t)^{-\frac{5}{2}} + C\int_0^t (1+t-\tau)^{-\frac{5}{2}}\big(\|(f_1, f_2 + g_2\nabla n_2, f_3, f_4 + g_2\nabla n_1)\|_{L^1}^2 \\ &\quad + \|\nabla(f_1, f_2 + g_2\nabla n_2, f_3, f_4 + g_2\nabla n_1)\|_{L^2}^2\big) d\tau \\ &\le C(1+t)^{-\frac{5}{2}} + C\int_0^t \delta(1+t-\tau)^{-\frac{5}{2}}\|\nabla(n_1,u_1,n_2,u_2)\|_{H^1}^2 d\tau \\ &\le C(1+t)^{-\frac{5}{2}} + C\delta M(t)\int_0^t (1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{5}{2}} d\tau \\ &\le C(1+t)^{-\frac{5}{2}} + C\delta M(t)(1+t)^{-\frac{5}{2}} \\ &\le C(1+t)^{-\frac{5}{2}}(1+\delta M(t)). \end{aligned} \tag{3.14}$$

where $M(t) = \sup\limits_{0\le\tau\le t}(1+t)^{\frac{5}{2}}\mathcal{E}_1^N(t)$

Inserting (3.14) into (3.11), it follows

$$\begin{aligned} \mathcal{E}_1^N(t) &\le \mathcal{E}_1^N(0)e^{-Ct} + \int_0^t e^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}}(1+\delta M(\tau)) d\tau \\ &\le \mathcal{E}_1^N(0)e^{-Ct} + C(1+\delta M(t))\int_0^t e^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}} d\tau \\ &\le \mathcal{E}_1^N(0)e^{-Ct} + C(1+\delta M(t))(1+t)^{-\frac{5}{2}} \\ &\le C(1+\delta M(t))(1+t)^{-\frac{5}{2}}. \end{aligned} \tag{3.15}$$

where we have used the fact:

$$\begin{aligned} &\int_0^t e^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}} d\tau \\ &= \int_0^{\frac{t}{2}} e^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}} d\tau + \int_{\frac{t}{2}}^t e^{-C(t-\tau)}(1+\tau)^{-\frac{5}{2}} d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq e^{-\frac{Ct}{2}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{5}{2}}d\tau+(1+\frac{t}{2})^{-\frac{5}{2}}\int_{\frac{t}{2}}^{t}e^{-C(t-\tau)}d\tau\\
&\leq C(1+t)^{-\frac{5}{2}}.
\end{aligned}
$$

Noticing the definition of $M(t)$), we get

$$M(t)\leq C(1+\delta M(t)).$$

which implies

$$M(t)\leq C. \tag{3.16}$$

since $\delta>0$ is sufficiently small.

Hence, we have the following decay rates

$$\|\nabla(n_1,u_1,n_2,u_2)(t)\|_{H^{N-1}}\leq C(1+t)^{-\frac{5}{4}}. \tag{3.17}$$

On the other hand, by (2.13) − (2.14), it is easy to deduce

$$
\begin{aligned}
\|(n_1,u_1,n_2,u_2)\|_{L^2}^2 &\leq (1+t)^{-\frac{3}{2}}+C\int_0^t(1+t-\tau)^{-\frac{3}{2}}\|(f_1,f_2+g_2\nabla n_2,f_3,f_4+g_2\nabla n_1)\|_{L^1\cap L^2}^2d\tau\\
&\leq C(1+t)^{-\frac{3}{2}}+C\int_0^t\delta(1+t-\tau)^{-\frac{3}{2}}\|\nabla(n_1,u_1,n_2,u_2)\|_{L^2}^2d\tau\\
&\leq C(1+t)^{-\frac{3}{2}}+C\int_0^t(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{5}{2}}d\tau\\
&\leq C(1+t)^{-\frac{3}{2}}+C\int_0^t(1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{3}{2}}d\tau\\
&\leq C(1+t)^{-\frac{3}{2}}.
\end{aligned} \tag{3.18}
$$

