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Sensitivity of dendrite maps

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ABSTRACT

Suppose that X is a dendrite and $f : X \to X$ is a sensitive continuous map. We show that (a) (X, f) contains a bilaterally transitive subsystem with nonempty interior; (b) the system (X, f) satisfies only one of the following two conditions: (b1) (X, f) contains a topologically transitive subsystem with nonempty interior; (b2) there exists an f-invariant nowhere dense closed subset A of X such that the attraction basin $\text{Basin}(A, f)$ contains a residual subset B of an open set and the strong attraction basin $\text{Sbasin}(A, f)$ is dense in B; (c) if X is completely regular, then (X, f) contains a relatively strongly mixing subsystem with nonempty interior, dense periodic points and positive topological entropy. Unlike for interval maps, we construct a sensitive dendrite map with zero topological entropy.

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1. Introduction

A *topological dynamical system* is a pair (X, f) where X is a compact metric space and $f : X \to X$ is a continuous map. A system (X, f) is called *sensitive*, or f is called *sensitive* for simplicity, if there exists a constant $c > 0$, called a *sensitivity constant* of the system (X, f), such that for any nonempty open set $U \subset X$, there is $n \in \mathbb{N}$ such that $\text{diam}(f^n(U)) > c$. Sensitivity is usually regarded as an important feature of chaotic systems, though nowadays there is no universal agreement on the definition of chaos. For example, sensitivity is a key ingredient in the definitions of Devaney chaos and Auslander–Yorke chaos (also called Ruelle–Takens chaos) (see [12,4]).

The relationships between sensitivity, topological transitivity, and topological entropy have been extensively studied. In [8] it is shown that a transitive system (X, f) with dense periodic points must be sensitive except that X is a finite set. This result was extended to transitive non-minimal systems with dense minimal points by Glasner and Weiss in [17]. Also, a transitive system with positive topological entropy must be

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sensitive (see [17]). Some simple examples can show that the converses of the above results are far from being true for general systems. However, the situation is completely different when we consider one-dimensional systems. It is known that sensitivity implies the existence of a transitive cycle of intervals for any interval map f (see [9]). This implies that f has positive topological entropy (see, e.g., [10]). Some stronger forms of sensitivity such as Li–Yorke sensitivity, strong sensitivity, syndetic sensitivity and cofinite sensitivity have been discussed in [3,32]. Sensitivity also played a key role in proving that Devaney's chaos implies Li–Yorke's chaos (see [19,24]).

The aim of this paper is to study sensitivity of dendrite maps. We mainly consider the relations between sensitivity, transitivity and topological entropy for dendrite maps. Recall that a *continuum* is a compact connected metric space, and a *dendrite* is a locally connected continuum containing no simple closed curves. By a *tree*, we mean a connected compact one-dimensional polyhedron which contains no simple closed curves. Clearly trees are dendrites by definition. Dynamical systems on dendrites appeared naturally in the study of complex dynamical systems and hyperbolic geometry. In recent years, many people started to study the dynamics of dendrite maps. Although dendrites possess many properties of trees, dynamical properties on dendrites are much more varied than that on trees. For example, it is well known that the $\overline{P} = \overline{R}$ property holds for trees (see [34]), but a counterexample for the Gehman dendrite was constructed by Kato in [21]. Further, Illanes proved that a dendrite X contains a Gehman dendrite if and only if X does not have the $\overline{P} = \overline{R}$ property in [20] (see also [6,25]). Recently, Hoehn and Mouron gave a map of the Wazewski's universal dendrite that is weakly mixing but not mixing (see [18]) and has a unique periodic point (see [1]), which also showed the sharp difference between the dynamics of tree maps and dendrite maps. One may refer to [5,15,22,29–31] for more results about the dynamics of dendrite maps.

Before the statement of the theorem, let us recall some definitions and notation. We denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ the sets of real numbers, integers and positive integers respectively. Let (X, f) be a topological dynamical system. For $x \in X$, the set $O^+(x, f) = \{f^n(x) : n \in \mathbb{N} \cup \{0\}\}$ is called the *forward orbit* (or usually, the *orbit*) of x under f, and $O^-(x, f) = \cup\{f^{-n}(x) : n \in \mathbb{N} \cup \{0\}\}$ is called the *backward orbit* of x under f. The set $O(x, f) = O^+(x, f) \cup O^-(x, f)$ is called the *bilateral orbit* of x under f. We should note that $f^{-n}(x)$ may be a set with more than one point if f is not injective, so the symbol $O(x, f)$ has different meanings from its usual ones.

Recall that the *ω-limit set* $\omega(x, f)$ of a point $x \in X$ is the set of all limit points of $O^+(x, f)$, i.e., $\omega(x, f) = \{y \in X :$ there is a sequence of positive integers $n_i \to +\infty$ s.t. $f^{n_i}(x) \to y\}$. A point $x \in X$ is a *nonwandering point of* f if for every neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points of f is denoted by $\Omega(X, f)$. A subset A of X is called *f-invariant* provided that $f(A) \subset A$. If $A \subset X$ is closed and f-invariant, then $(A, f|_A)$ is also a topological dynamical system, which is called a *subsystem* of (X, f). The *attraction basin* of an f-invariant closed set A is defined to be the set $\mathrm{Basin}(A, f) = \{x \in X : \omega(x, f) \cap A \neq \emptyset\}$ and the *strong attraction basin* of A is defined to be the set $\mathrm{Sbasin}(A, f) = \{x \in \mathrm{Basin}(A, f) : f^n(x) \in A$ for some $n \in \mathbb{N}\}$. The set A is called the *attracting set* of its attraction basin. To compare the notion of attracting set in the present paper with various definitions of attractors, one may consult [26,27].

In this paper, we will refer to the following kinds of transitivity. Let (X, f) be a topological dynamical system, then

(1) (X, f) is said to be *topologically transitive*, or *transitive* in short, if for every pair of nonempty open subsets U and V of X, there is $n \in \mathbb{N}$, such that $f^n(U) \cap V \neq \emptyset$;
(2) (X, f) is said to be *point transitive*, if there exists a point $x \in X$, such that the closure $\overline{O^+(x, f)} = X$, and the point x is said to be a *transitive point*;
(3) (X, f) is said to be *bilaterally transitive*, if there exists a point $x \in X$, such that $\overline{O(x, f)} = X$;
(4) (X, f) is said to be *strongly mixing* if for any pair of nonempty open subsets U and V of X, there is some $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.

