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Interlacing of zeros of certain weakly holomorphic modular forms for $\Gamma_0^+(2)$

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ABSTRACT

We prove that zeros of each basis element of the space of weakly holomorphic modular forms of weight k for the Fricke group $\Gamma_0^+(2)$ of level 2 interlace, extending the result for $SL_2(\mathbb{Z})$ of Jenkins and Pratt [4].

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1. Introduction and preliminaries

The location of zeros of various modular forms has been studied actively by many mathematicians. In particular, Rankin and Swinnerton-Dyer have showed in [8] that in the standard fundamental domain for the full modular group $SL_2(\mathbb{Z})$, the zeros of the Eisenstein series $E_k(z)$ of even weight $k \geq 4$ lie on the unit circle.

It would be helpful to understand the locations of zeros of modular forms if we find special relations and patterns such as ‘*interlacing*’ of zeros of a family of certain modular forms for some congruence groups.

First, we recall the definition of interlacing of zeros of functions.

Definition 1.1. ([4, p. 63]) When two functions have zeros that lie on the same arc, we say that the zeros of two functions *interlace* if every zero of one function is contained in an open interval whose endpoints are zeros of the other function, and each such interval contains exactly one zero.

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In [4], Jenkins and Pratt proved that the zeros of a family of extremal modular forms interlace. Referring to the introduction of [4], there are related preceding results as follows; Nozaki [7] proved that the zeros of the Eisenstein series $E_k(z)$ of weight k interlace with the zeros of $E_{k+12}(z)$ of weight $k+12$, and Asai, Kaneko, and Ninomiya [1] and Jermann [5] proved that the zeros of the modular function $j_n(z)$ interlace with the zeros of $j_{n+1}(z)$.

Let $\Gamma_0^+(2)$ be the Fricke group of level 2 and denote by $f_{k,m}$ each basis element of the space of weakly holomorphic modular forms of weight k for $\Gamma_0^+(2)$. In this paper, we consider the location of zeros of $f_{k,m}$ and show that the zeros of $f_{k,m}$ interlace with those of $f_{k,m+1}$ and with those of $f_{k+8,m}$, respectively for large enough k and m .

Our main result extends the result for $\mathrm{SL}_2(\mathbb{Z})$ of Jenkins and Pratt in [4] to the result for the Fricke group $\Gamma_0^+(2)$ and the proof of our result follows the idea of [4], but we include another approach to computations and detailed verifications in our case for $\Gamma_0^+(2)$ by analyzing the behavior of zeros of approximations of those basis forms, because the proofs for some cases have to be treated in a different way from ones in [4]. For example, the behavior of the zeros near $\frac{3\pi}{4}$ needs a different computational treatment from those near $\frac{\pi}{2}$. (See Lemma 4.1, Lemma 4.2, Lemma 4.3, and Lemma 4.4.)

2. The main theorem: interlacing zeros

Throughout this paper, we let $\mathfrak{F}^+(2)$ be the standard fundamental domain for $\Gamma_0^+(2)$ given by

$$\mathfrak{F}^+(2) := \left\{ z \in \mathbb{C} : |z| \geq 1/\sqrt{2}, -1/2 \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ z \in \mathbb{C} : |z| > 1/\sqrt{2}, 0 \leq \operatorname{Re}(z) < 1/2 \right\}$$

referring to [6, p. 694], and let I be the interval and S be a subset of $\mathfrak{F}^+(2)$ defined by

$$I := (\pi/2, 3\pi/4), \quad \text{and} \quad S := \left\{ \frac{1}{\sqrt{2}} e^{i\theta} : \theta \in I \right\}.$$

For a given even integer $k \in 2\mathbb{Z}$, we can write

$$k = 8\ell_k + r_k,$$

for a unique integer $\ell_k \in \mathbb{Z}$ and a unique integer $r_k \in \{0, 4, 6, 10\}$. We then have showed in [2] that the zeros in $\mathfrak{F}^+(2)$ of each $f_{k,m}$ of the canonical basis elements for the space of weakly holomorphic modular forms of weight k for $\Gamma_0^+(2)$ lie on the circle with radius $\frac{1}{\sqrt{2}}$ for $m \geq 2|\ell_k| - \ell_k + 8$. Here as in [3], $f_{k,m}$ is the unique weakly holomorphic modular form having the Fourier expansion of the form

$$f_{k,m}(z) = q^{-m} + \mathcal{O}(q^{\ell_k+1}).$$

The following is our main theorem and its proof is given in Section 4.

Theorem 2.1. *Let $\epsilon > 0$. Then for large enough $k > 0$ and each fixed $m \geq \ell_k + 8$ (for large enough $m \geq \ell_k + 8$ when k is fixed, respectively), the zeros of $f_{k,m}(z)$ interlace with the zeros of $f_{k+12,m}(z)$ (resp. $f_{k,m+1}(z)$) on the arc*

$$\mathcal{A}_\epsilon = \left\{ \frac{e^{i\theta}}{\sqrt{2}} : \frac{\pi}{2} < \theta < \frac{3\pi}{4} - \epsilon \right\}.$$

First we note that the zeros in $\mathfrak{F}^+(2)$ of $f_{k,m}$ for $\Gamma_0^+(2)$ are contained in $\{1/\sqrt{2}e^{i\theta} : \frac{\pi}{2} < \theta < \frac{3\pi}{4}\}$ by [2]. Thus we can take $\rho = \frac{3\pi}{4} - \epsilon$ for some ϵ such that $\frac{\pi}{2} < \rho < \frac{3\pi}{4}$ and the zeros in $\mathfrak{F}^+(2)$ of $f_{k,m}$ are contained in $\{1/\sqrt{2}e^{i\theta} : \frac{\pi}{2} < \theta < \rho\}$.

In particular, for $k > 0$ as assumed in [Theorem 2.1](#), if $\frac{\pi}{2} < \theta < 2$, then referring to [\[2, p. 309, Line 18\]](#), we let

$$D_1 := (0.5509743592)^{\ell_k} \frac{377.7378014}{(0.03690703328)(0.61658483)} (0.2172670797)^m,$$

and if $2 \leq \theta < \rho$, then since $\frac{1}{\sqrt{2}} < \sin \theta < 1$ and

$$-\frac{1}{\sqrt{2}} < \cos \rho < \cos \theta < \cos(2) < -0.4161468364,$$

referring to [\[2, p. 311, Eq. \(12\), p. 308, Lines 5–12\]](#), we let

$$D_2 := e^{-2\pi m(1+\sqrt{2}\cos\rho)/3} (3 + 2\sqrt{2}\cos\rho)^{-\frac{1}{2}} + e^{-2\pi m(1+0.4161468364)/5} (3 + 2\sqrt{2} \cdot 0.4161468364)^{-\frac{1}{2}} \\ + (2.314348553)^{\ell_k} \frac{5292.997457}{(0.02697058723)(9.145597363)} (0.286095432)^m.$$

Then since $\frac{377.7378014}{(0.03690703328)(0.61658483)} < \frac{5292.997457}{(0.02697058723)(9.145597363)}$, we have that $D_1 < D_2$. Let

$$D := e^{-2\pi m(1+\sqrt{2}\cos\rho)/3} (3 + 2\sqrt{2}\cos\rho)^{-\frac{1}{2}} + e^{-2\pi m(1+0.4161468364)/5} (3 + 2\sqrt{2} \cdot 0.4161468364)^{-\frac{1}{2}} \\ + 26.07168336(0.6586856845)^m < 1.960061437 < 2. \quad (1)$$

Then since $m \geq \ell_k + 8$, we have that $D_2 \leq D$, and hence $D_1 < D_2 \leq D$. Therefore, by [\[2, Lemma 2.3 and its proof\]](#), for all $\theta \in (\frac{\pi}{2}, \rho)$,

$$\left| e^{ik\theta/2} e^{-2\pi m \frac{1}{\sqrt{2}} \sin \theta} f_{k,m} \left(\frac{1}{\sqrt{2}} e^{i\theta} \right) - 2 \cos \left(\frac{k\theta}{2} - 2\pi m \frac{1}{\sqrt{2}} \cos \theta \right) \right| < D < 2. \quad (2)$$

So we first show the zeros of the cosine functions of the following functions interlace as in [\[4\]](#) in order to show the interlacing of zeros of $f_{k,m}$ with those of $f_{k+8,m}$ and those of $f_{k,m+1}$, respectively: We define for $\theta \in I$,

$$b(\theta) = \frac{k\theta}{2} - 2\pi m \frac{1}{\sqrt{2}} \cos \theta = \frac{k\theta}{2} - \sqrt{2}\pi m \cos \theta, \\ b_{k+8}(\theta) = \frac{(k+8)\theta}{2} - \sqrt{2}\pi m \cos \theta, \\ b_{m+1}(\theta) = \frac{k\theta}{2} - \sqrt{2}\pi(m+1) \cos \theta.$$

From now on, $b_*(\theta)$ denotes $b_{k+8}(\theta)$ or $b_{m+1}(\theta)$ unless otherwise specified.

3. Interlacing for cosine functions

In this section, we show that the zeros of the cosine functions of $b(\theta)$ interlace with those of the cosine function of $b_*(\theta)$, as in [\[4\]](#).

Lemma 3.1. *If $m \geq 2|\ell_k| - \ell_k + 8$, then*

- (a) *the first zero in I belongs to $\cos(b_*(\theta))$,*
- (b) *the last zero in I belongs to $\cos(b_*(\theta))$,*

- (c) the zeros of $\cos(b_*(\theta))$ and $\cos(b(\theta))$ in I are never equal, and
 (d) between two consecutive zeros of $\cos(b_*(\theta))$ there is exactly one zero of $\cos(b(\theta))$.

Hence we conclude that the zeros of $\cos(b(\theta))$ interlace on I with the zeros of $\cos(b_{k+8}(\theta))$ and with the zeros of $\cos(b_{m+1}(\theta))$ respectively.

Proof. We include the proof for our case here by modifying [4, Lemma 3.1] and giving a detailed new argument for some parts.

