J. Math. Anal. Appl. 453 (2017) 942–953

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Monotonicity and functional inequalities for the complete $p$-elliptic integrals

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ARTICLE INFO

*Article history:*
Received 30 December 2016
Available online 18 April 2017
Submitted by S. Cooper

*Keywords:*
Generalized trigonometric functions
Generalized complete elliptic integrals
Monotonicity
Inequalities

ABSTRACT

Takeuchi's generalized complete elliptic integrals related to generalized trigonometric functions are of importance in the computation of the generalized pi and in the elementary proof of Ramanujan's cubic transformation. In this paper we generalize some well-known results of the classical complete elliptic integrals to the case of Takeuchi's generalized complete elliptic integrals. We obtain sharp monotonicity and convexity results for combinations of these functions, as well as functional inequalities.



## 1. Introduction

It is well known from basic calculus that

$$\arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{1/2}}dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi}{2} = \arcsin(1) = \int_0^1 \frac{1}{(1-t^2)^{1/2}}dt.$$

We can define the function sin on $[0, \pi/2]$ as the inverse of arcsin and extend it on $(-\infty, \infty)$.

Let $1 < p < \infty$. We can generalize the above functions as follows:

$$\arcsin_p(x) \equiv \int_0^x \frac{1}{(1-t^p)^{1/p}}dt, \quad 0 \le x \le 1,$$

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http://dx.doi.org/10.1016/j.jmaa.2017.04.025

and

$$\frac{\pi_p}{2} = \arcsin_p(1) \equiv \int_0^1 \frac{1}{(1-t^p)^{1/p}} dt = \frac{\pi/p}{\sin(\pi/p)} = \frac{1}{p} B(1/p, 1-1/p),$$

where $B$ is the beta function. The inverse of $\arcsin_p$ on $[0, \pi_p/2]$ is called the *generalized sine function* and denoted by $\sin_p$. By standard extension procedures as the sine function we get a differentiable function on the whole of $(-\infty, \infty)$ which coincides with the sine function sin when $p = 2$. In the same way we can define the generalized cosine function, the generalized tangent function, and their inverses, and also the corresponding hyperbolic functions (see Section 2). The generalized sine function $\sin_p$ occurs as an eigenfunction of the Dirichlet problem for the one-dimensional $p$-Laplacian. There are several different definitions for these generalized trigonometric and hyperbolic functions [9,11–13]. Recently, these functions have been studied very extensively. In particular, the reader is referred to [11–14].

The well-known complete elliptic integrals of the first and second kind are respectively defined by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r^2 \sin^2 \theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}$$

and

$$\mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \theta} d\theta = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}} dt.$$

In 2014, S. Takeuchi [16] introduced a form of the generalized complete elliptic integrals as an application of generalized trigonometric functions. The complete $p$-elliptic integrals of the first and second kind are respectively defined as follows: for $p \in (1, \infty)$ and $r \in [0, 1)$,

$$\mathcal{K}_p(r) = \int_0^{\pi_p/2} \frac{d\theta}{(1 - r^p \sin_p^p \theta)^{1-1/p}} = \int_0^1 \frac{dt}{(1-t^p)^{1/p}(1-r^pt^p)^{1-1/p}} \tag{1.1}$$

and

$$\mathcal{E}_p(r) = \int_0^{\pi_p/2} (1 - r^p \sin_p^p \theta)^{1/p} d\theta = \int_0^1 \left(\frac{1-r^pt^p}{1-t^p}\right)^{1/p} dt. \tag{1.2}$$

The complete $p$-elliptic integrals can be represented by the Gaussian hypergeometric function [16, Proposition 2.8]: for $r \in [0, 1)$,

$$\mathcal{K}_p(r) = \frac{\pi_p}{2} F(1/p, 1-1/p; 1; r^p) \tag{1.3}$$

and

$$\mathcal{E}_p(r) = \frac{\pi_p}{2} F(1/p, -1/p; 1; r^p), \tag{1.4}$$

where $F(a, b; c; x)$ denotes the Gaussian hypergeometric function (see Section 2 for the definition).