We remark that topological transitivity and point transitivity are equivalent if the phase space X is a compact metric space with no isolated points, and it follows directly from the above definitions that point transitivity implies bilateral transitivity. Therefore, it is obvious that topological transitivity implies bilateral transitivity and strong mixing implies topological transitivity. When the system (X, f) is topologically transitive (respectively, point transitive, bilaterally transitive, strongly mixing), we also call that the map f is *topologically transitive* (respectively, *point transitive, bilaterally transitive, strongly mixing*). A set $D \subset X$ is *regularly closed* if it is the closure of its interior. A *regular periodic decomposition* for f is a finite sequence $\mathcal{D} = (D_0, \ldots, D_{m-1})$ of regularly closed subsets of X covering X such that $f(D_i) \subset D_{i+1(\mathrm{mod}\, m)}$ for $0 \leq i \leq m-1$ and $D_i \cap D_j$ is nowhere dense in X for $i \neq j$. We say, according to [7], that f is *strongly mixing relative to a regular periodic decomposition* $\mathcal{D}$ if f^m is strongly mixing on each of the sets D_i. Also, we say that f is *relatively strongly mixing* if it is strongly mixing relative to some of its regular periodic decompositions (see [14]). A continuum X is *completely regular* if every non-degenerate subcontinuum of X has a nonempty interior. A subset A of an open set V in a topological space X is called a *residual subset* of V, if it contains the intersection of a countable family of dense open subsets of V.

In this paper, we will prove the following theorem in Section 3.

Theorem 1.1. *Let X be a dendrite and $f : X \to X$ be a sensitive map. Then the following three statements hold:*

(1) (X, f) contains a bilaterally transitive subsystem $(Y, f|_Y)$ such that $\mathrm{Int}(Y) \neq \emptyset$;
(2) the system (X, f) satisfies only one of the following two conditions:
  (2.1) (X, f) contains a topologically transitive subsystem $(Z, f|_Z)$ such that $\mathrm{Int}(Z) \neq \emptyset$;
  (2.2) there exist an f-invariant closed subset A of X such that $\mathrm{Int}(A) = \emptyset$, an open subset V of X and a residual subset B of V such that $B \subset \mathrm{Basin}(A, f)$ and $\mathrm{Sbasin}(A, f)$ is dense in B.
(3) If X is completely regular then assertion (2.1) holds and, even more, (X, f) contains a topologically transitive subsystem $(Z, f|_Z)$ with $\mathrm{Int}(Z) \neq \emptyset$ so that $(Z, f|_Z)$ is relatively strongly mixing, has dense periodic points and positive topological entropy.

Unlike for interval maps, we construct a sensitive dendrite map with zero topological entropy in Section 4.

2. Dendrites

In this section, we give some topological properties of dendrites, which are needed in the proof of the main theorem.

Suppose that X is a compact metric space with metric d. Let $A, B \subset X$. We use the symbols $\overline{A}$ and $\mathrm{Int}(A)$ to denote the closure and the interior of A in X respectively. Define the diameter of A by $\mathrm{diam}(A) = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$. Write $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Let X be a continuum with metric d. The *hyperspace* of X is defined by

$$2^X = \{A : A \text{ is a nonempty closed subset of } X\}.$$

For $\epsilon > 0$ and $A \in 2^X$, let $N_d(\epsilon, A) = \{x \in X : d(x, a) < \epsilon, \text{ for some } a \in A\}$. For any $A, B \in 2^X$, define $d_H(A, B) = \inf\{\epsilon > 0 : A \subset N_d(\epsilon, B) \text{ and } B \subset N_d(\epsilon, A)\}$. Then d_H is a metric on 2^X and is called the *Hausdorff metric* induced by d (see [28, pp. 52–53]).

Let X be a dendrite and $x \in X$. Define $C_x = \{C \subset X - \{x\} : C \text{ is a component of } X - \{x\}\}$. We define the *order* of x in X, denoted by $\mathrm{ord}(x, X)$, as follows: if the cardinality $|C_x|$ of the family C_x is finite, we define $\mathrm{ord}(x, X) = |C_x|$. If $|C_x|$ is not finite then it is countable and the elements of C_x describe a sequence whose diameters tend to zero. In such case we define $\mathrm{ord}(x, X) = \omega$. If x is of order 1, then x is called an *endpoint*. The set of endpoints of X is denoted by $E(X)$. If $\mathrm{ord}(x, X) \geq 3$, then x is called a *branch point*. The set of branch points of X is denoted by $B(X)$.

The following properties of dendrites are well known: every connected subset of a dendrite is arcwise connected (a topological space X is *arcwise connected* if for any two different points $x, y \in X$, there exists an arc in X with endpoints x and y); every subcontinuum of a dendrite is still a dendrite; the intersection of any two subcontinua of a dendrite is still a dendrite. For any two different points x, y in a dendrite X, there is a unique arc from x to y which is denoted by $[x, y]$. Write $[x, y) = (y, x] = [x, y] - \{y\}$ and $(x, y) = [x, y) - \{x\}$. For more details, we refer to [28, Chapter X].

The following lemma follows from [28, Theorem 10.4, p. 167] and the fact that dendrites are hereditarily locally connected continua.

Lemma 2.1. *Let $C_1, C_2, \ldots$ be pairwise disjoint connected subsets of a dendrite X. Then $\lim_{i\to\infty} \mathrm{diam}(C_i) = 0$.*

Lemma 2.2. *Let X be a dendrite. Then for any $\epsilon > 0$, there exists a finite subset $F \subset X$ such that each component of $X - F$ has diameter $\leq \epsilon$.*

Proof. By Theorem 5 in [23, Chapter VI, p. 302], for each $\epsilon > 0$, there exist finitely many dendrites $D_1, \ldots, D_n$ such that $X = D_1 \cup \cdots \cup D_n$, $\mathrm{diam}(D_i) \leq \epsilon$, $D_i \cap D_j$ has at most one point, and $D_i \cap D_j \cap D_k = \emptyset$ for every $i \neq j \neq k \in \{1, \ldots, n\}$. Take $F = \{x : x \in D_i \cap D_j,\ i, j = 1, \ldots, n \text{ and } i \neq j\}$, then F is a finite subset of X. For each component $C \subset X - F$, by the connectedness of C, we have $C \subset D_i$ for some $i \in \{1, \ldots, n\}$. Hence $\mathrm{diam}(C) \leq \epsilon$. $\square$

The following lemma immediately follows from Theorem 4.11 in [28].