Note that $\cos(b_{k+8}(\theta) - \pi) = -\cos(b_{k+8}(\theta))$, hence we consider $\cos(b_{k+8}(\theta) - \pi)$ in order to consider the zeros of $\cos(b_{k+8}(\theta))$. At the end points of I , we have that

$$b\left(\frac{\pi}{2}\right) = \frac{k\pi}{4} = b_{m+1}\left(\frac{\pi}{2}\right) = \left(b_{k+8}\left(\frac{\pi}{2}\right) - \pi\right) - \pi,$$

and

$$b\left(\frac{3\pi}{4}\right) = \frac{3k\pi}{8} + \pi m = b_{m+1}\left(\frac{3\pi}{4}\right) - \pi = \left(b_{k+8}\left(\frac{3\pi}{4}\right) - \pi\right) - 2\pi.$$

And for each $\theta \in I$, we can easily show that

$$b(\theta) < b_{m+1}(\theta) < b(\theta) + \pi < b_{k+8}(\theta) - \pi < b(\theta) + 2\pi. \quad (3)$$

And since all derivatives of $b(\theta)$, $b_{k+12}(\theta)$ and $b_{m+1}(\theta)$ are positive on I , each of them is monotonically increasing on I .

To prove (a), since $b\left(\frac{\pi}{2}\right) = b_{m+1}\left(\frac{\pi}{2}\right) = \frac{k\pi}{4}$, we let α and β be the first zeros of $\cos(b_{m+1}(\theta))$ and $\cos(b(\theta))$ on I respectively so that $b_{m+1}(\alpha) = \frac{2n+1}{2}\pi = b(\beta)$ for the smallest integer n such that $2n+1$ greater than $\frac{k}{2}$. Then $b(\alpha) < \frac{2n+1}{2}\pi = b(\beta)$ by (3). Since $b(\theta)$ is increasing on I , $\alpha < \beta$. So the first zero of $\cos(b_{m+1}(\theta))$ occurs before the first zero of $\cos(b(\theta))$. Similarly, since $b\left(\frac{\pi}{2}\right) + \pi = b_{k+8}\left(\frac{\pi}{2}\right) - \pi = \frac{(k+4)\pi}{4}$, by letting α and β be the first zeros of $\cos(b_{k+8}(\theta) - \pi)$ and $\cos(b(\theta) + \pi)$ on I respectively so that $b_{k+8}(\alpha) - \pi = \frac{2n+1}{2}\pi = b(\beta) + \pi$ for the smallest integer n such that $2n+1$ greater than $\frac{k}{2}$, we show that the first zero of $\cos(b_{k+8}(\theta) - \pi)$ occurs before the first zero of $\cos(b(\theta) + \pi)$. Hence since $\cos(b_{k+8}(\theta) - \pi)$ and $\cos(b_{k+8}(\theta))$ have the same zeros and $\cos(b(\theta) + \pi)$ and $\cos(b(\theta))$ have the same zeros, both cases prove (a).

To prove (b), first we note that

$$b\left(\frac{3\pi}{4}\right) = b_{m+1}\left(\frac{3\pi}{4}\right) - \pi = \frac{3k\pi}{8} + \pi m.$$

The last zeros of $\cos(b(\theta))$ and $\cos(b_{m+1}(\theta) - \pi)$ on I occur when $b(\theta)$ and $b_{m+1}(\theta) - \pi$ have the same value $\frac{(2n+1)\pi}{2}$ for the largest odd integer $2n+1$ less than $\frac{3k}{4} + 2m$. Let α be the last zero of $\cos(b_{m+1}(\theta) - \pi)$ on I . Then $b_{m+1}(\alpha) - \pi = \frac{(2n+1)\pi}{2} < b(\alpha)$ by (3). Thus the last zero of $-\cos(b_{m+1}(\theta)) = \cos(b_{m+1}(\theta) - \pi)$ occurs after the last zero of $\cos(b(\theta))$.

Similarly, we note that

$$\frac{3k\pi}{8} + m\pi = b\left(\frac{3\pi}{4}\right) = \left(b_{k+8}\left(\frac{3\pi}{4}\right) - \pi\right) - 2\pi.$$

We proceed the same argument as above by replacing $b_{m+1}(\theta) - \pi$ by $b_{k+8}(\theta) - 3\pi$ and complete the proof of (b).

To prove (c), first we suppose $\cos(b(\alpha)) = 0 = \cos(b_{m+1}(\alpha))$ for some common zero $\alpha \in I$. Then

$$b_{m+1}(\alpha) - b(\alpha) = n\pi, \text{ for some integer } n.$$

But by (3), $0 < n\pi < \pi$. This is a contradiction. So $\cos(b(\theta))$ and $\cos(b_{m+1}(\theta))$ cannot have the same zero on I .

Secondly, we suppose $\cos(b(\alpha)) = 0 = \cos(b_{k+8}(\alpha))$ for some common zero $\alpha \in I$. Then since $\cos(b(\theta)) = -\cos(b(\theta) + \pi)$ and $\cos(b_{k+8}(\theta)) = -\cos(b_{k+8}(\theta) - \pi)$,

$$(b_{k+8}(\alpha) - \pi) - (b(\alpha) + \pi) = n\pi, \text{ for some integer } n,$$

which leads a contradiction again. So $\cos(b(\theta))$ and $\cos(b_{k+8}(\theta))$ cannot have the same zero on I .

To prove (d), first we consider the case when $b_*(\theta) = b_{m+1}(\theta)$, if α_1 and α_2 are two consecutive zeros of $\cos(b_{m+1}(\theta))$ so that $b_{m+1}(\alpha_1) = b_{m+1}(\alpha_2) - \pi = \frac{2n+1}{2}\pi$ for some integer n , then by (3), $b(\alpha_1) < \frac{2n+1}{2}\pi < b(\alpha_2)$, so there exists a zero β in the interval (α_1, α_2) such that $b(\beta) = \frac{2n+1}{2}\pi$ and $\cos(b(\beta)) = 0$. Hence there exists a zero of $\cos(b(\theta))$ between every two consecutive zeros of $\cos(b_{m+1}(\theta))$.

Next we give a detailed proof of the uniqueness of a zero β of $\cos(b(\theta))$ between every two consecutive zeros of $\cos(b_{m+1}(\theta))$. Suppose there exists another zero β^* of $\cos(b(\theta))$ such that β and β^* (or β^* and β) are consecutive zeros of $\cos(b(\theta))$.

If $\beta^* \in (\beta, \alpha_2)$, then $b(\beta^*) = b(\beta) + \pi = \frac{2n+1}{2}\pi + \pi = b_{m+1}(\alpha_2)$. Since $b_{m+1}(\alpha_2) = b(\beta^*) < b_{m+1}(\beta^*)$ by (3), we have that $\alpha_2 < \beta^*$, which is a contradiction.

If $\beta^* \in (\alpha_1, \beta)$, then $b(\beta^*) = b(\beta) - \pi = \frac{2n+1}{2}\pi - \pi = b_{m+1}(\alpha_1) - \pi < b(\alpha_1)$ by (3). Hence we have that $\beta^* < \alpha_1$, which is a contradiction.

We can show that there exists exactly one zero of $\cos(b(\theta))$ between two consecutive zeros of $\cos(b_{k+8}(\theta))$ by the same argument as in the case for $b_{m+1}(\theta)$. So this completes the proof of (d). \square

Lemma 3.2. Suppose $m \geq 2|\ell_k| - \ell_k + 8$. The distance between two consecutive zeros of $\cos(b(\theta))$ (resp. $\cos(b_{k+8}(\theta))$, $\cos(b_{m+1}(\theta))$) is less than or equal to $\frac{2\pi}{k+2m\pi}$ (resp. $\frac{2\pi}{k+8+2m\pi}$, $\frac{2\pi}{k+2(m+1)\pi}$).

Proof. We include a self-contained proof here by modifying [4, Proposition 3.2].

If z_1 and z_2 such that $z_1 < z_2$ are two consecutive zeros of $\cos(b(\theta))$, then $b(z_1)$ and $b(z_2)$ differ by π . Since $b(\theta)$ is increasing, we have that for some $\theta_0 \in (z_1, z_2) \subseteq I$,

$$\frac{\pi}{z_2 - z_1} = \frac{b(z_2) - b(z_1)}{z_2 - z_1} = b'(\theta_0) = \frac{k}{2} + \sqrt{2}m\pi \sin(\theta_0) \geq \frac{k}{2} + m\pi > 0.$$

Hence

$$z_2 - z_1 \leq \frac{2\pi}{k + 2m\pi}.$$

For $b_*(\theta)$ we can obtain our assertion by the same argument. \square

Now we prove that the difference between two consecutive zeros of $\cos(b(\theta))$ is greater than the difference between adjacent two consecutive zeros of $\cos(b_*(\theta))$.

Proposition 3.3. Suppose that $m \geq 2|\ell_k| - \ell_k + 8$. Let w_1, w_2, w_3 be three consecutive zeros of $\cos(b_*(\theta))$ in I and z_1, z_2 be two consecutive zeros of $\cos(b(\theta))$ in I such that $w_1 < z_1 < w_2 < z_2 < w_3$. Suppose $k > 0$. Then $z_1 - w_1 < z_2 - w_2$ and $w_2 - z_1 > w_3 - z_2$.

Hence the shortest distance between the zeros of $\cos(b_*(\theta))$ and the zeros of $\cos(b(\theta))$ on I is one between the first zero of $\cos(b_*(\theta))$ and the first zero of $\cos(b(\theta))$ or between the last zero of $\cos(b(\theta))$ and the last zero of $\cos(b_*(\theta))$.

Proof. We include the proof for our case here by modifying [4, Proposition 3.4].

