



Existence and multiplicity of solutions for $p(x)$ -curl systems arising in electromagnetism

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Abstract

In this paper, we study the existence and multiplicity of solutions to a class of $p(x)$ -curl systems arising in electromagnetism. The results obtained in this paper extend several contributions concerning the p -curl operator and we focus on new existence results which are due to the presence of variable exponent. To our best knowledge, our results are new even in the semilinear case.

Keywords: $p(x)$ -curl systems; Variational methods; Ground state solutions.

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1 Introduction and main results

In recent years, a great attention has been focused on the study of various mathematical problems with variable exponent growth conditions. The variable exponent problems arise in a quite natural way in many different applications, such as, nonlinear elastic [29], electrorheological fluids [20], image processing [6] and other physics phenomena [4, 28]. The literature on variable exponent Sobolev spaces and on their applications is quite large, here we just quote a few, see [11, 15, 16, 17, 21, 22, 23, 25] and the references therein. For the basic properties of variable exponent Sobolev spaces and their applications to **partial** differential equations, we refer the readers to [9, 24].

Let Ω be a bounded simply connected domain of \mathbb{R}^3 with a $C^{1,1}$ boundary denoted by $\partial\Omega$. In what follows, vector functions and spaces of vector functions will be denoted by boldface symbols. We will use \mathbf{n} to denote the outward unitary normal vector to $\partial\Omega$ and ∂_x to denote the partial derivative of a function with respect to the variable x .

To introduce our problem precisely, we first give some notations. Let $\mathbf{u} = (u_1, u_2, u_3)$ be a vector function on Ω . The divergence of \mathbf{u} is denoted by

$$\nabla \cdot \mathbf{u} = \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3$$

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and the curl of \mathbf{u} is defined by

$$\nabla \times \mathbf{u} = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1).$$

Then $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ satisfy the following identity

$$-\Delta \mathbf{u} = \nabla \times (\nabla \times \mathbf{u}) - \nabla (\nabla \cdot \mathbf{u}),$$

where $\Delta \mathbf{u} = (\Delta u_1, \Delta u_2, \Delta u_3)$ and $\Delta u_i = \nabla \cdot (\nabla u_i)$, $i = 1, 2, 3$.

Now we consider the following stationary $p(x)$ -curl systems:

$$\begin{aligned} \nabla \times (|\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u}) + a(x)|\mathbf{u}|^{p(x)-2} \mathbf{u} &= \mathbf{f}(x, \mathbf{u}), \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \\ |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \times \mathbf{n} &= 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.1)$$

Throughout this paper, unless special statement, we always assume that the exponent $p(x)$ is continuous on $\overline{\Omega}$ with

$$1 < p^- = \min_{x \in \Omega} p(x) \leq p^+ = \max_{x \in \Omega} p(x) < 3,$$

and $p(x)$ satisfies logarithmic continuity: there exists a function $\omega : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that

$$\forall x, y \in \overline{\Omega}, \quad |x - y| < 1, \quad |p(x) - p(y)| \leq \omega(|x - y|), \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} \omega(\tau) \log \frac{1}{\tau} = C < \infty. \quad (1.2)$$

In [2], *Antontsev, Miranda* and *Santos* studied the qualitative properties of solutions for the following $p(x, t)$ -curl systems:

$$\begin{aligned} \partial_t \mathbf{u} + \nabla \times (|\nabla \times \mathbf{u}|^{p(x,t)-2} \nabla \times \mathbf{u}) &= \mathbf{f}(\mathbf{u}), \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T), \\ |\nabla \times \mathbf{u}|^{p(x,t)-2} \nabla \times \mathbf{u} \times \mathbf{n} &= 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}(x, 0) &= \mathbf{u}_0(x) \quad \text{in } \Omega, \end{aligned} \quad (1.3)$$

where $\nabla \times (|\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u})$ is the $p(x, t)$ -curl operator, $\mathbf{f}(\mathbf{u}) = \lambda \mathbf{u} (\int_{\Omega} |\mathbf{u}|^2 dx)^{\frac{\rho-2}{2}}$ with $\lambda \in \{-1, 0, 1\}$, and $\rho > 0$ is constant. The authors introduced a suitable variable exponent Sobolev space and obtained the existence of local or global weak solutions for system (1.3) by using Galerkin's method. The authors also studied the blow-up and finite time extinction properties of solutions. When $p(x, t) \equiv p$, then problem (1.3) reduces to

$$\begin{aligned} \partial_t \mathbf{u} + \nabla \times (|\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u}) &= \mathbf{f}(\mathbf{u}), \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T), \\ |\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u} \times \mathbf{n} &= 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}(x, 0) &= \mathbf{u}_0(x) \quad \text{in } \Omega. \end{aligned} \quad (1.4)$$

In fact, model (1.4) comes from the generalized Maxwell's equations in the electromagnetic field theory. More precisely, \mathbf{u} denotes the magnetic field, $\nabla \times \mathbf{u}$ denotes the total current density, \mathbf{f} denotes an internal magnetic current, and $\nabla \times (|\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u})$ denotes the electric field, for more details about system (1.4) and the corresponding stationary problem, we refer to [3] and [13]. We also collect some recent results with respect to this kind of problems, see [5, 14, 27].

Motivated by the above works, we study the existence and multiplicity of solutions for systems (1.1) with general nonlinearities. **To the best of our knowledge**, this is the first time to deal with

the existence of steady-state solutions for systems (1.1) involving the $p(x)$ -curl operator by applying variational methods different with that used in [2]. It is of interest to exploit similar variational methods used here studying nonlocal problems as in [18, 19].

Now we impose the following assumptions on $a(x)$ and the source or sink term \mathbf{f} :

(A) $a \in L^\infty(\Omega)$ and there exist $a_0, a_1 > 0$ such that

$$a_0 \leq a(x) \leq a_1 \quad \text{for all } x \in \Omega.$$

(H₁) $F : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is differentiable with respect to $\mathbf{t} \in \mathbb{R}^3$, such that $\mathbf{f} = \partial_{\mathbf{t}} F(x, \mathbf{t}) : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a Carathéodory function.

(H₂) There exist constants $C > 0$, $q \in C(\overline{\Omega})$ and $1 < q(x) < 3p(x)/(3 - p(x))$ in $\overline{\Omega}$ such that

$$|\mathbf{f}(x, \mathbf{t})| \leq C(1 + |\mathbf{t}|^{q(x)-1}) \quad \text{for all } (x, \mathbf{t}) \in \Omega \times \mathbb{R}^3.$$

(H₃) There is a constant $\mu > p^+$ such that $0 < \mu F(x, \mathbf{t}) \leq \mathbf{f}(x, \mathbf{t}) \cdot \mathbf{t}$ for all $x \in \Omega$ and $\mathbf{t} \in \mathbb{R}^3 \setminus \{0\}$.

(H₄) $\lim_{\mathbf{t} \rightarrow 0} \frac{|\mathbf{f}(x, \mathbf{t})|}{|\mathbf{t}|^{p(x)-1}} = 0$ uniformly in $x \in \Omega$.

(H₅) $\inf_{x \in \Omega, \mathbf{t} \in \mathbb{R}^3, |\mathbf{t}|=1} F(x, \mathbf{t}) > 0$

The variational structure of this problem leads us to introduce the following space

$$\mathbf{W}^{p(x)}(\Omega) = \{\mathbf{v} \in \mathbf{L}^{p(x)}(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^{p(x)}(\Omega), \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$

see Section 2 for more details. We first give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$ is a (weak) solution of problem (1.1), if

$$\int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} dx + \int_{\Omega} a(x) |\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx = \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx,$$

for any $\mathbf{v} \in \mathbf{W}^{p(x)}(\Omega)$.