Note that there are several different forms of the generalized elliptic integrals, see [6,1,7,8,15]. For example, the Borweins [6, Section 5.5] defined the generalized complete elliptic integrals of the first and of the second kind by

$$\mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2) \tag{1.5}$$

$$\mathcal{E}_a(r) = \frac{\pi}{2} F(a-1, 1-a; 1; r^2) \tag{1.6}$$

for $a \in (0,1)$ and $r \in (0,1)$. The Borweins' generalized elliptic integrals are of importance in the study of Ramanujan's modular equations and approximations to $\pi$. For $a = 1/3$ and for some $\delta \in (0,1)$, they have proved the identity

$$\mathcal{K}_{1/3}\left(\sqrt{1-\left(\frac{1-r}{1+2r}\right)^2}\right) = (1+2r)\mathcal{K}_{1/3}(r^{3/2}), \quad r \in (0,\delta),$$

and used it to derive a cubically convergent algorithm for the computation of $\pi$. Many noteworthy monotonicity and convexity properties of this form of the generalized elliptic integrals (1.5) and (1.6) have been obtained in [1]. A general form of the generalized elliptic integrals with more parameters were introduced in [8] and [7], and the results in [1] were extended to these generalized elliptic integrals.

Takeuchi's complete $p$-elliptic integrals are in the Legendre–Jacobi standard form with generalized trigonometric functions. As it shows in [16], the advantage of using the complete $p$-elliptic integrals lies in the fact that it is possible to prove formulas of the generalized complete elliptic integrals simply as well as that of the classical complete elliptic integrals. The complete $p$-elliptic integrals have been used to establish computation formulas of the generalized $\pi$ in terms of the arithmetic-geometric mean and to prove Ramanujan's cubic transformation in an elementary way [16,17].

In this paper we will continue Takeuchi's study following the ideas of [1,2,8,7] and generalize some well-known results of the classical complete elliptic integrals to the case of complete $p$-elliptic integrals. We obtain sharp monotonicity and convexity results for combinations of these functions, as well as functional inequalities. This article is organized as follows. In Section 2 we introduce the necessary notation and the functions studied, as well as known results used in the sequel. In Section 3 we obtain various generalizations of monotonicity results and sharp functional inequalities for certain combinations of the complete $p$-elliptic integrals.

## 2. Preliminaries and definitions

The *generalized cosine function* $\cos_p$ is defined as

$$\cos_p(x) \equiv \frac{d}{dx}\sin_p(x).$$

It is clear from the definitions that

$$\cos_p(x) = (1-\sin_p(x)^p)^{1/p}, \qquad x \in [0, \pi_p/2],$$

and

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \qquad x \in \mathbb{R}.$$

It is easy to see that

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}, \qquad x \in [0, \pi_p/2].$$

Similarly, the *generalized inverse hyperbolic sine function*

$$\mathrm{arsh}_p(x) \equiv \begin{cases} \int_0^x \frac{1}{(1+t^p)^{1/p}}dt, & x \in [0,\infty), \\ -\mathrm{arsh}_p(-x), & x \in (-\infty, 0) \end{cases}$$

generalizes the classical inverse hyperbolic sine function. The inverse of $\mathrm{arsh}_p$ is called the *generalized hyperbolic sine function* and denoted by $\mathrm{sh}_p$. The *generalized hyperbolic cosine function* is defined as

$$\mathrm{ch}_p(x) \equiv \frac{d}{dx}\mathrm{sh}_p(x).$$

The definitions show that

$$\mathrm{ch}_p(x)^p - |\mathrm{sh}_p(x)|^p = 1, \qquad x \in \mathbb{R},$$

and

$$\frac{d}{dx}\mathrm{ch}_p(x) = \mathrm{ch}_p(x)^{2-p}\mathrm{sh}_p(x)^{p-1}, \qquad x \geq 0.$$

The *generalized hyperbolic tangent function* is defined as

$$\mathrm{th}_p(x) \equiv \frac{\mathrm{sh}_p(x)}{\mathrm{ch}_p(x)},$$

and hence we have

$$\frac{d}{dx}\mathrm{th}_p(x) = 1 - |\mathrm{th}_p(x)|^p.$$

The inverse of generalized hyperbolic tangent function is denoted by $\mathrm{arth}_p$, and it is easy to see that

$$\frac{d}{dx}\mathrm{arth}_p(x) = \frac{1}{1-x^p}.$$

For real numbers $a, b$ and $c$ with $c \neq 0, -1, -2\cdots$, the *Gaussian hypergeometric function* is defined by

$$F(a,b;c;x) ={}_2 F_1(a,b;c;x) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}, \quad |x| < 1.$$

Here $(a,0) = 1$ for $a \neq 0$, and $(a,n)$ is the *shifted factorial function*

$$(a,n) \equiv a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{N} \equiv \{k : k \text{ is a positive integer}\}$.

