



An index of summability for pairs of Banach spaces [☆]



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ABSTRACT

We introduce the notion of index of summability for pairs of Banach spaces; for Banach spaces E, F , this index plays the role of a kind of measure of how the m -homogeneous polynomials from E to F are far from being absolutely summing. In some cases the optimal index of summability is computed.

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1. Introduction and background

For $1 \leq q \leq p < \infty$ and Banach spaces E, F over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we recall that a continuous linear operator $u : E \rightarrow F$ is absolutely (p, q) -summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{k=1}^n \|u(x_k)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{E^*}} \left(\sum_{k=1}^n |\varphi(x_k)|^q \right)^{\frac{1}{q}} \quad (1)$$

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for every $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$. Above, and from now on, the topological dual of E and its closed unit ball are denoted by E^* and B_{E^*} , respectively.

The space of absolutely (p, q) -summing linear operators from E to F is denoted by $\Pi_{(p,q)}(E; F)$. The (p, q) -summing norm of u , defined as the infimum of the constants C in (1), is represented by $\pi_{p,q}(u)$. If $p = q$ the operator u is simply called absolutely p -summing and we write $\Pi_p(E; F)$ and $\pi_p(u)$ for the space of absolutely p -summing operators and the p -summing norm of u , respectively. For the theory of absolutely summing operators we refer to [5].

When only sequences $(x_j)_{j=1}^n$ of fixed length n are considered, the infimum over all C satisfying (1) is denoted by $\pi_{p,q}^{(n)}(u)$ (or $\pi_p^{(n)}(u)$ when $p = q$). Of course, $\pi_{p,q}^{(n)}(u) \leq \pi_{p,q}(u)$. In [15,16] the authors investigated in depth estimates of the type

$$\pi_{p,q}(u) \leq c\pi_{p,q}^{(n)}(u),$$

where c is a positive constant. These estimates show that the (p, q) -summing norm of an operator can be sometimes well-approximated using only “few” vectors in the definition of the (p, q) -summing norm. The following results of finite-dimensional nature will be crucial in this paper:

Theorem 1.1. (See Szarek [15].) *There exists a universal constant C such that whenever $u : E \rightarrow F$ (E, F are Banach spaces) is a finite rank linear operator (say $\text{rank}(u) = n$) and $q \geq 2$, then*

$$\pi_{q,2}(u) \leq C\pi_{q,2}^{(n)}(u).$$

Theorem 1.2. (See König, Retherford, Tomczak-Jaegermann [9].) *Let id_{X_n} denote the identity on a n -dimensional space X_n . For $q > 2$, we have*

$$(2e)^{-1}n^{\frac{1}{q}} \leq \pi_{q,2}(\text{id}_{X_n}).$$

From now on, as usual, given $x_1, \dots, x_n \in E$, we define

$$\|(x_k)_{k=1}^n\|_{w,p} := \sup_{\varphi \in B_{E^*}} \left(\sum_{k=1}^n |\varphi(x_k)|^p \right)^{\frac{1}{p}}.$$

Let $m \in \mathbb{N}$ and $E_1, \dots, E_m$ be Banach spaces over $\mathbb{K}$. By $\mathcal{L}(E_1, \dots, E_m; F)$ we denote the Banach space of all bounded m -linear operators from $E_1 \times \dots \times E_m$ into F . In the case $E_1 = \dots = E_m = E$, we will simply write $\mathcal{L}(^m E; F)$, whereas $\mathcal{L}(E; F)$ is the usual Banach space of all continuous linear operators from E to F . For the theory of multilinear operators and polynomials between Banach spaces we refer to the excellent books of Dineen [6] and Mujica [12].

For $1 \leq q \leq p < \infty$, an m -linear operator $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is called *multiple (p, q) -summing* ([11, 13]) if there is a constant $C \geq 0$ such that

$$\left(\sum_{k_1, \dots, k_m=1}^n \left\| T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)}) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{i=1}^m \left\| (x_{k_i}^{(i)})_{k_i=1}^n \right\|_{w,q} \quad (2)$$

for all positive integers n and all $x_k^{(i)} \in E_i$, with $1 \leq k \leq n$ and $1 \leq i \leq m$. The vector space of all multiple (p, q) -summing operators is denoted by $\Pi_{(p,q)}^{mult}(E_1, \dots, E_m; F)$. The infimum $\pi_{(p,q)}^{mult}(T)$, taken over all possible constants C satisfying (2), defines a complete norm in $\Pi_{(p,q)}^{mult}(E_1, \dots, E_m; F)$. When $E_1 = \dots = E_m = E$, we write $\Pi_{(p,q)}^{mult}(^m E; F)$. When $k_1 = \dots = k_m = k$, we recover the definition of the class of absolutely (p, q) -summing m -linear operators that will be denoted by $(\Pi_{(p,q)}(E_1, \dots, E_m; F); \pi_{(p,q)}(\cdot))$.

For polynomials, let $\mathcal{P}(^m E; F)$ denote the Banach space of all continuous m -homogeneous polynomials from E into F . We recall that given $1 \leq p, q < \infty$, with $p \geq \frac{q}{m}$, a polynomial $P \in \mathcal{P}(^m E; F)$ is *absolutely* (p, q) -*summing* if there is a constant $C \geq 0$ such that

$$\left(\sum_{k=1}^n \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq C \|(x_k)_{k=1}^n\|_{w,q}^m \quad (3)$$

for all positive integers n and all $x_k \in E$, with $1 \leq k \leq n$. We denote by $\mathcal{P}_{(p,q)}(^m E; F)$ the Banach space of all absolutely (p, q) -summing polynomials from E to F .

Of course, when (2) or (3) is not valid, this means that such a constant C does not exist. However it is not obvious at a first glance that there exists a constant C_n depending on n satisfying (2) or (3), since at least formally it could happen that varying the vectors $x_1, \dots, x_n$ the constant could tend to infinity. But it is not difficult to prove that this is not the case and when (2) or (3) fails there will exist a constant C_n that makes the inequality true. We shall also observe that in all cases a constant $C_n = C_1 n^s$ can be found for a certain s depending on p, q, m . Note that the number s plays the role of a kind of index of (non) summability: when $s = 0$ the operator is multiple (p, q) -summing and when s cannot be chosen to be zero, the map is not multiple (p, q) -summing and the “optimal” value of s can be naturally identified as an index of (non) summability. In this case, as the “optimal” value of s grows, we can say how far is the map to be multiple (p, q) -summing. We shall adopt a slightly different approach. Instead of defining the index of summability s as we have just remarked we shall define the index of summability of a pair $(E_1 \times \dots \times E_m, F)$ as follows. The following definition is inspired by the paper [1], where a kind of index of summability was investigated for Hardy–Littlewood type inequalities. Another recent reference in this line of classical inequalities is the very interesting and deep paper [7] of Galicer, Mansilla and Muro.