Remark 1.1. Let \mathbf{u} be a classical solution of (1.1). Let $\mathbf{e} = |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u}$ and \mathbf{v} be a smooth function in Ω , then we obtain

$$\nabla \cdot (\mathbf{e} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{e} - \mathbf{e} \cdot \nabla \times \mathbf{v}. \quad (1.5)$$

Multiplying the first equation of (1.1) by \mathbf{v} and integrating over Ω , we get

$$\int_{\Omega} \nabla \times \mathbf{e} \cdot \mathbf{v} dx + \int_{\Omega} a(x) |\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx = \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx.$$

Using (1.5) and the boundary conditions in (1.1) and integrating by parts, we have

$$\int_{\Omega} \mathbf{e} \cdot \nabla \times \mathbf{v} dx + \int_{\Omega} a(x) |\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx = \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx,$$

which means that Definition 1.1 is rational.

Now we are in a position to state the main results. For the superlinear case, i.e. $p(x) < q(x) < 3p(x)/(3 - p(x))$, the first result reads as follows.

Theorem 1.1 (Superlinear case). *Assume that a satisfies (\mathcal{A}) , and \mathbf{f} satisfies (H_1) – (H_5) . If $p(x) < q(x) < 3p(x)/(3 - p(x))$ for all $x \in \overline{\Omega}$, then (1.1) admits one nontrivial mountain pass solution $\mathbf{u}_0 \in \mathbf{W}^{p(x)}(\Omega)$. Furthermore, problem (1.1) has a ground state $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$ with positive energy.*

Note that a ground state solution is a nontrivial solution that has the least energy among all nontrivial solutions. A simple example for \mathbf{f} is given by $\mathbf{f}(x, \mathbf{t}) = |\mathbf{t}|^{q(x)-2}\mathbf{t}$ with $p(x) < q(x) < 3p(x)/(3 - p(x))$.

We also consider the multiplicity of solutions in the superlinear case. To this aim, we need the following hypothesis:

(H_6) $F(x, -\mathbf{t}) = F(x, \mathbf{t})$ for all $(x, \mathbf{t}) \in \Omega \times \mathbb{R}^3$.

Theorem 1.2 (Multiplicity of solutions). *Assume that a satisfies (\mathcal{A}) , and \mathbf{f} satisfies (H_1) – (H_4) and (H_6) . If $p(x) < q(x) < 3p(x)/(3 - p(x))$ for all $x \in \overline{\Omega}$, then problem (1.1) has infinitely many nontrivial solutions in $\mathbf{W}^{p(x)}(\Omega)$.*

For the sublinear case, we need the following additional assumption.

(H_7) There exist $\delta > 0$, $C_0 > 0$ and a nonempty open set $\Omega_0 \subset \Omega$ such that

$$|\mathbf{f}(x, \mathbf{t})| \geq C_0 |\mathbf{t}|^{q(x)-1}, \quad \text{for all } x \in \Omega_0 \text{ and } |\mathbf{t}| \in (0, \delta).$$

Obviously, a typical example for \mathbf{f} is given by $\mathbf{f}(x, \mathbf{t}) = (1+x^2)^{-(2-q(x))/2} |\mathbf{t}|^{q(x)-2}\mathbf{t}$ with $1 < q(x) < p(x)$.

Theorem 1.3 (Sublinear case). *Assume that a satisfies (\mathcal{A}) , and \mathbf{f} satisfies (H_1) , (H_2) and (H_7) . If $1 < q(x) < p(x)$ for all $x \in \overline{\Omega}$, then problem (1.1) admits one nontrivial solution $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$, which is a ground state of (1.1).*

This article is organized as follows. In Section 2, we recall some necessary definitions and properties of variable exponent Lebesgue spaces and Sobolev spaces and the space $\mathbf{W}^{p(x)}(\Omega)$. In Section 3, we obtain some preliminary results. In Section 4, the existence of ground state solutions and infinitely many solutions for (1.1) with the nonlinearity \mathbf{f} satisfying superlinear growth conditions is obtained by combining the mountain pass theorem with the Nehari manifold method, and a variant of the mountain pass theorem respectively. In Section 5, we obtain the existence of ground state solutions of (1.1) if nonlinear term \mathbf{f} satisfies sublinear growth conditions.

2 Function spaces with variable exponent

In this section, we recall some necessary properties of variable exponent spaces and the space $\mathbf{W}^{p(x)}(\Omega)$ of divergence free vector functions belonging to $L^{p(x)}(\Omega)$ with curl in $L^{p(x)}(\Omega)$, see [2, 9, 10, 12] for more details.

2.1 Variable exponent Lebesgue and Sobolev spaces

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. Let $p \in C(\overline{\Omega})$. Set

$$p^- = \min_{x \in \Omega} p(x) \text{ and } p^+ = \max_{x \in \Omega} p(x)$$

and assume that $1 < p^- \leq p^+ < \infty$.

Define

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u|u : \Omega \rightarrow \mathbb{R} \text{ is a measurable function, } \rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{p(x)}(\lambda^{-1}u) \leq 1 \right\},$$

becomes a separable, reflexive Banach space. The dual space $(L^{p(x)}(\Omega))'$ can be identified with $L^{p'(x)}(\Omega)$, where the conjugate exponent p' is defined by $p'(x) = \frac{p(x)}{p(x)-1}$ for all $x \in \bar{\Omega}$. In variable exponent Lebesgue space $L^{p(x)}(\Omega)$, we have the following relations

$$\min \left\{ \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right\} \leq \rho_{p(\cdot)}(u) \leq \max \left\{ \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right\}, \quad (2.1)$$

which **imply** that the norm convergence is equivalent to convergence with respect to the modular $\rho_{p(x)}$.

In the variable exponent Lebesgue space, Hölder's inequality is still valid. For all $u \in L^{p(x)}(\Omega)$, $v \in L^{p(x)'}(\Omega)$, the following inequality holds

$$\int_{\Omega} |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

We also have that for $1 \leq q(x) \leq p(x)$, there exists $C > 0$ such that $\|u\|_{L^{q(x)}(\Omega)} \leq C \|u\|_{L^{p(x)}(\Omega)}$ for all $u \in L^{p(x)}(\Omega)$.

Now we give the definition of variable exponent Sobolev space. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Then $W^{1,p(x)}(\Omega)$ is a separable, reflexive Banach space. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{1,p(x)}(\Omega)}$.

Theorem 2.1. (see [10, Theorem 2.3]) *Let $q \in C(\bar{\Omega})$ satisfy*

$$1 \leq q(x) < \frac{3p(x)}{3-p(x)}$$

for all $x \in \bar{\Omega}$. Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

2.2 The space $\mathbf{W}^{p(x)}(\Omega)$

Let $\mathbf{L}^{p(x)}(\Omega) = L^{p(x)}(\Omega) \times L^{p(x)}(\Omega) \times L^{p(x)}(\Omega)$ and define

$$\mathbf{W}^{p(x)}(\Omega) = \left\{ \mathbf{v} \in \mathbf{L}^{p(x)}(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^{p(x)}(\Omega), \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \right\},$$

where \mathbf{n} denotes the outward unitary normal vector to $\partial\Omega$. Equip $\mathbf{W}^{p(x)}(\Omega)$ with the norm

$$\|\mathbf{v}\|_{\mathbf{W}^{p(x)}(\Omega)} = \|\mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)} + \|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

If $p^- > 1$, by Theorem 2.1 of [2], $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_n^{1,p(x)}(\Omega)$, where

$$\mathbf{W}_n^{1,p(x)}(\Omega) = \left\{ \mathbf{v} \in \mathbf{W}^{1,p(x)}(\Omega) : \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \right\}$$

and

$$\mathbf{W}^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega).$$

Thus, we have the following theorem.

