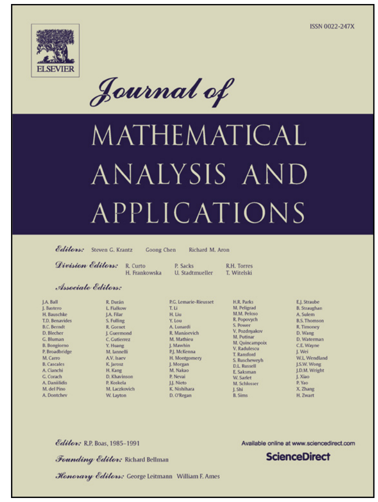
Accepted Manuscript



A level 16 analogue of Ramanujan series for $1/\pi$

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PII: S0022-247X(17)30644-3
DOI: http://dx.doi.org/10.1016/j.jmaa.2017.06.082
Reference: YJMAA 21524

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 30 November 2016

A LEVEL 16 ANALOGUE OF RAMANUJAN SERIES FOR $1/\pi$

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ABSTRACT. The modular function

$$h(\tau) = q \prod_{n=1}^{\infty} \frac{(1-q^{16n})^2(1-q^{2n})}{(1-q^n)^2(1-q^{8n})}$$

is called *a level 16 analogue of Ramanujan's series for* $1/\pi$. We prove that $h(\tau)$ generates the field of modular functions on $\Gamma_0(16)$ and find its modular equation of level n for any positive integer n. Furthermore, we construct the ray class field $K(h(\tau))$ modulo 4 over an imaginary quadratic field K for $\tau \in K \cap \mathfrak{H}$ such that $\mathbb{Z}[4\tau]$ is the integral closure of $\mathbb{Z}$ in K, where $\mathfrak{H}$ is the complex upper half plane. For any $\tau \in K \cap \mathfrak{H}$, it turns out that the value $1/h(\tau)$ is integral, and we can also explicitly evaluate the values of $h(\tau)$ if the discriminant of K is divisible by 4.

1. INTRODUCTION

In [21], Ramanujan studied the series converging to $1/\pi$:

$$\frac{1}{\pi} = \frac{1}{16} \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n}}.$$

The coefficient $\binom{2n}{n}^3$ appearing in the above series is the same as the coefficient appearing in the following identity

$$q\frac{d}{dq} \log\left(\frac{\omega}{1-16\omega}\right) = \sum_{n=0}^{\infty} \binom{2n}{n}^3 (\omega(1-16\omega))^n,$$

where

$$\omega(\tau) = q \prod_{n=1}^{\infty} \frac{(1-q^n)^8(1-q^{4n})^{16}}{(1-q^{2n})^{24}},$$

$q = e^{2\pi i\tau}$ and $\tau \in \mathfrak{H} := \{\tau \in \mathbb{C} : Im(\tau) > 0\}$. Actually, $\omega(\tau)$ generates the field of modular function on $\Gamma_0(4)$ with a simple zero at ∞ and a simple pole at $1/2$.

These types of identities for $1/\pi$ have been studied: for levels $1, 2$ and 3 by Berndt-Bhargava-Garvan [1], for levels $5, 6, 8$ and 9 by Chan-Cooper [3], for levels $7, 10$ and 18 by Cooper [6, 7], for levels 11 and 23 by Cooper-Ge-Ye [8] and for levels $12, 13, 14$ and 15 by Cooper-Ye [9, 10, 11].

2010 *Mathematics Subject Classification.* 11Y65, 11F03, 11R37, 11R04, 14H55.

Key words and phrases. Ramanujan's series for $1/\pi$, modular function, modular equation, ray class field.

The first named author is supported by Basic Science Research Program through the National Research Foundation of Korea (KRF) funded by the Ministry of Education (2009-0093827) and also by the Korea government (MEST)(2014-002731). The second named author is supported by RP-Grant 2016 of Ewha Womans University and Basic Science Research Program through the National Research Foundation of Korea (KRF) funded by the Ministry of Education (2017R1D1A1B03029519).

Recently, Ye studied a modular function $h(\tau)$ of level 16 [23]:

$$h(\tau) = q \prod_{n=1}^{\infty} \frac{(1-q^{16n})^2(1-q^{2n})}{(1-q^n)^2(1-q^{8n})}. \tag{1.1}$$

This modular function $h(\tau)$ is called *a level 16 analogue of Ramanujan's series for* $1/\pi$. He showed that the coefficient A_n appearing in the series (1.3) converging to $1/\pi$ is exactly the same as the coefficient A_n appearing in (1.2), where

$$q\frac{d}{dq}\log\left(\frac{h}{(1+2h)(1+4h)}\right) = \sum_{n=0}^{\infty} A_n \left(\frac{h(1+2h)(1+4h)}{(1-8h^2)^2}\right)^n, \tag{1.2}$$

$$\frac{1}{\pi} = \frac{1}{8\sqrt{14}} \sum_{n=0}^{\infty} A_n \frac{48n+13}{56^n}. \tag{1.3}$$

For its proof, he proved several identities between $h(\tau)$ and some η-quotients, and he found modular equations of $h(\tau)$ of levels $2, 4$ and 8.

In this paper we study the modular function $h(\tau)$ of level 16. We first prove that $h(\tau)$ generates the field of modular functions on $\Gamma_0(16)$ (Theorem 1.1), and we find the modular equations of level n for any positive integer n (Theorem 1.2). We find examples for levels $2, 3, 5, 7$ and 11 in Table 1 using MAPLE program. On the other hand, we use $h(\tau)$ to get the ray class field modulo 4 over an imaginary quadratic field K (Theorem 1.3, Corollary 1.4). We show that the value $1/h(\tau)$ is an algebraic integer in a certain number field (Theorem 1.5). If $h(\tau)$ can be written in terms of radicals, then we can write $h(r\tau)$ in terms of radicals for any positive rational number r by using the algorithm in [17, Algorithm 1.6]. Furthermore, we can get the value $h(\tau)$ for $\tau \in K \cap \mathfrak{H}$ when K has discriminant divisible by 4 (Theorem 1.6).

We state our main results as follows.

Theorem 1.1. *Let $h(\tau)$ be defined in (1.1). Then $h(\tau)$ is a modular function on $\Gamma_0(16)$ and the field of modular functions on $\Gamma_0(16)$ is $\mathbb{C}(h(\tau))$.*

Theorem 1.2. *For any positive integer n, we can obtain a modular equation $F_n(X,Y)$ of $h(\tau)$ of level n in an explicit way.*

Theorem 1.3. *Let K be an imaginary quadratic field with discriminant d_K. Let $\tau \in K \cap \mathfrak{H}$ be a root of the equation $16ax^2 + 4bx + c = 0$ such that $b^2 - 4ac = d_K, (a,b,c) = 1$ and $(a,2) = 1$, where $a,b,c \in \mathbb{Z}$. Then $K(h(\tau))$ is the ray class field modulo 4 over K.*

Corollary 1.4. *Let K be an imaginary quadratic field. If $\mathbb{Z}[4\tau]$ is the integral closure of $\mathbb{Z}$ in K, then $K(h(\tau))$ is the ray class field modulo 4 over K.*

Theorem 1.5. *Let K be an imaginary quadratic field. Then $1/h(\tau)$ is an algebraic integer for any $\tau \in K \cap \mathfrak{H}$.*

Theorem 1.6. *We can explicitly evaluate the value of $h(\tau)$ for any $\tau \in K \cap \mathfrak{H}$ if the discriminant of K is divisible by 4. If $h(\tau)$ is expressed in terms of radicals, then we can express $h(r\tau)$ in terms of radicals for any positive rational number r. In particular, if $n = 1$ or n is a square free positive integer with $n \not\equiv 3 \pmod 4$, then we can evaluate $h(r\sqrt{-n})$ for any positive rational number r.*

2. A modular function of level 16

We recall some definitions and properties in the theory of modular functions. Let $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. For a positive integer N, the congruence subgroup $\Gamma_0(N)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N \mid c \right\}.$$

A element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\Gamma_0(N)$ acts on $\mathfrak{H}^*$ by a linear fractional transformation: $\gamma\tau = (a\tau + b)/(c\tau + d)$. We call an element $s \in \mathbb{Q} \cup \{\infty\}$ a *cusp*. If there exists $\gamma \in \Gamma_0(N)$ satisfying $\gamma s_1 = s_2$, then two cusps s_1 and s_2 are equivalent under $\Gamma_0(N)$. In fact, there are at most finitely many inequivalent cusps of $\Gamma_0(N)$. The *width* h of a cusp s in $\Gamma_0(N)\backslash\mathfrak{H}^*$ is the smallest positive integer satisfying that $\rho^{-1}\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\rho \in \Gamma_0(N)$ for some $\rho \in \mathrm{SL}_2(\mathbb{Z})$ with $\rho(s) = \infty$. Indeed, the width of the cusp s depends only on the equivalent class of s under $\Gamma_0(N)$, and it does not depend on the choice of ρ.