Lemma 2.3. *Suppose that X is a compact metric space with metric d, $Y_1 \supset Y_2 \supset \cdots \supset Y_n \supset \cdots$ is a sequence of decreasing closed sets in X, and $Y = \bigcap_{n\in\mathbb{N}} Y_n$. Then $\lim_{n\to\infty} d_H(Y_n, Y) = 0$.*

Lemma 2.4. *Let X be a dendrite with metric d, $Y_1 \supset Y_2 \supset \cdots \supset Y_n \supset \cdots$ be a sequence of closed subsets in X, and $Y = \bigcap_{n\in\mathbb{N}} Y_n$. Then for every $\epsilon > 0$, there exists some $n \in \mathbb{N}$ such that each component of $Y_n - Y$ has diameter $\leq \epsilon$.*

Proof. Assume, to the contrary, that there is $\epsilon > 0$ so that for each $n \in \mathbb{N}$ there is a component $C = C(n) \subset Y_n - Y$ with $\mathrm{diam}(C) > \epsilon$. Then we will get a sequence of pairwise disjoint arcs $\{[a_i, b_i] : i \in \mathbb{N}\}$ in X such that $\mathrm{diam}([a_i, b_i]) \geq \epsilon$ as follows.

First, let $Y_{n_1} = Y_1$, and take a component $C_1 \subset Y_{n_1} - Y$ with $\mathrm{diam}(C_1) > \varepsilon$. Then there are two points $a_1, b_1 \in C_1$ such that $d(a_1, b_1) \geq \epsilon$. Thus we have that $\mathrm{diam}([a_1, b_1]) \geq \epsilon$. Since $[a_1, b_1] \cap Y = \emptyset$, we have $d([a_1, b_1], Y) > 0$. Write $d_1 = d([a_1, b_1], Y)$. By Lemma 2.3, there is $n_2 \in \mathbb{N}$ such that $d_H(Y_{n_2}, Y) < d_1$. Therefore, $Y_{n_2} \cap [a_1, b_1] = \emptyset$.

Next, take a component $C_2 \subset Y_{n_2} - Y$ such that $\mathrm{diam}(C_2) > \varepsilon$. Then there are two points $a_2, b_2 \in C_2$ such that $d(a_2, b_2) \geq \epsilon$. Thus we have that $\mathrm{diam}([a_2, b_2]) \geq \epsilon$ and $[a_1, b_1] \cap [a_2, b_2] = \emptyset$. Since $[a_2, b_2] \cap Y = \emptyset$, $d([a_2, b_2], Y) > 0$. Write $d_2 = d([a_2, b_2], Y)$.

Now suppose that we have got $[a_i, b_i] \subset C_i \subset Y_{n_i} - Y$ and d_i for $i = 1, 2, \ldots, k$ satisfying:

(1) C_i is a component of $Y_{n_i} - Y$ with $\mathrm{diam}(C_i) > \epsilon$;
(2) $[a_i, b_i]$ is an arc in C_i such that $\mathrm{diam}([a_i, b_i]) \geq \epsilon$;
(3) $d([a_i, b_i], Y) = d_i$ and $d_H(Y_{n_{i+1}}, Y) < d_i$, where $d_1 > d_2 > \cdots > d_k > 0$;
(4) the arcs $[a_i, b_i], i = 1, \ldots, k$, are pairwise disjoint.

Then for $i = k+1$, there exists $Y_{n_{k+1}}$ such that $d_H(Y_{n_{k+1}}, Y) < d_k$ by Lemma 2.3. So $Y_{n_{k+1}} \cap [a_i, b_i] = \emptyset$ for all $i = 1, \cdots, k$. Take a component $C_{k+1} \subset Y_{n_{k+1}} - Y$ with $\mathrm{diam}(C_{k+1}) > \varepsilon$. Then there are two points $a_{k+1}, b_{k+1} \in C_{k+1}$ such that $d(a_{k+1}, b_{k+1}) \geq \epsilon$. Thus we have that $\mathrm{diam}([a_{k+1}, b_{k+1}]) \geq \epsilon$, and $[a_{k+1}, b_{k+1}] \cap [a_i, b_i] = \emptyset$ for $i = 1, \ldots, k$.

By induction, we get pairwise disjoint arcs $\{[a_i, b_i] : i \in \mathbb{N}\}$ with $\mathrm{diam}([a_i, b_i]) \geq \epsilon$, which contradicts Lemma 2.1. Thus, we finish the proof. □

3. Proof of the main theorem

For a topological space X, we call an open set J a *free interval* of X if it is an open subset of X homeomorphic to an open interval in the real line. We denote the topological entropy of (X, f) by $h_{top}(X, f)$. Topological entropy is an important conjugate invariant in dynamical systems, which was first defined by Alder, Konheim and McAndrew in [2], and later extended by Dinaburg (see [13]) and Bowen (see [11]). We assume that the reader is familiar with the definition and the basic properties of topological entropy (see, e.g., [33, Chapter 7] for details).

The following lemma is taken from [14, Theorem C], which is needed in the proof of Theorem 1.1.

Lemma 3.1 *(**Dichotomy for transitive maps**). Let X be a compact metrizable space with a free interval and let $f : X \to X$ be a transitive map. Then exactly one of the following two statements holds.*

(1) The map f is relatively strongly mixing, non-invertible, has positive topological entropy and dense periodic points.
(2) The space X is a disjoint union of finitely many circles, $X = \bigoplus_{i=0}^{n-1} \mathbb{S}_i^1$, which are cyclically permuted by f and, on each of them, f^n is topologically conjugate to the same irrational rotation.

Obviously, if the space X in the above theorem contains no simple closed curves, then necessarily the case (1) holds.