First, consider the case when $b_*(\theta) = b_{k+8}(\theta)$. Note that

$$b'(\theta) = \frac{k}{2} + \sqrt{2}m\pi \sin \theta, \text{ and } b'_{k+8}(\theta) = \frac{k+8}{2} + \sqrt{2}m\pi \sin \theta = b'(\theta) + 4.$$

Note that $m > 0$ and $\sin \theta$ decreases from 1 to $\frac{1}{\sqrt{2}}$ on I . Hence both $b'(\theta)$ and $b'_{k+8}(\theta)$ are decreasing functions of θ on I . If $b'(\theta) > b'_{k+8}(\theta + \epsilon)$ for some $\epsilon > 0$, then b'_{k+8} has decreased by at least 4 on the interval $(\theta, \theta + \epsilon)$ since $b'_{k+8}(\theta) = b'(\theta) + 4$. Since $b_{k+8}^{(2)}(\theta) = \sqrt{2}\pi m \cos \theta > -m\pi$ on I , we have that $\epsilon > \frac{4}{m\pi}$ by the Mean Value Theorem. Indeed,

$$\frac{-4}{\epsilon} = \frac{b'(\theta) - (b'(\theta) + 4)}{\epsilon} = \frac{b'(\theta) - b'_{k+8}(\theta)}{\epsilon} > \frac{b'_{k+8}(\theta + \epsilon) - b'_{k+8}(\theta)}{\epsilon} = b_{k+8}^{(2)}(\theta_1) > -m\pi,$$

for some $\theta_1 \in [\theta, \theta + \epsilon]$.

Since $z_1 < w_2 < z_2$, by Lemma 3.2,

$$w_2 - z_1 < z_2 - z_1 < \frac{2\pi}{k + 2m\pi} < \frac{4}{m\pi},$$

when $k \geq 0$. Thus we have that $b'_{k+8}(w_2) > b'(z_1)$. Since $b'(\theta)$ and $b'_{k+8}(\theta)$ are decreasing,

$$b'_{k+8}(\theta_1) > b'(\theta_2) \quad \text{for all } \theta_1 \in (w_1, w_2) \text{ and for all } \theta_2 \in (z_1, z_2).$$

Since $b_{k+8}(w_2) - b_{k+8}(w_1) = \pi$ and $b(z_2) - b(z_1) = \pi$, we conclude that $w_2 - w_1 < z_2 - z_1$ and $z_1 - w_1 < z_2 - w_2$.

We now prove that $w_2 - z_1 > w_3 - z_2$. Noticing that $|b_{k+8}^{(2)}(\theta)| < m\pi$ and by Lemma 3.2, we have

$$w_3 - w_2 < \frac{2\pi}{k + 8 + 2\pi m}.$$

So for all $\theta_1, \theta_2 \in (w_2, w_3)$,

$$\left| \frac{b'_{k+8}(\theta_1) - b'_{k+8}(\theta_2)}{w_3 - w_2} \right| < \left| \frac{b'_{k+8}(w_3) - b'_{k+8}(w_2)}{w_3 - w_2} \right| = |b_{k+8}^{(2)}(\theta_3)| < m\pi,$$

for some $\theta_3 \in (w_3, w_2)$.

Hence for all $\theta_1, \theta_2 \in (w_2, w_3)$,

$$\left| b'_{k+8}(\theta_1) - b'_{k+8}(\theta_2) \right| < \frac{2m\pi^2}{k + 8 + 2m\pi}.$$

Moreover we have $b'_{k+8}(\theta) \geq b'(\theta) + 4$ on I and so we see that for $k > 0$,

$$b'_{k+8}(\theta_1) - b'(\theta_1) \geq 4 > \frac{2m\pi^2}{k + 8 + 2m\pi} > b'_{k+8}(\theta_1) - b'_{k+8}(\theta_2),$$

which imply that $b'_{k+8}(\theta_2) > b'(\theta_1)$ for all $\theta_1, \theta_2 \in (w_2, w_3)$. Since $b_{k+8}(w_3) - b_{k+8}(w_2) = \pi$ and $b(z_2) - b(z_1) = \pi$, we get that $w_3 - w_2 < z_2 - z_1$.

Next, we consider the case when $b_*(\theta) = b_{m+1}(\theta)$.

Note that

$$b'(\theta) = \frac{k}{2} + \sqrt{2}m\pi \sin \theta, \text{ and } b'_{m+1}(\theta) = \frac{k}{2} + \sqrt{2}(m+1)\pi \sin \theta = b'(\theta) + \sqrt{2}\pi \sin(\theta) > b'(\theta) + \pi.$$

Note that $m > 0$ and $\sin \theta$ decreases from 1 to $\frac{1}{\sqrt{2}}$ on I . Hence both $b'(\theta)$ and $b'_{m+1}(\theta)$ are decreasing functions of θ on I . If $b'(\theta) > b'_{m+1}(\theta + \epsilon)$ for some $\epsilon > 0$, then $\epsilon > \frac{1}{m+1}$. Indeed,

$$\frac{-\pi}{\epsilon} = \frac{b'(\theta) - (b'(\theta) + \pi)}{\epsilon} > \frac{b'(\theta) - b'_{m+1}(\theta)}{\epsilon} > \frac{b'_{m+1}(\theta + \epsilon) - b'_{m+1}(\theta)}{\epsilon} = b_{m+1}^{(2)}(\theta_1) > -(m+1)\pi,$$

for some $\theta_1 \in (\theta, \theta + \epsilon)$. Hence $\epsilon > \frac{1}{m+1}$.

Since $z_1 < w_2 < z_2$, by [Lemma 3.2](#),

$$w_2 - z_1 < z_2 - z_1 < \frac{2\pi}{k + 2(m+1)\pi} < \frac{1}{m+1},$$

when $k \geq 0$. Thus we must have $b'_{m+1}(w_2) > b'(z_1)$. Since $b'(\theta)$ and $b'_{m+1}(\theta)$ are decreasing,

$$b'(\theta_2) < b'(z_1) < b'_{m+1}(w_2) < b'_{m+1}(\theta_1),$$

for all $\theta_1 \in (w_1, w_2)$ and for all $\theta_2 \in (z_1, z_2)$. Since $b_{m+1}(w_2) - b_{m+1}(w_1) = \pi$ and $b(z_2) - b(z_1) = \pi$, we conclude that $w_2 - w_1 < z_2 - z_1$ and $z_1 - w_1 < z_2 - w_2$.

We now prove that $w_2 - z_1 > w_3 - z_2$. Noticing that $|b_{m+1}^{(2)}(\theta)| < (m+1)\pi$ and by [Lemma 3.2](#), we have

$$w_3 - w_2 < \frac{2\pi}{k + 2\pi(m+1)}.$$

So for any $\theta, \theta' \in (w_2, w_3)$,

$$|b'_{m+1}(\theta) - b'_{m+1}(\theta')| < \frac{2\pi^2(m+1)}{k + 2\pi(m+1)}.$$

Moreover we have $b'_{m+1}(\theta) > b'(\theta) + \pi$ on I and

$$\pi - \frac{2\pi^2(m+1)}{k + 2\pi(m+1)} > 0$$

for $k > 0$. So we see that

$$b'_{m+1}(\theta_1) - b'(\theta_1) > \pi > \frac{2\pi^2(m+1)}{k + 2\pi(m+1)} > b'_{m+1}(\theta_1) - b'_{m+1}(\theta_2),$$

which imply that $b'_{m+1}(\theta_2) > b'(\theta_1)$ for all $\theta_1, \theta_2 \in (w_2, w_3)$. Since $b_{m+1}(w_3) - b_{m+1}(w_2) = \pi$ and $b(z_2) - b(z_1) = \pi$, we get the desired inequalities. \square

4. The proof of [Theorem 2.1](#)

4.1. Zeros near $\frac{\pi}{2}$

Throughout this section, we suppose $k > 0$ to prove [Theorem 2.1](#).

As in the proof of [4, Sec. 5], we estimate the bound for zeros near $\theta = \frac{\pi}{2}$. The linear approximation by the Taylor series for $b(\theta) = \frac{k\theta}{2} - \sqrt{2}\pi m \cos \theta$ is given by

$$L_{k,m}(\theta) = \frac{k\pi}{4} + \frac{k + 2\sqrt{2}m\pi}{2} \left(\theta - \frac{\pi}{2} \right).$$

Similarly, the linear approximations for $b_{k+8}(\theta)$ and $b_{m+1}(\theta)$ are given by

$$L_{k+8,m}(\theta) = \frac{(k+8)\pi}{4} + \frac{k+8+2\sqrt{2}m\pi}{2} \left(\theta - \frac{\pi}{2} \right),$$

and

$$L_{k,m+1}(\theta) = \frac{k\pi}{4} + \frac{k+2\sqrt{2}(m+1)\pi}{2} \left(\theta - \frac{\pi}{2} \right),$$

respectively.

Since the second derivatives of $b(\theta)$ and $b_{k+8}(\theta)$ are the same, their linear approximations have the same error term $R_m(\theta)$. Let $b(\theta) = L_{k,m}(\theta) - R_m(\theta)$ and $b_{k+8}(\theta) = L_{k+8,m}(\theta) - R_m(\theta)$. Then since $R'_m(\theta) > 0$ for all $\theta \in I$ and $R_m(\frac{\pi}{2}) = 0$, $R_m(\theta)$ is increasing and positive on I .

Let α_1, α_2 and α_3 be the first zeros of $\cos(L_{k+8,m}(\theta))$, $\cos(L_{k,m}(\theta))$ and $\cos(L_{k,m+1}(\theta))$ in I , respectively, and let β_1, β_2 and β_3 be the first zeros of $\cos(b_{k+8}(\theta))$, $\cos(b(\theta))$ and $\cos(b_{m+1}(\theta))$ in I , respectively.

We note that by Lemma 3.1 (a),

$$\beta_2 > \beta_1, \text{ and } \beta_2 > \beta_3.$$

Also we can get $\alpha_1, \alpha_2, \alpha_3$ explicitly as follows, letting

$$c = \begin{cases} \frac{\pi}{2}, & \text{if } k \equiv 0 \pmod{4}, \\ \pi, & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

Lemma 4.1. For $i = 1, 2, 3$,

$$\alpha_i = \frac{\pi}{2} + \frac{2c}{g_i(k, m)},$$

$$\text{where } g_i(k, m) = \begin{cases} k + 8 + 2\sqrt{2}m\pi, & \text{if } i = 1, \\ k + 2\sqrt{2}m\pi, & \text{if } i = 2, \\ k + 2\sqrt{2}(m+1)\pi, & \text{if } i = 3. \end{cases}$$

Hence

$$\alpha_2 > \alpha_1.$$

Proof. First, for $i = 1$, we note that $\cos\left(L_{k+8,m}\left(\frac{\pi}{2}\right)\right) = \begin{cases} \pm 1, & \text{if } k \equiv 0 \pmod{4}, \\ 0, & \text{if } k \equiv 2 \pmod{4}, \end{cases}$ and since α_1 is the first zero such that $L_{k+8,m}(\alpha_1) = \frac{2n+1}{2}\pi$ for some integer n , we have that

$$L_{k+8,m}(\alpha_1) = L_{k+8,m}\left(\frac{\pi}{2}\right) + c$$

in case of k modulo 4. By solving this for α_1 , we have that

$$\alpha_1 = \frac{\pi}{2} + \frac{2c}{k+8+2\sqrt{2}m\pi}.$$

(In fact, the period of the function $y = \cos(L_{k+8,m}(\theta))$ is $\frac{4\pi}{k+8+2\sqrt{2}m\pi}$.)