A *modular function* $f(\tau)$ on $\Gamma_0(N)$ is a $\mathbb{C}$-valued function of $\mathfrak{H}$ satisfying the following three conditions:

(1) $f(\tau)$ is meromorphic on $\mathfrak{H}$.
(2) $f(\tau)$ is invariant under $\Gamma_0(N)$, i.e., $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma_0(N)$.
(3) $f(\tau)$ is meromorphic at all cusps of $\Gamma_0(N)$.

The order of $f(\tau)$ at a cusp is calculated as follows. Let s be a cusp of $\Gamma_0(N)$, $f(\tau)$ be a modular function on $\Gamma_0(N)$, h be the width of s, and ρ be an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\rho(s) = \infty$. Note that

$$(f \circ \rho^{-1})(\tau + h) = \left(f \circ \rho^{-1} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rho \right)(\rho^{-1}\tau) = (f \circ \rho^{-1})(\tau)$$

and $f \circ \rho^{-1}$ has a Laurent series expansion in $q_h = e^{2\pi i\tau/h}$, i.e., $(f \circ \rho^{-1})(\tau) = \sum_{n \geq n_0} a_n q_h^n$ with $a_{n_0} \neq 0$. Then we call n_0 the *order* at the cusp s of $f(\tau)$ and denote by $\mathrm{ord}_s f(\tau)$.

Let $A_0(\Gamma_0(N))$ be the field of all modular functions on $\Gamma_0(N)$ and $A_0(\Gamma_0(N))_{\mathbb{Q}}$ be the subfield of $A_0(\Gamma_0(N))$ which consists of all modular functions $f(\tau)$ whose Fourier coefficients belong to $\mathbb{Q}$. The field $A_0(\Gamma_0(N))$ may be identified with the field $\mathbb{C}(\Gamma_0(N)\backslash\mathfrak{H}^*)$ of all meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathfrak{H}^*$. When $f(\tau) \in A_0(\Gamma_0(N))$ is nonconstant, the extension degree $[A_0(\Gamma_0(N)) : \mathbb{C}(f(\tau))]$ is equal to the total degree of poles of $f(\tau)$. Hence, if a modular function $f(\tau)$ of $\Gamma_0(N)$ has neither zeros nor poles on $\mathfrak{H}$, then

$$[A_0(\Gamma_0(N)) : \mathbb{C}(f(\tau))] = - \sum_{\substack{s \text{ is a cusp of } \Gamma_0(N), \\ \mathrm{ord}_s f(\tau) < 0}} \mathrm{ord}_s f(\tau).$$

From the following lemma we can find the set of all inequivalent cusps of $\Gamma_0(N)$ and the width of each cusp.

Lemma 2.1. *Let $a, c, a', c' \in \mathbb{Z}$ be such that $(a, c) = 1$ and $(a', c') = 1$. We understand that $\pm 1/0 = \infty$. We denote by $S_{\Gamma_0(N)}$ a set of all the inequivalent cusps of $\Gamma_0(N)$. Then*

(1) *a/c and a'/c' are equivalent under $\Gamma_0(N)$ if and only if there exist $\overline{s} \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $n \in \mathbb{Z}$ such that $(a', c') \equiv (\overline{s}^{-1} \cdot a + nc, \overline{s} \cdot c) \pmod{N}$.*

(2) *We can take $S_{\Gamma_0(N)}$ as the following set*

$$\begin{aligned} S_{\Gamma_0(N)} \quad = \quad & \left\{ \frac{a_{c,j}}{c} \in \mathbb{Q} : 0 < c \mid N, 0 < a_{c,j} \le N, (a_{c,j}, N) = 1, \right. \\ & \left. a_{c,j} = a_{c,j'} \overset{def.}{\Leftrightarrow} a_{c,j} \equiv a_{c,j'} \pmod{(c, N/c)} \right\}. \end{aligned}$$

(3) *The width of the cusp $a/c \in S_{\Gamma_0(N)}$ is $N/(N, c^2)$.*

Proof. See [5, Corollary 4 (1)]. □

The *Dedekind eta function* is defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

The following two lemmas present some information on the modularity and the behavior of an eta-quotient.

Lemma 2.2. *Let $f(\tau) = \prod_{\delta | N} \eta(\delta\tau)^{r_\delta}$ be an eta-quotient, where $k = \frac{1}{2}\sum_{\delta | N} r_\delta \in \mathbb{Z}$ and $s = \prod_{\delta|N} \delta^{r_\delta}$, with the additional properties that*

(1) *$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$, and*

(2) *$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$.*

Then $f(\tau)$ satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{(-1)^k s}{d}\right) (c\tau + d)^k f(\tau)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

Lemma 2.3. *Let c, d and N be positive integers with $d \mid N$ and $(c, d) = 1$. If $f(\tau)$ is an eta-quotient satisfying the conditions of Lemma 2.2, then the order of vanishing of $f(\tau)$ at the cusp c/d is*

$$\frac{N}{24} \sum_{\delta|N} \frac{(d, \delta)^2 \cdot r_\delta}{(d, N/d) \cdot d\delta}.$$

The proof of Lemma 2.2 is found in [13, 19, 20], and the proof of Lemma 2.3 is given in [2, 16, 18].

Proof of Theorem 1.1. Note that $h(\tau)$ is written as $\eta(\tau)^{-2}\eta(2\tau)\eta(8\tau)^{-1}\eta(16\tau)^2$. By Lemma 2.2, for any $\gamma \in \Gamma_0(16)$, $h(\gamma\tau) = h(\tau)$. Since $\eta(\tau)$ is holomorphic on $\mathfrak{H}^*$ with only zero at cusps, $h(\tau)$ is a modular function on $\Gamma_0(16)$. By Lemma 2.1 (2), the set $S_{\Gamma_0(16)}$ of inequivalent cusps of $\Gamma_0(16)$ can be taken as

$$S_{\Gamma_0(16)} = \left\{ \infty, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8} \right\}.$$

Furthermore, Lemma 2.3 shows that $h(\tau)$ has the only simple pole at 0 and the only simple zero at ∞. Hence $[A_0(\Gamma_0(16)) : \mathbb{C}(h(\tau))] = 1$ and the field $A_0(\Gamma_0(16))$ is generated by $h(\tau)$. □

For the simplicity, we call the *Hauptmodul* the normalized generator of a genus zero function field with q-series $q^{-1} + 0 + \sum_{n=1}^{\infty} c_n q^n$. Since $1/h(\tau)$ has a pole at ∞ and its q-series is $q^{-1} - 2 + 2q^3 - q^7 + \cdots$, we see that $1/h(\tau) + 2$ is the Hauptmodul of $\Gamma_0(16)$. By Theorem 1.1 we get another proof of [23, Theorem 3.3 (1)].

Proposition 2.4.

$$h\left(\tau + \frac{1}{2}\right) = -\frac{h(\tau)}{1 + 4h(\tau)}.$$

Proof. By the definition of $h(\tau)$,

$$H(\tau) := h\left(\tau + \frac{1}{2}\right) = -\frac{\eta^2(\tau)\eta^2(4\tau)\eta^2(16\tau)}{\eta^5(2\tau)\eta(8\tau)}.$$

Using Lemma 2.3, $H(\tau)$ is a modular function on $\Gamma_0(16)$ with only zero at ∞ and only pole at $1/2$. From that $1/H(\tau) = -q^{-1} - 2 - 2q^3 + q^7 + \cdots$, we note that $-1/H(\tau) - 2$ is the Hauptmodul on $\Gamma_0(16)$. By solving $1/h(\tau) + 2 = -1/H(\tau) - 2$, we obtain that

$$h\left(\tau + \frac{1}{2}\right) = -\frac{h(\tau)}{1 + 4h(\tau)}.$$

□

We need the following lemma to have the existence of an affine plane model defined over $\mathbb{Q}$, which will be called the *modular equation*.