Definition 1.3. The *multilinear m -index of (p, q) -summability* of a pair $(E_1 \times \dots \times E_m, F)$ is defined as

$$\eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) = \inf s_{m,p,q},$$

where $s_{m,p,q} \geq 0$ satisfies the following:

There is a constant $C \geq 0$ (not depending on n) satisfying

$$\left(\sum_{k_1, \dots, k_m=1}^n \left\| T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)}) \right\|^p \right)^{\frac{1}{p}} \leq C n^{s_{m,p,q}} \prod_{i=1}^m \left\| (x_{k_i}^{(i)})_{k_i=1}^n \right\|_{w,q}$$

for every $T \in \mathcal{L}(E_1, \dots, E_m; F)$ and all positive integers n and $x_{k_i}^{(i)} \in E_i$, with $1 \leq k_i \leq n$ and $1 \leq i \leq m$.

When $E_1 = \dots = E_m = E$, we write $\eta_{(p,q)}^{m-mult}(E; F)$ instead of $\eta_{(p,q)}^{m-mult}(E, \dots, E; F)$.

Similarly the *polynomial m -index of (p, q) -summability* of a pair of Banach spaces (E, F) is defined as

$$\eta_{(p,q)}^{m-pol}(E, F) = \inf s_{m,p,q},$$

where $s_{m,p,q} \geq 0$ satisfies the following:

There is a constant $C \geq 0$ (not depending on n) satisfying

$$\left(\sum_{j=1}^n \|P(x_j)\|^p \right)^{\frac{1}{p}} \leq C n^{s_{m,p,q}} \|(x_j)_{j=1}^n\|_{w,q}^m$$

for every $P \in \mathcal{P}(^m E; F)$, all positive integers n and all $x_j \in E$, with $1 \leq j \leq n$.

When $m = 1$, we have $\Pi_{(p,q)}^{mult}({}^1E; F) = \mathcal{P}_{(p,q)}({}^1E; F) = \Pi_{(p,q)}(E; F)$ and in this case we will simply write $\eta_{(p,q)}(E; F)$.

This paper is divided in four sections. In the second section we show that there always exist finite upper estimates for the index of summability. In Sections 3 and 4 the main results are proved, presenting upper and lower estimates for the index of summability in several special cases.

2. Basic results

One of the cornerstones of the theory of absolutely p -summing linear operators is the Dvoretzky–Rogers Theorem. A weak version of this theorem asserts that if $p \geq 1$ and E is a Banach space, then the identity operator on E , denoted by id_E , is absolutely p -summing if and only if E is finite dimensional. The main goal of this section is to certify that the index of summability is always finite. The next result provides the 2-summing norm of the identity operator when E is finite dimensional and will be very important for us:

Theorem 2.1. (See Pietsch [14].) *If E is a Banach space and $\dim E = n$, then $\pi_2(id_E) = \sqrt{n}$.*

We highlight the following corollary of the above theorem for future reference. Note that below we extrapolate the notion of absolutely p -summing operators to $p > 0$.

Corollary 2.2. *Let $0 < p < \infty$. If E is a normed space and $\dim E = n$, then*

$$\pi_p(id_E) \leq n^{\max\{\frac{1}{p}, \frac{1}{2}\}}. \quad (4)$$

Proof. Let $0 < p < 2$ and $r > 0$ be such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. Thus, given $x_1, \dots, x_n \in E$ and using Hölder's Inequality we obtain

$$\begin{aligned} \left(\sum_{j=1}^n \|id_E(x_j)\|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{j=1}^n \|id_E(x_j)\|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^n |1|^r \right)^{\frac{1}{r}} \\ &\leq \pi_2(id_E) \|(x_j)_{j=1}^n\|_{w,2} n^{\frac{1}{r}} \\ &\stackrel{\text{Theorem 2.1}}{\leq} n^{\frac{1}{2} + \frac{1}{r}} \|(x_j)_{j=1}^n\|_{w,p}. \end{aligned}$$

Therefore

$$\pi_p(id_E) \leq n^{\frac{1}{p}}.$$

For the case $p \geq 2$ we use Inclusion Theorem for absolutely p -summing operators (see [5, Theorem 2.8]) to obtain

$$\pi_p(id_E) \leq \pi_2(id_E) = n^{\frac{1}{2}}. \quad \square$$

Remark 2.3. Clearly, if X is a subspace of an n -dimensional normed space E , then

$$\pi_p(id_X) \leq (\dim X)^{\max\{\frac{1}{p}, \frac{1}{2}\}} \leq n^{\max\{\frac{1}{p}, \frac{1}{2}\}}.$$

Although multiple (p, q) -summing operators are defined for $p, q \geq 1$, the next result is also valid for $p, q > 0$.

Proposition 2.4. Let $0 < p < \infty$ and $E_1, \dots, E_m, F$ be Banach spaces. Then

$$\eta_{(p,p)}^{m-mult}(E_1, \dots, E_m; F) \leq \frac{m}{p} \text{ for } 0 < p \leq 2;$$

$$\eta_{(p,p)}^{m-mult}(E_1, \dots, E_m; F) \leq \frac{m}{2} \text{ for } p \geq 2.$$

Proof. Let $T \in \mathcal{L}(E_1, \dots, E_m; F)$, $x_{k_i}^{(i)} \in E_i$ and $X_i = \text{span}\{x_{1_i}^{(i)}, \dots, x_{n_i}^{(i)}\} \subset E_i$ with $k_i = 1, \dots, n$ and $i = 1, \dots, m$. Then

$$\begin{aligned} \left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}} &\leq \|T\| \left(\sum_{k_1=1}^n \|x_{k_1}^{(1)}\|^p \right)^{\frac{1}{p}} \cdots \left(\sum_{k_m=1}^n \|x_{k_m}^{(m)}\|^p \right)^{\frac{1}{p}} \\ &= \|T\| \left(\sum_{k_1=1}^n \|id_{X_1}(x_{k_1}^{(1)})\|^p \right)^{\frac{1}{p}} \cdots \left(\sum_{k_m=1}^n \|id_{X_m}(x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Since id_{X_i} is absolutely p -summing, for each $i = 1, \dots, m$, we have

$$\left(\sum_{k_i=1}^n \|id_{X_i}(x_{k_i}^{(i)})\|^p \right)^{\frac{1}{p}} \leq \pi_p(id_{X_i}) \sup_{\psi \in B_{X_i^*}} \left(\sum_{k_i=1}^n |\psi(x_{k_i}^{(i)})|^p \right)^{\frac{1}{p}}.$$

By the Hahn–Banach Theorem, for each $\psi \in X_i^*$ there is an extension $\bar{\psi} \in E_i^*$ such that $\|\psi\| = \|\bar{\psi}\|$. Thus

$$\begin{aligned} \left(\sum_{k_i=1}^n \|id_{X_i}(x_{k_i}^{(i)})\|^p \right)^{\frac{1}{p}} &\leq \pi_p(id_{X_i}) \sup_{\bar{\psi} \in B_{E_i^*}} \left(\sum_{k_i=1}^n |\bar{\psi}(x_{k_i}^{(i)})|^p \right)^{\frac{1}{p}} \\ &\leq \pi_p(id_{X_i}) \sup_{\varphi \in B_{E_i^*}} \left(\sum_{k_i=1}^n |\varphi(x_{k_i}^{(i)})|^p \right)^{\frac{1}{p}} \\ &= \pi_p(id_{X_i}) \left\| \left(x_{k_i}^{(i)} \right)_{k_i=1}^n \right\|_{w,p}, \end{aligned}$$

and hence

$$\left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}} \leq \|T\| \pi_p(id_{X_1}) \left\| \left(x_{k_1}^{(1)} \right)_{k_1=1}^n \right\|_{w,p} \cdots \pi_p(id_{X_m}) \left\| \left(x_{k_m}^{(m)} \right)_{k_m=1}^n \right\|_{w,p}.$$