We shall also need the function

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \quad R(a) = R(a, 1-a), \quad R(1/2) = \log 16,$$

where $\psi$ is the psi function and $\gamma = 0.577215\ldots$ is the Euler–Mascheroni constant.

The complete $p$-elliptic integrals can be represented by the Gaussian hypergeometric function as (1.3) and (1.4) show. As is traditional, we always use the notation $r' = (1 - r^p)^{1/p}$ for $r \in [0, 1]$. The complementary integrals $\mathcal{K}_p{}'(r)$ and $\mathcal{E}_p{}'(r)$ are defined by $\mathcal{K}_p{}'(r) = \mathcal{K}_p(r')$ and $\mathcal{E}_p{}'(r) = \mathcal{E}_p(r')$. Then we have the following beautiful Legendre relation [16, Theorem 1.1]:

$$\mathcal{K}_p(r)\mathcal{E}_p{}'(r) + \mathcal{K}_p{}'(r)\mathcal{E}_p(r) - \mathcal{K}_p(r)\mathcal{K}_p{}'(r) = \frac{\pi_p}{2}.$$

The functions $\mathcal{K}_p$ and $\mathcal{E}_p$ satisfy a system of differential equations [16, Proposition 2.1]:

$$\frac{d\mathcal{K}_p}{dr} = \frac{\mathcal{E}_p - r'^{\,p}\mathcal{K}_p}{rr'^{\,p}}, \quad \frac{d\mathcal{E}_p}{dr} = \frac{\mathcal{E}_p - \mathcal{K}_p}{r}. \tag{2.1}$$

From (2.1), it is easy to get the following derivative formula:

$$\frac{d}{dr}(\mathcal{E}_p - r'^{\,p}\mathcal{K}_p) = (p-1)r^{p-1}\mathcal{K}_p, \quad \frac{d}{dr}(\mathcal{K}_p - \mathcal{E}_p) = \frac{r^{p-1}\mathcal{E}_p}{r'^{\,p}}. \tag{2.2}$$

We define two related functions $m_p$ and $\mu_p$ as follows: for $0 < r < 1$,

$$m_p(r) = \frac{2}{\pi_p} r'^{\,p}\mathcal{K}_p(r)\mathcal{K}_p{}'(r), \tag{2.3}$$

$$\mu_p(r) = \frac{\pi_p}{2}\frac{\mathcal{K}_p{}'(r)}{\mathcal{K}_p(r)}. \tag{2.4}$$

For $p = 2$, these functions reduce to well-known special cases. The function $\mu(r) = \mu_2(r)$ is the modulus of the Grötzsch ring domain in the plane, which has numerous applications in the conformal invariants and the theory of quasiconformal mappings [2,10]. The function $\mu(r)$ also appears in the classical modular equations [5,6]. Many noteworthy monotonicity and convexity properties of functions defined in terms of the modulus of the Grötzsch ring are presented in the monograph [2]. Applications of these results lead to various sharp functional inequalities for the function $\mu$. These sharp inequalities of the functions $m_2(r)$ and $\mu_2(r)$ can be used to deduce very good estimates of quasiconformal distortion functions.

## 3. Monotonicity, convexity and inequalities

The l'Hôpital Monotone Rule (LMR), Lemma 3.1, is a key tool in proofs of our generalizations. For $p = 2$, our results in this section reduce to the results for the classical functions, see [2, Chapter 3].

**3.1. Lemma.** [3] (l'Hôpital Monotone Rule). *Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on $(a, b)$, with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$. If $f'/g'$ is increasing (decreasing) on $(a, b)$, then so is $f/g$.*

Some other applications of the l'Hôpital Monotone Rule (LMR) in special functions one is referred to the survey [4].

**3.2. Lemma.** [2, Theorem 1.52(2)] *The function $r \mapsto B(a, b)F(a, b; a + b; r) + \log(1 - r)$ is strictly decreasing from $(0, 1)$ onto $(R(a, b), B(a, b))$.*

**3.3. Lemma.** *The function $f(r) = \mathcal{K}_p(r) + \log r'$ is decreasing from $(0, 1)$ onto $(R(1/p)/p, \pi_p/2)$.*