Lemma 2.5. *Let n be a positive integer. Then we get*

$$\mathbb{Q}(h(\tau), h(n\tau)) = A_0(\Gamma_0(16n))_{\mathbb{Q}}.$$

Proof. For any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, $h(\alpha\tau) = h(\tau)$ if and only if $\alpha \in \mathbb{Q}^\times \cdot \Gamma_0(16)$ since $\mathbb{Q}(h(\tau)) = A_0(\Gamma_0(16))_{\mathbb{Q}}$. For $\beta_n = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we get that

$$\Gamma_0(16) \cap \beta_n^{-1}\Gamma_0(16)\beta_n = \Gamma_0(16n).$$

Hence $h(\tau)$ and $h(n\tau) = h \circ \beta_n(\tau)$ belong to $A_0(\Gamma_0(16n))$.

It is sufficient to show that

$$\mathbb{Q}(h(\tau), h(n\tau)) \supset A_0(\Gamma_0(16n)).$$

We choose $M_i \in \Gamma_0(16)$ and write $\Gamma_0(16) = \bigcup_i \Gamma_0(16n) \cdot M_i$ as a disjoint union. Let $x(\tau) := h(n\tau) = (h \circ \beta_n)(\tau)$.

Suppose that we can choose distinct indices i and j such that

$$x \circ M_i = x \circ M_j. \tag{2.1}$$

Then

$$\begin{aligned} & h \circ \beta_n \circ M_i = h \circ \beta_n \circ M_j \\ \Rightarrow\ & h = h \circ \beta_n M_j M_i^{-1} \beta_n^{-1} \\ \Rightarrow\ & \beta_n M_j M_i^{-1} \beta_n^{-1} \in \mathbb{Q}^\times \cdot \Gamma_0(16) \\ \Rightarrow\ & M_j M_i^{-1} \in \beta_n^{-1} \Gamma_0(16) \beta_n. \end{aligned}$$

As $M_i, M_j \in \Gamma_0(16)$, we have $M_j M_i^{-1} \in \Gamma_0(16n)$, and it is a contradiction to (2.1). Therefore all functions $x \circ M_i$ are distinct, and $h(\tau)$ and $h(n\tau)$ generate the field $A_0(\Gamma_0(16n))_{\mathbb{Q}}$ over $\mathbb{Q}$. □

Lemma 2.6. *Let $a, c, a', c' \in \mathbb{Z}$ and $h(\tau)$ as above. Then we have the following assertions.*

(1) *$h(\tau)$ has a pole at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a,c) = 1$ if and only if $(a,c) = 1$ and $(2,c) = 1$.*
(2) *$h(n\tau)$ has a pole at $a'/c' \in \mathbb{Q} \cup \{\infty\}$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a',c') = 1$ and $(2,c) = 1$.*
(3) *$h(\tau)$ has a zero at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a,c) = 1$ if and only if $(a,c) = 1$ and $16 \mid c$.*
(4) *$h(n\tau)$ has a zero at $a'/c' \in \mathbb{Q} \cup \{\infty\}$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a',c') = 1$ and $16 \mid c$.*

Proof. We note that $h(\tau)$ has the only pole at $a/c \in \mathbb{Q} \cup \{\infty\}$ such that a/c is equivalent to 0 under $\Gamma_0(16)$. By Lemma 2.1, $(a,c) \equiv (n, \overline{s}) \pmod{16}$ for some $\overline{s} \in (\mathbb{Z}/16\mathbb{Z})^\times$ and $n \in \mathbb{Z}$. Hence $(a,c) = 1$, $(2,c) = 1$ and we have (1) and (2).

We use that $h(\tau)$ has the only zero at $a/c \in \mathbb{Q} \cup \{\infty\}$ such that a/c is equivalent to ∞ under $\Gamma_0(16)$. For this we have the pair (a,c) such that $(a,c) \equiv (\overline{s}, 0) \pmod{16}$ for some $\overline{s} \in (\mathbb{Z}/16\mathbb{Z})^\times$. In other words, $h(\tau)$ has the zero at a/c such that $(a,c) = 1$ and $c \equiv 0 \pmod{16}$. So, we get (3) and (4). □

Let d_m be the total degree of poles of $h(m\tau)$ for any positive integer m. We focus on finding a modular equation which gives the relation between $h(\tau)$ and $h(n\tau)$. Then the following lemma gives a polynomial $F_n(X,Y)$ and some information on its coefficients. In detail,

$$F_n(X,Y) = \sum_{\substack{0 \le i \le d_n \\ 0 \le j \le d_1}} C_{i,j} X^i Y^j \in \mathbb{Q}[X,Y].$$

In [14], Ishida-Ishii shows the following lemma using the theory of algebraic functions. This lemma is useful in checking which coefficient $C_{i,j}$ is zero or nonzero in $F_n(X,Y)$.

Lemma 2.7. *For any congruence subgroup Γ, let $f_1(\tau)$ and $f_2(\tau)$ be nonconstants such that $\mathbb{C}(f_1(\tau), f_2(\tau)) = A_0(\Gamma)$ with the total degree D_k of poles of $f_k(\tau)$ for $k = 1,2$. Let*

$$F(X,Y) = \sum_{\substack{0 \le i \le D_2 \\ 0 \le j \le D_1}} C_{i,j} X^i Y^j \in \mathbb{C}[X,Y]$$

be such that $F(f_1(\tau), f_2(\tau)) = 0$. Define the subsets $S_{k,0}$ and $S_{k,\infty}$ of the set S_Γ of all inequivalent cusps of Γ by

$$S_{k,0} := \{s \in S_\Gamma : f_k(\tau) \text{ has zeros at } s\}$$

and

$$S_{k,\infty} := \{s \in S_\Gamma : f_k(\tau) \text{ has poles at } s\}$$

for $k = 1,2$. Let

$$a = -\sum_{s \in S_{1,\infty} \cap S_{2,0}} ord_s f_1(\tau) \text{ and } b = \sum_{s \in S_{1,0} \cap S_{2,0}} ord_s f_1(\tau).$$

We assume that a (respectively, b) is zero if $S_{1,\infty} \cap S_{2,0}$ (respectively, $S_{1,0} \cap S_{2,0}$) is empty. Then we obtain the following assertions.

(1) *$C_{D_2,a} \neq 0$. In addition, if $S_{1,\infty} \subset S_{2,\infty} \cup S_{2,0}$, then $C_{D_2,j} = 0$ for any $j \neq a$.*
(2) *$C_{0,b} \neq 0$. In addition, if $S_{1,0} \subset S_{2,\infty} \cup S_{2,0}$, then $C_{0,j} = 0$ for any $j \neq b$.*
(3) *$C_{i,D_1} = 0$ for $0 \le i < |S_{1,0} \cap S_{2,\infty}|, D_2 - |S_{1,\infty} \cap S_{2,\infty}| < i \le D_2$.*

(4) *$C_{i,0} = 0$ for $0 \leq i < |S_{1,0} \cap S_{2,0}|, D_2 - |S_{1,\infty} \cap S_{2,0}| < i \leq D_2$.*
If we interchange the roles of $f_1(\tau)$ and $f_2(\tau)$, then we may have more properties similar to (1)-(4). Suppose that there exist $r \in \mathbb{R}$ and $N, n_1, n_2 \in \mathbb{Z}$ with $N > 0$ such that
$$f_j(\tau + r) = \zeta_N^{n_j} f_j(\tau)$$
for $j = 1, 2$, where $\zeta_N = e^{2\pi i/N}$. Then we get the following assertion:

(5) *If $n_1 i + n_2 j \not\equiv n_1 D_2 + n_2 a \pmod N$, then $C_{i,j} = 0$. Here note that $n_2 b \equiv n_1 D_2 + n_2 a \pmod N$.*