Proof of Theorem 1.1. Let c be a sensitivity constant of the system (X, f). To show (1) note that, since X is a dendrite, by Lemma 2.2, we have a finite set $\{a_1, a_2, \ldots, a_n\}$ in X such that

$$\text{each component of } X - \{a_1, a_2, \ldots, a_n\} \text{ has diameter } \leq c. \tag{$*$}$$

We claim that $\overline{\bigcup_{i=1}^n O^-(a_i, f)} = X$. Indeed, otherwise there exists a connected open set $U \subset X - \overline{\bigcup_{i=1}^n O^-(a_i, f)}$ whose orbit misses all points $a_1, \ldots, a_n$; by the choice of $a_1, \ldots, a_n$, this implies that $\mathrm{diam}(f^k(U)) \leq c$ for each $k \in \mathbb{N}$, which is a contradiction.

By the claim, there is a point $a_i \in \{a_1, \ldots, a_n\}$ such that $\mathrm{Int}\left(\overline{O^-(a_i, f)}\right) \neq \emptyset$. Denote $Y = \overline{O(a_i, f)} = \overline{\bigcup_{j=-\infty}^{+\infty} f^j(a_i)}$. Then we have

$$f(Y) = f\left(\overline{\bigcup_{j=-\infty}^{+\infty} f^j(a_i)}\right) = \overline{f\left(\bigcup_{j=-\infty}^{+\infty} f^j(a_i)\right)} = \overline{\bigcup_{j=-\infty}^{+\infty} f^{j+1}(a_i)} = Y.$$

Thus Y is closed and f-invariant. So $(Y, f|_Y)$ is a bilaterally transitive subsystem of (X, f) with nonempty interior. This shows assertion (1) of Theorem 1.1.

Now, to show assertion (2) let the family

$$\mathcal{F} = \{F \subset X : F \text{ is closed and } f\text{-invariant with } \mathrm{Int}(F) \neq \emptyset\}.$$

Clearly $\mathcal{F}$ is nonempty since $X \in \mathcal{F}$. We endow $\mathcal{F}$ with the partial order given by the inclusion relation "$\supset$". We will consider two cases.

Case 1. For any totally ordered family $\{Y_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{F}$, $\mathrm{Int}\left(\bigcap_{\lambda\in\Lambda} Y_\lambda\right) \neq \emptyset$.

In this case, we see that any totally ordered family $\{Y_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{F}$ admits a lower bound $\bigcap_{\lambda\in\Lambda} Y_\lambda$. According to Zorn's Lemma, there exists a minimal element Z in $\mathcal{F}$. By the definition of $\mathcal{F}$, we see that $\mathrm{Int}(Z) \neq \emptyset$. Now we prove that $(Z, f|_Z)$ is a topologically transitive subsystem. Denote $U = \mathrm{Int}(Z)$. We have the following

Claim 1. For any two nonempty open sets $V, W \subset U$, $\left(\bigcup_{n=0}^{+\infty} f^{-n}(V)\right) \cap W \neq \emptyset$.

Assume, to the contrary, that there are nonempty open sets V and W contained in U such that $\left(\bigcup_{n=0}^{+\infty} f^{-n}(V)\right) \cap W = \emptyset$. Then we have $\left(\bigcup_{n=0}^{+\infty} f^{n}(W)\right) \cap V = \emptyset$, and so $\overline{\bigcup_{n=0}^{+\infty} f^{n}(W)} \cap V = \emptyset$. It follows that $\overline{\bigcup_{n=0}^{+\infty} f^{n}(W)}$ is a proper subset of Z. Since $W \subset \overline{\bigcup_{n=0}^{+\infty} f^{n}(W)}$ and W is an open set, then $\mathrm{Int}\left(\overline{\bigcup_{n=0}^{+\infty} f^{n}(W)}\right) \neq \emptyset$. Thus $\overline{\bigcup_{n=0}^{+\infty} f^{n}(W)} \in \mathcal{F}$, which contradicts the minimality of Z. So the claim holds.

Let $\{U_i\}_{i=1}^{+\infty}$ be a countable basis of neighborhoods of U. For each U_i, according to Claim 1, $\left(\bigcup_{n=0}^{+\infty} f^{-n}(U_i)\right) \cap W \neq \emptyset$ for any open subset $W \subset U$, which implies that $\left(\bigcup_{n=0}^{+\infty} f^{-n}(U_i)\right) \cap U$ is dense in U. Hence, $\left(\bigcap_{i=1}^{+\infty}\left(\bigcup_{n=0}^{+\infty} f^{-n}(U_i)\right)\right) \cap U$ is a dense G_δ set in U. Take a point $x \in \left(\bigcap_{i=1}^{+\infty}\left(\bigcup_{n=0}^{+\infty} f^{-n}(U_i)\right)\right) \cap U$. Then, for every $i \in \mathbb{N}$, there exists an $n(i) \in \mathbb{N}$ such that $f^{n(i)}(x) \in U_i$. It implies that $O^+(x, f) \cap U_i \neq \emptyset$ for every $i \in \mathbb{N}$. Therefore, we have that $U \subset \overline{O^+(x, f)}$. By the minimality of Z, we have $\overline{O^+(x, f)} = Z$. Then $(Z, f|_Z)$ is a topologically transitive subsystem with nonempty interior. This shows that assertion (2.1) of Theorem 1.1 holds.

Case 2. There is a totally ordered family $\{Y_\lambda\}_{\lambda\in\Lambda}$ of sets from $\mathcal{F}$ such that $\mathrm{Int}\left(\bigcap_{\lambda\in\Lambda} Y_\lambda\right) = \emptyset$.

Since X is a compact metric space, by Theorem 1.1.14 in [16, p. 34] we can take a subsequence $\{Y_{\lambda_i}\}_{i=1}^{+\infty} \subset \{Y_\lambda\}_{\lambda\in\Lambda}$ such that $\bigcap_{i=1}^{+\infty} Y_{\lambda_i} = \bigcap_{\lambda\in\Lambda} Y_\lambda$. Denote $A_i = Y_{\lambda_i}$ for every $i \in \mathbb{N}$. Let $A = \bigcap_{i\in\mathbb{N}} A_i = \bigcap_{\lambda\in\Lambda} Y_\lambda$. Then A is a closed f-invariant nowhere dense set. By Lemma 2.4, there exists some $k \in \mathbb{N}$ such that

$$\text{each component of } A_k - A \text{ has diameter } \leq c. \tag{$**$}$$

Define $V = \mathrm{Int}(A_k)$. Note that V is an open subset of X.