By the previous corollary, we have:

1) If $0 < p \leq 2$, then

$$\begin{aligned} \left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}} &\stackrel{(4)}{\leq} \|T\| \left(n^{\frac{1}{p}} \right)^m \prod_{i=1}^m \left\| \left(x_{k_i}^{(i)} \right)_{k_i=1}^n \right\|_{w,p} \\ &= \|T\| n^{\frac{m}{p}} \prod_{i=1}^m \left\| \left(x_{k_i}^{(i)} \right)_{k_i=1}^n \right\|_{w,p}, \end{aligned}$$

and

$$\eta_{(p,p)}^{m-mult}(E_1, \dots, E_m; F) \leq \frac{m}{p}.$$

2) If $p \geq 2$, then, analogously,

$$\eta_{(p,p)}^{m-mult}(E_1, \dots, E_m; F) \leq \frac{m}{2}. \quad \square$$

The next result shows that the above estimates can not be improved, keeping its universality.

Corollary 2.5. $\eta_{(2,2)}^{m-mult}(\ell_2; c_0) = \frac{m}{2}$.

Proof. Let t be a positive real number such that for each $T \in \mathcal{L}^m(\ell_2; c_0)$ there is a constant $C \geq 0$ such that

$$\left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^2 \right)^{\frac{1}{2}} \leq C n^t \prod_{i=1}^m \left\| (x_{k_i}^{(i)})_{k_i=1}^n \right\|_{w,2} \quad (5)$$

for all positive integers n and all $x_{k_i}^{(i)} \in \ell_2$, with $1 \leq k_i \leq n$.

Now, let $T \in \mathcal{L}^m(\ell_2; c_0)$ be defined by

$$T(x^{(1)}, \dots, x^{(m)}) = \left(x_{j_1}^{(1)} \cdots x_{j_m}^{(m)} \right)_{j_1, \dots, j_m=1}^n.$$

Of course $\|T\| = 1$ and

$$\left(\sum_{j_1, \dots, j_m=1}^n \|T(e_{j_1}, \dots, e_{j_m})\|^2 \right)^{\frac{1}{2}} = n^{\frac{m}{2}}.$$

Since $\|(e_{j_i})_{j_i=1}^n\|_{w,2} = 1$, the latter condition together with (5) implies

$$n^{\frac{m}{2}} \leq C n^t$$

and thus $t \geq \frac{m}{2}$. The converse inequality is given by the previous proposition and the proof is done. $\square$

If $q < p$ it is plain that

$$\eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) \leq \eta_{(p,p)}^{m-mult}(E_1, \dots, E_m; F).$$

The next results provide better estimates for $q \geq 2$.

Proposition 2.6. Let $1 \leq q \leq p < \infty$ and $E_1, \dots, E_m, F$ be Banach spaces. Then

$$\begin{aligned} \eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) &\leq \frac{m}{p} \text{ for } 1 \leq q \leq 2; \\ \eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) &\leq \frac{mq}{2p} \text{ for } q \geq 2. \end{aligned}$$

Proof. Note that

$$\left(\sum_{k_1, \dots, k_m=1}^n \left\| T \left(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leq \|T\| \left(\sum_{k_1=1}^n \left\| x_{k_1}^{(1)} \right\|^p \right)^{\frac{1}{p}} \cdots \left(\sum_{k_m=1}^n \left\| x_{k_m}^{(m)} \right\|^p \right)^{\frac{1}{p}}$$

for every $x_{k_i}^{(i)} \in E_i$, $1 \leq k_i \leq n$, $1 \leq i \leq m$.

Let $X_i := \text{span} \{x_{1_i}^{(i)}, \dots, x_{n_i}^{(i)}\} \subset E_i$ with $i = 1, \dots, m$. Since X_i is a finite dimensional Banach space it follows that id_{X_i} is absolutely q -summing. So, by [8, Corollary 16.3.1] we have

$$\pi_{p,q}(\text{id}_{X_i}) \leq \pi_q(\text{id}_{X_i})^{\frac{q}{p}}. \quad (6)$$

Thus, for each $i = 1, \dots, m$, we obtain

$$\left(\sum_{k_i=1}^n \left\| x_{k_i}^{(i)} \right\|^p \right)^{\frac{1}{p}} \leq \pi_{p,q}(\text{id}_{X_i}) \left\| (x_{k_i})_{k_i=1}^n \right\|_{w,q} \stackrel{(6)}{\leq} \pi_q(\text{id}_{X_i})^{\frac{q}{p}} \left\| (x_{k_i})_{k_i=1}^n \right\|_{w,q}$$

and, for $q \geq 2$, we have

$$\left(\sum_{k_i=1}^n \left\| x_{k_i}^{(i)} \right\|^p \right)^{\frac{1}{p}} \leq \left(n^{\frac{1}{2}} \right)^{\frac{q}{p}} \left\| (x_{k_i})_{k_i=1}^n \right\|_{w,q} = n^{\frac{q}{2p}} \left\| (x_{k_i})_{k_i=1}^n \right\|_{w,q}.$$

Therefore

$$\eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) \leq \frac{mq}{2p}.$$

Analogously, when $1 \leq q \leq 2$ we conclude that

$$\left(\sum_{k_1, \dots, k_m=1}^n \left\| T \left(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} = \|T\| n^{\frac{m}{p}} \prod_{i=1}^m \left\| (x_{k_i}^{(i)})_{k_i=1}^n \right\|_{w,q}$$

and

$$\eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) \leq \frac{m}{p}. \quad \square$$

It is well-known that the notion of multiple $(p; q)$ -summing operators has no sense when $p < q$. But, curiously, in our context it makes sense to extrapolate the definition. As a matter of fact, it is interesting to investigate the case $0 < p < q$ since, in this case, for any pairs of Banach spaces, the index of summability is not trivially estimated.