Theorem 2.2. (see [2, Theorem 2.1]) *Assume that $1 < p^- \leq p^+ < \infty$ and p satisfies (1.2). Then $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_n^{1,p(x)}(\Omega)$. Moreover, if $p^- > 6/5$, then $\|\nabla \times \cdot\|_{\mathbf{L}^{p(x)}(\Omega)}$ is a norm in $\mathbf{W}^{p(x)}(\Omega)$ and there exists $C = C(N, p^-, p^+) > 0$ such that*

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p(x)}(\Omega)} \leq C \|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

Remark 2.1. By Theorems 2.1–2.2, we know the embedding $\mathbf{W}^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact, with $1 < p^- \leq p^+ < 3$, $q \in C(\bar{\Omega})$ and $1 \leq q(x) < 3p(x)/(3 - p(x))$ for all $x \in \bar{\Omega}$. Moreover, $(\mathbf{W}^{p(x)}(\Omega), \|\cdot\|_{\mathbf{W}^{p(x)}(\Omega)})$ is a reflexive Banach space.

3 Preliminary results

The functional associated with problem (1.1) is defined as

$$I(\mathbf{u}) = J(\mathbf{u}) - H(\mathbf{u}), \quad \text{for all } \mathbf{u} \in \mathbf{W}^{p(x)}(\Omega),$$

where

$$J(\mathbf{u}) = \int_{\Omega} \frac{|\nabla \times \mathbf{u}|^{p(x)} + a(x)|\mathbf{u}|^{p(x)}}{p(x)} dx \quad \text{and} \quad H(\mathbf{u}) = \int_{\Omega} F(x, \mathbf{u}) dx$$

Lemma 3.1. *The functional J is of class C^1 and*

$$\langle J'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} + a(x)|\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{p(x)}(\Omega)$. For each $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$, $J'(\mathbf{u}) \in (\mathbf{W}^{p(x)}(\Omega))^*$, where $(\mathbf{W}^{p(x)}(\Omega))^*$ is the dual space of $\mathbf{W}^{p(x)}(\Omega)$. Moreover, J is a convex functional in $\mathbf{W}^{p(x)}(\Omega)$.

Proof. It is easy to see that for each $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$, $J'(\mathbf{u}) \in (\mathbf{W}^{p(x)}(\Omega))^*$, I is a convex functional in $\mathbf{W}^{p(x)}(\Omega)$ and I is Gâteaux-differentiable in $\mathbf{W}^{p(x)}(\Omega)$ and

$$\langle I'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} + a(x) |\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx,$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{p(x)}(\Omega)$.

Set $J(\mathbf{u}) = L(\mathbf{u}) + K(\mathbf{u})$, where

$$L(\mathbf{u}) = \int_{\Omega} \frac{|\nabla \times \mathbf{u}|^{p(x)}}{p(x)} dx$$

and

$$K(\mathbf{u}) = \int_{\Omega} \frac{a(x) |\mathbf{u}|^{p(x)}}{p(x)} dx.$$

Then

$$\langle L'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} dx$$

and

$$\langle K'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx.$$

Next we prove that L' is continuous. Let $\{\mathbf{u}_n\}_n \subset \mathbf{W}^{p(x)}(\Omega)$ such that $\mathbf{u}_n \rightarrow \mathbf{u}$ strongly in $\mathbf{W}^{p(x)}(\Omega)$. Then $\nabla \times \mathbf{u}_n \rightarrow \nabla \times \mathbf{u}$ strongly in $\mathbf{L}^{p(x)}(\Omega)$. Without loss of generality, we may assume that $\nabla \times \mathbf{u}_n \rightarrow \nabla \times \mathbf{u}$ a.e. in Ω . For any measurable subset $U \subset \Omega$, with Lebesgue measure $|U| < 1$, we have by the Hölder inequality and (2.1)

$$\begin{aligned} \int_U |\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n \Big|^{p'(x)} dx &= \int_U |\nabla \times \mathbf{u}_n|^{p(x)-1} dx \\ &\leq 2 \left\| |\nabla \times \mathbf{u}_n|^{p(x)-1} \right\|_{\mathbf{L}^{p'(x)}(\Omega)} \|1\|_{\mathbf{L}^{p(x)}(U)} \leq C |U|^{\frac{1}{p-1}}, \end{aligned}$$

this means that the sequence $\{|\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n\}^{p'(x)}_n$ is equi-integrable on $L^1(\Omega)$. Hence the Vitali convergence theorem (see [9, Theorem 1.4.12, p. 16]) implies that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left| |\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n - |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \right|^{p'(x)} dx = 0,$$

which together with (2.1) yields that

$$|\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n \rightarrow |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \text{ strongly in } \mathbf{L}^{p'(x)}(\Omega),$$

as $n \rightarrow \infty$. Then by the Hölder inequality, we get

$$\begin{aligned} &\|L'(\mathbf{u}_n) - L'(\mathbf{u})\| \\ &= \sup_{\mathbf{v} \in \mathbf{W}^{p(x)}(\Omega), \|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}=1} \left| \int_{\Omega} \left[|\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n - |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \right] \cdot \nabla \times \mathbf{v} dx \right| \\ &\leq \left\| |\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n - |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \right\|_{\mathbf{L}^{p'(x)}(\Omega)} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence L' is continuous in $\mathbf{W}^{p(x)}(\Omega)$. Then we obtain that L is of class C^1 .

Similarly, we can prove that K is of class C^1 and

$$\langle K'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} a(x) |\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx.$$

Therefore, the desired conclusion follows. \square

Lemma 3.2. *Assume \mathbf{f} satisfies $(H_1) - (H_2)$. Then H is of class C^1 and*

$$\langle H'(\mathbf{u}), v \rangle = \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx,$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{p(x)}(\Omega)$. Moreover, H is weakly continuous in $\mathbf{W}^{p(x)}(\Omega)$.

Proof. By using a similar discussion as for Lemma 3.1, we can prove that H is of class C^1 and

$$\langle H'(\mathbf{u}), v \rangle = \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx,$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{p(x)}(\Omega)$.

In the sequel, we prove that H is weakly continuous in $\mathbf{W}^{p(x)}(\Omega)$. Let $\{\mathbf{u}_n\}_n \subset \mathbf{W}^{p(x)}(\Omega)$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $\mathbf{W}^{p(x)}(\Omega)$. By Remark 2.1, up to a subsequence, we assume that $\mathbf{u}_n \rightarrow \mathbf{u}$ in $\mathbf{L}^{q(x)}(\Omega)$ and $\mathbf{u}_n \rightarrow \mathbf{u}$ a.e. in Ω . By (H_2) and $F(x, \mathbf{u}) = \int_0^1 \mathbf{f}(x, t\mathbf{u}) \cdot \mathbf{u} dt$, we get

$$|F(x, \mathbf{u}_n)| \leq C(|\mathbf{u}_n| + |\mathbf{u}_n|^{q(x)})$$

It is easy to verify that sequence $\{F(x, \mathbf{u}_n) - F(x, \mathbf{u})\}_n$ is equi-integrable on $L^1(\Omega)$, since $1 \leq q(x) < \frac{3p(x)}{3-p(x)}$ in $\bar{\Omega}$. By the continuity of $F(x, \cdot)$, we have $F(x, \mathbf{u}_n) \rightarrow F(x, \mathbf{u})$ a.e. in Ω . Hence the Vitali convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |F(x, \mathbf{u}_n) - F(x, \mathbf{u})| dx = 0,$$

which implies that functional H is weakly continuous in $\mathbf{W}^{p(x)}(\Omega)$. \square

By Lemmas 3.1–3.2, we know that I is of class C^1 and

$$\langle I'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} + a(x) |\mathbf{u}|^{p(x)-2} \mathbf{u} \cdot \mathbf{v} dx - \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} dx,$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{p(x)}(\Omega)$. Hence a critical point of I is a (weak) solution of (1.3).