Claim 2. $\left(\bigcup_{n=1}^{+\infty} f^{-n}(A)\right) \cap V$ is dense in V.

In fact, for any nonempty connected open set $U \subset A_k$, we have $\mathrm{diam}(f^n(U)) > c$ for some $n \in \mathbb{N}$ by the sensitivity of (X, f). Since $f^n(U) \subset A_k$ and $f^n(U)$ is connected, we have $f^n(U) \cap A \neq \emptyset$ by ($**$). So, $U \cap f^{-n}(A) \neq \emptyset$ and Claim 2 holds by the arbitrariness of U.

For each $i \in \mathbb{N}$, define $U_i = \{x \in X : d(x, A) < 1/i\}$. Since $A \subset U_i$, we have that $\left(\bigcup_{n=1}^{+\infty} f^{-n}(U_i)\right) \cap V$ is dense in V by Claim 2. Thus $\left(\bigcap_{i=1}^{+\infty}\left(\bigcup_{n=1}^{+\infty} f^{-n}(U_i)\right)\right) \cap V$ is a dense G_δ set in V. Let $B = \left(\bigcap_{i=1}^{+\infty}\left(\bigcup_{n=1}^{+\infty} f^{-n}(U_i)\right)\right) \cap V$. Note that B is residual in V. We consider both $\mathrm{Basin}(A, f)$ and $\mathrm{Sbasin}(A, f)$ in V, i.e., we define $\mathrm{Basin}(A, f) = \{x \in V : \omega(x, f) \cap A \neq \emptyset\}$ and $\mathrm{Sbasin}(A, f) = \{x \in \mathrm{Basin}(A, f) : f^n(x) \in A \text{ for some } n \in \mathbb{N}\}$. Then we have the following

Claim 3. $B \subset \mathrm{Basin}(A, f)$ and $\mathrm{Sbasin}(A, f)$ is dense in B.

For each $x \in B$, we have $x \in \bigcup_{n=1}^{+\infty} f^{-n}(U_i)$ for each $i \in \mathbb{N}$. Then there exists some $n_i \in \mathbb{N}$ such that $f^{n_i}(x) \in U_i$. Thus, by the definition of U_i, we have $d(f^{n_i}(x), A) < 1/i$ for each $i \in \mathbb{N}$. This implies that $\omega(x, f) \cap A \neq \emptyset$. Therefore $x \in \mathrm{Basin}(A, f)$. By the arbitrariness of x, we have $B \subset \mathrm{Basin}(A, f)$. In addition, for each $x \in \left(\bigcup_{n=1}^{+\infty} f^{-n}(A)\right) \cap V$, there exists some $n \in \mathbb{N}$ such that $f^n(x) \in A$. Since A is an f-invariant closed set, then $f^{n+k}(x) \in A$ for all $k \in \mathbb{N}$ and $\omega(x,f) \cap A \neq \emptyset$. Thus $x \in \mathrm{Sbasin}(A,f)$. So it implies that $\mathrm{Sbasin}(A,f) = \left(\bigcup_{n=1}^{+\infty} f^{-n}(A)\right) \cap V$. Then $\mathrm{Sbasin}(A,f)$ is dense in B by Claim 2. Hence the claim holds.

This shows that assertion (2.2) of Theorem 1.1 holds. Hence the proof of assertion (2) is complete.

To show assertion (3) assume that X is a completely regular dendrite. Define $\mathcal{F}$ as that in the proof of assertion (2). Let $\{Y_\lambda\}_{\lambda\in\Lambda}$ be any totally ordered family of $\mathcal{F}$. Take a subsequence $Y_{\lambda_1} \supset Y_{\lambda_2} \supset \cdots \supset Y_{\lambda_i} \supset \cdots$ of $\{Y_\lambda\}_{\lambda\in\Lambda}$ such that $\bigcap_{i=1}^{+\infty} Y_{\lambda_i} = \bigcap_{\lambda\in\Lambda} Y_\lambda$. Since X is completely regular, there exists a subcontinuum $A_{\lambda_i} \subset Y_{\lambda_i}$ with $\mathrm{Int}(A_{\lambda_i}) \neq \emptyset$ for every $i \in \mathbb{N}$. As (X,f) is sensitive, for every A_{λ_i}, there is some $n(\lambda_i) \in \mathbb{N}$ such that $\mathrm{diam}\left(f^{n(\lambda_i)}(A_{\lambda_i})\right) > c$. Denote $L_{\lambda_i} = f^{n(\lambda_i)}(A_{\lambda_i})$. Since the hyperspace 2^X is compact under the Hausdorff topology, there exists a subsequence of $\{L_{\lambda_i}\}_{i\in\mathbb{N}}$ which converges to a subcontinuum L of X. Obviously, $\mathrm{diam}(L) \geq c$ and $L \subset \bigcap_{\lambda\in\Lambda} Y_\lambda$. Because X is completely regular, L contains nonempty interior, which implies that $\mathrm{Int}(\bigcap_{\lambda\in\Lambda} Y_\lambda) \neq \emptyset$. So, assertion (2.1) holds. Thus we get a topologically transitive subsystem $(Z, f|_Z)$ with nonempty interior. Using the complete regularity of X again, it is easy to see that Z has a free interval. Since Z contains no simple closed curves, we see that $(Z, f|_Z)$ is relatively strongly mixing, has positive topological entropy and has dense periodic points by Lemma 3.1. Thus we complete the proof. □

As a supplement of Theorem 1.1, Example 3.2 shows that assertion (2.2) of Theorem 1.1 would occur indeed.

Example 3.2. First we construct a dendrite in the plane $\mathbb{R}^2$. Let $Y_0 = [0,1] \times \{0\}$. For each $n \in \mathbb{N}$ and $i = 1, 2, \ldots, 2^{n-1}$, let $I_{ni} = \{\frac{2i-1}{2^n}\} \times [0, \frac{1}{2^n}]$. Define $X = Y_0 \cup \left(\cup\{I_{ni} : n \in \mathbb{N}, i = 1, 2, \ldots, 2^{n-1}\}\right)$. Obviously X is a dendrite under the subspace topology of $\mathbb{R}^2$.