Proposition 2.7. *Let $0 < p < q < \infty$ and $E_1, \dots, E_m, F$ be Banach spaces. Then*

$$\begin{aligned} \eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) &\leq \frac{m}{p} \text{ for } 0 < q \leq 2; \\ \eta_{(p,q)}^{m-mult}(E_1, \dots, E_m; F) &\leq \frac{(qp - 2p + 2q)m}{2qp} \text{ for } q \geq 2. \end{aligned}$$

Proof. Note that

$$\left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}} \leq \|T\| \left(\sum_{k_1=1}^n \|x_{k_1}^{(1)}\|^p \right)^{\frac{1}{p}} \cdots \left(\sum_{k_m=1}^n \|x_{k_m}^{(m)}\|^p \right)^{\frac{1}{p}}.$$

For all $i = 1, \dots, m$, the Hölder inequality tells us that

$$\begin{aligned} \left(\sum_{k_i=1}^n \|x_{k_i}^{(i)}\|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{k_i=1}^n \|x_{k_i}^{(i)}\|^q \right)^{\frac{1}{q}} \left(\sum_{k_i=1}^n |1|^{\frac{pq}{q-p}} \right)^{\frac{q-p}{pq}} \\ &\leq \left(\sum_{k_i=1}^n \|x_{k_i}^{(i)}\|^q \right)^{\frac{1}{q}} n^{\frac{q-p}{qp}}. \end{aligned}$$

Hence, for $q \geq 2$, we have

$$\begin{aligned} \left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}} &\stackrel{(4)}{\leq} \|T\| \left(n^{\frac{1}{2}} \left\| \left(x_{k_1}^{(1)} \right)_{k_1=1}^n \right\|_{w,q} n^{\frac{q-p}{qp}} \right) \cdots \left(n^{\frac{1}{2}} \left\| \left(x_{k_m}^{(m)} \right)_{k_m=1}^n \right\|_{w,q} n^{\frac{q-p}{qp}} \right) \\ &= \|T\| n^{\frac{(qp-2p+2q)m}{2qp}} \prod_{i=1}^m \left\| \left(x_{k_i}^{(i)} \right)_{k_i=1}^n \right\|_{w,q} \end{aligned}$$

and, for $0 < q \leq 2$, we get

$$\begin{aligned} \left(\sum_{k_1, \dots, k_m=1}^n \|T(x_{k_1}^{(1)}, \dots, x_{k_m}^{(m)})\|^p \right)^{\frac{1}{p}} &\stackrel{(4)}{\leq} \|T\| \left(n^{\frac{1}{q}} \left\| \left(x_{k_1}^{(1)} \right)_{k_1=1}^n \right\|_{w,q} n^{\frac{q-p}{qp}} \right) \cdots \left(n^{\frac{1}{q}} \left\| \left(x_{k_m}^{(m)} \right)_{k_m=1}^n \right\|_{w,q} n^{\frac{q-p}{qp}} \right) \\ &= \|T\| n^{\frac{m}{p}} \prod_{i=1}^m \left\| \left(x_{k_i}^{(i)} \right)_{k_i=1}^n \right\|_{w,q}. \quad \square \end{aligned}$$

It is plain that the polynomial m -index of (p, q) -summability can be estimated using the estimates for the multilinear m -index of (p, q) -summability. Below we present more accurate estimates.

Proposition 2.8. Let E, F be Banach spaces, m be a natural number, $q > 0$ and $p < \frac{q}{m}$. Then

$$\begin{aligned} \eta_{(p,q)}^{m-pol}(E; F) &\leq \frac{1}{p} \text{ for } 0 < q \leq 2; \\ \eta_{(p,q)}^{m-pol}(E; F) &\leq \frac{1}{p} + \frac{m(q-2)}{2q} \text{ for } q \geq 2. \end{aligned}$$

Proof. For any $P \in \mathcal{P}(^m E; F)$, by virtue of the Hölder inequality we have

$$\begin{aligned} \left(\sum_{k=1}^n \|P(x_k)\|^p \right)^{\frac{1}{p}} &\leq \|P\| \left(\sum_{k=1}^n \|x_k\|^{mp} \right)^{\frac{1}{p}} \\ &\leq \|P\| \left[\left(\sum_{k=1}^n (\|x_k\|^{mp})^{\frac{q}{mp}} \right)^{\frac{mp}{q}} \left(\sum_{k=1}^n |1|^{\left(\frac{q}{mp}\right)^*} \right)^{\frac{1}{\left(\frac{q}{mp}\right)^*}} \right]^{\frac{1}{p}} \end{aligned}$$

$$= \|P\| \left(\sum_{k=1}^n \|x_k\|^q \right)^{\frac{m}{q}} n^{\frac{q-mp}{qp}}.$$

Hence, for $0 < q \leq 2$, we have

$$\begin{aligned} \left(\sum_{k=1}^n \|P(x_k)\|^p \right)^{\frac{1}{p}} &\leq \|P\| n^{\frac{m}{q}} n^{\frac{q-mp}{qp}} \|(x_k)_{k=1}^n\|_{w,q}^m \\ &= \|P\| n^{\frac{1}{p}} \|(x_k)_{k=1}^n\|_{w,q}^m, \end{aligned}$$

and

$$\eta_{(p,q)}^{m-pol}(E; F) \leq \frac{1}{p}.$$

For $q \geq 2$ we obtain

$$\begin{aligned} \left(\sum_{k=1}^n \|P(x_k)\|^p \right)^{\frac{1}{p}} &\leq \|P\| n^{\frac{m}{2}} n^{\frac{q-mp}{qp}} \|(x_k)_{k=1}^n\|_{w,q}^m \\ &= \|P\| n^{\frac{mpq+2q-2pm}{2pq}} \|(x_k)_{k=1}^n\|_{w,q}^m \end{aligned}$$

and thus

$$\eta_{(p,q)}^{m-pol}(E; F) \leq \frac{1}{p} + \frac{m(q-2)}{2q}. \quad \square$$

3. Main results: vector-valued maps

We begin this section with a (simple) technical lemma.

Lemma 3.1. *Let E be an n -dimensional Banach space. If $1 \leq d \leq s \leq 2$, then there exists a constant $K > 0$ such that*

$$Kn^{\frac{2d+s(d-2)}{2sd}} \leq \pi_{s,d}^{(n)}(id_E).$$

Proof. Using the Inclusion Theorem [5, Theorem 10.4] we have

$$\pi_{\frac{2sd}{2d+s(d-2)}, 2}^{(n)}(id_E) \leq \pi_{s,d}^{(n)}(id_E)$$

and by invoking Theorem 1.1 we know that there is a constant $C > 0$ such that

$$\frac{1}{C} \pi_{\frac{2sd}{2d+s(d-2)}, 2}^{(n)}(id_E) \leq \pi_{\frac{2sd}{2d+s(d-2)}, 2}^{(n)}(id_E).$$

Theorem 1.2 assures the existence of a constant $A > 0$ such that

$$An^{\frac{1}{2d+s(d-2)}} \leq \pi_{\frac{2sd}{2d+s(d-2)}, 2}^{(n)}(id_E).$$

Therefore

$$Kn^{\frac{2d+s(d-2)}{2sd}} \leq \pi_{s,d}^{(n)}(id_E),$$

where $K = \frac{A}{C}$. $\square$

We recall that for $2 \leq q \leq \infty$, a Banach space E has cotype q if there is a constant $C \geq 0$ such that no matter how we select finitely many vectors $x_1, \dots, x_n$ from E ,

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{\frac{1}{q}} \leq C \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \right)^{\frac{1}{2}},$$

where r_k denotes the k -th Rademacher function, that is, given $k \in \mathbb{N}$ and $t \in [0, 1]$, we have $r_k(t) = \text{sign} [\sin (2^k \pi t)]$. When $q = \infty$, the left hand side will be replaced by the sup norm. It is plain that if $q_1 \leq q_2$, then E has cotype q_1 implies that E has cotype q_2 ; thus, henceforth, we will denote $\inf\{q : E \text{ has cotype } q\}$ by $\text{cot}(E)$.