Definition 3.1. *We say that I satisfies the (PS) condition in $\mathbf{W}^{p(x)}(\Omega)$, if any (PS) sequence $\{\mathbf{u}_n\} \subset \mathbf{W}^{p(x)}(\Omega)$, i.e., $\{I(\mathbf{u}_n)\}$ is bounded and $I'(\mathbf{u}_n) \rightarrow 0$ as $n \rightarrow \infty$, admits a strongly convergent subsequence in $\mathbf{W}^{p(x)}(\Omega)$.*

Lemma 3.3. ((PS) condition) *Let (\mathcal{A}) and $(H_1) - (H_4)$ hold. Then I satisfies the (PS) condition.*

Proof. Let $\{\mathbf{u}_n\}_n$ be a (PS) sequence in $\mathbf{W}^{p(x)}(\Omega)$. Then there exists $C > 0$ such that

$$|I(\mathbf{u}_n)| \leq C \quad \text{and} \quad |\langle I'(\mathbf{u}_n), \mathbf{u}_n \rangle| \leq C \|\mathbf{u}_n\|_{\mathbf{W}^{p(x)}(\Omega)}.$$

Thus, (H_2) yields

$$\begin{aligned} C + C \|\mathbf{u}_n\|_{\mathbf{W}^{p(x)}(\Omega)} &\geq I(\mathbf{u}_n) - \frac{1}{\mu} \langle I'(\mathbf{u}_n), \mathbf{u}_n \rangle \\ &= \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{\mu} \right) (|\nabla \times \mathbf{u}_n|^{p(x)} + a(x)|\mathbf{u}_n|^{p(x)}) dx \\ &\quad - \frac{1}{\mu} \int_{\mathbb{R}^N} (\mu F(x, \mathbf{u}_n) - \mathbf{f}(x, \mathbf{u}_n) \cdot \mathbf{u}_n) dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu} \right) \min\{1, a_0\} \int_{\Omega} |\nabla \times \mathbf{u}_n|^{p(x)} + |\mathbf{u}_n|^{p(x)} dx. \end{aligned}$$

Then by (2.1), we have

$$C + C \|\mathbf{u}_n\|_{\mathbf{W}^{p(x)}(\Omega)} \geq \left(\frac{1}{p^+} - \frac{1}{\mu} \right) \min \left\{ \|\mathbf{u}_n\|_{\mathbf{W}^{p(x)}(\Omega)}^{p^-}, \|\mathbf{u}_n\|_{\mathbf{W}^{p(x)}(\Omega)}^{p^+} \right\}$$

which implies that $\{\mathbf{u}_n\}_n$ is bounded in $\mathbf{W}^{p(x)}(\Omega)$, since $\mu > p^+$ and $p^+ \geq p^- > 1$. By the reflexivity of $\mathbf{W}^{p(x)}(\Omega)$ and Remark 2.1, up to a subsequence, there exists $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$ such that

$$\begin{aligned} \mathbf{u}_n &\rightharpoonup \mathbf{u} \quad \text{in } \mathbf{W}^{p(x)}(\Omega), \quad \mathbf{u}_n \rightarrow \mathbf{u} \quad \text{a.e. in } \mathbb{R}^N, \\ \mathbf{u}_n &\rightarrow \mathbf{u} \quad \text{in } \mathbf{L}^{q(x)}(\Omega) \cap \mathbf{L}^{p(x)}(\Omega). \end{aligned} \tag{3.1}$$

By Lemma 3.1, $J'(\mathbf{u}) \in (\mathbf{W}^{p(x)}(\Omega))^*$, which together with the weak convergence in (3.1) yields

$$\lim_{n \rightarrow \infty} \langle J'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle = 0. \tag{3.2}$$

It follows from (H_4) that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|\mathbf{f}(x, \mathbf{t})| \leq \varepsilon |\mathbf{t}|^{p(x)-1}, \quad \text{for all } x \in \Omega \text{ and } 0 \leq |\mathbf{t}| \leq \delta.$$

By (H_2) , for all $x \in \Omega$ and $|\mathbf{t}| > \delta$, we have

$$|\mathbf{f}(x, \mathbf{t})| \leq C \left(1 + \frac{1}{\delta^{q(x)-1}} \right) |\mathbf{t}|^{q(x)-1}.$$

The above two inequalities imply that

$$|\mathbf{f}(x, \mathbf{t})| \leq \varepsilon |\mathbf{t}|^{p(x)-1} + C_{\varepsilon} |\mathbf{t}|^{q(x)-1} \quad \text{for all } x \in \Omega \text{ and } \mathbf{t} \in \mathbb{R}^3, \tag{3.3}$$

where $C_{\varepsilon} = C(1 + \max\{\frac{1}{\delta^{q^- - 1}}, \frac{1}{\delta^{q^+ - 1}}\})$. Using (3.3) with $\varepsilon = 1$ and the Hölder inequality, we obtain

$$\begin{aligned} &\int_{\Omega} |(\mathbf{f}(x, \mathbf{u}_n) - \mathbf{f}(x, \mathbf{u})) \cdot (\mathbf{u}_n - \mathbf{u})| dx \\ &\leq \int_{\Omega} \left[(|\mathbf{u}_n|^{p(x)-1} + |\mathbf{u}|^{p(x)-1}) + C_1 (|\mathbf{u}_n|^{q(x)-1} + |\mathbf{u}|^{q(x)-1}) \right] |\mathbf{u}_n - \mathbf{u}| dx \\ &\leq \left\| |\mathbf{u}_n|^{p(x)-1} + |\mathbf{u}|^{p(x)-1} \right\|_{\mathbf{L}^{p'(x)}(\Omega)} \|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^{p(x)}(\Omega)} \\ &\quad + C_1 \left\| |\mathbf{u}_n|^{q(x)-1} + |\mathbf{u}|^{q(x)-1} \right\|_{\mathbf{L}^{q'(x)}(\Omega)} \|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^{q(x)}(\Omega)} \\ &\leq C (\|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^{p(x)}(\Omega)} + \|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^{q(x)}(\Omega)}). \end{aligned} \tag{3.4}$$

Here we use the following fact:

$$\left\| |\mathbf{u}_n|^{p(x)-1} \right\|_{L^{p'(x)}(\Omega)} \leq \max \left\{ \left(\int_{\Omega} |\mathbf{u}_n|^{p(x)} dx \right)^{(p^- - 1)/(p^-)}, \left(\int_{\Omega} |\mathbf{u}_n|^{p(x)} dx \right)^{(p^+ - 1)/p^+} \right\} \leq C.$$

By Remark 2.1, we have $\mathbf{u}_n \rightarrow \mathbf{u}$ in $L^{p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. Hence (3.4) implies that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\mathbf{f}(x, \mathbf{u}_n) - \mathbf{f}(x, \mathbf{u})) \cdot (\mathbf{u}_n - \mathbf{u}) dx = 0. \quad (3.5)$$