Proof. See [14, Lemmas 3 and 6]. □

The modular equation of level 2. In Lemma 2.7, let $f_1(\tau) = h(\tau)$ and $f_2(\tau) = h(2\tau)$. By Lemma 2.5, $\mathbb{Q}(f_1(\tau), f_2(\tau)) = A_0(\Gamma_0(32))$, and we may take the set $S_{\Gamma_0(32)}$ of inequivalent cusps of $\Gamma_0(32)$ as follows:
$$S_{\Gamma_0(32)} = \left\{\infty, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{1}{16}\right\}.$$
From Lemma 2.3, we have
$$\mathrm{ord}_\infty f_1(\tau) = 1, \mathrm{ord}_{1/16} f_1(\tau) = 1, \mathrm{ord}_0 f_1(\tau) = -2,$$
$$\mathrm{ord}_\infty f_2(\tau) = 2, \mathrm{ord}_0 f_2(\tau) = -1, \text{ and } \mathrm{ord}_{1/2} f_2(\tau) = -1.$$
Hence we have the polynomial
$$F_2(X, Y) = \sum_{0 \leq i,j \leq 2} C_{i,j} X^i Y^j$$
such that $F_2(f_1(\tau), f_2(\tau)) = 0$. Since there is no point s such that $\mathrm{ord}_s f_1(\tau) < 0$ and $\mathrm{ord}_s f_2(\tau) > 0$, $C_{2,0} \neq 0$, and $C_{2,1} = C_{2,2} = 0$. In a similar way, by using that both $f_1(\tau)$ and $f_2(\tau)$ have a zero at ∞ with $\mathrm{ord}_\infty f_1(\tau) = 1$, we obtain that $C_{0,1} \neq 0$. Without loss of generality, we may assume that $C_{2,0} = 1$. Replacing X and Y by q-expansions of $h(\tau)$ and $h(2\tau)$ in
$$F_2(X, Y) = X^2 + C_{1,2} XY^2 + C_{1,1} XY + C_{1,0} X + C_{0,2} Y^2 + C_{0,1} Y + C_{0,0},$$
then we can determine the remaining coefficients $C_{i,j}$ as follows:
$$C_{0,1} = -1, C_{0,2} = -2, C_{1,1} = -4, C_{1,2} = -8, C_{1,0} = 0, C_{0,0} = 0.$$
Thus, we get
$$F_2(X, Y) = X^2 - Y(1 + 4X)(1 + 2Y).$$
So the modular equation of level 2 of $h(\tau)$ is
$$\frac{h^2(\tau)}{1 + 4h(\tau)} = h(2\tau)(1 + 2h(2\tau)).$$

Theorem 2.8. *Let p be an odd prime and $F_n(X, Y)$ be the irreducible polynomial satisfying $F_n(h(\tau), h(n\tau)) = 0$. Then*
$$F_p(X, Y) = \sum_{0 \leq i,j \leq p+1} C_{i,j} X^i Y^j \in \mathbb{Q}[X, Y]$$
and

(1) *$C_{p+1,0} \neq 0$ and $C_{0,p+1} \neq 0$.*
(2) *$C_{p+1,j} = C_{j,p+1} = 0$ if $j = 1, \ldots, p+1$.*

(3) $C_{0,j} = C_{j,0} = 0$ *if* $j = 0, \dots, p$.

Proof. For an odd prime p, let $S_{\Gamma_0(16p)}$ be the set of all inequivalent cusps of $\Gamma_0(16p)$. Then by Lemma 2.1 all the elements of $S_{\Gamma_0(16p)}$ are $\infty, 0, 1/2, 1/4, 3/4, 1/8, 1/16, 1/p, 1/2p, 1/4p, 3/4p$ and $1/8p$ with widths $1, 16p, 4p, p, p, p, p, 16, 4, 1, 1$ and 1, respectively. Since $\mathbb{Q}(h(\tau), h(p\tau)) = A_0(\Gamma_0(16p))$, we have

$$\mathrm{ord}_0 h(\tau) = -p, \mathrm{ord}_{1/p} h(\tau) = -1, \mathrm{ord}_{1/16} h(\tau) = p, \mathrm{ord}_{1/16p} h(\tau) = 1$$

and

$$\mathrm{ord}_0 h(p\tau) = -1, \mathrm{ord}_{1/p} h(p\tau) = -p, \mathrm{ord}_{1/16} h(p\tau) = 1, \mathrm{ord}_{1/16p} h(p\tau) = p.$$

Thus, the modular equation $F_p(X, Y)$ is

$$F_p(X, Y) = \sum_{0 \le i,j \le p+1} C_{i,j} X^i Y^j.$$

Consider $f_1(\tau) = h(\tau)$ and $f_2(\tau) = h(p\tau)$ in Lemma 2.7. Then we can take $a = 0$ and $b = p + 1$, so we get

(1) $C_{p+1,0} \neq 0$ and $C_{p+1,1} = C_{p+1,2} = \cdots = C_{p+1,p+1} = 0$,
(2) $C_{0,p+1} \neq 0$ and $C_{0,0} = C_{0,1} = \cdots = C_{0,p} = 0$.

Similarly, if we let $f_1(\tau) = h(p\tau)$ and $f_2(\tau) = h(\tau)$, then $a = 0$ and $b = p + 1$. Moreover,

(1) $C_{0,p+1} \neq 0$ and $C_{1,p+1} = C_{2,p+1} = \cdots = C_{p+1,p+1} = 0$,
(2) $C_{p+1,0} \neq 0$ and $C_{0,0} = C_{1,0} = \cdots = C_{p,0} = 0$.

Hence we proved (1), (2) and (3). $\square$

Proof of Theorem 1.2. Let $f_1(\tau)$ and $f_2(\tau)$ be nonconstant modular functions satisfying $\mathbb{C}(f_1(\tau), f_2(\tau)) = A_0(\Gamma)$ for some congruence subgroup Γ. Then

$$[\mathbb{C}(f_1(\tau), f_2(\tau)) : \mathbb{C}(f_j(\tau))] = d_j,$$

where d_j is the total degree of poles of $f_j(\tau)$ on $\Gamma \backslash \mathfrak{H}^*$ and $j = 1, 2$. Hence there exists a polynomial $\Phi(X, Y) \in \mathbb{C}[X, Y]$ such that $\Phi(f_1(\tau), Y)$ (respectively, $\Phi(X, f_2(\tau))$) is a minimal polynomial of $f_2(\tau)$ (respectively, $f_1(\tau)$) over $\mathbb{C}(f_1(\tau))$ (respectively, $\mathbb{C}(f_2(\tau))$) with degree d_1 (respectively, d_2). Let $f_1(\tau) = h(\tau)$ and $f_2(\tau) = h(n\tau)$. Then by using Lemma 2.5, we may consider a polynomial $F_n(X, Y) \in \mathbb{Q}[X, Y]$ such that $F_n(h(\tau), h(n\tau)) = 0$ for any positive integer n. Here, $\deg_X F_n(X, Y) = d_2$ and $\deg_Y F_n(X, Y) = d_1$. This polynomial $F_n(X, Y)$ is the modular equation of $h(\tau)$ of level n for every positive integer n. $\square$

Using Theorem 2.8, we get the modular equations of level p which are presented in Table 1 when $p = 2, 3, 5, 7$ and 11.

TABLE 1. The modular equations $F_p(X,Y)$ of $h(\tau)$ of levels $2, 3, 5, 7$ and 11