Now we state and prove the main result of this section. The arguments are based in ideas taken from [4,10]:

Theorem 3.2. *Let E, F be infinite dimensional Banach spaces and $r := \text{cot}(F)$.*

(a) *For $1 \leq q \leq 2$ and $0 < p \leq \frac{rq}{mr+q}$, we have*

$$\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; F).$$

(b) *For $1 \leq q \leq 2$ and $\frac{rq}{mr+q} \leq p \leq \frac{2r}{mr+2}$, we have*

$$\frac{mp+2}{2p} - \frac{mr+q}{rq} \leq \eta_{(p,q)}^{m-pol}(E; F).$$

(c) *For $2 \leq q < \infty$ and $0 < p \leq \frac{2r}{mr+2}$, we have*

$$\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; F).$$

(d) *For $2 \leq q < \infty$ and $\frac{2r}{mr+2} < p < r$, we have*

$$\frac{r-p}{pr} \leq \eta_{(p,q)}^{m-pol}(E; F).$$

Proof. Since F is infinite dimensional, from [5, Theorem 14.5] we have

$$\text{cot}(F) = \sup\{2 \leq s \leq \infty : F \text{ finitely factors the formal inclusion } \ell_s \hookrightarrow \ell_\infty\},$$

and from [5, p.304] we know that this supremum is attained. So F finitely factors the formal inclusion $\ell_r \hookrightarrow \ell_\infty$, that is, there exist $C_1, C_2 > 0$ such that for every $n \in \mathbb{N}$, there are $y_1, \dots, y_n \in F$ so that

$$C_1 \left\| (a_j)_{j=1}^n \right\|_\infty \leq \left\| \sum_{j=1}^n a_j y_j \right\| \leq C_2 \left(\sum_{j=1}^n |a_j|^r \right)^{\frac{1}{r}} \quad (7)$$

for every $a_1, \dots, a_n \in \mathbb{K}$.

Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$. Consider $x_1^*, \dots, x_n^* \in B_{E^*}$ such that $x_j^*(x_j) = \|x_j\|$ for every $j = 1, \dots, n$. Let $a_1, \dots, a_n$ be scalars such that $\sum_{j=1}^n |a_j|^{\frac{r}{p}} = 1$ and define

$$P_n: E \longrightarrow F, \quad P_n(x) = \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x)^m y_j.$$

Then for every $x \in E$, by (7)

$$\|P_n(x)\| = \left\| \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x)^m y_j \right\| \leq C_2 \left(\sum_{j=1}^n \left| |a_j|^{\frac{1}{p}} x_j^*(x)^m \right|^r \right)^{\frac{1}{r}} \leq C_2 \left(\sum_{j=1}^n |a_j|^{\frac{r}{p}} \right)^{\frac{1}{r}} \|x\|^m = C_2 \|x\|^m,$$

and thus

$$\|P_n\| \leq C_2. \quad (8)$$

Note that for $k = 1, \dots, n$, from (7), we have

$$\begin{aligned} \|P_n(x_k)\| &= \left\| \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x_k)^m y_j \right\| \geq C_1 \left\| \left(|a_j|^{\frac{1}{p}} x_j^*(x_k)^m \right)_{j=1}^n \right\|_{\infty} \geq C_1 |a_k|^{\frac{1}{p}} x_k^*(x_k)^m \\ &= C_1 |a_k|^{\frac{1}{p}} \|x_k\|^m. \end{aligned} \quad (9)$$

Hence,

$$\begin{aligned} \left(\sum_{j=1}^n \|x_j\|^{mp} |a_j| \right)^{\frac{1}{p}} &= \left(\sum_{j=1}^n \left(\|x_j\|^m |a_j|^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\ &= \frac{1}{C_1} \left(\sum_{j=1}^n \left(C_1 \|x_j\|^m |a_j|^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\ &\stackrel{(9)}{\leq} \frac{1}{C_1} \left(\sum_{j=1}^n \|P_n(x_j)\|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Suppose that there exists $t \geq 0$ and $D > 0$ such that

$$\left(\sum_{j=1}^n \|P_n(x_j)\|^p \right)^{\frac{1}{p}} \leq D n^t \|P_n\| \left\| (x_j)_{j=1}^n \right\|_{w,q}^m,$$

hence

$$\left(\sum_{j=1}^n \|x_j\|^{mp} |a_j| \right)^{\frac{1}{p}} \leq \frac{D}{C_1} \|P_n\| n^t \left\| (x_j)_{j=1}^n \right\|_{w,q}^m. \quad (10)$$

Since this last inequality holds whenever $\sum_{j=1}^n |a_j|^{\frac{r}{p}} = 1$ and $p < r$, we have

$$\left(\sum_{j=1}^n \|x_j\|^{mp \left(\frac{r}{p}\right)^*} \right)^{\frac{1}{\left(\frac{r}{p}\right)^*}} = \sup \left\{ \left\| \sum_{j=1}^n a_j \|x_j\|^{mp} \right\| : \sum_{j=1}^n |a_j|^{\frac{r}{p}} = 1 \right\}$$

$$\begin{aligned}
&\leq \sup \left\{ \sum_{j=1}^n |a_j| \|x_j\|^{mp} : \sum_{j=1}^n |a_j|^{\frac{r}{p}} = 1 \right\} \\
&\stackrel{(10)}{\leq} \left(\frac{D}{C_1} \|P_n\| n^t \|(x_j)_{j=1}^n\|_{w,q}^m \right)^p \\
&\stackrel{(8)}{\leq} \left(\frac{DC_2}{C_1} n^t \|(x_j)_{j=1}^n\|_{w,q}^m \right)^p
\end{aligned}$$

and thus, denoting $\frac{DC_2}{C_1} := Q$, we obtain

$$\frac{\left(\sum_{j=1}^n \|x_j\|^{mp\left(\frac{r}{p}\right)^*} \right)^{\frac{1}{\left(\frac{r}{p}\right)^*}}}{\|(x_j)_{j=1}^n\|_{w,q}^{mp}} \leq n^{tp} Q^p.$$

Therefore

$$\frac{\left(\sum_{j=1}^n \|x_j\|^{mp\left(\frac{r}{p}\right)^*} \right)^{\frac{1}{mp\left(\frac{r}{p}\right)^*}}}{\|(x_j)_{j=1}^n\|_{w,q}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}. \quad (11)$$

Note that (11) is valid for any $x_1, \dots, x_n$. So, for any n -dimensional subspace X of E we have

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^{mp\left(\frac{r}{p}\right)^*} \right)^{\frac{1}{mp\left(\frac{r}{p}\right)^*}}}{\|(x_j)_{j=1}^n\|_{w,q}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}, \quad (12)$$

for all $x_1, \dots, x_n \in X$.