Clearly,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot (\mathbf{u}_n - \mathbf{u}) dx = 0,$$

which together with (3.2) yields that $\langle I'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle \rightarrow 0$. Obviously, $\langle I'(\mathbf{u}_n), \mathbf{u}_n - \mathbf{u} \rangle \rightarrow 0$ as $n \rightarrow \infty$, since $\{\mathbf{u}_n - \mathbf{u}\}_n$ is bounded in $\mathbf{W}^{p(x)}(\Omega)$ and $I'(\mathbf{u}_n) \rightarrow 0$ in the dual space of $\mathbf{W}^{p(x)}(\Omega)$. Thus,

$$\begin{aligned} o(1) &= \langle I'(\mathbf{u}_n) - I'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle \\ &= \langle J'(\mathbf{u}_n) - J'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle - \int_{\Omega} (\mathbf{f}(x, \mathbf{u}_n) - \mathbf{f}(x, \mathbf{u}))(\mathbf{u}_n - \mathbf{u}) dx. \end{aligned}$$

This, together with (3.5) implies that

$$\lim_{n \rightarrow \infty} \langle J'(\mathbf{u}_n) - J'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle = 0. \quad (3.6)$$

Now we divide Ω into $\Omega_1 = \{x \in \Omega : p(x) \geq 2\}$ and $\Omega_2 = \{x \in \Omega : 1 < p(x) < 2\}$. Let us first recall the well-known Simon inequalities (see [26]):

$$\begin{aligned} &|\xi - \eta|^{p(x)} \\ &\leq \begin{cases} C \left(|\xi|^{p(x)-2} \xi - |\eta|^{p(x)-2} \eta \right) \cdot (\xi - \eta) & \text{for } x \in \Omega_1 \\ C \left[\left(|\xi|^{p(x)-2} \xi - |\eta|^{p(x)-2} \eta \right) \cdot (\xi - \eta) \right]^{\frac{p(x)}{2}} (|\xi|^{p(x)} + |\eta|^{p(x)})^{\frac{2-p(x)}{2}} & \text{for } x \in \Omega_2, \end{cases} \end{aligned} \quad (3.7)$$

for all $\xi, \eta \in \mathbb{R}^3$, where C is a positive constant depending only on p^-, p^+ , see for example [7]. Then, (3.7) and (3.6) imply that

$$\int_{\Omega_1} |\nabla \times (\mathbf{u}_n - \mathbf{u})|^{p(x)} dx \leq C \langle J'(\mathbf{u}_n) - J'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

and

$$\begin{aligned} &\int_{\Omega_2} |\nabla \times (\mathbf{u}_n - \mathbf{u})|^{p(x)} dx \\ &\leq C \int_{\Omega_2} [(|\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n - |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u}) \cdot (\nabla \times (\mathbf{u}_n - \mathbf{u}))]^{\frac{p(x)}{2}} \\ &\quad \cdot (|\nabla \times \mathbf{u}_n|^{p(x)} + |\nabla \times \mathbf{u}|^{p(x)})^{\frac{2-p(x)}{2}} dx \\ &\leq C \left\| [(|\nabla \times \mathbf{u}_n|^{p(x)-2} \nabla \times \mathbf{u}_n - |\nabla \times \mathbf{u}|^{p(x)-2} \nabla \times \mathbf{u}) \cdot (\nabla \times (\mathbf{u}_n - \mathbf{u}))]^{p(x)/2} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

thanks to (2.1). Hence, the proof is complete. \square

To apply the mountain pass theorem of *Ambrosetti* and *Rabinowitz* in [1], we first give some necessary preliminary results.

Lemma 3.4 (Mountain Pass Geometry I). *Let (\mathcal{A}) , (H_1) , (H_2) and (H_4) hold and assume $q^- > p^+$. Then there exist constants $\varrho, \alpha > 0$ such that $I(\mathbf{u}) \geq \alpha$ for all $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$ with $\|\nabla \times \mathbf{u}\|_{\mathbf{L}^{p(x)}(\Omega)} = \varrho$.*

Proof. By (3.3) and $F(x, \mathbf{t}) = \int_0^1 \mathbf{f}(x, \tau \mathbf{t}) \cdot \mathbf{t} d\tau$, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(x, \mathbf{t}) \leq \varepsilon |\mathbf{t}|^{p(x)} + C_\varepsilon |\mathbf{t}|^{q(x)},$$

for all $x \in \Omega$ and $\mathbf{t} \in \mathbb{R}^3$. Thus, we obtain for all $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$

$$\begin{aligned} I(\mathbf{u}) &= \int_\Omega \frac{1}{p(x)} |\nabla \times \mathbf{u}|^{p(x)} + \frac{a(x)}{p(x)} |\mathbf{u}|^{p(x)} dx - \int_\Omega F(x, \mathbf{u}) dx \\ &\geq \int_\Omega \frac{|\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)}}{p(x)} dx - \varepsilon \int_\Omega |\mathbf{u}|^{p(x)} dx - C_\varepsilon \int_\Omega |\mathbf{u}|^{q(x)} dx. \end{aligned}$$

Let $c_1 := \frac{1}{2} \min_{x \in \overline{\Omega}} \{q(x) - p(x)\}$. Then $c_1 > 0$. Since $p, q \in C(\overline{\Omega})$, there exist m open hypercubes Ω_i ($i = 1, 2, \dots, m$) which have no common points such that $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega}_i$,

$$p_i^+ = \max_{x \in \overline{\Omega}_i} p(x) < q_i^- = \min_{x \in \overline{\Omega}_i} q(x),$$

and $q_i^- - p_i^+ > c_1$ for $i = 1, 2, \dots, m$, see [7] for further discussion. Then

$$\begin{aligned} &\int_\Omega \frac{|\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)}}{p(x)} dx - \varepsilon \int_\Omega |\mathbf{u}|^{p(x)} dx - C_\varepsilon \int_\Omega |\mathbf{u}|^{q(x)} dx \\ &= \sum_{i=1}^m \int_{\Omega_i} \left(\frac{|\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)}}{p(x)} - \varepsilon |\mathbf{u}|^{p(x)} - C_\varepsilon |\mathbf{u}|^{q(x)} \right) dx. \end{aligned}$$

Taking $\varepsilon = a_0/(2p^+)$, we have

$$\begin{aligned} &\int_\Omega \frac{|\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)}}{p(x)} dx - \varepsilon \int_\Omega |\mathbf{u}|^{p(x)} dx - C_\varepsilon \int_\Omega |\mathbf{u}|^{q(x)} dx \\ &\geq \sum_{i=1}^m \left[\frac{\min\{1, a_0\}}{2p^+} \int_{\Omega_i} |\nabla \times \mathbf{u}|^{p(x)} + |\mathbf{u}|^{p(x)} dx - C_0 \int_{\Omega_i} |\mathbf{u}|^{q(x)} dx \right]. \end{aligned}$$

By Theorem 2.1 of [2], there exists $\tilde{C} > 1$ such that

$$\|\mathbf{u}\|_{\mathbf{L}^{q(x)}(\Omega_i)} \leq \tilde{C} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)} \quad \text{for } i = 1, 2, \dots, m.$$