In order to give a sensitive map on X, we shall define two maps h and g as follows. For every $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, 2^{n-1}\}$, let $h_{ni} : I_{ni} \to [\frac{2i-1}{2^n}, \frac{i}{2^{n-1}}] \times \{0\}$ be the isometric map with $h_{ni}((\frac{2i-1}{2^n}, 0)) = (\frac{2i-1}{2^n}, 0)$. Define

$$h(x) = \begin{cases} h_{ni}(x), & x \in I_{ni}, \text{ for each } n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, 2^{n-1}; \\ x, & x \in Y_0. \end{cases}$$

Let $g : Y_0 \to Y_0$ be the tent map on the interval Y_0, that is

$$g((x,0)) = \begin{cases} (2x, 0), & 0 \leq x < \frac{1}{2}; \\ (2-2x, 0), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Since $h(X) \subset Y_0$, we can let $f = g \circ h$. It is easy to see that f is sensitive by the definition and Y_0 is a nowhere dense attracting set, whose attraction basin is the whole space X. □

Remark. It is well known that sensitive interval maps have positive topological entropy. Though the sensitive system (X,f) in Example 3.2 also has positive topological entropy, we will construct a sensitive dendrite map with zero topological entropy in Section 4, which shows that the dynamics of dendrite maps is more varied than that of interval maps.

4. An example

In this section, we will give a sensitive dendrite map with zero topological entropy. The following lemma is obvious.

Lemma 4.1. *Let $I = [a,b]$ and $J = [\alpha, \beta]$ be two nondegenerate closed intervals of the real line $\mathbb{R}$. Suppose $B = \{x_i : i \in \mathbb{N}\}$ is a sequence of distinct points which is dense in J, and $L = \{l_i : i \in \mathbb{N}\}$ is a sequence of*

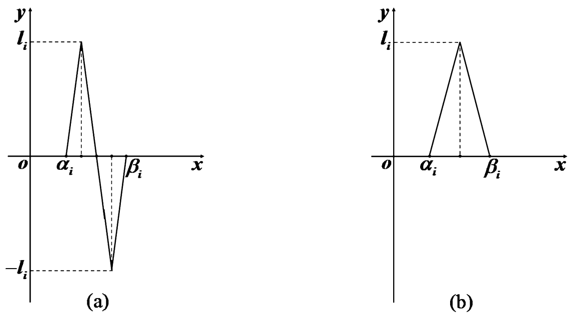

Fig. 1. τ_i.

positive numbers. Then there is a sequence of closed intervals $\{[\alpha_i, \beta_i] : i \in \mathbb{N}\}$ *in* I *satisfying the following conditions:*

(1) $\beta_i < \alpha_j$ *whenever* $x_i < x_j$*, for any* $i, j \in \mathbb{N}$*;*
(2) if $x_i = \alpha$*, then* $[\alpha_i, \beta_i] = [a, \beta_i]$*, and if* $x_i = \beta$*, then* $[\alpha_i, \beta_i] = [\alpha_i, b]$*;*
(3) $|\beta_i - \alpha_i| < \frac{1}{2} l_i$ *for every* $i \in \mathbb{N}$*;*
(4) $[\alpha_i, \beta_i] \cap [\alpha_j, \beta_j] = \emptyset$ *for any* $i \neq j$*;*
(5) $\bigcup_{i \in \mathbb{N}} [\alpha_i, \beta_i]$ *is dense in* I*.*

Remark. Note that the elements of the sequence $L = \{l_i : i \in \mathbb{N}\}$ in the above lemma does not have to be pairwise different.

Lemma 4.2. *Let* $I = [a, b]$*,* $J = [\alpha, \beta]$*,* $B = \{x_i : i \in \mathbb{N}\}$*, and* $L = \{l_i : i \in \mathbb{N}\}$ *be as in Lemma 4.1 with the additional assumption that* $l_i \to 0$ *as* $i \to \infty$*. For each* $i \in \mathbb{N}$*, if* $x_i \neq \beta$*, let* $A_i = \{(x_i, y) | -l_i \leq y \leq l_i\}$ *be an arc in* $\mathbb{R}^2$ *perpendicular to the x-axis; if* $x_i = \beta$*, let* $A_i = \{(x_i, y) | 0 \leq y \leq l_i\}$*. Then the plane point set* $Y = (J \times \{0\}) \cup (\bigcup_{i \in \mathbb{N}} A_i)$ *is a dendrite and there exist a sequence of closed intervals* $\{[\alpha_i, \beta_i] : i \in \mathbb{N}\}$ *in* I *with properties (1) to (5) of Lemma 4.1 and a continuous map* $f : I \to Y$ *such that*

(1) f *is surjective,* $f(a) = (\alpha, 0)$*, and* $f(b) = (\beta, 0)$*;*
(2) for every $i \in \mathbb{N}$*,* $f^{-1}(A_i) = [\alpha_i, \beta_i]$*;*
(3) for any subarc $K \subset [\alpha_i, \beta_i]$*, we have* $|f(K)| \geq 2|K|$*. (Here,* $|K|$ *denote the length of the arc* K *under the Euclidean metric.)*

Proof. It is easy to see that Y is compact, connected and contains no simple closed curves by definition. Since $l_i \to 0$ as $i \to \infty$, Y is locally connected. So, Y is a dendrite.