(a) Since

$$0 < p \leq \frac{rq}{mr + q},$$

we have

$$mp \left(\frac{r}{p} \right)^* \leq q,$$

and

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^q \right)^{\frac{1}{q}}}{\|(x_j)_{j=1}^n\|_{w,q}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

So

$$\pi_q^{(n)}(id_X) \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

Since $q \leq 2$, by [5, Theorem 2.8] we get

$$\pi_2^{(n)}(id_X) \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}. \quad (13)$$

Now Theorem 1.1 assures that there is a constant $C > 0$ such that

$$\pi_2(id_X) \leq C\pi_2^{(n)}(id_X). \quad (14)$$

Using (13), (14) and Theorem 2.1 we obtain

$$\frac{1}{C}n^{\frac{1}{2}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

Thus

$$t \geq \frac{m}{2}.$$

Therefore

$$\eta_{(p,q)}^{m-pol}(E; F) \geq \frac{m}{2}.$$

(b) By (12), we have

$$\pi_{mp\left(\frac{r}{p}\right)^*, q}^{(n)}(id_X) \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}. \quad (15)$$

Since $\frac{rq}{mr+q} \leq p \leq \frac{2r}{mr+2}$ and $mp\left(\frac{r}{p}\right)^* = \frac{mpr}{r-p}$, we have $q \leq mp\left(\frac{r}{p}\right)^* \leq 2$. From Lemma 3.1, there is a constant $K > 0$ such that

$$Kn^{\frac{2q+mp\left(\frac{r}{p}\right)^*(q-2)}{2mp\left(\frac{r}{p}\right)^*q}} \leq \pi_{mp\left(\frac{r}{p}\right)^*, q}^{(n)}(id_X). \quad (16)$$

From (15) and (16) it follows that

$$Kn^{\frac{2q+mp\left(\frac{r}{p}\right)^*(q-2)}{2mp\left(\frac{r}{p}\right)^*q}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

Thus

$$\frac{t}{m} \geq \frac{mp+2}{2mp} - \frac{mr+q}{mrq}$$

and we conclude that

$$t \geq \frac{mp+2}{2p} - \frac{mr+q}{rq}.$$

Therefore

$$\eta_{(p,q)}^{m-pol}(E; F) \geq \frac{mp+2}{2p} - \frac{mr+q}{rq}.$$

(c) Since $q \geq 2$, it follows from (12) and from the canonical inclusion between the $\ell_p^w(X)$ -norms that

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^{mp\left(\frac{r}{p}\right)^*}\right)^{\frac{1}{mp\left(\frac{r}{p}\right)^*}}}{\|(x_j)_{j=1}^n\|_{w,2}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}},$$

for all $x_1, \dots, x_n \in X$. But $\frac{2r}{mr+2} \geq p$ implies that $mp\left(\frac{r}{p}\right)^* \leq 2$, and thus

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^2\right)^{\frac{1}{2}}}{\|(x_j)_{j=1}^n\|_{w,2}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

Therefore

$$\pi_2^{(n)}(id_X) \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

From [Theorem 1.1](#) it follows that

$$\pi_2(id_X) \leq C\pi_2^{(n)}(id_X).$$

By [Theorem 2.1](#), we have

$$\frac{1}{C}n^{\frac{1}{2}} = \frac{1}{C}\pi_2(id_X) \leq n^{\frac{t}{m}} Q^{\frac{1}{m}},$$

and we conclude that

$$t \geq \frac{m}{2},$$

that is

$$\eta_{(p,q)}^{m-pol}(E; F) \geq \frac{m}{2}.$$

(d) Since $q \geq 2$, as in (c) it follows that

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^{mp\left(\frac{r}{p}\right)^*}\right)^{\frac{1}{mp\left(\frac{r}{p}\right)^*}}}{\|(x_j)_{j=1}^n\|_{w,2}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}},$$

for all $x_1, \dots, x_n \in X$. Then

$$\pi_{mp\left(\frac{r}{p}\right)^*, 2}^{(n)}(id_X) \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

But $\frac{2r}{mr+2} < p$ implies that $mp\left(\frac{r}{p}\right)^* > 2$, and from [Theorem 1.1](#) it follows that

$$\pi_{mp\left(\frac{r}{p}\right)^*, 2}^{(n)}(id_X) \leq C\pi_{mp\left(\frac{r}{p}\right)^*, 2}^{(n)}(id_X).$$

By [Theorem 1.2](#), there is a constant $A > 0$ such that

$$A \cdot n^{\frac{1}{mp(\frac{r}{p})^*}} \leq \pi_{mp(\frac{r}{p})^*, 2}(id_X),$$

and thus

$$\frac{A}{C} n^{\frac{r-p}{m.p^r}} \leq n^{\frac{t}{m}} Q^{\frac{1}{m}}.$$

Finally, we obtain

$$t \geq \frac{r-p}{pr},$$

and

$$\eta_{(p,q)}^{m-pol}(E; F) \geq \frac{r-p}{pr}. \quad \square$$

Remark 3.3. In the above result, there is a kind of continuity. In fact, when $p = \frac{rq}{mr+q}$, from (a) we have

$$\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; F).$$

On the other hand, considering $p = \frac{rq}{mr+q}$ it follows from (b) that

$$\frac{mp+2}{2p} - \frac{mr+q}{rq} = \frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; F).$$

Now, when $p = \frac{2r}{mr+2}$, by (c) we have

$$\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; F). \quad (17)$$

Given $\epsilon > 0$ and taking $p_\epsilon = \frac{2r}{mr+2} + \epsilon$ it follows by (d) that

$$\frac{r-p_\epsilon}{p_\epsilon r} \leq \eta_{(p_\epsilon,q)}^{m-pol}(E; F). \quad (18)$$

Again, there is a continuity between the lower estimates (17) and (18), because letting ϵ tend to zero, we have

$$p_\epsilon \rightarrow \frac{2r}{mr+2} \quad \text{and} \quad \frac{r-p_\epsilon}{p_\epsilon r} \rightarrow \frac{m}{2}.$$

The same behavior happens when $q = 2$.

The next two results provide optimality of $\eta_{(p,q)}^{m-pol}(E; F)$ in some cases:

Corollary 3.4. If $\frac{2}{2m+1} \leq p < \frac{2}{m+1}$, then $\eta_{(p,1)}^{m-pol}(\ell_1; \ell_2) = \frac{1}{p} - \frac{m+1}{2}$.