**Proof.** It follows from (1.3) and Lemma 3.2. $\square$

**3.4. Lemma.** *Let $p>1$ in the parts (1)–(7), and $1<p\le 2$ in the part (8). Then the function*

*(1) $f_1(r)=(\mathcal{E}_p-r'^{\,p}\mathcal{K}_p)/r^p$ is increasing from $(0,1)$ onto $((p-1)\pi_p/(2p),1)$.*
*(2) $f_2(r)=r'^{\,p}\mathcal{K}_p/\mathcal{E}_p$ is decreasing from $(0,1)$ onto $(0,1)$.*
*(3) $f_3(r)=(\mathcal{E}_p-r'^{\,p}\mathcal{K}_p)/(r^p\mathcal{K}_p)$ is decreasing from $(0,1)$ onto $(0,(p-1)/p)$.*
*(4) $f_4(r)=(\mathcal{K}_p-\mathcal{E}_p)/(r^p\mathcal{K}_p)$ is increasing from $(0,1)$ onto $(1/p,1)$.*
*(5) $f_5(r)=r'^{\,p}(\mathcal{K}_p-\mathcal{E}_p)/(r^p\mathcal{E}_p)$ is decreasing from $(0,1)$ onto $(0,1/p)$.*
*(6) $f_6(r)=r'^{\,c}\mathcal{K}_p$ is decreasing from $(0,1)$ if and only if $c\ge(p-1)/p$, in which case the range of $f_6$ is $(0,\pi_p/2)$. Moreover, $r\mapsto r'\mathcal{K}_p$ is decreasing for each $p\in(1,\infty)$.*
*(7) $f_7(r)=r'^{\,c}\mathcal{E}_p$ is increasing from $(0,1)$ if and only if $c\le -1/p$, in which case the range of $f_7$ is $(\pi_p/2,\infty)$.*
*(8) $f_8(r)=((\mathcal{K}_p-\mathcal{E}_p)-(\mathcal{E}_p-r'^{\,p}\mathcal{K}_p))/r^p$ is increasing from $(0,1)$ onto $((2-p)\pi_p/(2p),\infty)$.*

**Proof.** (1) Write $f_{11}(r)=\mathcal{E}_p-r'^{\,p}\mathcal{K}_p$ and $f_{12}(r)=r^p$. Then $f_{11}(0)=0=f_{12}(0)$ and

$$\frac{f_{11}'(r)}{f_{12}'(r)}=\frac{p-1}{p}\mathcal{K}_p$$

which is increasing, with $f_1(0+)=(p-1)\pi_p/(2p)$. The monotonicity of $f_1$ follows from Lemma 3.1.

(2) Since $f_2(r)=1-(\mathcal{E}_p-r'^{\,p}\mathcal{K}_p)/\mathcal{E}_p$, the result follows from the facts that $\mathcal{E}_p-r'^{\,p}\mathcal{K}_p$ is increasing and $\mathcal{E}_p$ is decreasing.

(3) Write $f_{31}(r)=\mathcal{E}_p-r'^{\,p}\mathcal{K}_p$ and $f_{32}(r)=r^p\mathcal{K}_p$. Then $f_{31}(0)=0=f_{32}(0)$ and

$$\frac{f_{31}'(r)}{f_{32}'(r)}=\frac{(p-1)r'^{\,p}\mathcal{K}_p}{(p-1)r'^{\,p}\mathcal{K}_p+\mathcal{E}_p}=\frac{1}{1+\mathcal{E}_p/((p-1)r'^{\,p}\mathcal{K}_p)}$$

which is decreasing by (2). Hence (3) follows from Lemma 3.1.

(4) This follows from (3), since $f_4(r)=1-f_3(r)$.

(5) This follows from (1), since $f_5(r)=1-f_1(r)/\mathcal{E}_p$.

(6) By differentiation, we have that

$$\begin{aligned}f_6'(r)&=cr'^{\,c-1}\left(-\left(\frac{r}{r'}\right)^{p-1}\right)\mathcal{K}_p+r'^{\,c}\frac{\mathcal{E}_p-r'^{\,p}\mathcal{K}_p}{rr'^{\,p}}\\&=\frac{r'^{\,c-p}}{r}\left(\mathcal{E}_p-r'^{\,p}\mathcal{K}_p-cr^p\mathcal{K}_p\right)\\&=r'^{\,c-p}r^{p-1}\mathcal{K}_p\left(\frac{\mathcal{E}_p-r'^{\,p}\mathcal{K}_p}{r^p\mathcal{K}_p}-c\right)\end{aligned}$$

which is nonpositive if and only if

$$c\ge\sup_r\frac{\mathcal{E}_p-r'^{\,p}\mathcal{K}_p}{r^p\mathcal{K}_p}=(p-1)/p,$$

by (3). Therefore, $f_6$ is decreasing from $(0,1)$ if and only if $c\ge(p-1)/p$.