where we have used the inequality [1]:

$$\int_0^t(1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{3}{2}}d\tau\leq C(1+t)^{-\frac{3}{2}}.$$

which together with (3.17) implies (3.8). □

Lemma 3.3. *Under the assumptions of Theorem 1.1, the global solution* (n_1,u_1,n_2,u_2) *of problem* (1.4) *satisfies*

$$\|\nabla^l(n_1,u_1,n_2,u_2)(t)\|_{H^{N-l}}\leq C(1+t)^{-\frac{3+2l}{4}},\quad for\ \ l=0,1,\cdots,N-1. \tag{3.19}$$

Proof. We are ready to prove (3.19) by induction. When $l=0,1$, inequality (3.19) has been established in Lemma 3.2, suppose (3.19) holds for the case $l=k-1$, and $k=2,3,\cdots,N-1$, that is

$$\|\nabla^{k-1}(n_1,u_1,n_2,u_2)(t)\|_{H^{N-k+1}}\leq C(1+t)^{-\frac{1+2k}{4}}. \tag{3.20}$$

We need show (3.19) holds for $l = k$. Let $l = k$ and $m = N$ in the estimates (2.36) , we have

$$\frac{d}{dt}\mathcal{E}_k^N(t) + \|\nabla^{k+1}(n_1, n_2)\|_{H^{N-k-1}}^2 + \|\nabla^{k+1}(u_1, u_2)\|_{H^{N-k}}^2 \leq 0. \tag{3.21}$$

Adding $\|\nabla^{k+1}(n_1, n_2)(t)\|_{L^2}^2$ to both sides of (3.21) gives

$$\frac{d}{dt}\mathcal{E}_k^N(t) + C(\|\nabla^{k+1}(n_1, n_2)\|_{L^2}^2 + \|\nabla^{k+1}(n_1, n_2)\|_{H^{N-k-1}}^2 + \|\nabla^{k+1}(u_1, u_2)\|_{H^{N-k}}^2) \leq 0. \tag{3.22}$$

As in [17], we define

$$S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \leq (\frac{a}{1+t})^{\frac{1}{2}}\}.$$

for a constant a that will be specified bellow. Then

$$\begin{aligned}
\|\nabla^{k+1} n_i\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\xi|^{2(k+1)} |\hat{n_i}|^2 d\xi \geq \int_{\mathbb{R}^3/S} |\xi|^{2(k+1)} |\hat{n_i}|^2 d\xi \\
&\geq \frac{a}{1+t} \int_{\mathbb{R}^3/S} |\xi|^{2k} |\hat{n_i}|^2 d\xi \\
&\geq \frac{a}{1+t} \int_{\mathbb{R}^3} |\xi|^{2k} |\hat{n_i}|^2 d\xi - \frac{a^2}{(1+t)^2} \int_S |\xi|^{2(k-1)} |\hat{n_i}|^2 d\xi \\
&\geq \frac{a}{1+t} \int_{\mathbb{R}^3} |\xi|^{2k} |\hat{n_i}|^2 d\xi - \frac{a^2}{(1+t)^2} \int_{\mathbb{R}^3} |\xi|^{2(k-1)} |\hat{n_i}|^2 d\xi.
\end{aligned}$$

Thus, we have

$$\|\nabla^{k+1} n_i\|_{L^2}^2 \geq \frac{a}{1+t}\|\nabla^k n_i\|_{L^2}^2 - \frac{a^2}{(1+t)^2}\|\nabla^{k-1} n_i\|_{L^2}^2, \quad i = 1, 2. \tag{3.23}$$

Similarly, one has

$$\|\nabla^{k+1}(u_1, u_2)\|_{L^2}^2 \geq \frac{a}{1+t}\|\nabla^k(u_1, u_2)\|_{L^2}^2 - \frac{a^2}{(1+t)^2}\|\nabla^{k-1}(u_1, u_2)\|_{L^2}^2. \tag{3.24}$$