Proof. Considering $q = 1$ and $r = 2$ in the previous theorem (item (b)) we have

$$\eta_{(p,1)}^{m-pol}(\ell_1; \ell_2) \geq \frac{1}{p} - \frac{m+1}{2}. \quad (19)$$

Let us show that (19) is sharp. From [2] we know that every continuous m -homogeneous polynomial from ℓ_1 to ℓ_2 is absolutely $(\frac{2}{m+1}, 1)$ -summing. Since $\frac{2}{2m+1} \leq p < \frac{2}{m+1}$, let $w > 0$ be such that

$$\frac{1}{p} = \frac{1}{\frac{2}{m+1}} + \frac{1}{w}.$$

Given $P \in \mathcal{P}(^m \ell_1; \ell_2)$, from the Hölder's inequality we have

$$\left(\sum_{k=1}^n \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{w}} \left(\sum_{k=1}^n \|P(x_k)\|^{\frac{2}{m+1}} \right)^{\frac{m+1}{2}} \leq D n^{\frac{1}{p} - \frac{m+1}{2}} \|(x_k)_{k=1}^n\|_{w,1}^m. \quad \square$$

Corollary 3.5. *Let K be a compact Hausdorff space and F be an infinite dimensional Banach space, with $\cot(F) = r$. If $\frac{2r}{r+2} < p < r$, then*

$$\eta_{(p,2)}(C(K); F) = \frac{1}{p} - \frac{1}{r}.$$

Proof. By the previous theorem (item (d)), if $q = 2$ and $\cot(F) = r$ we have

$$\eta_{(p,2)}(C(K); F) \geq \frac{1}{p} - \frac{1}{r}. \quad (20)$$

Let us show that (20) is sharp. From [5, Theorem 11.14] we know that every continuous linear operator from $C(K)$ to F , with $\cot(F) = r$, is absolutely $(r, 2)$ -summing.

Let $\frac{2r}{r+2} < p < r$ and let $w > 0$ be such that

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{w}.$$

Given $T \in \mathcal{L}(C(K); F)$, from the Hölder's inequality we have

$$\left(\sum_{k=1}^n \|T(x_k)\|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{w}} \left(\sum_{k=1}^n \|T(x_k)\|^r \right)^{\frac{1}{r}} \leq D n^{\frac{1}{p} - \frac{1}{r}} \|(x_k)_{k=1}^n\|_{w,2},$$

i.e.,

$$\eta_{(p,2)}(C(K); F) \leq \frac{1}{p} - \frac{1}{r}. \quad \square$$

4. Main results: real-valued maps

The following result complements the results of the previous section (its proof is inspired in techniques found in [3]); now we consider the case in which m is even and $F = \mathbb{R}$.

Theorem 4.1. *Let m be an even positive integer and E be an infinite dimensional real Banach space.*

(a) *If $1 \leq q \leq 2$ and $0 < p \leq \frac{q}{m+q}$, then*

$$\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}).$$

(b) If $1 \leq q \leq 2$ and $\frac{q}{m+q} \leq p \leq \frac{2}{m+2}$, then

$$\frac{mp+2}{2p} - \frac{m+q}{q} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}).$$

(c) If $2 \leq q < \infty$ and $0 < p \leq \frac{2}{m+2}$, then

$$\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}).$$

(d) If $2 \leq q < \infty$ and $\frac{2}{m+2} < p < 1$, then

$$\frac{1-p}{p} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}).$$

Proof. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$. Consider $x_1^*, \dots, x_n^* \in B_{E^*}$ such that $x_j^*(x_j) = \|x_j\|$ for every $j = 1, \dots, n$. Let $a_1, \dots, a_n$ be real numbers such that $\sum_{j=1}^n |a_j|^{\frac{1}{p}} = 1$ and define

$$P_n: E \longrightarrow \mathbb{R}, \quad P_n(x) = \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x)^m, \quad \text{for every } x \in E.$$

Since m is even, it follows that $P_n(x) \geq 0$, for every $x \in E$. Hence

$$|P_n(x)| = P_n(x) = \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x)^m \geq |a_k|^{\frac{1}{p}} x_k^*(x)^m, \quad \text{for every } x \in E \text{ and } k = 1, \dots, n,$$

and

$$|P_n(x_k)| = P_n(x_k) = \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x_k)^m \geq |a_k|^{\frac{1}{p}} x_k^*(x_k)^m = |a_k|^{\frac{1}{p}} \|x_k\|^m, \quad \text{for } k = 1, \dots, n. \quad (21)$$

Furthermore, for every $x \in E$, we have

$$|P_n(x)| = \left| \sum_{j=1}^n |a_j|^{\frac{1}{p}} x_j^*(x)^m \right| \leq \left(\sum_{j=1}^n |a_j|^{\frac{1}{p}} \right) \|x\|^m = \|x\|^m,$$

and thus

$$\|P_n\| \leq 1. \quad (22)$$

Therefore,

$$\begin{aligned} \left(\sum_{j=1}^n \|x_j\|^{mp} |a_j| \right)^{\frac{1}{p}} &= \left(\sum_{j=1}^n \left(\|x_j\|^m |a_j|^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\ &\stackrel{(21)}{\leq} \left(\sum_{j=1}^n |P_n(x_j)|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Suppose that there exists $t \geq 0$ and $D > 0$ such that

$$\begin{aligned} \left(\sum_{j=1}^n \|P_n(x_j)\|^p \right)^{\frac{1}{p}} &\leq D \|P_n\| n^t \|(x_j)_{j=1}^n\|_{w,q}^m \\ &\stackrel{(22)}{\leq} D n^t \|(x_j)_{j=1}^n\|_{w,q}^m. \end{aligned}$$

Hence

$$\left(\sum_{j=1}^n \|x_j\|^{mp} |a_j| \right)^{\frac{1}{p}} \leq D n^t \|(x_j)_{j=1}^n\|_{w,q}^m, \quad (23)$$

and since this last inequality holds whenever $\sum_{j=1}^n |a_j|^{\frac{1}{p}} = 1$ and $p < 1$, we have

$$\begin{aligned} \left(\sum_{j=1}^n \|x_j\|^{\frac{mp}{1-p}} \right)^{1-p} &= \sup \left\{ \left| \sum_{j=1}^n a_j \|x_j\|^{mp} \right| : \sum_{j=1}^n |a_j|^{\frac{1}{p}} = 1 \right\} \\ &\leq \sup \left\{ \sum_{j=1}^n |a_j| \|x_j\|^{mp} : \sum_{j=1}^n |a_j|^{\frac{1}{p}} = 1 \right\} \\ &\stackrel{(23)}{\leq} \left(D n^t \|(x_j)_{j=1}^n\|_{w,q}^m \right)^p. \end{aligned}$$

Therefore

$$\frac{\left(\sum_{j=1}^n \|x_j\|^{\frac{mp}{1-p}} \right)^{\frac{1-p}{mp}}}{\|(x_j)_{j=1}^n\|_{w,q}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}}. \quad (24)$$

See that (24) is valid for any $x_1, \dots, x_n$. So, for any n -dimensional subspace X of E we have

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^{\frac{mp}{1-p}} \right)^{\frac{1-p}{mp}}}{\|(x_j)_{j=1}^n\|_{w,q}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}}, \quad (25)$$

for all $x_1, \dots, x_n \in X$.

Now we prove each item separately.