Choose $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} \leq 1/\tilde{C}$. Clearly, $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)} \leq 1/\tilde{C} < 1$ and $\|\mathbf{u}\|_{\mathbf{L}^{q(x)}(\Omega_i)} \leq 1$ for $i = 1, 2, \dots, m$. Hence,

$$\begin{aligned} &\sum_{i=1}^m \left[\frac{\min\{1, a_0\}}{2p^+} \int_{\Omega_i} |\nabla \times \mathbf{u}|^{p(x)} + |\mathbf{u}|^{p(x)} dx - C_0 \int_{\Omega_i} |\mathbf{u}|^{q(x)} dx \right] \\ &\geq \sum_{i=1}^m \left(\frac{\min\{1, a_0\}}{2p^+} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{p_i^+} - C_0 \tilde{C}^{q^+} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{q_i^-} \right), \end{aligned}$$

due to (2.1). Set

$$g(t) := \frac{\min\{1, a_0\}}{2p^+} t^{p_i^+} - C_0 \tilde{C}^{q^+} t^{q_i^-}.$$

Since $p_i^+ < q_i^-$, there exist $t_i \in (0, 1)$ small enough such that $g(t) > 0$ for all $t \in (0, t_i]$ and $i = 1, 2, \dots, m$. For $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$ with $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} = \rho$, there exists i_0 such that $\rho/m^2 \leq \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_{i_0})} < \rho$, since $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} \leq m \sum_{i=1}^m \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)}$. Then for $0 < \rho \leq \min_{1 \leq i \leq m} \{t_i\}$, we have

$$\begin{aligned} I(\mathbf{u}) &\geq \sum_{i=1}^m \left(\frac{\min\{1, a_0\}}{2p^+} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{p_i^+} - C_0 \tilde{C}^{q^+} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{q_i^-} \right) \\ &\geq \frac{\min\{1, a_0\}}{2p^+} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_{i_0})}^{p_{i_0}^+} - C_0 \tilde{C}^{q^+} \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega_{i_0})}^{q_{i_0}^-} \\ &\geq \frac{\min\{1, a_0\}}{2p^+} \left(\frac{\rho}{m^2} \right)^{p_{i_0}^+} - C_0 \tilde{C}^{q^+} \rho^{q_{i_0}^-}. \end{aligned}$$

Choosing

$$\rho = \min \left\{ \left(\frac{\min\{1, a_0\}}{2p^+} \frac{1}{m^{2p_{i_0}^+} C_0 \tilde{C}} \right)^{\frac{1}{q_{i_0}^- - p_{i_0}^+}}, t_i \right\}, \quad i = 1, 2, \dots, m.$$

we get

$$I(\mathbf{u}) \geq \alpha := \left(\frac{\min\{1, a_0\}}{2p^+ m^{2p_{i_0}^+}} - C_0 \tilde{C}^{q^+} \rho^{q_{i_0}^- - p_{i_0}^+} \right) \rho^{p_{i_0}^+} > 0,$$

for all $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$, with $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} = \rho$. □

Lemma 3.5 (Mountain Pass Geometry II). *Let (\mathcal{A}) and (H_1) – (H_5) hold. Then there exists $\mathbf{e} \in (C_0^\infty(\Omega))^3$, with $\|\mathbf{e}\|_{\mathbf{W}^{p(x)}(\Omega)} > \rho$, such that $I(\mathbf{e}) < 0$, where $\rho > 0$ defined in Lemma 3.4.*

Proof. For any $(x, \xi) \in \Omega \times \mathbb{R}^3$, set $k(t) := F(x, t^{-1}\xi)t^\mu$ for all $t \geq 1$. Then one can deduce from (H_5)

$$k'(t) = t^{\mu-1} [\mu F(x, t^{-1}\xi) - t^{-1} \xi \mathbf{f}(x, t^{-1}\xi)] \leq 0,$$

that k is nondecreasing on $[1, \infty)$. Then for any $|\xi| \geq 1$, we have $k(1) \geq k(|\xi|)$, that is,

$$F(x, \xi) \geq F(x, |\xi|^{-1}\xi) |\xi|^\mu \geq c_F |\xi|^\mu, \quad \text{for all } x \in \Omega \text{ and } |\xi| \geq 1, \quad (3.8)$$

where $c_F = \inf_{x \in \Omega, |\xi|=1} F(x, \xi) > 0$ by assumption (H_5) . By (H_4) , there exists $\delta \in (0, 1)$ such that $|\mathbf{f}(x, \xi)| \leq |t|^{p(x)-1}$, for all $x \in \Omega$ and $|\xi| \in [0, \delta]$. Furthermore, by (H_2) , for all $x \in \Omega$ and all $\xi \in \mathbb{R}^3$, with $\delta < |\xi| \leq 1$, we have that $|\mathbf{f}(x, \xi)| \leq 2C |\xi|^{p(x)-1}$. Thus,

$$|\mathbf{f}(x, \xi) \cdot \xi| \geq -(2C + 1) |\xi|^{p(x)}$$

for $x \in \Omega$ and $0 \leq |\xi| \leq 1$. Hence,

$$F(x, \xi) = \int_0^1 \mathbf{f}(x, \tau \xi) \cdot \xi d\tau \geq -\frac{1+2C}{p(x)} |\xi|^{p(x)}, \quad (3.9)$$

for all $x \in \Omega$ and $0 \leq |\xi| \leq 1$. Combining (3.8) with (3.9), we obtain

$$F(x, \xi) \geq c_F |\xi|^\mu - M_1 |\xi|^{p(x)} \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^3, \quad (3.10)$$

where $M_1 = c_F + (1 + 2C)/p^-$. Therefore, for $t \geq 1$, and $\mathbf{u} \in (C_0^\infty(\Omega))^3$ with $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} = 1$, we deduce from (3.10) that

$$\begin{aligned} I(t\mathbf{u}) &\leq \frac{1}{p^-} t^{p^+} \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} + a_1 |\mathbf{u}|^{p(x)} dx - c_F t^\mu \int_{\Omega} |\mathbf{u}|^\mu dx + M_1 t^{p^+} \int_{\Omega} |\mathbf{u}|^{p(x)} dx \\ &\leq \frac{1}{p^-} t^{p^+} \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} dx + \left(\frac{a_1}{p^-} + M_1 \right) t^{p^+} \int_{\Omega} |\mathbf{u}|^{p(x)} dx - c_F C_\mu^\mu t^\mu \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)}^\mu \\ &\leq \left(\frac{1}{p^-} + \frac{a_1}{p^-} + M_1 \right) t^{p^+} - c_F C_\mu^\mu t^\mu. \end{aligned}$$

It follows from $\mu > p^+$ that $I(t\mathbf{u}) \rightarrow -\infty$ as $t \rightarrow \infty$. Then the assertion follows by letting $\mathbf{e} := T_0 \mathbf{u}$, with $T_0 > 0$ large enough. \square

4 Proof of Theorems 1.1

The following standard mountain pass theorem will be used to get our first result. For the reader's convenience, we would like to state as follows.

Theorem 4.1. *Let Q be a functional on a Banach space E and $Q \in C^1(E, \mathbb{R})$. Let us assume that there exists $\alpha, \rho > 0$ such that*

- (i) $Q(u) \geq \alpha$, for all $u \in E$ with $\|u\| = \rho$,
- (ii) $Q(0) = 0$ and $Q(e) < \alpha$ for some $e \in E$ with $\|e\| > \rho$.

Let us define $\Gamma = \{\gamma \in C([0, 1]; E) : \gamma(0) = 0, \gamma(1) = e\}$, and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} Q(\gamma(t)).$$

Then there exists a sequence $\{u_n\}_n \subset E$ such that $Q(u_n) \rightarrow c$ and $Q'(u_n) \rightarrow 0$ in E' (dual of E).