Summing up the estimates (3.24) for k from k to N, one has

$$\|\nabla^{k+1}(u_1, u_2)\|_{H^{N-k}}^2 \geq \frac{a}{1+t}\|\nabla^k(u_1, u_2)\|_{H^{N-k}}^2 - \frac{a^2}{(1+t)^2}\|\nabla^{k-1}(u_1, u_2)\|_{H^{N-k}}^2. \tag{3.25}$$

Substituting the inequalities (3.23), (3.25) into (3.22), applying (3.20), it follows

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_k^N(t) + \frac{Ca}{1+t}(\|\nabla^k(n_1, n_2)(t)\|_{L^2}^2 + \|\nabla^{k+1}(n_1, n_2)(t)\|_{H^{N-k-1}}^2 + \|\nabla^k(u_1, u_2)\|_{H^{N-k}}^2) \\
&\leq \frac{Ca^2}{(1+t)^2}(\|\nabla^{k-1}(n_1, n_2)(t)\|_{L^2}^2 + \|\nabla^{k-1}(u_1, u_2)\|_{H^{N-k}}^2) \\
&\leq C(1+t)^{-\frac{5+2k}{2}}.
\end{aligned}$$

where we have used

$$\frac{a}{1+t}\|\nabla^{k+1}(n_1, n_2)(t)\|_{H^{N-k-1}}^2 \leq \|\nabla^k(n_1, n_2)(t)\|_{H^{N-k-1}}^2,$$

for some sufficiently large time $t \geq a-1$, such that $\frac{a}{1+t} \leq 1$.

This, together with the definition of $\mathcal{E}_k^N(t)$, implies that

$$\frac{d}{dt}\mathcal{E}_k^N(t) + \frac{Ca}{1+t}\mathcal{E}_k^N(t) \leq C(1+t)^{-\frac{5+2k}{2}}. \tag{3.26}$$

Choosing

$$a = \frac{k+2}{C},$$

and multiplying both sides of (3.26) by $(1+t)^{k+2}$, we get

$$\frac{d}{dt}[(1+t)^{k+2}\mathcal{E}_k^N(t)] \leq C(1+t)^{-\frac{1}{2}}. \tag{3.27}$$

Solving the inequality direcly yields

$$\|\nabla^k(n_1,u_1,n_2,u_2)(t)\|_{H^{N-k}}^2 \leq C(1+t)^{-\frac{3+2k}{2}}. \tag{3.28}$$

Hence, we have verified that (3.8) holds on for the case $l = k$, this concludes the proof of the lemma.

□

With Lemma 3.2 and Lemma 3.3 in hand, we are ready to proof Theorem 1.1:

Proof. With the help of Lemma 3.2 and Lemma 3.3, it is easy to obtain the conclusion (1.8) . As for (1.9), by (1.8) and the Gagliardo-Nirenberg inequality,

$$\|\nabla^l f(t)\|_{L^p} \leq C\|\nabla^l f(t)\|_{L^2}^{\theta}\|f(t)\|_{L^2}^{1-\theta}, \quad \frac{1}{p} - \frac{l}{3} = (\frac{1}{2} - \frac{l}{3})\theta + \frac{1}{2}(1-\theta),$$

the claim follows. □

Acknowledgements. This work is partially supported by NNSFC(Grant No.11271381)and NSF(Grant No.11501373), China 973 Program(Grant No. 2011CB808002); Guangdong Provincial culture of seedling of China (no. 2013LYM0081); Guangdong Provincial Department of Education research platform project (Guangdong Young Creative Talents (no.2014KQNCX208); the Shaoguan Science and Technology Foundation (no. 313140546) and Education Reform Project of Shaoguan University(no. SYJY20121361 and SYJY20141576).