Applying Lemma 4.1 to the intervals I and J, point sequence $\{x_i : i \in \mathbb{N}\}$, and positive number sequence $\{l_i : i \in \mathbb{N}\}$, we can get a sequence of closed intervals $[\alpha_i, \beta_i] \subset I$ satisfying the five conditions in Lemma 4.1. Now we start to construct a map $f : I \to Y$ satisfying the requirements. For each $i \in \mathbb{N}$, define a map $\tau_i : [\alpha_i, \beta_i] \to [-l_i, l_i]$ (see Fig. 1-(a)) or $\tau_i : [\alpha_i, \beta_i] \to [0, l_i]$ (see Fig. 1-(b)) as follows:

If $x_i \neq \beta$, let

$$\tau_i(x) = \begin{cases} \frac{4l_i}{\beta_i - \alpha_i}(x - \alpha_i), & \alpha_i \leq x \leq \alpha_i + \frac{\beta_i - \alpha_i}{4}; \\ l_i - \frac{4l_i}{\beta_i - \alpha_i}(x - \alpha_i - \frac{\beta_i - \alpha_i}{4}), & \alpha_i + \frac{\beta_i - \alpha_i}{4} \leq x \leq \alpha_i + \frac{3(\beta_i - \alpha_i)}{4}; \\ -l_i + \frac{4l_i}{\beta_i - \alpha_i}(x - \alpha_i - \frac{3(\beta_i - \alpha_i)}{4}), & \alpha_i + \frac{3(\beta_i - \alpha_i)}{4} \leq x \leq \beta_i. \end{cases}$$

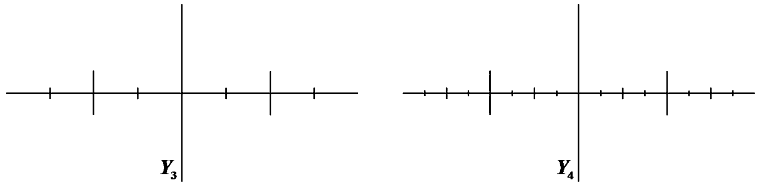

Fig. 2. Y_3 and Y_4.

If $x_i = \beta$, let

$$\tau_i(x) = \begin{cases} \frac{2l_i}{\beta_i-\alpha_i}(x-\alpha_i), & \alpha_i \le x \le \alpha_i + \frac{\beta_i-\alpha_i}{2}; \\ l_i - \frac{2l_i}{\beta_i-\alpha_i}(x-\alpha_i-\frac{\beta_i-\alpha_i}{2}), & \alpha_i + \frac{\beta_i-\alpha_i}{2} \le x \le \beta_i. \end{cases}$$

Now, define $\tilde{f} : \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i] \to Y$ by letting $\tilde{f}|_{[\alpha_i,\beta_i]}(x) = (x_i, \tau_i(x))$ for every $x \in [\alpha_i, \beta_i]$. As $\bigcup_{i=1}^{\infty}[\alpha_i, \beta_i]$ is dense in I, $\tilde{f}|_{\bigcup_{i=1}^{\infty}[\alpha_i,\beta_i]}$ can naturally be extended to a continuous map $f : I \to Y$. It is easy to see that f meets the requirements. □

Example 4.3. First, we define inductively a sequence of trees Y_i in the plane $\mathbb{R}^2$ as follows. Let $Y_0 = [0, 1]\times\{0\}$, $Y_1 = Y_0 \cup A_{11}$, where $A_{11} = \{(\frac{1}{2}, y)| -\frac{1}{4} \le y \le \frac{1}{4}\}$, and $Y_2 = Y_1 \cup A_{21} \cup A_{22}$, where $A_{21} = \{(\frac{1}{2^2}, y)| -\frac{1}{4^2} \le y \le \frac{1}{4^2}\}$, $A_{22} = \{(\frac{3}{2^2}, y)| -\frac{1}{4^2} \le y \le \frac{1}{4^2}\}$. Assume that Y_{n-1} has been defined, then let

$$Y_n = Y_{n-1} \cup \left(\bigcup_{i=1}^{2^{n-1}} A_{ni}\right),$$

where

$$A_{ni} = \left\{\left(\frac{2i-1}{2^n}, y\right) | -\frac{1}{4^n} \le y \le \frac{1}{4^n}\right\}, \text{ for } i = 1, 2, \dots, 2^{n-1}.$$

Please see Fig. 2 for Y_3 and Y_4. Clearly, $Y_0 \subset Y_1 \subset Y_2 \cdots$, and $X = \bigcup_{n=0}^{\infty} Y_n$ is a dendrite under the subspace topology of the plane $\mathbb{R}^2$.

For any plane point set $A \subset \mathbb{R}^2$, denote $A^+ = A \cap \{(x, y)|y \ge 0\}$, and $A^- = A \cap \{(x, y)|y \le 0\}$. Then $X^+ = Y_0 \cup \left(\bigcup_{n\in\mathbb{N}} \left(\bigcup_{i=1}^{2^{n-1}} A^+_{ni}\right)\right)$, and $X^- = Y_0 \cup \left(\bigcup_{n\in\mathbb{N}} \left(\bigcup_{i=1}^{2^{n-1}} A^-_{ni}\right)\right)$.

Now, we start to define a sensitive map $f : X \to X$ by 5 steps.

Step 1. For any $n \in \mathbb{N}$, we write $\mathcal{A}_{ni} = \{A_{pq} : A_{pq} \subset (\frac{2i-1}{2^n}, \frac{2i+1}{2^n}) \times (-\frac{1}{4^n}, \frac{1}{4^n}), p \in \mathbb{N}, q \in \{1, \dots, 2^{p-1}\}\}$ if $i \in \{1, 2, \dots, 2^{n-1}-1\}$, and write $\mathcal{A}_{n2^{n-1}} = \{A_{pq} : A_{pq} \subset (\frac{2^n-1}{2^n}, 1) \times (-\frac{1}{4^n}, \frac{1}{4^n}), p \in \mathbb{N}, q \in \{1, \dots, 2^{p-1}\}\}$. Then let

$$Y_{ni} = \left(\left[\frac{2i-1}{2^n}, \frac{2i+1}{2^n}\right] \times \{0\}\right) \cup (\cup\mathcal{A}_{ni}) \cup A^+_{n,i+1}, \text{ for } i \in \{1, 2, \dots, 2^{n-1}-1\},$$

and let

$$Y_{n2^{n-1}} = \left(\left[\frac{2^n-1}{2^n}, 1\right] \times \{0\}\right) \cup (\cup\mathcal{A}_{n2^{n-1}}).$$

Then Y_{ni} is a subdendrite of X (the shadowing parts in Fig. 3 are Y_{21} and Y_{22}).