Similarly, we get the statement for $i = 2, 3$. Also, in either case,

$$\alpha_2 - \alpha_1 = 2c \left(\frac{1}{k+2\sqrt{2}m\pi} - \frac{1}{k+8+2\sqrt{2}m\pi} \right) > 0.$$

So this completes the proof. \square

Now we prove that $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$, which is a part of the assumption of Proposition 3.3, and we find lower bounds for differences of zeros, $\beta_2 - \beta_1$ and $\beta_2 - \beta_3$.

Lemma 4.2. *For some integers n_1 and n_2 , we have the following.*

(a)

$$\begin{cases} b_{k+8}(\beta_1) &= \frac{2n_1+1}{2}\pi = L_{k+8,m}(\alpha_1), \\ b_{k+8}(\alpha_1) &= \frac{2n_1+1}{2}\pi - R_m(\alpha_1), \\ b(\beta_2) &= \frac{2n_2+1}{2}\pi = L_{k,m}(\alpha_2), \\ b(\alpha_2) &= \frac{2n_2+1}{2}\pi - R_m(\alpha_2), \\ b_{m+1}(\beta_3) &= L_{k,m+1}(\alpha_3). \end{cases}$$

In fact, $n_2 = n_1 - 2$.

(b) $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ and $\alpha_3 < \beta_3$.

(c) $\beta_1 < \alpha_2$.

(d) A lower bound of $\beta_2 - \beta_1$ is $\left(\frac{2c}{k+2\sqrt{2}m\pi} - \frac{2c}{k+8+2\sqrt{2}m\pi} \right) \geq \pi \left(\frac{1}{k+2\sqrt{2}m\pi} - \frac{1}{k+8+2\sqrt{2}m\pi} \right)$.

(e) A lower bound of $\beta_2 - \beta_3$ is $\frac{\pi(k+2\sqrt{2}m\pi)+4c}{2(k+2\sqrt{2}m\pi)} - \left(\frac{\pi(k+2\sqrt{2}(m+1)\pi)+4c}{2(k+2\sqrt{2}(m+1)\pi)} + \frac{8\sqrt{2}\pi^4(m+1)}{(k+2\pi(m+1))^4} \right)$.

Proof. First note that

$$b_{k+8}\left(\frac{3\pi}{4}\right) - b_{k+8}\left(\frac{\pi}{2}\right) = \frac{k+8+8m}{8}\pi > \pi.$$

Hence if we let n_1 be the smallest integer such that

$$L_{k+8,m}\left(\frac{\pi}{2}\right) = b_{k+8}\left(\frac{\pi}{2}\right) < \frac{2n_1+1}{2}\pi < b_{k+8}\left(\frac{3\pi}{4}\right) < L_{k+8,m}\left(\frac{3\pi}{4}\right),$$

then by the minimality of α_1 , we have

$$L_{k+8,m}(\alpha_1) = \frac{2n_1+1}{2}\pi.$$

Then since $R_m(\theta)$ is positive in I , it follows that

$$b_{k+8}\left(\frac{\pi}{2}\right) < b_{k+8}(\alpha_1) < L_{k+8,m}(\alpha_1) < b_{k+8}\left(\frac{3\pi}{4}\right).$$

Since $b_{k+8}(\theta)$ is continuous on I , there exists $\beta \in I$ such that

$$b_{k+8}(\beta) = L_{k+8,m}(\alpha_1) = \frac{2n_1 + 1}{2}\pi.$$

Since β_1 is the first zero of $\cos(b_{k+8}(\theta))$ on I , we have that $\beta_1 \leq \beta$, and if we let $b_{k+8}(\beta_1) = \frac{2n+1}{2}\pi$ for some integer n , then since $b_{k+8}(\theta)$ is increasing on I ,

$$L_{k+8,m}\left(\frac{\pi}{2}\right) = b_{k+8}\left(\frac{\pi}{2}\right) \leq b_{k+8}(\beta_1) \leq b_{k+8}(\beta) = L_{k+8,m}(\alpha_1) \leq L_{k+8,m}\left(\frac{3\pi}{4}\right).$$

Then again, there exists $\alpha \in I$ such that $L_{k+8,m}(\alpha) = b_{k+8}(\beta_1) = \frac{2n+1}{2}\pi \leq L_{k+8,m}(\alpha_1)$. By the minimality of n_1 , we have $n_1 = n$ and $\alpha = \alpha_1$. Hence we have that

$$\begin{aligned} b_{k+8}(\beta_1) &= \frac{2n_1 + 1}{2}\pi = L_{k+8,m}(\alpha_1), \\ b_{k+8}(\alpha_1) &= \frac{2n_1 + 1}{2}\pi - R_m(\alpha_1). \end{aligned}$$

Similarly, we can show that for $b_{m+1}(\theta)$ and $L_{k,m+1}(\theta)$,

$$\begin{aligned} b(\beta_2) &= \frac{2n_2 + 1}{2}\pi = L_{k,m}(\alpha_2), \\ b(\alpha_2) &= \frac{2n_2 + 1}{2}\pi - R_m(\alpha_2). \end{aligned}$$

In order to get the relation between n_1 and n_2 , note that $b_{k+8}(\theta) = b(\theta) + 4\theta$. Hence $b_{k+8}\left(\frac{\pi}{2}\right) = b\left(\frac{\pi}{2}\right) + 2\pi$. So

$$b\left(\frac{\pi}{2}\right) = b_{k+8}\left(\frac{\pi}{2}\right) - 2\pi < \frac{2n_1 + 1}{2}\pi - 2\pi = \frac{2(n_1 - 2) + 1}{2}\pi,$$

which shows that $n_2 = n_1 - 2$ by the minimality of n_1 and n_2 such that $L_{k,m}\left(\frac{\pi}{2}\right) = b\left(\frac{\pi}{2}\right) < \frac{2n_2 + 1}{2}\pi$. Hence (a) is proved.

(b) follows from (a), since $b_{k+8}(\theta)$, $b(\theta)$ and $b_{m+1}(\theta)$ are convex down and increasing on I .

For (c), to prove the inequality $\beta_1 < \alpha_2$, we note that β_1 is near $\frac{\pi}{2}$ since it is the first zero of $\cos(b_{k+8}(\theta))$ in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Let $\xi = \beta_1 - \pi/2 > 0$. Since

$$b_{k+8}\left(\frac{\pi}{2}\right) = \frac{k\pi}{4} + 2\pi = \begin{cases} t\pi + 2\pi, & \text{if } k \equiv 0 \pmod{4}, \\ t\pi + 2\pi + \pi/2, & \text{if } k \equiv 2 \pmod{4}, \end{cases}$$

for some integer t , and $b_{k+8}(\theta)$ is strictly increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$, we get the following together with (a):

$$\frac{2n_1 + 1}{2}\pi = b_{k+8}(\beta_1) = \frac{k\pi}{4} + 2\pi + c. \quad (4)$$

Hence $n_1 = \frac{k+8}{4}$ if $k \equiv 0 \pmod{4}$ and $n_1 = \frac{k+10}{4}$ if $k \equiv 2 \pmod{4}$. On the other hand,

$$b_{k+8}(\beta_1) = b_{k+8}\left(\xi + \frac{\pi}{2}\right) = \left(\frac{k+8}{2}\right)\left(\xi + \frac{\pi}{2}\right) + \sqrt{2}\pi m \sin(\xi). \quad (5)$$

Since $b'_{k+8}(\theta)$ is bounded below by $\frac{k+8}{2} + \pi m$, we have that in each case of k modulo 4,

$$\left(\frac{k+8}{2} + \pi m\right)\xi \leq \int_{\pi/2}^{\beta_1} b'_{k+8}(x)dx = b_{k+8}(\beta_1) - b_{k+8}(\pi/2) = c. \quad (6)$$

So we have that ξ approaches to 0 as k goes to infinity or as m goes to infinity when k is fixed.

Also,

$$\sin(\xi) = \xi - \frac{1}{3!}\xi^3 + \frac{1}{5!}\xi^5 + \cdots = \xi - \frac{1}{6}\xi^3 g(\xi),$$

where $g(\xi) = 1 + h(\xi)$ and $h(\xi)$ converges to 0 as ξ approaches to 0. Hence by equating (4) and (5), we get that

$$c = \sqrt{2}\pi m \sin(\xi) + \left(\frac{k+8}{2}\right)\xi = \sqrt{2}\pi m \left(\xi - \frac{1}{6}\xi^3 g(\xi)\right) + \left(\frac{k+8}{2}\right)\xi.$$

Hence we have that

$$\xi = \left(c + \frac{\sqrt{2}\pi m}{6}\xi^3 g(\xi)\right) / \left(\sqrt{2}\pi m + \frac{k+8}{2}\right). \quad (7)$$

Hence by Lemma 4.1 and (7) and (6), since ξ goes to 0 as k goes to infinity or as m goes to infinity when k is fixed, and $g(\xi) = 1 + h(\xi)$ where $h(\xi)$ approaches to 0 as ξ goes to 0,

$$\begin{aligned} \alpha_2 - \beta_1 &= \frac{2c}{k + 2\sqrt{2}m\pi} - \xi = \frac{16c - \frac{\sqrt{2}\pi m}{3}(k + 2\sqrt{2}m\pi)\xi^3 g(\xi)}{(k + 2\sqrt{2}m\pi)(k + 8 + 2\sqrt{2}m\pi)} \\ &\geq \frac{16c - \frac{\sqrt{2}\pi m}{3}(k + 2\sqrt{2}m\pi) \left(c \cdot \frac{2}{k+8+2\sqrt{2}m\pi}\right)^3 g(\xi)}{(k + 2\sqrt{2}m\pi)(k + 8 + 2\sqrt{2}m\pi)} \\ &> 0 \end{aligned} \quad (8)$$

for large enough $k > 0$ and each fixed $m \geq \ell_k + 8$ (so for small enough ξ) or for large enough $m \geq \ell_k + 8$ when k is fixed.