p	The modular equation $F_p(X,Y)$ of $h(\tau)$ of level p
2	$X^2 - Y(1+4X)(1+2Y)$
3	$(X^3-Y)(X-Y^3)$ $-3\,XY\left(2\,Y+4\,Y^2+2\,X+10\,XY+16\,XY^2+4\,X^2+16\,X^2Y+21\,X^2Y^2\right)$
5	$(X^5-Y)(X-Y^5)$ $-5\,XY\,(\,16\,Y^3+8\,X^2+2\,Y+8\,Y^2+14\,Y^4+2\,X+20\,XY+80\,XY^2+157\,XY^3+128\,XY^4$ $+80\,X^2Y+324\,X^2Y^2+640\,X^2Y^3+512\,X^2Y^4+16\,X^3+157\,X^3Y+640\,X^3Y^2+1280\,X^3Y^3$ $+1024\,X^3Y^4+14\,X^4+128\,X^4Y+512\,X^4Y^2+1024\,X^4Y^3+819\,X^4Y^4\,)$
7	$(X^7-Y)(X-Y^7)$ $-7\,XY\,(\,2\,X+2\,Y+12\,Y^2+40\,Y^3+84\,Y^5+40\,Y^6+78\,Y^4+12\,X^2+40\,X^3+78\,X^4$ $+84\,X^5+40\,X^6+92160\,X^5Y^4+672\,XY^6+180\,X^2Y+20480\,X^6Y^3+29\,XY+180\,XY^2$ $+620\,XY^3+1252\,XY^4+1396\,XY^5+1156\,X^2Y^2+4120\,X^2Y^3+8632\,X^2Y^4+10016\,X^2Y^5$ $+4992\,X^2Y^6+620\,X^3Y+4120\,X^3Y^2+15190\,X^3Y^3+32960\,X^3Y^4+39680\,X^3Y^5+20480\,X^3Y^6$ $+1252\,X^4Y+8632\,X^4Y^2+32960\,X^4Y^3+73984\,X^4Y^4+92160\,X^4Y^5+49152\,X^4Y^6$ $+1396\,X^5Y+10016\,X^5Y^2+39680\,X^5Y^3+118784\,X^5Y^5+65536\,X^5Y^6+672\,X^6Y+4992\,X^6Y^2$ $+49152\,X^6Y^4+65536\,X^6Y^5+37449\,X^6Y^6\,)$
11	$(X^{11}-Y)(X-Y^{11})$ $-11\,XY\,(\,2\,Y+2\,X+20\,Y^2+120\,Y^3+478\,Y^4+1316\,Y^5+2520\,Y^6+3280\,Y^7$ $+2729\,Y^8+1270\,Y^9+236\,Y^{10}+20\,X^2+120\,X^3+478\,X^4+1316\,X^5+2520\,X^6$ $+3280\,X^7+2729\,X^8+1270\,X^9+236\,X^{10}+407502848\,X^6Y^9+480\,X^2Y+6242560\,X^9Y^3$ $+10545728\,X^8Y^3+1243548\,X^2Y^8+10617267\,X^3Y^7+36489216\,X^4Y^9+335544320\,X^{10}Y^8$ $+43122688\,X^5Y^{10}+1006632960\,X^9Y^8+1006632960\,X^8Y^9+206867200\,X^7Y^5+3000\,X^3Y$ $+771751936\,X^9Y^9+945184768\,X^7Y^7+786432000\,X^9Y^7+86758\,X^8Y+126141088\,X^6Y^5$ $+230758\,X^3Y^3+531390464\,X^6Y^7+916956\,X^6Y^2+531390464\,X^7Y^6+251658240\,X^7Y^{10}$ $+694064\,X^9Y^2+10321920\,X^{10}Y^4+268435456\,X^9Y^{10}+71268\,X^6Y+35647\,X^5Y$ $+97612893\,X^{10}Y^{10}+46\,XY+480\,XY^2+3000\,XY^3+12436\,XY^4+35647\,XY^5+71268\,XY^6$ $+97540\,XY^7+86758\,XY^8+44862\,XY^9+10160\,XY^{10}+5223\,X^2Y^2+34014\,X^2Y^3$ $+146872\,X^2Y^4+438876\,X^2Y^5+916956\,X^2Y^6+1318216\,X^2Y^7+694064\,X^2Y^9+174656\,X^2Y^{10}$ $+34014\,X^3Y^2+1037872\,X^3Y^4+3232300\,X^3Y^5+7050856\,X^3Y^6+10545728\,X^3Y^8$ $+6242560\,X^3Y^9+1679360\,X^3Y^{10}+12436\,X^4Y+146872\,X^4Y^2+1037872\,X^4Y^3$ $+4860966\,X^4Y^4+15767636\,X^4Y^5+35860024\,X^4Y^6+56406848\,X^4Y^7+58685184\,X^4Y^8$ $+10321920\,X^4Y^{10}+438876\,X^5Y^2+3232300\,X^5Y^3+15767636\,X^5Y^4+53252516\,X^5Y^5$ $+126141088\,X^5Y^6+206867200\,X^5Y^7+224704512\,X^5Y^8+146010112\,X^5Y^9+7050856\,X^6Y^3$ $+35860024\,X^6Y^4+311101824\,X^6Y^6+601587712\,X^6Y^8+125304832\,X^6Y^{10}+97540\,X^7Y$ $+1318216\,X^7Y^2+10617267\,X^7Y^3+56406848\,X^7Y^4+1114570752\,X^7Y^8+786432000\,X^7Y^9$ $+1243548\,X^8Y^2+58685184\,X^8Y^4+224704512\,X^8Y^5+601587712\,X^8Y^6+1114570752\,X^8Y^7$ $+1369178112\,X^8Y^8+335544320\,X^8Y^{10}+44862\,X^9Y+36489216\,X^9Y^4+146010112\,X^9Y^5$ $+407502848\,X^9Y^6+10160\,X^{10}Y+174656\,X^{10}Y^2+1679360\,X^{10}Y^3+43122688\,X^{10}Y^5$ $+125304832\,X^{10}Y^6+251658240\,X^{10}Y^7+268435456\,X^{10}Y^9\,)$

All modular equations $F_p(X,Y)$ in Table 1 except the case $p=2$ satisfies the congruence relation

$$F_p(X,Y) \equiv (X^p - Y)(X - Y^p) \pmod{p}$$

for $p = 3, 5, 7$ and 11, which is called the *Kronecker's congruence*. Including this property, we discuss the modular equations of $h(\tau)$.

Let $\Gamma = \Gamma_0(16)$. For any integer a with $(a,2)=1$, we choose $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \pmod{16}$ and $\sigma_a \in \Gamma_0(16)$. Under this condition, we may choose σ_a as

$$\sigma_{\pm 1} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_{\pm 3} = \pm\begin{pmatrix} -5 & -1 \\ 16 & 3 \end{pmatrix}, \sigma_{\pm 5} = \pm\begin{pmatrix} -3 & -1 \\ 16 & 5 \end{pmatrix}, \sigma_{\pm 7} = \pm\begin{pmatrix} 7 & 3 \\ 16 & 7 \end{pmatrix}.$$

For any integer with $(n,2)=1$, by [22, Proposition 3.36] we have

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma = \bigcup_{0<a|n} \bigcup_{0 \le b < n/a} \Gamma \sigma_a \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}$$

which is a disjoint union and $|\Gamma \backslash \Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) \Gamma| = n \prod_{p|n}(1+1/p)$. Let $\alpha_{a,b} = \sigma_a \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}$. Consider the polynomial

$$\Phi_n(X,\tau) := \prod_{0<a|n} \prod_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} (X - (h \circ \alpha_{a,b})(\tau))$$

of degree $n\prod_{p|n}(1+1/p)$. Note that the coefficients of $\Phi_n(X,\tau)$ are elementary symmetric functions of $f \circ \alpha_{a,b}$. Hence these are invariant under Γ, so they belong to $A_0(\Gamma) = \mathbb{C}(h(\tau))$. Thus, $\Phi_n(X,\tau) \in \mathbb{C}(h(\tau))[X]$ and we may write $\Phi_n(X, h(\tau))$ instead of $\Phi_n(X,\tau)$. Since $\alpha_{1,0} = \sigma_1 \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$ and $(h \circ \alpha_{1,0})(\tau) = h(\tau/n)$, we get $\Phi_n(h(\tau/n), h(\tau)) = 0$ and $\Phi_n(h(\tau), h(n\tau)) = 0$.

Let $S_{m,\infty}$ (respectively, $S_{m,0}$) be the set of cusps which are poles (respectively, zeros) of $h(m\tau)$. In Lemma 2.7, we write a as

$$a = -\sum_{s \in S_{1,\infty} \cap S_{n,0}} \mathrm{ord}_s h(\tau).$$

Multiplying $\Phi_n(X, h(\tau))$ by a suitable power of $h(\tau)$, we get a polynomial in $\mathbb{C}[X, h(\tau)]$. However, Lemma 2.6 shows that $S_{1,\infty} \cap S_{n,0} = \phi$ for any positive odd integer n and $a = 0$. Thus, regarding $\Phi_n(X, h(\tau))$ as a polynomial of X and $h(\tau)$, we prove the following theorem.

Theorem 2.9. *With the notation as above, let $\Phi_n(X,Y)$ be a polynomial such that $\Phi_n(h(\tau), h(n\tau)) = 0$ for a positive odd integer n. Then we obtain the following assertions:*

1. $\Phi_n(X,Y) \in \mathbb{Z}[X,Y]$ *and* $\deg_X \Phi_n(X,Y) = \deg_Y \Phi_n(X,Y) = n\prod_n(1+1/p)$.
2. $\Phi_n(X,Y)$ *is irreducible both as a polynomial in X over $\mathbb{C}(Y)$ and as a polynomial in Y over $\mathbb{C}(X)$.*
3. $\Phi_n(X,Y) = \Phi_n(Y,X)$.
4. *If n is not a square, then $\Phi_n(X,Y)$ is a polynomial of degree > 1 whose leading coefficient is ± 1.*
5. *(Kronecker's congruence) Let p be an odd prime. Then*

$$\Phi_p(X,Y) \equiv (X^p - Y)(X - Y^p) \pmod{p\mathbb{Z}[X,Y]}.$$

Proof. Let $\Gamma = \Gamma_0(16)$. Since $h(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{-2}(1-q^{2n})(1-q^{8n})^{-1}(1-q^{16n})^2$, we write

$$h(\tau) = \sum_{m=1}^{\infty} c_m q^m,$$

where $c_m \in \mathbb{Z}$.