(a) Since

$$0 < p \leq \frac{q}{m+q},$$

we have

$$\frac{mp}{1-p} \leq q$$

and thus

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^q\right)^{\frac{1}{q}}}{\|(x_j)_{j=1}^n\|_{w,q}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}}.$$

So

$$\pi_q^{(n)}(id_X) \leq D^{\frac{1}{m}} n^{\frac{t}{m}}.$$

Since $1 \leq q \leq 2$ by [5, Theorem 2.8] we have

$$\pi_2^{(n)}(id_X) < D^{\frac{1}{m}} n^{\frac{t}{m}}, \quad (26)$$

and from Theorem 1.1 we conclude that

$$\pi_2(id_X) \leq C\pi_2^{(n)}(id_X). \quad (27)$$

By Theorem 2.1 we know that

$$\pi_2(id_X) = n^{\frac{1}{2}}$$

and thus, from (26) and (27), it follows that

$$\frac{1}{C} n^{\frac{1}{2}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}}.$$

Hence

$$t \geq \frac{m}{2},$$

i.e.,

$$\eta_{(p,q)}^{m-pol}(E; \mathbb{R}) \geq \frac{m}{2}.$$

(b) By (25), we have

$$\pi_{\frac{mp}{1-p},q}^{(n)}(id_X) \leq n^{\frac{t}{m}} D^{\frac{1}{m}}. \quad (28)$$

Since $\frac{q}{m+q} \leq p \leq \frac{2}{m+2}$ we have $q \leq \frac{mp}{1-p} \leq 2$. From (28) and Lemma 3.1, there is a constant $K > 0$ such that

$$Kn^{\frac{2q + \frac{mp}{1-p}(q-2)}{2 \frac{mp}{1-p} q}} \leq n^{\frac{t}{m}} D^{\frac{1}{m}}.$$

Thus

$$\frac{t}{m} \geq \frac{mp+2}{2mp} - \frac{m+q}{mq}$$

and we conclude that

$$t \geq \frac{mp+2}{2p} - \frac{m+q}{q},$$

that is,

$$\eta_{(p,q)}^{m-pol}(E; \mathbb{R}) \geq \frac{mp+2}{2p} - \frac{m+q}{q}.$$

(c) Since $q \geq 2$, we have

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^{\frac{mp}{1-p}}\right)^{\frac{1-p}{mp}}}{\|(x_j)_{j=1}^n\|_{w,2}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}},$$

for all $x_1, \dots, x_n \in X$. But $\frac{2}{m+2} \geq p$ implies that $\frac{mp}{1-p} \leq 2$; hence

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^2\right)^{\frac{1}{2}}}{\|(x_j)_{j=1}^n\|_{w,2}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}},$$

and thus

$$\pi_2^{(n)}(id_X) \leq D^{\frac{1}{m}} n^{\frac{t}{m}}.$$

From [Theorem 1.1](#) we have

$$\pi_2(id_X) \leq C\pi_2^{(n)}(id_X)$$

and from [Theorem 2.1](#), we have

$$\frac{1}{C}n^{\frac{1}{2}} = \frac{1}{C}\pi_2(id_X) \leq n^{\frac{t}{m}}D^{\frac{1}{m}}.$$

So, we conclude that

$$t \geq \frac{m}{2}$$

and

$$\eta_{(p,q)}^{m-pol}(E; \mathbb{R}) \geq \frac{m}{2}.$$

(d) Since $q \geq 2$, we obtain

$$\frac{\left(\sum_{j=1}^n \|id_X(x_j)\|^{\frac{mp}{1-p}}\right)^{\frac{1-p}{mp}}}{\|(x_j)_{j=1}^n\|_{w,2}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}},$$

for all $x_1, \dots, x_n \in X$.

So

$$\pi_{\frac{mp}{1-p},2}^{(n)}(id_X) \leq n^{\frac{t}{m}}D^{\frac{1}{m}}.$$

But $\frac{2}{m+2} < p$ implies that $\frac{mp}{1-p} > 2$, and from [Theorem 1.1](#)

$$\pi_{\frac{mp}{1-p}, 2}(id_X) \leq C \pi_{\frac{mp}{1-p}, 2}^{(n)}(id_X).$$

By [Theorem 1.2](#), there is a constant $A > 0$ such that

$$A n^{\frac{1-p}{mp}} \leq \pi_{\frac{mp}{1-p}, 2}(id_X),$$

thus

$$\frac{A}{C} n^{\frac{1-p}{mp}} \leq n^{\frac{t}{m}} D^{\frac{1}{m}},$$

we conclude that

$$t \geq \frac{1-p}{p},$$

so

$$\eta_{(p,q)}^{m-pol}(E; \mathbb{R}) \geq \frac{1-p}{p}. \quad \square$$

Remark 4.2. (i) As in the previous theorem, in this result we have a clear “continuity” in our estimates.

(ii) We stress that [Theorem 4.1](#) cannot be compared with [Theorem 3.2](#), since [Theorem 3.2](#) is valid only when F is infinite dimensional.

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References

- [1] G. Araújo, D. Pellegrino, Optimal Hardy–Littlewood inequalities for m -linear forms on ℓ_p spaces with $1 \leq p \leq m$, *Arch. Math.* 106 (2015) 285–295.
- [2] A.T.L. Bernardino, On cotype and a Grothendieck-type theorem for absolutely summing multilinear operators, *Quaest. Math.* 34 (2011) 513–519.
- [3] G. Botelho, D. Pellegrino, P. Rueda, Dominated bilinear forms and 2-homogeneous polynomials, *Publ. Res. Inst. Math. Sci.* 46 (2010) 201–208.
- [4] G. Botelho, D. Pellegrino, P. Rueda, Cotype and absolutely summing linear operators, *Math. Z.* 267 (2011) 1–7.
- [5] J. Diestel, H. Jarchow, A. Tonge, *Absolutely Summing Operators*, Cambridge University Press, Cambridge, 1995.
- [6] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London, 1999.
- [7] D. Galicer, M. Mansilla, S. Muro, The sup-norm vs. the norm of the coefficients: equivalence constants for homogeneous polynomials, *arXiv:1602.01735v1 [math.FA]*.
- [8] D.J.H. Garling, *Inequalities: A Journey into Linear Analysis*, Cambridge University Press, Cambridge, 2007.
- [9] H. König, J.R. Retherford, N. Tomczak-Jaegermann, On the eigenvalues of $(p, 2)$ -summing operators and constants associated with normed spaces, *J. Funct. Anal.* 37 (1980) 88–126.
- [10] J. Lindenstrauss, A. Pelczynski, Absolutely summing operators in $\mathcal{L}_p$ -spaces and their applications, *Studia Math.* 29 (1968) 275–326.
- [11] M.C. Matos, Fully absolutely summing and Hilbert–Schmidt multilinear mappings, *Collect. Math.* 54 (2003) 111–136.
- [12] J. Mujica, *Complex Analysis in Banach Spaces*, Dover Publications, 2010.
- [13] D. Pérez-García, *Operadores multilineales absolutamente sumantes*, Thesis, Universidad Complutense de Madrid, 2003.
- [14] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam/New York/Oxford, 1980.
- [15] S.J. Szarek, Computing summing norms and type constants on few vectors, *Studia Math.* 98 (1991) 148–156.
- [16] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite Dimensional Operator Ideals*, Longman, Harlow, and Wiley, New York, 1988.