(7) We have that

$$\begin{aligned}f_7'(r)&=cr'^{\,c-1}\left(-\left(\frac{r}{r'}\right)^{p-1}\right)\mathcal{E}_p-r'^{\,c}\frac{\mathcal{K}_p-\mathcal{E}_p}{r}\\&=r'^{\,c-p}r^{p-1}\mathcal{E}_p\left(-\frac{r'^{\,p}(\mathcal{K}_p-\mathcal{E}_p)}{r^p\mathcal{E}_p}-c\right)\end{aligned}$$

which is nonnegative if and only if

$$c \leq -\sup_r \frac{r'^p(\mathcal{K}_p - \mathcal{E}_p)}{r^p \mathcal{E}_p} = -\frac{1}{p},$$

by part (5). Therefore, $f_7$ is increasing from $(0,1)$ if and only if $c \leq -1/p$.

(8) Write $f_{81}(r) = (\mathcal{K}_p - \mathcal{E}_p) - (\mathcal{E}_p - r'^p \mathcal{K}_p)$ and $f_{82}(r) = r^p$. Then $f_{81}(0) = 0 = f_{82}(0)$ and

$$\frac{f'_{81}(r)}{f'_{82}(r)} = \frac{\mathcal{E}_p - r'^p \mathcal{K}_p}{p r'^p} + \frac{2-p}{p}\mathcal{K}_p,$$

which is increasing since $1 < p \leq 2$. Hence the monotonicity of $f_8$ follows from Lemma 3.1. $\square$

**3.5. Lemma.** *Let $p \in (1,\infty)$. Then the function*

*(1) $f_1(r) = r\mathcal{K}_p/\mathrm{arth}_p r$ is decreasing from $(0,1)$ onto $(1, \pi_p/2)$. In particular, for $r \in (0,1)$ and $p \in (1,\infty)$,*

$$\frac{\mathrm{arth}_p r}{r} \leq \mathcal{K}_p \leq \frac{\pi_p}{2}\frac{\mathrm{arth}_p r}{r}.$$

*(2) $f_2(r) = r^p \mathcal{K}_p/\log(1/r')$ is decreasing from $(0,1)$ onto $(1, p\pi_p/2)$. In particular, for $r \in (0,1)$ and $p \in (1,\infty)$,*

$$\frac{1}{p}\frac{\log(1/(1-r^p))}{r^p} \leq \mathcal{K}_p \leq \frac{\pi_p}{2}\frac{\log(1/(1-r^p))}{r^p}.$$

**Proof.** (1) Write $f_1(r) = f_{11}(r)/f_{12}(r)$, where $f_{11}(r) = r\mathcal{K}_p$ and $f_{12}(r) = \mathrm{arth}_p r$. Then $f_{11}(0) = 0 = f_{12}(0)$ and

$$\frac{f'_{11}(r)}{f'_{12}(r)} = \frac{\mathcal{K}_p + r(\mathcal{E}_p - r'^p\mathcal{K}_p)/(r r'^p)}{1/(r'^p)} = \mathcal{E}_p$$

which is strictly decreasing on $(0,1)$. Hence the monotonicity of $f_1$ follows from Lemma 3.1. The limiting values follow from l'Hôpital rule.

(2) Write $f_2(r) = f_{21}(r)/f_{22}(r)$, where $f_{21}(r) = r^p\mathcal{K}_p$ and $f_{22}(r) = \log(1/r')$. Then $f_{21}(0) = 0 = f_{22}(0)$ and

$$\frac{f'_{21}(r)}{f'_{22}(r)} = \mathcal{E}_p + (p-1)r'^p\mathcal{K}_p,$$

which is strictly decreasing on $(0,1)$. Hence the monotonicity of $f_2$ follows from Lemma 3.1. The limiting values follow from l'Hôpital rule. $\square$

**3.6. Theorem.** *Let $p \in (1,2]$. Then*

*(1) the function $f_1(x) = \mathcal{K}_p(1/\mathrm{ch}x)$ is strictly decreasing and convex on $(0,\infty)$. In particular, for $s,t \in (0,1)$,*

$$2\mathcal{K}_p\left(\sqrt{\frac{2st}{1+st+\sqrt{(1-s^2)(1-t^2)}}}\right) \leq \mathcal{K}_p(s) + \mathcal{K}_p(t), \tag{3.7}$$