We first prove (1) and (2). Let ψ_k be an automorphism of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ such that $\psi_k(\zeta_n) = \zeta_n^k$ for k relatively prime to n. By observing the action of $\left(\begin{smallmatrix} a & b \\ 0 & n/a \end{smallmatrix}\right)$ on h, we see that

$$\begin{aligned}\left(h\circ\begin{pmatrix} a & b \\ 0 & n/a\end{pmatrix}\right)(\tau) &= h\left(\frac{a\tau+b}{n/a}\right) = h\left(\frac{a^2\tau+ab}{n}\right)\\ &= \sum_{m=1}^{\infty} c_m \zeta_n^{abm} q^{a^2m/n},\end{aligned}$$

it is natural that ψ_k induces an automorphism ψ_k of $\mathbb{Q}(\zeta_n)((q^{1/n}))$ over $\mathbb{Q}(\zeta_n)$ as

$$\psi_k\left(h\circ\begin{pmatrix} a & b \\ 0 & n/a\end{pmatrix}(\tau)\right) = \sum_{m=1}^{\infty} c_m \zeta_n^{abkm} q^{a^2m/n}.$$

Let b' be an integer with $0 \le b' < a$ and $b' \equiv bk \pmod{n/a}$. Then $ab' \equiv abk \pmod{n}$ and

$$\begin{aligned}\psi_k(h\circ\alpha_{a,b}) &= \psi_k\left(h\circ\sigma_a\begin{pmatrix} a & b \\ 0 & n/a\end{pmatrix}\right) = \psi_k\left(h\circ\begin{pmatrix} a & b \\ 0 & n/a\end{pmatrix}\right)\\ &= h\circ\begin{pmatrix} a & b' \\ 0 & n/a\end{pmatrix} = h\circ\sigma_a\begin{pmatrix} a & b' \\ 0 & n/a\end{pmatrix} = h\circ\alpha_{a,b'}.\end{aligned}$$

So, $\psi_k(\Phi_n(X, h(\tau))) = \Phi_n(X, h(\tau))$ and $\Phi_n(X, h(\tau)) \in \mathbb{Q}((q^{1/n}))[X]$. Moreover, we already check that $\Phi_n(h(\tau/n), h(\tau)) = 0$ and $[\mathbb{C}(h(\tau/n), h(\tau)) : \mathbb{C}(h(\tau))] \le d$, where $d = n\prod_{p|n}(1+1/p)$.

For fixed a and b, since $\Gamma\alpha_{a,b} \subset \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & n\end{smallmatrix}\right)\Gamma$, we can choose $\gamma, \gamma', \gamma_{a,b} \in \Gamma$ satisfying $\gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & n\end{smallmatrix}\right)\gamma_{a,b} = \gamma'\alpha_{a,b}$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & n\end{smallmatrix}\right)\gamma_{a,b}\alpha_{a,b}^{-1} \in \Gamma = \Gamma_0(16)$.

Let $\xi_{a,b}$ be an embedding of $\mathbb{C}(h(\tau/n), h(\tau))$ to the field of all meromorphic function on $\mathfrak{H}$ containing $\mathbb{C}(h(\tau/n), h(\tau))$ over $\mathbb{C}(h(\tau))$ defined as $\xi_{a,b}(f) = f\circ\gamma_{a,b}$. Then

$$\xi_{a,b}(h) = h,$$

and

$$\xi_{a,b}(h(\tau/n)) = \xi_{a,b}\left(h\circ\begin{pmatrix} 1 & 0 \\ 0 & n\end{pmatrix}(\tau)\right) = h\circ\begin{pmatrix} 1 & 0 \\ 0 & n\end{pmatrix}\gamma_{a,b}(\tau) = h\circ\alpha_{a,b}(\tau).$$

It means that $h\circ\alpha_{a,b} \ne h\circ\alpha_{a',b'}$ for $(a,b) \ne (a',b')$ and there exist distinct d embeddings $\xi_{a,b}$ of $\mathbb{C}(h(\tau/n), h(\tau))$ over $\mathbb{C}(h(\tau))$. Hence we have

$$\left[\mathbb{C}\left(h\left(\frac{\tau}{n}\right), h(\tau)\right) : \mathbb{C}(h(\tau))\right] = d = n\prod_{p|n}\left(1+\frac{1}{p}\right)$$

and $\Phi_n(X, h(\tau))$ is irreducible over $\mathbb{C}(h(\tau))$.

Let $F(X,Y)$ be the polynomial in Lemma 2.7. When $f_1(\tau) = h(\tau)$ and $f_2(\tau) = h(n\tau)$, let $a = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s h(\tau)$. Then

$$F(X,Y) = C_{d_n,a}X^n + \sum_{\substack{0\le i\le d_n\\ 0\le j\le d_1}} C_{i,j}X^iY^j,$$

where d_1 (respectively, d_n) is the total degree of poles of $h(\tau)$ (respectively, $h(n\tau)$). Since $F(X,h(\tau))$ is the minimal polynomial of $h(\tau/n)$ over $\mathbb{C}(h(\tau))$ and $F(h(\tau/n),Y)$ is the minimal polynomial of $h(\tau)$ over $\mathbb{C}(h(\tau/n))$, we have

$$h(\tau)^a\Phi_n(X,h(\tau)) = C_{d_n,a}^{-1}F(X,h(\tau)).$$

By using Lemma 2.6, we have $a = 0$ and $F(X,Y) \in \mathbb{Z}[X,Y]$. Hence $\Phi_n(X,Y) \in \mathbb{Z}[X,Y]$, so (1) and (2) are proved.

(3) Since $(h\circ\alpha_{n,0})(\tau) = h(n\tau)$, $\Phi_n(h(n\tau),h(\tau)) = 0$. Thus, we have that $\Phi_n(h(\tau),h(\tau/n)) = 0$ and $h(\tau/n)$ is a root of $\Phi_n(h(\tau),X) = 0$. By using that $\Phi_n(X,h(\tau)) \in \mathbb{Z}[X,h(\tau)]$ and $\Phi_n(X,h(\tau))$ is irreducible, we can take a polynomial $G(X,h(\tau))$ such that

$$\Phi_n(h(\tau),X) = G(X,h(\tau))\Phi_n(X,h(\tau)). \tag{2.2}$$

If we change the places of variables in (2.2) and multiplying it by $G(X,h(\tau))$, then

$$G(X,h(\tau)) \times [G(h(\tau),X)\Phi_n(h(\tau),X)] = G(X,h(\tau)) \times \Phi_n(X,h(\tau)) = \Phi_n(h(\tau),X).$$

So, $G(X,Y) = 1$ or -1. Suppose that $G(X,Y) = -1$. Then (2.2) is written as $\Phi_n(h(\tau),X) = -\Phi_n(X,h(\tau))$. If we substitute $X = h(\tau)$, then $\Phi_n(h(\tau),h(\tau)) = -\Phi_n(h(\tau),h(\tau))$ and $\Phi_n(h(\tau),h(\tau)) = 0$. So, $h(\tau)$ is a root of $\Phi_n(X,h(\tau))$ and $X - h(\tau)$ is a factor of the irreducible polynomial $\Phi_n(X,h(\tau))$ with degree > 1. It is a contradiction, thus we have $G(X,Y) = 1$. Therefore, $\Phi_n(h(\tau),X) = \Phi_n(X,h(\tau))$.

(4) Note that

$$h(\tau) - (h\circ\alpha_{a,b})(\tau) = q - \zeta_n^{ab}q^{a^2/n} + c_2(q^2 - \zeta_n^{2ab}q^{2a^2/n}) + \cdots. \tag{2.3}$$

Assume that n is not a square. Then the coefficient of the lowest degree term of $\Phi_n(h(\tau),h(\tau))$ is the product of 1 or $-\zeta_n^{ab}$ from (2.3); so it is a unit.

(5) Let p be an odd prime. We denote

$$f(\tau) \equiv g(\tau) \pmod{\alpha}$$

when $f(\tau) - g(\tau) \in \alpha\mathbb{Z}[\zeta_p]((q^{\frac{1}{p}}))$ for $f(\tau), g(\tau) \in \mathbb{Z}[\zeta_p]((q^{\frac{1}{p}}))$.