References

[1] R.Duan, S.Ukai, T.Yang, H.J.Zhao, Optimal convergence rate for compressible Navier-Stokes equations with potential force, Math.Models.Methods Appl. Sci. 17 (2007) 737-758.

[2] R.Duan and C.H.Zhou, On the compactness of the reduced-gravity two-and-a-half layer equations, J. Differential Equations, 252 (2012) 3506-3519.

[3] H.B.Cui, L.Yao and Z.A.Yao, Global Existence and Optimal Decay rates of solutions to a reduced gravity two and a half layer model, Commun. Pure Appl. Anal., 14(2015) 981-1000.

[4] Z.H.Guo, Z.L.Li and L.Yao, Existence of global weak solution for a reduced gravity two and half layer model, J. Math. Phys., 54 (2013) 1-19.

[5] T.Kobayashi, Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in R^3, J. Differential Equations, 184 (2002) 587–619.

[6] H.L.Li and T.Zhang, Large time behavior of isentropic compressible Navier-Stokes system in R^3, Math. Meth. Appl. Sci., 34 (2011) 670-682.

[7] N.Ju, Existence and uniqueness of the solution to the dissipative 2D Quasi-Geostrophic equations in the Sobolev space, Comm. Math. Phys. 251 (2004) 365-376.

[8] L.Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa, 13 (1959) 115-162.

[9] G.K.Vallis, Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-scale Circulation, Cambridge University Press, 2006.

[10] G.J.Zhang, H.L.Li and C. J. Zhu, Optimal decay rate of the non-isentropic compressible Navier-Stokes-Poisson system in R3, J. Differential Equations, 250 (2011) 866-891.

[11] K.Deckelnick, Decay estimates for the compressible Navier-Stokes equations in unbounded domains, Math. Z. 209(1992) 115-130.

[12] K.Deckelnick, L^2-decay for the compressible Navier-Stokes equations in unbounded domains, Commun. Partial Differ. Equ. 18 (1993)1445-1476.

[13] R.Duan, S.Ukai, T.Yang and H.J.Zhao, Optimal $L^p - L^q$ convergence rate for the compressible Navier-Stokes equations with potential force, J. Differential Equations 238 (2007) 220-223.

[14] Y.Guo and Y.J.Wang, Decay of dissipative equations and negative Sobolev spaces, Comm. Partial Differential Equations. 37 (2012) 2165-2208.

[15] T.Kobayashi and Y.Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in R^3, Commun.Math.Phys. 200 (1999) 621-659.

[16] T.Kobayashi and Y.Shibata, Remark on the rate of decay of solutions to linearized compressible Navier-Stokes equations, Pacific J. Math. 207 (2002) 199-234.

[17] M.E.Schonbek, On the decay of higher-order norms of the solutions of Navier-Stokes equations, Proceedings of the Royal Society of Edinburgh, 126A (1996) 677-685.

[18] M.E.Schonbek, L^2 decay of weak solutions of the Navier-Stokes equations, Arch. Ration. Mech. Anal. 88 (1985) 209-222.

[19] M.E.Schonbek, Large time behaviour of solutions of the Navier-Stokes equations, Comm. Partial Differential Equations 11 (1986) 733-763.

[20] Y.J.Wang, Decay of the Navier-Stokes-Poisson equations, J. Differential Equations, 253 (2012) 273-297.

[21] Y.J.Wang, Z.Tan, Global existence and optimal decay rate for the strong solutions in H^2 to the compressible Navier-Stokes equations, Appl. Math. Lett. 24 (2011) 1778-1784.

[22] R.Y. Wei,Y. Li and Z.A. Yao. Decay of the Liquid Crystal System, Mathematical Methods in the Applied Sciences DOI: 10.1002/mma.3494.

[23] J.C.Gao, Q. Tao,Z.A. Yao.Long-time Behavior of Solution for the Compressible Nematic Liquid Crystal Flows in R^3,arXiv:1503.02865.