*with equality if and only if $s = t$.*

*(2) the function $f_2(x) = \mathcal{K}_p(\mathrm{th}x)$ is strictly increasing and convex on $(0, \infty)$. In particular, for $s, t \in (0, 1)$,*

$$2\mathcal{K}_p\left(\frac{s(1+\sqrt{1-t^2})+t(1+\sqrt{1-s^2})}{st+(1+\sqrt{1-s^2})(1+\sqrt{1-t^2})}\right) \leq \mathcal{K}_p(s)+\mathcal{K}_p(t), \tag{3.8}$$

*with equality if and only if $s = t$.*

**Proof.** 1. Let $r = 1/\cosh x$. Then $dr/dx = -r(1-r^2)^{1/2}$ and

$$f_1'(x) = -(\mathcal{E}_p(r) - r'^p\mathcal{K}_p(r))\frac{(1-r^2)^{1/2}}{1-r^p},$$

which is negative and strictly decreasing in $r$, hence increasing in $x$ for any $1 < p \leq 2$. Therefore, $f_1$ is strictly decreasing and convex on $(0, \infty)$. In particular, we have $f_1((x + y)/2) \leq (f_1(x) + f_1(y))/2$, with equality if and only if $x = y$. Set $s = 1/\mathrm{ch}x$ and $t = 1/\mathrm{ch}y$. Now

$$\mathrm{ch}\left(\frac{x+y}{2}\right) = \sqrt{\frac{1+st+\sqrt{(1-s^2)(1-t^2)}}{2st}}.$$

Hence

$$f_1\left(\frac{x+y}{2}\right) \leq \frac{f_1(x)+f_1(y)}{2}$$

gives

$$2\mathcal{K}_p\left(\sqrt{\frac{2st}{1+st+\sqrt{(1-s^2)(1-t^2)}}}\right) \leq \mathcal{K}_p(s)+\mathcal{K}_p(t)$$

with equality if and only if $s = t$.

2. Let $s = \mathrm{th}x$ and $t = \mathrm{th}y$. Then

$$f_2'(x) = \frac{\mathcal{E}_p(s) - s'^p\mathcal{K}_p(s)}{s}\frac{1-s^2}{1-s^p},$$

which is positive and strictly increasing in $s$, hence increasing in $x$. Therefore, $f_2$ is strictly increasing and convex on $(0, \infty)$. In particular, we have $f((x+y)/2) \leq (f(x)+f(y))/2$, with equality if and only if $x = y$. Now

$$\mathrm{th}\left(\frac{x+y}{2}\right) = \frac{s(1+\sqrt{1-t^2})+t(1+\sqrt{1-s^2})}{st+(1+\sqrt{1-s^2})(1+\sqrt{1-t^2})}.$$

Hence

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

gives

$$2\mathcal{K}_p\left(\frac{s(1+\sqrt{1-t^2})+t(1+\sqrt{1-s^2})}{st+(1+\sqrt{1-s^2})(1+\sqrt{1-t^2})}\right) \leq \mathcal{K}_p(s)+\mathcal{K}_p(t),$$

with equality if and only if $s = t$. $\square$

**3.9. Theorem.** *For each $p \in (1,\infty)$ and $r \in (0,1)$, the function $f(\alpha) = \mathcal{K}_p(r^\alpha)/\alpha$ is strictly decreasing and strictly log-convex on $(0,\infty)$.*

**Proof.** Let $t = r^\alpha$. By logarithmic differentiation, we obtain

$$\frac{d}{d\alpha}\log f_1(\alpha) = \left(\frac{\mathcal{E}_p(t)}{t'^{\,p}\mathcal{K}_p(t)} - 1 + \frac{1}{\log(1/t)}\right)\log r,$$

which is negative and strictly decreasing in $t$ and hence strictly increasing in $\alpha$ by Lemma 3.4(2). Therefore we get the monotonicity and log-convexity of $f$. □

**3.10. Corollary.** *Let $p > 1$. For all $s, t \in (0,1)$,*

$$\frac{\mathcal{K}_p(\sqrt{st})}{\log(1/\sqrt{st})} \le \sqrt{\frac{\mathcal{K}_p(s)\mathcal{K}_p(t)}{\log(1/s)\log(1/t)}}$$