For $h(\tau) = \sum_{m=1}^{\infty} c_mq^m$ ($c_m \in \mathbb{Z}$), we have

$$\begin{aligned}(h\circ\alpha_{1,b})(\tau) &= \sum_{m=1}^{\infty} c_m\zeta_p^{bm}q^{\frac{m}{p}} \equiv \sum_{m=1}^{\infty} c_mq^{\frac{m}{p}} \pmod{1-\zeta_p}\\ &= (h\circ\alpha_{1,0})(\tau),\end{aligned}$$

$$(h\circ\alpha_{p,0})(\tau) = \sum_{m=1}^{\infty} c_mq^{pm} \equiv \sum_{m=1}^{\infty} c_m^pq^{pm} \equiv h(\tau)^p \pmod{p}$$

and

$$\begin{aligned}(h\circ\alpha_{1,0})(\tau)^p &= \left(\sum_{m=1}^{\infty} c_m q^{\frac{m}{p}}\right)^p \equiv \sum_{m=1}^{\infty} c_m^p q^m \equiv \sum_{m=1}^{\infty} c_m q^m \pmod{1-\zeta_p}\\ &= h(\tau).\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\Phi_p(X,h(\tau)) &= \left[\prod_{0\le b<p}(X-(h\circ\alpha_{1,b})(\tau))\right](X-(h\circ\alpha_{p,0})(\tau))\\ &\equiv (X-(h\circ\alpha_{1,0})(\tau))^p(X-h(\tau)^p) \pmod{1-\zeta_p}\\ &\equiv (X^p-(h\circ\alpha_{1,0})(\tau)^p)(X-h(\tau)^p) \pmod{1-\zeta_p}\\ &\equiv (X^p-h(\tau))(X-h(\tau)^p) \pmod{1-\zeta_p}.\end{aligned}$$

In other words,

$$\Phi_n(X,h(\tau))-(X^p-h(\tau))(X-h(\tau)^p) \tag{2.4}$$

is contained in $(1-\zeta_p)\mathbb{Z}[X,h(\tau)]$. Since $\Phi_n(X,Y)\in\mathbb{Z}[X,Y]$, all coefficients of (2.4) are integers divisible by $1-\zeta_p$; so they are divisible by p. Thus, (5) is proved. □

3. The ray class field and evaluation of $h(\tau)$

Let K be an imaginary quadratic field and d_K be the discriminant of K. For a positive integer N, let $K_{(N)}$ be the ray class field modulo N over K. In this section we first prove that $h(\tau/4)$ generates $K_{(4)}$ over K for given $\tau\in K\cap\mathfrak{H}$ satisfying certain conditions. We then investigate the value $h(\tau)$ as an algebraic number.

Lemma 3.1. *Let K be an imaginary quadratic field with discriminant d_K and $\tau\in K\cap\mathfrak{H}$ be a root of the primitive equation $ax^2+bx+c=0$ such that $b^2-4ac=d_K$, and let Γ' be any congruence subgroup such that $\Gamma(N)\subset\Gamma'\subset\Gamma_1(N)$. Suppose that $(N,a)=1$. Then the field generated over K by all the values $h(\tau)$, where $h\in A_0(\Gamma')_{\mathbb{Q}}$ is defined and finite at τ, is the ray class field modulo N over K.*

Proof. See [4, Corollary 5.2]. □

In the proof of Theorem 1.3, we use the modular function $h(\tau/4)$ which is a generator of the field of modular functions on $\Gamma(4)$.

Proof of Theorem 1.3. Let $f(\tau):=h(\tau/4)$. Then by Theorem 1.1, $\mathbb{C}(f(\tau))=A_0(\Gamma(4))$. For given $\tau\in K\cap\mathfrak{H}$ satisfying $16a\tau^2+4b\tau+c=0$, where $b^2-4ac=d_K$, $(a,2)=1$ and $a,b,c\in\mathbb{Z}$, let $\tau_0=4\tau$. Then $\tau_0\in K\cap\mathfrak{H}$ and τ_0 satisfies $a\tau_0^2+b\tau_0+c=0$, where $b^2-4ac=d_K$, $(a,2)=1$ and $a,b,c\in\mathbb{Z}$. By Lemma 3.1, $K(f(\tau_0))$ is the ray class field modulo 4 over K and $K(f(\tau_0))=K(h(\tau_0/4))=K(h(\tau))$. Hence $K(h(\tau))$ is the ray class field modulo 4 over K for τ satisfying that $16a\tau^2+4b\tau+c=0$, $b^2-4ac=d_K$, $(a,2)=1$ and $a,b,c\in\mathbb{Z}$. □

Proof of Corollary 1.4. When $\mathbb{Z}[4\tau]$ is the integral closure of $\mathbb{Z}$ in K, there exist $b,c\in\mathbb{Z}$ such that $16\tau^2+4b\tau+c=0$ and $b^2-4c=d_K$. By Theorem 1.3, $K(h(\tau))$ is the ray class field modulo 4 over K. □

Lemma 3.2. *The Hauptmodul of* $A_0(\Gamma_1(4))$ *is* $(1/h(\tau/4)+2)^4-8$.

Proof. Let $H(\tau)=\eta^{-8}(\tau)\eta^{24}(2\tau)\eta^{-16}(4\tau)$. By Lemmas 2.2 and 2.3, $H(\tau)$ generates the field of modular functions on $\Gamma_0(4)$. Since $\Gamma_0(4)=\pm\Gamma_1(4)$, we have $\mathbb{C}(H(\tau))=A_0(\Gamma_1(4))$.

Consider $H(4\tau)$ as a modular function on $\Gamma_0(16)$. Then by Theorem 1.1, the field $\mathbb{C}(H(4\tau),h(\tau))=A_0(\Gamma_0(16))$. By checking the behavior of $H(4\tau)$ at the cusps using Lemma 2.3, we get that

$$\mathrm{ord}_\infty H(4\tau)=-4 \text{ and } \mathrm{ord}_{1/8}H(4\tau)=4.$$

Lemma 2.7 gives us the polynomial

$$F(X,Y)=\sum_{\substack{0\le i\le 1\\ 0\le j\le 4}} C_{i,j}X^iY^j, \tag{3.1}$$

where $F(H(4\tau),h(\tau))=0$ and $C_{i,j}\in\mathbb{C}$. Hence we may assume that $C_{1,4}=1$ from Lemma 2.7 (1). When we substitute the q-expansions of $H(4\tau)$ and $h(\tau)$ for X and Y, respectively, in (3.1); we obtain

$$F(X,Y)=-1-8Y-24Y^2-32Y^3-16Y^4+XY^4$$

and

$$H(4\tau)=\left(2+\frac{1}{h(\tau)}\right)^4.$$

Since the q-expansion of $H(\tau)$ is $q^{-1}+8+20q+O(q^2)$,

$$H(\tau)-8=\left(2+\frac{1}{h(\tau/4)}\right)^4-8$$

is the Hauptmodul on $\Gamma_1(4)$. □

Proposition 3.3. *Let K be an imaginary quadratic field with discriminant d_K and $t=\mathcal{N}(j_{1,N})$ be the Hauptmodul of $A_0(\Gamma_1(N))$. Let s be a cusp of $\Gamma_1(N)$ whose width is h_s and $S_{\Gamma_1(N)}$ be the set of inequivalent cusps of $\Gamma_1(N)\backslash\mathfrak{H}^*$. If $t\in q^{-1}\mathbb{Z}[[q]]$ and $\prod_{s\in S_{\Gamma_1(n)}-\{\infty\}}(t(z)-t(s))^{h_s}$ is a polynomial in $\mathbb{Z}[t]$, then $t(\tau)$ is an algebraic integer for $\tau\in K\cap\mathfrak{H}$.*

Proof. See [15, Theorem 5]. □

Proof of Theorem 1.5. As in the proof of Lemma 3.2 , let $H(\tau)=\eta^{-8}(\tau)\eta^{24}(2\tau)\eta^{-16}(4\tau)$. We already know that

$$\mathrm{ord}_0 H(\tau)=0 \text{ and } \mathrm{ord}_{1/2}H(\tau)=1.$$

In detail, we have

$$\begin{aligned} H(0) &= \lim_{\tau\to 0} H(\tau)=\lim_{\tau\to\infty} H|\begin{pmatrix}0&-1\\1&0\end{pmatrix}(\tau)=\lim_{\tau\to\infty} H\left(-\frac{1}{\tau}\right)\\ &= \lim_{\tau\to\infty}\frac{\eta^{24}\left(-\frac{1}{\tau/2}\right)}{\eta^8\left(-\frac{1}{\tau}\right)\eta^{16}\left(-\frac{1}{\tau/4}\right)}=\lim_{\tau\to\infty}16\cdot\prod_{n=1}^{\infty}\frac{(1-q^{\frac{n}{2}})^{24}}{(1-q^n)^8(1-q^{\frac{n}{4}})^{16}}=16. \end{aligned}$$