Proof of Theorem 1.1. Taking into account Lemmas 3.4 and 3.5, by Theorem 4.1 there exists a sequence $\{\mathbf{u}_n\}_n \subset \mathbf{W}^{p(x)}(\Omega)$ such that $I(\mathbf{u}_n) \rightarrow c > 0$ and $I'(\mathbf{u}_n) \rightarrow 0$ as $n \rightarrow \infty$. Then, in view of Lemma 3.3, there exists a nontrivial critical point $\mathbf{u}_0 \in \mathbf{W}^{p(x)}(\Omega)$ of I with $I(\mathbf{u}_0) = c > 0 = I(\mathbf{0})$.

Set $\mathcal{N} := \{\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega) \setminus \{\mathbf{0}\} : I'(\mathbf{u}) = 0\}$. Then $\mathbf{u}_0 \in \mathcal{N} \neq \emptyset$. Next we show that I is coercive and bounded from below on \mathcal{N} . Indeed, by $I'(\mathbf{u}) = 0$ and (H_3) , we get

$$\int_{\Omega} F(x, \mathbf{u}) dx \leq \frac{1}{\mu} \int_{\Omega} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{u} dx = \frac{1}{\mu} \left(\int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)} dx \right). \quad (4.1)$$

By using $\mu > p^+$ and (4.1), we obtain

$$\begin{aligned} I(\mathbf{u}) &\geq \frac{1}{p^+} \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)} dx - \frac{1}{\mu} \left(\int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)} dx \right) \\ &= \left(\frac{1}{p^+} - \frac{1}{\mu} \right) \left(\int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} + a(x) |\mathbf{u}|^{p(x)} dx \right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu} \right) \min\{1, a_0\} \int_{\Omega} |\nabla \times \mathbf{u}|^{p(x)} + |\mathbf{u}|^{p(x)} dx, \end{aligned} \quad (4.2)$$

which implies that $I(\mathbf{u})$ is coercive and bounded from below on \mathcal{N} . Define

$$c_{\min} = \inf\{I(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\}.$$

Clearly, $0 \leq c_{\min} \leq I(\mathbf{u}_0) = c$. Let $\{\mathbf{u}_n\}_n$ be a minimizing sequence for c_{\min} , namely $I(\mathbf{u}_n) \rightarrow c_{\min}$ and $\langle I'(\mathbf{u}_n), \mathbf{u}_n \rangle = 0$. Then, since \mathcal{N} is a complete metric space, by Ekeland's variational principle we can find a new minimizing sequence, still denoted by $\{\mathbf{u}_n\}_n$, which is a (PS) sequence for I at the level c_{\min} . Moreover, by Lemma 3.3, $\{\mathbf{u}_n\}_n$ has a convergence subsequence, still denoted by $\{\mathbf{u}_n\}_n$, such that $\mathbf{u}_n \rightarrow \mathbf{u}$ in $\mathbf{W}^{p(x)}(\Omega)$. Then $c_{\min} = I(\mathbf{u})$ and $I'(\mathbf{u}) = 0$. Now we claim that $c_{\min} > 0$. By contradiction, there is $\{\mathbf{v}_n\}_n \subset \mathbf{W}^{p(x)}(\Omega) \setminus \{\mathbf{0}\}$ with $I'(\mathbf{v}_n) = 0$ and $I(\mathbf{v}_n) \rightarrow 0$. This, (4.2) and (2.1) imply that $\|\mathbf{v}_n\|_{\mathbf{W}^{p(x)}(\Omega)} \rightarrow 0$. On the other hand, by (3.3), we have for any $\varepsilon \in (0, a_0)$

$$\begin{aligned} \int_{\Omega} |\nabla \times \mathbf{v}_n|^{p(x)} + a(x)|\mathbf{v}_n|^{p(x)} dx &= \int_{\Omega} \mathbf{f}(x, \mathbf{v}_n) \cdot \mathbf{v}_n dx \\ &\leq \varepsilon \int_{\Omega} |\mathbf{v}_n|^{p(x)} dx + C_{\varepsilon} \int_{\Omega} |\mathbf{v}_n|^{q(x)} dx. \end{aligned}$$

Thus, we have

$$\int_{\Omega} |\nabla \times \mathbf{v}_n|^{p(x)} dx + (a_0 - \varepsilon) \int_{\Omega} |\mathbf{v}_n|^{p(x)} dx \leq C_{\varepsilon} \int_{\Omega} |\mathbf{v}_n|^{q(x)} dx.$$

Taking $\varepsilon = \frac{a_0}{2}$, we deduce

$$\frac{\min\{1, a_0\}}{2} \int_{\Omega} |\nabla \times \mathbf{v}_n|^{p(x)} dx + |\mathbf{v}_n|^{p(x)} dx - C \int_{\Omega} |\mathbf{v}_n|^{q(x)} dx \leq 0 \quad (4.3)$$

Similar to Lemma 3.4, there exist m open hypercubes Ω_i ($i = 1, 2, \dots, m$) which have no common points such that $\bar{\Omega} = \bigcup_{i=1}^m \bar{\Omega}_i$,

$$p_i^+ = \max_{x \in \Omega_i} p(x) < q_i^- = \min_{x \in \Omega_i} q(x),$$

and $q_i^- - p_i^+ > c_1$ for $i = 1, 2, \dots, m$. Then

$$\begin{aligned} &\frac{\min\{1, a_0\}}{2} \int_{\Omega} |\nabla \times \mathbf{v}_n|^{p(x)} dx + |\mathbf{v}_n|^{p(x)} dx - C \int_{\Omega} |\mathbf{v}_n|^{q(x)} dx \\ &= \sum_{i=1}^m \left[\frac{\min\{1, a_0\}}{2} \int_{\Omega_i} |\nabla \times \mathbf{v}_n|^{p(x)} + |\mathbf{v}_n|^{p(x)} dx - C \int_{\Omega_i} |\mathbf{v}_n|^{q(x)} dx \right] \\ &\geq \sum_{i=1}^m \left(\frac{\min\{1, a_0\}}{2} \|\mathbf{v}_n\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{p_i^+} - C \tilde{C}^{q^+} \|\mathbf{v}_n\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{q_i^-} \right), \end{aligned}$$

Note that $q_i^- > p_i^+$ and $\|\mathbf{v}_n\|_{\mathbf{W}^{p(x)}(\Omega_i)}^{p_i^+} \leq \|\mathbf{v}_n\|_{\mathbf{W}^{p(x)}(\Omega)}^{p_i^+} \rightarrow 0$ as $n \rightarrow \infty$. Thus, we have

$$\frac{\min\{1, a_0\}}{2} \int_{\Omega} |\nabla \times \mathbf{v}_n|^{p(x)} dx + |\mathbf{v}_n|^{p(x)} dx - C \int_{\Omega} |\mathbf{v}_n|^{q(x)} dx > 0$$

for n large enough, which contradicts (4.3). Thus the claim follows. Therefore, \mathbf{u} is a nontrivial critical point of I , with $I(\mathbf{u}) = c_{\min} > 0$, that is, \mathbf{u} is a ground state solution of (1.1). \square

5 Proof of Theorem 1.2

In this section, we will apply the following symmetric mountain pass theorem to obtain our second result, see for example [8, Theorem 2.2].