*with equality if and only if $s = t$.*

**Proof.** Let $r \in (0,1)$. It follows from Theorem 3.9 that the function $\alpha \mapsto \mathcal{K}_p(r^\alpha)/\alpha$ is strictly log-convex. Hence the log-convexity implies that

$$\frac{\mathcal{K}_p\left(r^{(\alpha+\beta)/2}\right)}{(\alpha+\beta)/2} \le \sqrt{\frac{\mathcal{K}_p(r^\alpha)\mathcal{K}_p(r^\beta)}{\alpha\beta}} \tag{3.11}$$

with the equality if and only if $\alpha = \beta$. Set $s = r^\alpha$ and $t = r^\beta$. Then

$$r^{(\alpha+\beta)/2} = \sqrt{st},$$
$$(\alpha+\beta)/2 = \log(1/\sqrt{st})/\log(1/r),$$

and

$$\sqrt{\alpha\beta} = \sqrt{\log(1/s)\log(1/t)}/\log(1/r).$$

Then from (3.11) we conclude that

$$\frac{\mathcal{K}_p(\sqrt{st})}{\log(1/\sqrt{st})} \le \sqrt{\frac{\mathcal{K}_p(s)\mathcal{K}_p(t)}{\log(1/s)\log(1/t)}}$$

with equality if and only if $s = t$. □

**3.12. Lemma.** *Let $p > 1$. For $0 < r < 1$,*

$$\frac{d\,m_p(r)}{d\,r} = \frac{\pi_p - 4\mathcal{K}_p(r)\mathcal{E}_p{}'(r) - 2(p-2)r^p\mathcal{K}_p(r)\mathcal{K}_p{}'(r)}{\pi_p r} \tag{3.13}$$

$$= -\frac{1}{r} - \frac{4\mathcal{K}_p{}'(r)(\mathcal{K}_p(r) - \mathcal{E}_p(r)) + 2(p-2)r^p\mathcal{K}_p(r)\mathcal{K}_p{}'(r)}{\pi_p r}. \tag{3.14}$$

$$\frac{d\,\mu_p(r)}{d\,r} = -\frac{\pi_p^2}{4rr'^{\,p}\mathcal{K}_p(r)^2}. \tag{3.15}$$

**Proof.** The formulas for the derivatives of $m_p$ and $\mu_p$ follow easily from (2.1) and Legendre relation. $\square$

**3.16. Theorem.** *Let $p > 1$. Then the function*

*(1) $f_1(r) = m_p(r) + \log r$ is decreasing and concave from $(0, 1)$ onto $(0, R(1/p)/p)$.*
*(2) $f_2(r) = m_p(r)/\log(1/r)$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$.*
*(3) $f_3(r) = \mu_p(r) + \log r$ is decreasing from $(0, 1)$ onto $(0, R(1/p)/p)$.*
*(4) $f_4(r) = \mu_p(r)/\log(1/r)$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$.*

**Proof.** (1) For $p \geq 2$, by the formula (3.14) we have

$$f_1'(r) = -\frac{4\mathcal{K}_p{}'(\mathcal{K}_p - \mathcal{E}_p) + 2(p-2)r^p\mathcal{K}_p\mathcal{K}_p{}'}{\pi_p r} \tag{3.17}$$

$$= -\left(\frac{4r^{p-1}\mathcal{K}_p{}'}{\pi_p}\frac{\mathcal{K}_p - \mathcal{E}_p}{r^p} + \frac{2(p-2)}{\pi_p}r^{p-1}\mathcal{K}_p{}'\mathcal{K}_p\right) \tag{3.18}$$

which is negative and decreasing.

For $1 < p < 2$, we have

$$f_1'(r) = -\left(\frac{2}{\pi_p}\frac{2\mathcal{K}_p{}'(\mathcal{K}_p - \mathcal{E}_p)}{r} - \frac{2}{\pi_p}\frac{r^p\mathcal{K}_p{}'\mathcal{K}_p}{r} + \frac{2(p-1)}{\pi_p}r^{p-1}\mathcal{K}_p{}'\mathcal{K}_p\right) \tag{3.19}$$

$$= -\left(\frac{2}{\pi_p}\frac{r^{p-1}\mathcal{K}_p{}'\left((\mathcal{K}_p - \mathcal{E}_p) - (\mathcal{E}_p - r'^p\mathcal{K}_p)\right)}{r^p} + \frac{2(p-1)}{\pi_p}r^{p-1}\mathcal{K}_p{}'\mathcal{K}_p\right) \tag{3.20}$$

which is negative and decreasing.

Hence, for $1 < p$, the function $f_1$ is decreasing and concave on $(0, 1)$. The limiting values are from Lemma 3.3 and Lemma 3.4(6).