To prove (d), we note that the derivative of $b_*(\theta)$ is greater than the derivative of $b(\theta)$ in I and it is strictly decreasing for k large enough. Thus for large k , since $\beta_1 < \alpha_2$ by (b), we have that

$$\frac{R_m(\alpha_1)}{\beta_1 - \alpha_1} > \frac{R_m(\alpha_2)}{\beta_2 - \alpha_2}.$$

Note that $\alpha_1 < \alpha_2$ by Lemma 4.1. So since R_m is increasing, we have that

$$\frac{R_m(\alpha_2)}{\beta_1 - \alpha_1} > \frac{R_m(\alpha_1)}{\beta_1 - \alpha_1} > \frac{R_m(\alpha_2)}{\beta_2 - \alpha_2}.$$

Hence we have that $\beta_2 - \alpha_2 > \beta_1 - \alpha_1$. Hence by Lemma 4.1,

$$\beta_2 - \beta_1 > \alpha_2 - \alpha_1 = \left(\frac{2c}{k + 2\sqrt{2}m\pi} - \frac{2c}{k + 8 + 2\sqrt{2}m\pi}\right) \geq \pi \left(\frac{1}{k + 2\sqrt{2}m\pi} - \frac{1}{k + 8 + 2\sqrt{2}m\pi}\right),$$

which implies that the distance between the first zeros of $\cos(L_{k,m}(\theta))$ and $\cos(L_{k+8,m}(\theta))$ is less than the distance between the first zeros of $\cos(b(\theta))$ and $\cos(b_{k+8}(\theta))$. Hence we get the desired lower bound for $\beta_2 - \beta_1$.

To prove (e), first by the Mean Value Theorem, there exists $\gamma \in (\frac{\pi}{3}, \beta_3)$ such that

$$\frac{b_{m+1}(\beta_3) - b_{m+1}(\frac{\pi}{2})}{\beta_3 - \frac{\pi}{2}} = b'_{m+1}(\gamma) = \frac{k + 2\sqrt{2}(m+1)\pi \sin \gamma}{2} > \frac{k + 2(m+1)\pi}{2}.$$

Also since β_3 is the first zero of $\cos(b_{m+1}(\theta))$ in I , $b_{m+1}(\beta_3) - b_{m+1}(\frac{\pi}{2}) \leq \pi$. Therefore,

$$\frac{\pi}{\beta_3 - \frac{\pi}{2}} > \frac{b_{m+1}(\beta_3) - b_{m+1}(\frac{\pi}{2})}{\beta_3 - \frac{\pi}{2}} > \frac{k + 2(m+1)\pi}{2},$$

which implies that the first zero β_3 of $\cos(b_{m+1}(\theta))$ in I satisfies that

$$\beta_3 < \frac{\pi}{2} + \frac{2\pi}{k + 2(m+1)\pi}.$$

By Taylor's Theorem, for $\theta \in (\frac{\pi}{2}, \beta_3) \subset (\frac{\pi}{2}, \frac{\pi}{2} + \frac{2\pi}{k+2(m+1)\pi})$, we have that

$$b_{m+1}(\theta) = L_{k,m+1}(\theta) + \frac{b''_{m+1}(\xi)}{2} \left(\theta - \frac{\pi}{2} \right)^2,$$

for some $\xi \in (\frac{\pi}{2}, \frac{\pi}{2} + \frac{2\pi}{k+2(m+1)\pi})$. Here we see that for $0 < \alpha < \frac{2\pi}{k+2(m+1)\pi}$,

$$\begin{aligned} -b''_{m+1}(\xi) &= -\sqrt{2}(m+1)\pi \cos(\xi) = -\sqrt{2}(m+1)\pi \cos\left(\frac{\pi}{2} + \alpha\right) \\ &= \sqrt{2}(m+1)\pi \sin \alpha \leq \sqrt{2}(m+1)\pi \alpha \\ &\leq \sqrt{2}(m+1)\pi \frac{2\pi}{k + 2(m+1)\pi}. \end{aligned}$$

Hence we have that

$$L_{k,m+1}(\theta) \leq b_{m+1}(\theta) + \frac{\sqrt{2}(m+1)\pi}{2} \left(\frac{2\pi}{k + 2(m+1)\pi} \right)^3,$$

for large enough m when k is fixed or for large enough k .

By Lemma 4.1, we have that

$$\alpha_2 = \frac{\pi(k + 2\sqrt{2}m\pi) + 4c}{2(k + 2\sqrt{2}m\pi)}, \text{ and } \alpha_3 = \frac{\pi(k + 2\sqrt{2}(m+1)\pi) + 4c}{2(k + 2\sqrt{2}(m+1)\pi)}.$$

On the other hand, since $\alpha_3 < \beta_3$ by (a), there exists $\gamma \in (\alpha_3, \beta_3)$ such that

$$\frac{b_{m+1}(\beta_3) - b_{m+1}(\alpha_3)}{\beta_3 - \alpha_3} = b'_{m+1}(\gamma) > \frac{k}{2} + \pi(m+1).$$

Since $b_{m+1}(\beta_3) = L_{k,m+1}(\alpha_3) \leq b_{m+1}(\alpha_3) + \frac{\sqrt{2}\pi(m+1)}{2} \left(\frac{2\pi}{k+2(m+1)\pi} \right)^3$, we have shown that the first zero β_3 of $\cos(b_{m+1}(\theta))$ in I satisfies that

$$\beta_3 < \alpha_3 + \frac{b_{m+1}(\beta_3) - b_{m+1}(\alpha_3)}{\frac{k}{2} + \pi(m+1)} \leq \frac{\pi(k + 2\sqrt{2}(m+1)\pi) + 4c}{2(k + 2\sqrt{2}(m+1)\pi)} + \frac{\frac{\sqrt{2}\pi(m+1)}{2} \left(\frac{2\pi}{k+2(m+1)\pi} \right)^3}{\frac{k}{2} + \pi(m+1)}.$$

Since $\alpha_2 < \beta_2$, a lower bound on the distance between β_2 and β_3 is

$$\beta_2 - \beta_3 > \frac{\pi(k + 2\sqrt{2}m\pi) + 4c}{2(k + 2\sqrt{2}m\pi)} - \left(\frac{\pi(k + 2\sqrt{2}(m+1)\pi) + 4c}{2(k + 2\sqrt{2}(m+1)\pi)} + \frac{\frac{\sqrt{2}\pi(m+1)}{2} \left(\frac{2\pi}{k+2(m+1)\pi} \right)^3}{\frac{k}{2} + \pi(m+1)} \right),$$

which is positive for large enough k or for large enough m when k is fixed. \square

4.2. Zeros near $\frac{3\pi}{4}$

By a similar argument, we need to find the lower bounds near $\theta = \frac{3\pi}{4}$. We give concrete lower bounds in this case.

The linear approximation by the Taylor series for $b(\theta) = \frac{k\theta}{2} - \sqrt{2}\pi m \cos \theta$ near $\theta = \frac{3\pi}{4}$ is given by

$$U_{k,m}(\theta) = \frac{(3k + 8m)\pi}{8} + \frac{k + 2m\pi}{2} \left(\theta - \frac{3\pi}{4} \right).$$

Similarly, the linear approximations for $b_{k+8}(\theta)$ and $b_{m+1}(\theta)$ are given by

$$U_{k+8,m}(\theta) = \frac{(3(k+8) + 8m)\pi}{8} + \frac{k+8 + 2m\pi}{2} \left(\theta - \frac{3\pi}{4} \right),$$

and

$$U_{k,m+1}(\theta) = \frac{(3k + 8(m+1))\pi}{8} + \frac{k + 2(m+1)\pi}{2} \left(\theta - \frac{3\pi}{4} \right),$$

respectively.

Since the second derivatives of $b(\theta)$ and $b_{k+8}(\theta)$ are the same, their linear approximations have the same error term $R_m(\theta)$. Let $b(\theta) = U_{k,m}(\theta) - R_m(\theta)$ and $b_{k+8}(\theta) = U_{k+8,m}(\theta) - R_m(\theta)$. Then since $R'_m(\theta) = m\pi(1 - \sqrt{2}\sin \theta) < 0$ for all $\theta \in I$ and $R_m(\frac{3\pi}{4}) = 0$, $R_m(\theta)$ is decreasing and positive on I .

Now let $\gamma_1, \gamma_2, \gamma_3$ be the last zeros of $\cos(U_{k+8,m}(\theta))$, $\cos(U_{k,m}(\theta))$ and $\cos(U_{k,m+1}(\theta))$ in I , respectively, and let μ_1, μ_2 and μ_3 be the last zeros of $\cos(b_{k+8}(\theta))$, $\cos(b(\theta))$ and $\cos(b_{m+1}(\theta))$ in I , respectively.