Theorem 5.1. *Let X be a real infinite dimensional Banach space and $K \in C^1(X)$ a functional satisfying the (PS) condition as well as the following three properties:*

- (a) $K(0) = 0$ and there exist two constant $\rho, \alpha > 0$ such that $K|_{\partial B_\rho} \geq \alpha$;
- (b) K is even;
- (c) for all finite dimensional subspaces $Y \subset X$, there exists $R = R(Y) > 0$ such that

$$K(u) \leq 0 \quad \text{for all } u \in X \setminus B_R(Y),$$

where $B_R(Y) = \{u \in Y : \|u\| \leq R\}$. Then K posses an unbounded sequence of critical values characterized by a minimax argument.

Lemma 5.1. *Suppose that f satisfies (H_1) – (H_6) . Then for any finite dimensional subspace E of $\mathbf{W}^{p(x)}(\Omega)$ there exists $R_0 = R_0(E) > 0$ such that*

$$I(\mathbf{u}) < 0 \quad \text{for all } \mathbf{u} \in \mathbf{W}^{p(x)}(\Omega) \setminus B_{R_0}(E),$$

where $B_{R_0}(E) = \{\mathbf{u} \in E : \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} < R_0\}$.

Proof. Let E be a fixed finite dimensional subspace of $\mathbf{W}^{p(x)}(\Omega)$. For any $\mathbf{u} \in E$, with $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} = 1$, and for all $t \geq 1$ we conclude from (3.5) and (3.8) that

$$\begin{aligned} I(t\mathbf{u}) &\leq t^{p^+} \int_{\Omega} \frac{|\nabla \times \mathbf{u}|^{p(x)} + a|\mathbf{u}|^{p(x)}}{p(x)} dx - \int_{\Omega} F(x, t\mathbf{u}) dx \\ &\leq t^{p^+} \int_{\Omega} \frac{|\nabla \times \mathbf{u}|^{p(x)} + a|\mathbf{u}|^{p(x)}}{p(x)} dx - c_F t^\mu \int_{\Omega} |\mathbf{u}|^\mu dx + M_1 t^{p^+} \int_{\Omega} |\mathbf{u}|^{p(x)} dx \\ &\leq \left(\frac{1}{p^-} + \frac{a_1}{p^-} + M_1 \right) t^{p^+} - c_F C_E t^\mu, \end{aligned}$$

where $C_E > 0$ such that $\|\mathbf{u}\|_{L^\mu(\Omega)} \geq C_E \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)}$ for all $u \in E$. Moreover, since $\mu > p^+$, we have

$$I(t\mathbf{u}) \leq \left(\frac{1}{p^-} + \frac{a_1}{p^-} + M_1 \right) t^{p^+} - c_F C_E t^\mu \rightarrow -\infty, \quad \text{as } t \rightarrow \infty.$$

Therefore, as $R \rightarrow \infty$,

$$\sup_{\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)}=R, \mathbf{u} \in E} I(\mathbf{u}) = \sup_{\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)}=1, \mathbf{u} \in E} I(R\mathbf{u}) \rightarrow -\infty.$$

Hence, there exists $R_0 > 0$ so large such that $I(\mathbf{u}) < 0$ for all $\mathbf{u} \in E$, with $\|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} = R$ and $R \geq R_0$. This completes the proof. \square

Proof of Theorem 1.2. By (H_3) , we know that I is even. Since $I(0) = 0$, we conclude from Lemma 3.3, Lemma 3.4, Lemma 5.1 and Theorem 5.1, that there exists an unbounded sequence of weak solutions of problem (1.1). \square

6 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. By Lemma 3.1, J is a convex functional in $\mathbf{W}^{p(x)}(\Omega)$, so J is weakly lower semi-continuous in $\mathbf{W}^{p(x)}(\Omega)$. This together with Lemma 3.2 yields that I is weakly lower semi-continuous in $\mathbf{W}^{p(x)}(\Omega)$. By (H_2) , we have for all $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$

$$I(\mathbf{u}) \geq \frac{1}{p^+} \int_{\Omega} |\nabla \times \mathbf{u}_n|^{p(x)} + a(x) |\mathbf{u}_n|^{p(x)} dx - C|\Omega| - 2C \int_{\Omega} |\mathbf{u}|^{q(x)} dx.$$

Since $q(x) < p(x)$ in $\bar{\Omega}$, we have $\Lambda := \inf_{\bar{\Omega}}(p(x) - q(x)) > 0$. Thus, Young's inequality implies that

$$\begin{aligned} |\mathbf{u}|^{q(x)} &\leq \frac{a_0}{4Cp^+} |\mathbf{u}|^{p(x)} + \left(\frac{a_0}{4Cp^+} \right)^{-\frac{p(x)}{p(x)-q(x)}} \\ &\leq \frac{a_0}{4Cp^+} |\mathbf{u}|^{p(x)} + \left(\frac{4Cp^+}{a_0} \right)^{\frac{p^+}{\Lambda}} + 1. \end{aligned}$$

Hence, we arrive at

$$I(\mathbf{u}) \geq \frac{1}{p^+} \int_{\Omega} |\nabla \times \mathbf{u}_n|^{p(x)} dx + \frac{a_0}{2p^+} \int_{\Omega} |\mathbf{u}_n|^{p(x)} dx - 2C \left[\left(\frac{4Cp^+}{a_0} \right)^{\frac{p^+}{\Lambda}} + 1 \right] |\Omega|.$$

This implies that I is coercive and bounded from below on $\mathbf{W}^{p(x)}(\Omega)$. **Therefore** there exists $\mathbf{u}_0 \in \mathbf{W}^{p(x)}(\Omega)$ such that

$$I(\mathbf{u}_0) = \inf \{ I(\mathbf{u}) : \mathbf{u} \in \mathbf{W}^{p(x)}(\Omega) \}.$$

Next we show $\mathbf{u}_0 \neq \mathbf{0}$. Let $x_0 \in \Omega_0$. **Since** $p, q \in C(\bar{\Omega})$ and $q(x) < p(x)$ for all $x \in \bar{\Omega}$, we can choose $R > 0$ small enough such that $B_R(x_0) \subset \Omega_0$ and $p_0^- = \max_{x \in B_R(x_0)} p(x) > q_0^+ = \min_{x \in B_R(x_0)} \{q(x)\}$. Choose $\mathbf{v} \in (C_0^\infty(B_R(x_0)))^3$ with $0 \leq |\mathbf{v}| \leq 1$, $\|\mathbf{v}\|_{\mathbf{W}^{p(x)}(B_R(x_0))} \leq C(R)$ and $\|\mathbf{v}\|_{L^{q(x)}(B_R(x_0))} > 0$. Then for $0 < t < \delta$,

$$\begin{aligned} I(t\mathbf{v}) &\leq \frac{1}{p^-} \int_{B_R(x_0)} t^{p(x)} |\nabla \times \mathbf{v}|^{p(x)} + a(x) t^{p(x)} |\mathbf{v}|^{p(x)} dx - \int_{B_R(x_0)} C_0 t^{q(x)} |\mathbf{v}|^{q(x)} dx \\ &\leq \frac{\max\{1, a_1\}}{p^-} t^{p_0^-} \max \left\{ C(R)^{p_0^+}, C(R)^{p_0^-} \right\} - t^{q_0^+} C_0 \int_{B_R(x_0)} |\mathbf{v}|^{q(x)} dx. \end{aligned}$$

Since $q_0^+ < p_0^-$, we get $I(\bar{t}\mathbf{v}) < 0$ by taking $\bar{t} > 0$ small enough. Hence $I(\mathbf{u}_0) \leq I(\bar{t}\mathbf{v}) < 0$. Then \mathbf{u}_0 is a nontrivial critical point, and hence a nontrivial ground state solution of (1.1).

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