(2) Write $f_{21}(r) = m_p(r)$ and $f_{22}(r) = \log(1/r)$. Then $f_{21}(1) = 0$ and $f_{22}(1) = 0$. By differentiation,

$$\frac{f_{21}'(r)}{f_{22}'(r)} = 1 + \frac{4r^p\mathcal{K}_p{}'}{\pi_p}\frac{\mathcal{K}_p - \mathcal{E}_p}{r^p} + \frac{2(p-2)}{\pi_p}r^p\mathcal{K}_p{}'\mathcal{K}_p \tag{3.21}$$

$$= 1 + \frac{2}{\pi_p}r^p\mathcal{K}_p{}'\left(\frac{(\mathcal{K}_p - \mathcal{E}_p) - (\mathcal{E}_p - r'^p\mathcal{K}_p)}{r^p} + (p-1)\mathcal{K}_p\right) \tag{3.22}$$

which is strictly increasing by Lemma 3.4(6) and (8). Hence, the function $f_2$ is strictly increasing by Lemma 3.1. $\square$

**3.23. Theorem.** *Let $p > 1$. Then the function*

*(1) $f_1(r) = \mu_p(1 - e^{-x})$ is decreasing and convex from $(0, \infty)$ onto $(0, \infty)$. In particular, for $a, b \in (0, 1)$*

$$\mu_p(1 - \sqrt{(1-a)(1-b)}) \leq \frac{\mu_p(a) + \mu_p(b)}{2}$$

*with equality if and only if $a = b$.*

*(2) $f_2(r) = \mu_p(e^{-x})$ is strictly increasing and concave from $(0, \infty)$ onto $(0, \infty)$. In particular, for $a, b \in (0, 1)$*

$$\frac{\mu_p(a) + \mu_p(b)}{2} \leq \mu_p(\sqrt{ab}),$$

*with equality if and only if $a = b$.*

*(3)* $f_3(r) = \mathrm{th}\mu_p(e^{-x})$ *is strictly increasing and concave on* $(0,\infty)$*. In particular, for* $a, b \in (0,1)$

$$\mathrm{th}\mu_p\left(\sqrt{ab}\right) \geq \frac{\mathrm{th}\mu_p(a) + \mathrm{th}\mu_p(b)}{2}$$

*with equality if and only if* $a = b$*.*

**Proof.** (1) With $r = 1 - e^{-x}$, we have

$$f_1'(x) = -\frac{\pi_p^2}{4r\mathcal{K}_p{}^2}\frac{1-r}{r'^{\,p}},$$

which is negative and strictly increasing in $r$, hence in $x$, so $f_1$ is strictly decreasing and convex in $x$. Hence

$$f_1\left(\frac{x+y}{2}\right) \leq \frac{f_1(x) + f_1(y)}{2}$$

for all $x, y \in (0,\infty)$, with equality if and only if $x = y$. Setting $1 - e^{-x} = a$, $1 - e^{-y} = b$, we get

$$\mu_p(1 - \sqrt{(1-a)(1-b)}) \leq \frac{\mu_p(a) + \mu_p(b)}{2}.$$

(2) With $r = e^{-x}$, we get

$$f_2'(x) = \frac{\pi_p^2}{4r'^{\,p}\mathcal{K}_p(r)^2},$$

which is positive and strictly increasing in $r$ by Lemma 3.4(6), hence strictly decreasing in $x$, so $f_2$ is strictly increasing and concave in $x$. Hence

$$f_2\left(\frac{x+y}{2}\right) \geq \frac{f_2(x) + f_2(y)}{2}$$

for all $x, y \in (0,\infty)$, with equality if and only if $x = y$. Setting $e^{-x} = a$, $e^{-y} = b$, we get

$$\mu_p(\sqrt{ab}) \geq \frac{\mu_p(a) + \mu_p(b)}{2}.$$

(3) With $r = e^{-x}$, we have

$$f_3'(x) = \frac{\pi_p^2}{4r'^{\,p}\mathcal{K}_p(r)^2\mathrm{ch}^2\mu_p(r)},$$

which is strictly increasing in $r$, hence decreasing in $x$, so $f_3$ is concave. Consequently,

$$\mathrm{th}\mu_p\left(e^{-(x+y)/2}\right) > \frac{\mathrm{th}\mu_p(e^{-x}) + \mathrm{th}\mu_p(e^{-x})}{2}$$

for all distinct $x$ and $y$. Now set $e^{-x} = a$ and $e^{-y} = a$, and the desired inequality follows. □

## Acknowledgments

This research was supported by National Natural Science Foundation of China (NNSF of China, No. 11601485) and Science Foundation of Zhejiang Sci-Tech University (ZSTU, No. 16062023-Y). The author is indebted to the anonymous referee for the valuable suggestions.

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