We note that by Lemma 3.1 (b),

$$\mu_2 < \mu_1, \text{ and } \mu_2 < \mu_3.$$

Also we can get $\gamma_1, \gamma_2, \gamma_3$ explicitly as follows, letting

$$a = \begin{cases} \frac{\pi}{2}, & \text{if } k \equiv 0 \pmod{8}, \\ \pi, & \text{if } k \equiv 4 \pmod{8}, \\ \frac{\pi}{4}, & \text{if } k \equiv 2 \pmod{8}, \\ \frac{3\pi}{4}, & \text{if } k \equiv 6 \pmod{8}. \end{cases}$$

Lemma 4.3. For $i = 1, 2, 3$,

$$\gamma_i = \frac{3\pi}{4} - \frac{2a}{g_i(k, m)},$$

$$\text{where } g_i(k, m) = \begin{cases} k + 8 + 2m\pi, & \text{if } i = 1, \\ k + 2m\pi, & \text{if } i = 2, \\ k + 2(m+1)\pi, & \text{if } i = 3. \end{cases}$$

Hence

$$\gamma_1 > \gamma_2.$$

Proof. First, for $i = 1$, we note that

$$\cos\left(U_{k+8,m}\left(\frac{3\pi}{4}\right)\right) = \cos\left(\frac{3(k+8)}{8}\pi + m\pi\right) = \begin{cases} \pm 1, & \text{if } k \equiv 0 \pmod{8}, \\ 0, & \text{if } k \equiv 4 \pmod{8}, \\ \pm \frac{1}{\sqrt{2}}, & \text{if } k \equiv 2 \text{ or } 6 \pmod{8}. \end{cases}$$

In particular, since γ_1 in I is the last zero such that $U_{k+8,m}(\gamma_1) = \frac{2n+1}{2}\pi$ for some integer n , we have that in each case of $k \pmod{8}$,

$$U_{k+8,m}(\gamma_1) = U_{k+8,m}\left(\frac{3\pi}{4}\right) - a.$$

Solving this linear equation for γ_1 , then we get

$$\gamma_1 = \frac{3\pi}{4} - \frac{2a}{k+8+2m\pi}.$$

Similarly, we get the statement for $i = 2, 3$.

Also,

$$\gamma_1 - \gamma_2 = \left(\frac{3\pi}{4} - \frac{2a}{k+8+2\pi m}\right) - \left(\frac{3\pi}{4} - \frac{2a}{k+2\pi m}\right) > 0.$$

Hence this completes the proof. \square

Now we prove that $\gamma_2 < \mu_2 < \gamma_1 < \mu_1$, which is a part of the assumption of [Proposition 3.3](#), and find lower bounds for the differences of zeros.

Lemma 4.4. *For some integers n_1 and n_2 , we have the following.*

(a)

$$\begin{cases} b_{k+8}(\mu_1) &= \frac{2n_1+1}{2}\pi = U_{k+8,m}(\gamma_1), \\ b_{k+8}(\gamma_1) &= \frac{2n_1+1}{2}\pi - R_m(\gamma_1), \\ b(\mu_2) &= \frac{2n_2+1}{2}\pi = U_{k,m}(\gamma_2), \\ b(\gamma_2) &= \frac{2n_2+1}{2}\pi - R_m(\gamma_2). \end{cases}$$

In fact, $n_2 = n_1 - 3$.

(b) $\gamma_1 < \mu_1$ and $\gamma_2 < \mu_2$.

(c) $\mu_2 < \gamma_1$.

(d) A lower bound of $\mu_1 - \mu_2$ is $\frac{16a}{(k+8+2\sqrt{2}m\pi)(k+8+2m\pi)} \geq \frac{4\pi}{(k+8+2\sqrt{2}m\pi)(k+8+2m\pi)}.$

(e) A lower bound of $\mu_3 - \mu_2$ is $\frac{4a\pi}{(k+8+2k\pi)(k+2\sqrt{2}\pi(m+1))} \geq \frac{\pi^2}{(k+8+2k\pi)(k+2\sqrt{2}\pi(m+1))}.$

Proof. To prove (a), first we note that $b_{k+8}\left(\frac{3\pi}{4}\right) - b_{k+8}\left(\frac{\pi}{2}\right) \geq \pi$, and $U_{k+8,m}\left(\frac{3\pi}{4}\right) - U_{k+8,m}\left(\frac{\pi}{2}\right) \geq \pi$. Let n_1 be the largest integer such that

$$b_{k+8}\left(\frac{\pi}{2}\right) < U_{k+8,m}\left(\frac{\pi}{2}\right) \leq \frac{2n_1+1}{2}\pi < b_{k+8}\left(\frac{3\pi}{4}\right) = U_{k+8,m}\left(\frac{3\pi}{4}\right).$$

Then by the maximality of γ_1 , we have that

$$U_{k+8,m}(\gamma_1) = \frac{2n_1+1}{2}\pi.$$

Since $R_m(\theta)$ is positive on I and $b_{k+8}(\theta)$ is continuous on I , there exists $\mu \in I$ such that

$$b_{k+8}(\mu) = U_{k+8,m}(\gamma_1) = \frac{2n_1+1}{2}\pi.$$

Since μ_1 is the last zero of $\cos(b_{k+8}(\theta))$ on I , we have that $\mu \leq \mu_1$, and if we let $b_{k+8}(\mu_1) = \frac{2n+1}{2}\pi$ for some integer n , then since $b_{k+8}(\theta)$ is increasing on I , there exists $\gamma \in I$ such that $U_{k+8,m}(\gamma) = b_{k+8}(\mu_1) = \frac{2n+1}{2}\pi$. By the maximality of n_1 , we have $n_1 = n$ and $\gamma = \gamma_1$. Hence we have that

$$\begin{aligned} b_{k+8}(\mu_1) &= \frac{2n_1+1}{2}\pi = U_{k+8,m}(\gamma_1), \\ b_{k+8}(\gamma_1) &= \frac{2n_1+1}{2}\pi - R_m(\gamma_1). \end{aligned}$$

Similarly, we can show for $b(\theta)$ and $U_{k,m}(\theta)$, and hence we have that for some integer n_2 ,

$$\begin{aligned} b(\mu_2) &= \frac{2n_2+1}{2}\pi = U_{k,m}(\gamma_2), \\ b(\gamma_2) &= \frac{2n_2+1}{2}\pi - R_m(\gamma_2). \end{aligned}$$

In order to get the relation between n_1 and n_2 , since $b_{k+8}(\theta) - b(\theta) = 4\theta$, we have that $b_{k+8}\left(\frac{3\pi}{4}\right) = b\left(\frac{3\pi}{4}\right) + 3\pi$. Hence

$$b\left(\frac{3\pi}{4}\right) = b_{k+8}\left(\frac{3\pi}{4}\right) - 3\pi > \frac{2n_1+1}{2}\pi - 3\pi = \frac{2n_1-5}{2}\pi.$$

Then by the maximality of n_2 for μ_2 of $b(\mu_2) = \frac{2n_2+1}{2}\pi$,

$$\frac{2n_2+1}{2}\pi \geq \frac{2n_1-5}{2}\pi.$$

Hence $n_2 \geq n_1 - 3$. Then, the maximality of n_1 for $b_{k+8}(\theta)$ implies that $n_2 = n_1 - 3$. Hence (a) is proved.

(b) follows from (a), since $b_{k+8}(\theta)$ and $b(\theta)$ are increasing and $R_m(\theta)$ is positive on I .

To prove (c), we let $\xi = \frac{3\pi}{4} - \mu_2$, since $\mu_2 \in I$ is near $\frac{3\pi}{4}$. Since $b(\theta)$ is strictly increasing on I and $b\left(\frac{3\pi}{4}\right) = \frac{3k\pi}{8} + m\pi$ and μ_2 is the last zero in I of $\cos(b(\theta))$, we have that in each case of k modulo 8,

$$b(\mu_2) = b\left(\frac{3\pi}{4}\right) - a. \quad (9)$$

On the other hand,

$$\begin{aligned} b(\mu_2) &= b\left(\frac{3\pi}{4} - \xi\right) = \frac{k}{2}\left(\frac{3\pi}{4} - \xi\right) - \sqrt{2}m\pi \cos\left(\frac{3\pi}{4} - \xi\right) \\ &= \frac{3k\pi}{8} - \frac{k}{2}\xi + \pi m(\cos \xi - \sin \xi). \end{aligned} \quad (10)$$

Since the derivative $b'(\theta)$ is bounded below by $\frac{k}{2} + m\pi$, we have that

$$\left(\frac{k}{2} + \pi m\right)\xi \leq \int_{\mu_2}^{3\pi/4} b'(x)dx = b\left(\frac{3\pi}{4}\right) - b(\mu_2) = a. \quad (11)$$

So we have that ξ approaches to 0 as m hence k goes to infinity. Also,

$$\begin{aligned} \cos(\xi) - \sin(\xi) &= 1 - \xi - \frac{1}{2!}\xi^2 + \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 - \frac{1}{5!}\xi^5 - \dots \\ &= 1 - \left(\xi + \xi^2\left(\frac{1}{2!} - \frac{1}{3!}\xi - \frac{1}{4!}\xi^2 + \frac{1}{5!}\xi^3 + \dots\right)\right) \\ &= 1 - (\xi + \xi^2 g(\xi)), \end{aligned} \quad (12)$$

where $g(\xi) = \frac{1}{2} + h(\xi)$ and $h(\xi)$ approaches to 0 as ξ goes to 0.

Hence by equating (9) and (10), we get that in each case of $k \pmod{8}$,

$$\frac{k}{2}\xi + m\pi(\xi + \xi^2 g(\xi)) = a.$$

Solving this for ξ , we get that

$$\xi = \frac{a - m\pi\xi^2 g(\xi)}{\frac{k}{2} + m\pi}. \quad (13)$$

Hence by Lemma 4.3 and (11) and (13), since ξ goes to 0 as k goes to infinity or as m goes to infinity when k is fixed, and $g(\xi) = \frac{1}{2} + h(\xi)$ where $h(\xi)$ approaches to 0 as ξ goes to 0,

$$\begin{aligned} \gamma_1 - \mu_2 &= \left(\frac{3\pi}{4} - \frac{2a}{k+8+2m\pi}\right) - \left(\frac{3\pi}{4} - \xi\right) \\ &= \frac{a - m\pi\xi^2 g(\xi)}{\frac{k}{2} + m\pi} - \frac{2a}{k+8+2m\pi} \\ &\geq \frac{16a - 2m\pi(k+8+2m\pi)\left(\frac{2a}{k+2\pi m}\right)^2 g(\xi)}{(k+2m\pi)(k+8+2m\pi)} \\ &> 0 \end{aligned} \quad (14)$$

for large enough k or for large enough m when k is fixed (so for small enough ξ).