For any fixed $n \in \mathbb{N}$ and $i \in \{1, \dots, 2^{n-1}\}$, denote $I = \{\frac{2i-1}{2^n}\} \times [0, \frac{1}{4^n}]$; if $i \in \{1, \dots, 2^{n-1}-1\}$, let $J = [\frac{2i-1}{2^n}, \frac{2i+1}{2^n}] \times \{0\}$ and let B be the branch point set of Y_{ni} together with the point $(\frac{2i+1}{2^n}, 0)$; if $i = 2^{n-1}$,

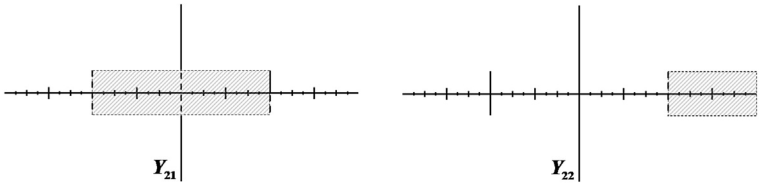

Fig. 3. Y_{21} and Y_{22}.

let $J = [\frac{2i-1}{2^n}, 1] \times \{0\}$ and let B be the branch point set of Y_{ni}; let $L = \{|A^+_{pq}| : A^+_{pq} \subset Y_{ni}, p \in \mathbb{N}, q \in \{1, \dots, 2^{p-1}\}\}$.

Applying Lemma 4.2 to I, J, B and L, we get a surjective map $f^+_{ni} : A^+_{ni} \to Y_{ni}$ satisfying that $f^+_{ni}((\frac{2i-1}{2^n}, 0)) = (\frac{2i-1}{2^n}, 0)$, $f^+_{ni}((\frac{2i-1}{2^n}, \frac{1}{4^n})) = (\frac{2i+1}{2^n}, 0)$ (particularly, $f^+_{ni}((\frac{2i-1}{2^n}, \frac{1}{4^n})) = (1, 0)$ if $i = 2^{n-1}$) and $(f^+_{ni})^{-1}(A_{pq})$ is a closed arc $\{\frac{2i-1}{2^n}\} \times [\alpha_j, \beta_j]$ in I for every $A_{pq} \subset Y_{ni}$. Also, $(f^+_{ni})^{-1}(A^+_{n,i+1})$ is a closed arc $\{\frac{2i-1}{2^n}\} \times [\alpha_j, \beta_j]$ in I. In addition, we have

$$|f(K)| \geq 2|K|, \qquad \text{for any closed arc } K \subset \left\{\frac{2i-1}{2^n}\right\} \times [\alpha_j, \beta_j]. \tag{$*$}$$

Step 2. Define $f^+_n : \bigcup_{i=1}^{2^{n-1}} A^+_{ni} \to X$ by letting $f^+_n|_{A^+_{ni}} = f^+_{ni}$, for each $i \in \{1, 2, \dots, 2^{n-1}\}$.

Step 3. Define $f^+ : X^+ \to X$ by

$$f^+(z) = \begin{cases} z, & z \in [0,1] \times \{0\}; \\ f^+_1(z), & z \in A^+_{11}; \\ \cdots & \\ f^+_n(z), & z \in \bigcup_{i=1}^{2^{n-1}} A^+_{ni}; \\ \cdots & \end{cases}$$

Step 4. Let φ be a symmetry of X about the point $(\frac{1}{2}, 0)$, i.e., $\varphi(x, y) = (1 - x, -y)$ for any $(x, y) \in X$. Then, define $f^- : X^- \to X$ by $f^-(z) = \varphi \circ f \circ \varphi(z)$ for any $z \in X^-$.

Step 5. Since $f^+(z) = f^-(z) = z$ for any $z \in [0,1] \times \{0\}$ by the definitions of f^+ and f^-, we have naturally a continuous map $f : X \to X$ defined by

$$f(z) = \begin{cases} f^+(z), & z \in X^+; \\ f^-(z), & z \in X^-. \end{cases}$$

Proposition 4.4. *The map $f : X \to X$ defined in Example 4.3 is sensitive and has zero topological entropy.*

Proof. Claim 1. For any $n \in \mathbb{N}$ and $i \in \{1, 2, \dots, 2^{n-1}\}$, we have

$$f^m(A^+_{ni}) \supset \left[\frac{2i-1}{2^n}, 1\right] \times \{0\}, \text{ for sufficiently large } m \in \mathbb{N}.$$

In fact, from the definition of f, we can see that

$$f(A^+_{ni}) \supset Y_{ni} \supset A^+_{n,i+1} \cup \left(\left[\frac{2i-1}{2^n}, \frac{2i+1}{2^n}\right] \times \{0\}\right), \text{ for } 1 \leq i \leq 2^{n-1} - 1;$$

$$f(A^+_{n,2^{n-1}}) \supset Y^+_{n,2^{n-1}} \supset \left[1 - \frac{1}{2^n}, 1\right] \times \{0\};$$

$$f(z) = z, \text{ for all } z \in \left[\frac{2i-1}{2^n}, 1\right] \times \{0\}.$$

This implies that $f^m(A_{ni}^+) \supset [\frac{2i-1}{2^n}, 1] \times \{0\}$ for $m \geq 2^{n-1} - i + 1$.

Similarly, we have

Claim 2. When m is sufficiently large, $f^m(A_{ni}^-) \supset [0, \frac{2i-1}{2^n}] \times \{0\}$.

By Claim 1 and Claim 2, we immediately get the following claim

Claim 3. For any $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, 2^{n-1}\}$, we have

$$f^m(A_{ni}) \supset [0,1] \times \{0\}, \text{ for sufficiently large } m \in \mathbb{N}.$$

From $(*)$ in Example 4.3, we can easily get the following

Claim 4. For any $n \in \mathbb{N}$, $i \in \{1, 2, \ldots, 2^{n-1}\}$, and any nondegenerate closed arc $K \subset A_{ni}$, there is $m \in \mathbb{N}$ such that $f^m(K) \supset A_{pq}$ for some $p \in \mathbb{N}$ and $q \in \{1, 2, \ldots, 2^{p-1}\}$.

By Claim 3 and Claim 4, we get immediately that for any nonempty open set U, there exists $m \in \mathbb{N}$ such that $f^m(U) \supset [0,1]$. Hence (X, f) is sensitive.

From the construction, we see that $\Omega(X, f) = [0,1] \times \{0\}$, and $f|_{\Omega(X,f)} = \mathrm{Id}$. So, $h_{top}(X, f) = h_{top}(\Omega(X, f), f|_{\Omega(X,f)}) = 0$. $\square$

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