To prove (d), we note that since $b(\theta) = b_{k+8}(\theta) - 4\theta$ and by (a),

$$b_{k+8}(\mu_2) - 4\mu_2 = b(\mu_2) = \frac{2n_2+1}{2}\pi = \frac{2n_1+1}{2}\pi - 3\pi = b_{k+8}(\mu_1) - 3\pi.$$

Also by the Mean Value Theorem, there exists $z \in (\mu_2, \mu_1)$ such that $b'_{k+8}(z) = \frac{b_{k+8}(\mu_1) - b_{k+8}(\mu_2)}{\mu_1 - \mu_2}$. Hence

$$3\pi - 4\mu_2 = b_{k+8}(\mu_1) - b_{k+8}(\mu_2) = b'_{k+8}(z)(\mu_1 - \mu_2).$$

Since $b'_{k+8}(z)$ is bounded above by $\frac{k+8}{2} + \sqrt{2}m\pi$ on I and $\mu_2 < \gamma_1$ by (c) and $a \geq \frac{\pi}{4}$ in each case, we have that

$$\begin{aligned}\mu_1 - \mu_2 &= \frac{3\pi - 4\mu_2}{b'_{k+8}(z)} \geq \frac{3\pi - 4\gamma_1}{\frac{k+8}{2} + \sqrt{2}m\pi} \\ &= \frac{16a}{(k+8+2\sqrt{2}m\pi)(k+8+2m\pi)} \geq \frac{4\pi}{(k+8+2\sqrt{2}m\pi)(k+8+2m\pi)}.\end{aligned}\quad (15)$$

To prove (e), since $b\left(\frac{3\pi}{4}\right) = \frac{3k\pi}{8} + m\pi$ and μ_2 is the last zero in I of $\cos(b(\theta))$, we have that in each of k modulo 8,

$$b(\mu_2) = b\left(\frac{3\pi}{4}\right) - a.$$

Also since $b_{m+1}(\theta) = b(\theta) - \sqrt{2}\pi \cos \theta$, $b_{m+1}\left(\frac{3\pi}{4}\right) = b\left(\frac{3\pi}{4}\right) + \pi$ and so for the last zero μ_3 of $\cos(b_{m+1}(\theta))$,

$$b_{m+1}(\mu_3) = b_{m+1}\left(\frac{3\pi}{4}\right) - a.$$

By subtracting two equations, we have that

$$b(\mu_2) - b_{m+1}(\mu_3) = b\left(\frac{3\pi}{4}\right) - b_{m+1}\left(\frac{3\pi}{4}\right) = -\pi.$$

So

$$b_{m+1}(\mu_3) - b_{m+1}(\mu_2) = \pi(1 + \sqrt{2}\cos(\mu_2)).$$

By the Mean Value Theorem, there exists $\mu \in (\mu_3, \mu_2)$ such that $b'_{m+1}(\mu) = \frac{b_{m+1}(\mu_3) - b_{m+1}(\mu_2)}{\mu_3 - \mu_2}$. Therefore, since $b'_{m+1}(\theta)$ is bounded above by $\frac{k}{2} + \sqrt{2}\pi(m+1)$, and $\mu_2 < \gamma_1$ by (b),

$$\mu_3 - \mu_2 = \frac{\pi(1 + \sqrt{2}\cos(\mu_2))}{b'_{m+1}(\mu)} > \frac{\pi(1 + \sqrt{2}\cos(\gamma_1))}{\frac{k}{2} + \sqrt{2}\pi(m+1)}.$$

Here by Lemma 4.3, we have that

$$\begin{aligned}\cos(\gamma_1) &= \cos\left(\frac{3\pi}{4} - \frac{2a}{k+8+2m\pi}\right) \\ &= -\frac{1}{\sqrt{2}}\left(\cos\left(\frac{2a}{k+8+2m\pi}\right) - \sin\left(\frac{2a}{k+8+2m\pi}\right)\right) \\ &> \frac{1}{\sqrt{2}}\left(\frac{2a}{k+8+2m\pi} - 1\right).\end{aligned}\quad (16)$$

In fact, the last inequality can be obtained as follows. If we let $f(\theta) = \sin(\theta) - \cos(\theta) - \theta + 1$, then

$$f'(\theta) = \cos(\theta) + \sin(\theta) - 1 = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) - 1 > 0,$$

for all $0 < \theta < \frac{\pi}{2}$. Note that $0 < \theta < \frac{\pi}{2}$ implies that $-\frac{\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{4}$. Hence, $\sin(\theta) - \cos(\theta) > \theta - 1$, where $\theta = \frac{2a}{k+8+2m\pi}$. Then,

$$\mu_3 - \mu_2 \geq \frac{\pi \frac{2a}{k+8+2m\pi}}{\frac{k}{2} + \sqrt{2}\pi(m+1)} \geq \frac{\pi^2}{(k+8+2m\pi)(k+2\sqrt{2}\pi(m+1))}. \quad \square$$

4.3. The proof of Theorem 2.1

Recalling (2), we need to derive an upper bound in terms of D of the distance between a zero of $f_{k,m}(\frac{1}{\sqrt{2}}e^{i\theta})$ and a zero of $2\cos(b(\theta))$, where

$$\left| e^{\frac{ik\theta}{2}} e^{-2\pi m \frac{1}{\sqrt{2}} \sin \theta} f_{k,m}\left(\frac{1}{\sqrt{2}}e^{i\theta}\right) - 2\cos(b(\theta)) \right| < B_i < 2.$$

Suppose that $2\cos(b(\alpha^*)) = 0$ for some $\alpha^* \in I$. Note that

$$\frac{d}{d\theta}(2\cos(b(\theta))) = -\sin(b(\theta))(k + 2\sqrt{2}\pi m \sin \theta).$$

We examine an interval containing α^* over which $2\cos(b(\theta))$ has no extremes, that is, over which the values of $2\cos(b(\theta))$ is either entirely increasing or decreasing so that $2\cos(b(\theta))$ has only one zero over that interval.

Since $k > 0$, we have that $k + 2\sqrt{2}\pi m \sin \theta \neq 0$ for any $\theta \in I$. So if there exists $\theta \in I$ such that $\frac{d}{d\theta}(2\cos(b(\theta))) = 0$, then $\sin(b(\theta)) = 0$.

First, since $2\cos(b(\alpha^*)) = 0$, we have that $b(\alpha^*) = \left(n + \frac{1}{2}\right)\pi$ for some $n \in \mathbb{Z}$.

Since $n\pi < \left(n + \frac{1}{2}\right)\pi < (n+1)\pi$ and b is increasing on I , in order to have that $\sin(b(\theta)) = 0$ where θ in I is near α^* , we have $b(\theta) = (n+1)\pi$ for $\theta > \alpha^*$, or $b(\theta) = n\pi$ for $\theta < \alpha^*$.

Hence we have

$$\begin{cases} b(\theta) &= \frac{k\theta}{2} - \sqrt{2}\pi m \cos \theta = n\pi \text{ or } (n+1)\pi, \\ b(\alpha^*) &= \frac{k\alpha^*}{2} - \sqrt{2}\pi m \cos(\alpha^*) = \left(n + \frac{1}{2}\right)\pi. \end{cases}$$

So

$$\frac{k(\theta - \alpha^*)}{2} - 2\sqrt{2}\pi m(\cos \theta - \cos(\alpha^*)) = \pm \frac{1}{2}\pi.$$

Then since $\frac{\cos \theta - \cos(\alpha^*)}{\theta - \alpha^*} = -\sin(\theta_0)$ for some θ_0 between θ and α^* , we have that

$$\begin{aligned} \pi &= |k(\theta - \alpha^*) - 4\sqrt{2}\pi m(\cos \theta - \cos(\alpha^*))| \\ &\leq k|\theta - \alpha^*| + 4\sqrt{2}\pi m|\cos \theta - \cos(\alpha^*)| \\ &\leq (k + 4\sqrt{2}\pi m)|\theta - \alpha^*|. \end{aligned}$$

Hence $|\theta - \alpha^*| \geq \frac{\pi}{k + 4\sqrt{2}\pi m}$, if there exists $\theta \in I$ near α^* at which $2\cos(b(\theta))$ has an extreme value. Therefore, $2\cos(b(\theta))$ has no extreme value on the interval

$$\left(\alpha^* - \frac{\pi}{2(k + 4\sqrt{2}\pi m)}, \alpha^* + \frac{\pi}{2(k + 4\sqrt{2}\pi m)}\right). \quad (17)$$

Let

$$\begin{cases} x &= \frac{k\alpha^*}{2} - \sqrt{2}\pi m \cos(\alpha^*) \cos\left(\frac{\pi}{2(k + 4\sqrt{2}\pi m)}\right), \\ y &= \sqrt{2}\pi m \sin(\alpha^*) \sin\left(\frac{\pi}{2(k + 4\sqrt{2}\pi m)}\right) + \frac{k\pi}{4(k + 4\sqrt{2}\pi m)}. \end{cases}$$

Then

$$\sin \left(b \left(\alpha^* \pm \frac{\pi}{2(k+4\sqrt{2}\pi m)} \right) \right) = \sin x \cos y \pm \cos x \sin y.$$

When k or m is large enough, then $\cos \left(\frac{\pi}{2(k+4\sqrt{2}\pi m)} \right)$ is very near 1 and so

$$\cos x \approx \cos(b(\alpha^*)) = 0,$$

and

$$|\sin x| \approx |\sin(b(\alpha^*))| = \left| \sin \left(\left(n + \frac{1}{2} \right) \pi \right) \right| = 1.$$

Suppose that k is large enough or m is large enough when k is fixed. Then, we have that

$$\begin{aligned} 0 < \sqrt{2}\pi m \sin(\alpha^*) \sin \left(\frac{\pi}{2(k+4\sqrt{2}\pi m)} \right) &\leq \sqrt{2}\pi m \sin(\alpha^*) \left(\frac{\pi}{2(k+4\sqrt{2}\pi m)} \right) \\ &< \frac{\sqrt{2}\pi^2 m \sin(\alpha^*)}{4\sqrt{2}\pi m} \leq \frac{\pi}{4}. \end{aligned}$$

And as k is large enough or as m is large enough when k is fixed, we can see that since $k > 0$,

$$\begin{aligned} y &< \frac{\pi}{4} + \frac{k\pi}{4(k+4\sqrt{2}\pi m)} < \frac{\pi}{4} + \frac{k\pi}{4(k+4\sqrt{2}\pi \frac{k-r_k}{8})} \\ &< \frac{\pi}{4} + \frac{k\pi}{4(|k| + \frac{\sqrt{2}\pi k}{2})} = \frac{\pi}{4} + \frac{\pi}{2(\sqrt{2}\pi + 2)}. \end{aligned}$$

Hence an upper bound of y is close to $\frac{\pi}{4} + \frac{\pi}{2(\sqrt{2}\pi + 2)}$. Hence a lower bound of $\cos(y)$ is close to or greater than $\cos(\frac{\pi}{4} + \frac{\pi}{2(\sqrt{2}\pi + 2)}) \approx 0.5155032383$.

Therefore, $\left| \sin \left(b \left(\alpha^* \pm \frac{\pi}{2(k+4\sqrt{2}\pi m)} \right) \right) \right|$ is close to or greater than $\cos(\frac{\pi}{4} + \frac{\pi}{2(\sqrt{2}\pi + 2)}) \approx 0.5155032383$. Let

$$E := 0.5155032383.$$

Now we are ready to prove the main theorem. Referring to (1) and (2), for

$$\begin{aligned} D &:= e^{-2\pi m(1+\sqrt{2}\cos\rho)/3} (3+2\sqrt{2}\cos\rho)^{-\frac{1}{2}} + e^{-2\pi m(1+0.4161468364)/5} (3+2\sqrt{2} \cdot 0.4161468364)^{-\frac{1}{2}} \\ &\quad + 26.07168336(0.6586856845)^m < 1.960061437, \end{aligned}$$

and for some ρ such that $\frac{\pi}{2} < \rho < \frac{3\pi}{4}$, and for all $\theta \in (\frac{\pi}{2}, \rho)$,

$$\left| e^{ik\theta/2} e^{-2\pi m \frac{1}{\sqrt{2}} \sin\theta} f_{k,m} \left(\frac{1}{\sqrt{2}} e^{i\theta} \right) - 2 \cos(b(\theta)) \right| < D < 2. \quad (18)$$

By (17), since $2 \cos(b(\theta))$ has no extremes in $\left(\alpha^* - \frac{\pi}{2(|k|+4\sqrt{2}\pi m)}, \alpha^* + \frac{\pi}{2(|k|+4\sqrt{2}\pi m)} \right)$ and $2 \cos(b(\alpha^*)) = 0$, we can take sufficiently small $\epsilon_1, \epsilon_2 > 0$ such that $\cos(b(\theta))$ is increasing (decreasing, respectively) on the interval $(\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$ and

$$\begin{aligned} 2 \cos(b(\alpha^* - \epsilon_1)) &= D = -2 \cos(b(\alpha^* + \epsilon_2)) \\ (-2 \cos(b(\alpha^* - \epsilon_1)) &= D = 2 \cos(b(\alpha^* + \epsilon_2)), \text{ respectively}). \end{aligned} \quad (19)$$

Then as explained in [2, p. 303, The proof of Theorem 1.2], the function $f_{k,m}$ has $\ell_k + m$ number of zeros on I and there are exactly $\ell_k + m + 1$ values of θ in $[\frac{\pi}{2}, \frac{3\pi}{4}]$ where $2 \cos(b(\theta))$ has absolute value 2, alternating between 2 and -2 as θ increases. Thus considering (18) and (19), by the intermediate value theorem, $f_{k,m}$ has a unique zero in the interval $(\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$.

Moreover, for some $\theta_1 \in (\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$,

$$\begin{aligned} \left| 2 \sin(b(\theta_1)) \cdot b'(\theta_1) \right| &= \left| \frac{d}{d\theta} (2 \cos(b(\theta))) \Big|_{\theta=\theta_1} \right| \\ &= \left| \frac{2 \cos(b(\alpha^* + \epsilon_2)) - 2 \cos(b(\alpha^* - \epsilon_1))}{\epsilon_1 + \epsilon_2} \right| = \frac{2D}{\epsilon_1 + \epsilon_2}. \end{aligned} \quad (20)$$

Let $t = 2 \cos(b(\theta))$ for $\theta \in (\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$. Then $-D < t < D$ and

$$|2 \sin(b(\theta))| = \sqrt{4 - t^2} \geq \sqrt{4 - D^2}.$$

Recall that $|b'(\theta)| \geq \frac{k}{2} + m\pi$. So for sufficiently large k , for example, if k and m are large enough to have $\sqrt{\frac{4 - D^2}{4D^2}} > 8\sqrt{2}$, then from Eq. (20), we have that

$$\frac{2D}{\epsilon_1 + \epsilon_2} \geq \sqrt{4 - D^2} \left(\frac{k}{2} + m\pi \right),$$

which implies that for each $i = 1, 2$,

$$\epsilon_i < \epsilon_1 + \epsilon_2 \leq \frac{1}{\frac{\sqrt{4-D^2}}{2D} \left(\frac{k}{2} + \pi m \right)} < \frac{1}{8\sqrt{2} \left(\frac{k}{2} + \pi m \right)} < \frac{1}{8\sqrt{2} \left(\frac{k}{4\sqrt{2}} + \pi m \right)} = \frac{1}{2(k + 4\sqrt{2}\pi m)}.$$

Thus,

$$(\alpha^* - \epsilon_1, \alpha^* + \epsilon_2) \subseteq \left(\alpha^* - \frac{\pi}{2(k + 4\sqrt{2}\pi m)}, \alpha^* + \frac{\pi}{2(k + 4\sqrt{2}\pi m)} \right).$$

On the other hand, since $\cos(b(\theta))$ has no extremes in $\left(\alpha^* - \frac{\pi}{2(k + 4\sqrt{2}\pi m)}, \alpha^* + \frac{\pi}{2(k + 4\sqrt{2}\pi m)} \right)$ and $\cos(b(\alpha^*)) = 0$ and $b(\theta)$ is increasing, we have that $|\sin(b(\alpha^*))| = 1$ and so $|\sin(b(\theta))|$ on $\left[\alpha^* - \frac{\pi}{2(k + 4\sqrt{2}\pi m)}, \alpha^* + \frac{\pi}{2(k + 4\sqrt{2}\pi m)} \right]$ has minimum value at the endpoint of $\left[\alpha^* - \frac{\pi}{2(k + 4\sqrt{2}\pi m)}, \alpha^* + \frac{\pi}{2(k + 4\sqrt{2}\pi m)} \right]$. Hence

$$\begin{aligned} \left| \frac{d}{d\theta} (2 \cos(b(\theta))) \Big|_{\theta=\theta_1} \right| &= |2 \sin(b(\theta_1)) \cdot b'(\theta_1)| \\ &> \left| \sin \left(b \left(\alpha^* \pm \frac{\pi}{2(k + 4\sqrt{2}\pi m)} \right) \right) \right| \left| k + 2\sqrt{2}\pi m \frac{1}{\sqrt{2}} \right| \\ &> E \cdot (k + 2\pi m). \end{aligned}$$

Together with (20), we have that

$$0 < \epsilon_1 + \epsilon_2 < \frac{2D}{E \cdot (k + 2\pi m)}.$$

Let $M(k, m)$ be the minimum of four lower bounds on the distance between zeros given in Lemma 4.2 (d), (e) and Lemma 4.4 (d), (e). Then $M(k, m)$ is the minimum of lower bounds on the distance zeros between the zeros of $\cos(b(\theta))$ and the zeros of $\cos(b_*(\theta))$ by Proposition 3.3. Hence since $M(k, m)$ is a rational function of k and m and D has exponential decay in k and m , we can have that

$$\epsilon_1 + \epsilon_2 < \frac{2D}{E \cdot (k + 2m\pi)} < \frac{M(k, m)}{4},$$

for sufficiently large k or for sufficiently large m when k is fixed.

For the case of $f_{k+8,m}$ ($f_{k,m+1}$, respectively), let ϵ'_i , D' and E' be the corresponding values for ϵ_i , D and E , respectively. Then again we have that

$$\begin{aligned} \epsilon'_1 + \epsilon'_2 &< \frac{2D'}{E' \cdot (k + 8 + 2m\pi)} < \frac{M(k, m)}{4}, \\ \left(\epsilon'_1 + \epsilon'_2 < \frac{2D'}{E' \cdot (k + 2(m+1)\pi)} < \frac{M(k, m)}{4}, \text{ respectively} \right), \end{aligned}$$

for sufficiently large k (and hence m).

Recalling that there is only one zero of $f_{k,m}$ in $(\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$, now we show that the zeros of $f_{k+8,m}$ ($f_{k,m+1}$, resp.) do not lie in $(\alpha^* - \epsilon, \alpha^* + \epsilon)$. Hence by Lemma 3.1, the zeros of $f_{k+8,m}$ and $f_{k,m}$ ($f_{k,m+1}$ and $f_{k,m}$, resp.) interlace.

Let $b_*(\theta)$ be $b_{k+8}(\theta)$ or $b_{m+1}(\theta)$ and let $f_{k,m}^*$ be $f_{k+8,m}$ or $f_{k,m+1}$ respectively.

Let $z_0 \in (\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$ be the unique zero of $f_{k,m}$ in $(\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$ and $\beta^* \in (\beta^* - \epsilon'_1, \beta^* + \epsilon'_2)$ such that $\cos(b_*(\beta^*)) = 0$. If there exists a zero $z_0^* \in (\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$, then

$$|z_0^* - z_0| < \epsilon_1 + \epsilon_2.$$

On the other hand, since $M(k, m)$ is the minimum of lower bounds on the distance between each zero of $\cos(b(\theta))$ and each zero of $\cos(b_*(\theta))$ by Proposition 3.3, we have that

$$\begin{aligned} M(k, m) &\leq |\alpha^* - \beta^*| \leq |\alpha^* - z_0| + |z_0 - z_0^*| + |z_0^* - \beta^*| \\ &< (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon'_1 + \epsilon'_2) < \frac{3M(k, m)}{4}, \end{aligned}$$

which is a contradiction.

Hence this proves that the zeros of $f_{k+8,m}$ ($f_{k,m+1}$, resp.) do not lie in $(\alpha^* - \epsilon_1, \alpha^* + \epsilon_2)$. This completes the proof for the main theorem.

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