Assume that $t(\tau)$ is the Hauptmodul on $\Gamma_1(4)$. Then $t(\tau)=H(\tau)-8=(1/h(\tau/4)+2)^4-8$. The polynomial defined in Proposition 3.3 is

$$\prod_{s\in\{0,1/2\}}(t-t(s))^{h_s}=(t-(H(0)-8))^4(t-(H(1/2)-8))=(t-8)^4(t+8)\in\mathbb{Z}[t];$$

this is because $h_0 = 4$ and $h_{1/2} = 1$. Hence for $\tau \in K \cap \mathfrak{H}$, $t(\tau)$ and $(1/h(\tau/4)+2)^4$ are algebraic integers in a suitable number field. Therefore the result follows. □

Hereafter, to evaluate $h(\tau)$ we briefly present an algorithm by using Shimura's reciprocity law. For more details, one can refer to [4, 12], which explain the action of Galois group $\mathrm{Gal}(K_{(N)}/K)$ to find its class polynomial.

Let $\mathfrak{F}_N$ be the field of automorphic functions of level N whose Fourier coefficients with respect to $e^{2\pi i\tau/N}$ belong to $\mathbb{Q}(\zeta_N)$ and $\mathfrak{F} = \cup_{N=1}^{\infty}\mathfrak{F}_N$. For an imaginary quadratic field K with discriminant d_K, let

$$\theta = \begin{cases} \frac{\sqrt{d_K}}{2} & \text{if } d_K \equiv 0 \pmod 4, \\ \frac{-1+\sqrt{d_K}}{2} & \text{if } d_K \equiv 1 \pmod 4. \end{cases}$$

Then we may write $X^2 + BX + C = 0$ as the primitive equation of θ with $B, C \in \mathbb{Z}$. Define a group

$$W_{N,\theta} := \left\{ \begin{pmatrix} t-Bs & -Cs \\ s & t \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : t, s \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Then, over the Hilbert class field H of K there is a surjective homomorphism

$$\begin{array}{rcl} W_{N,\theta} & \to & \mathrm{Gal}(K_{(N)}/H) \\ \alpha^{-1} & \mapsto & (f(\theta) \mapsto f^{\alpha}(\theta)) \end{array}$$

for $f \in \mathfrak{F}_N$. Note that the kernel T of this homomorphism is

$$T := \begin{cases} \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & \text{otherwise.} \end{cases}$$

Let $Q = [a, b, c]$ be a primitive positive definite quadratic form of discriminant d_K and $\tau_Q = (-b + \sqrt{d_K})/2a \in \mathfrak{H}$. Then we define $u_Q = (u_p)_p \in \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ as follows:

- $d_K \equiv 0 \pmod 4$

$$u_p = \begin{cases} \begin{pmatrix} a & b/2 \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a, \\ \begin{pmatrix} -b/2 & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{pmatrix} -a-b/2 & -c-b/2 \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c, \end{cases}$$

- $d_K \equiv 1 \pmod 4$

$$u_p = \begin{cases} \begin{pmatrix} a & (-1+b)/2 \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a, \\ \begin{pmatrix} (-1-b)/2 & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{pmatrix} -a+(-1-b)/2 & -c-(-1+b)/2 \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c. \end{cases}$$

Now we have an isomorphism between the *form class group* $C(d_K)$ of discriminant d_K and the Galois group $\mathrm{Gal}(H/K)$ defined as

$$\begin{array}{rcl} C(d_K) & \to & \mathrm{Gal}(H/K) \\ Q^{-1} & \mapsto & (f(\theta) \mapsto f^{u_Q}(\tau_Q)) \end{array}$$

for any $f \in \mathfrak{F}$ satisfying $f(\theta) \in H$. From that $\mathrm{Gal}(K_{(N)}/K)/\mathrm{Gal}(K_{(N)}/H) \cong \mathrm{Gal}(H/K)$, we have the following conclusion.

Lemma 3.4. *With the notation as above, we deduce that for $f \in \mathfrak{F}_N$,*

$$\{f^{\alpha \cdot u_Q}(\tau_Q) : \alpha \in W_{N,\theta}/T, Q \in C(d_K)\}$$

is the set of all conjugates of $f(\theta)$ over K.

In this situation, for $f \in \mathfrak{F}_N$, the polynomial

$$F_N(X) := \prod_{\substack{\alpha \in W_{N,\theta}/T \\ Q \in C(d_K)}} (X - f^{\alpha \cdot u_Q}(\tau_Q)) \in K[X] \tag{3.2}$$

is the minimal polynomial of $f(\theta)$ over K. Let $f(\tau) = 1/h(\tau/4)$. Assume that K is an imaginary quadratic field with $d_K \equiv 0 \pmod 4$. Then $f(\theta) \in \mathbb{R}$ since $\theta = \sqrt{d_K}/2$ and $e^{2\pi i\theta/4} \in \mathbb{R}$. By applying $f(\tau)$ to the equation $F_4(X)$ defined in (3.2), we know that

$$0 = F_4(f(\theta)) = \overline{F_4(f(\theta))} = \overline{F_4}(\overline{f(\theta)}) = \overline{F}_4(f(\theta))$$

and $F_4(X) \in (K \cap \mathbb{R})[X] = \mathbb{Q}[X]$. Note that for $\tau \in K \cap \mathfrak{H}$, $f(\tau)$ is an algebraic integer by Theorem 1.5. Hence we may assume that $F_4(X) \in \mathbb{Z}[X]$, and it makes us easy to determine the minimal polynomial by approximation. The following is one example for finding $F_4(X)$.

Example 3.5. Let $K = \mathbb{Q}(\sqrt{-1})$ be an imaginary quadratic field, $K_{(4)}$ be the ray class field of K modulo 4 and $F_4(X)$ be the class polynomial with $F_4(1/h(\sqrt{-1}/4)) = 0$. Then $F_4(X) = X^2 - 8$.

Solution. For $K = \mathbb{Q}(\sqrt{-1})$, we have $d_K = -4$. We can take a positive definite quadratic form Q with discriminant -4 as $Q = [1, 0, 1]$. Let $\theta = \tau_Q = \sqrt{-1}$ and $u_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for a prime p. Since $\theta^2 + 1 = 0$, let $B = 0$ and $C = 1$. Then

$$W_{4,\sqrt{-1}} = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \pm \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \right\}$$

and

$$W_{4,\sqrt{-1}}/T = \left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right] \right\}.$$

Thus, the polynomial $F_4(X)$ is

$$\begin{aligned} & F_4(X) = (X - f(i)) \left(X - f\left(\frac{3+2i}{2+i} \right) \right) \\ \approx \quad & X^2 - 0.000000002\, X + 4.678495626 \cdot 10^{-12}\, iX - 7.999999996 - 1.323278393 \cdot 10^{-11}\, i \\ \approx \quad & X^2 - 8, \end{aligned}$$

where $f(\tau) = 1/h(\tau/4)$. $\square$

Proof of Theorem 1.6. Let $r \in \mathbb{Q}_{>0}$. First we explain the method for getting the value $h(r\tau)$ if the value $h(\tau)$ is given in terms of radicals. One can use the algorithm in [17, Algorithm 1.6]. Suppose that the value $h(\tau)$ is expressed in terms of radicals. We can write $h(r\tau)$ in terms of radicals by using the algorithm [17, Algorithm 1.6].

Assume that K is an imaginary quadratic field with discriminant $d_K \equiv 0 \pmod 4$. Let $\theta = \sqrt{d_K}/2$. For all $\tau \in K \cap \mathfrak{H}$, τ can be written as $\tau = (b + c\theta)/a$, where $a, b, c \in \mathbb{Z}$ and $a, c > 0$. By the algorithm using the Shimura's reciprocity law, we can find the value $h(\theta/4)$ in terms of radicals by the minimal polynomial of the ray class field $K_{(4)}$ modulo 4 over K. From the modular equation $F_4(X, Y)$ we obtain the value $h(\theta)$, and then by $F_c(X, Y)$, $h(c\theta) = h(b + c\theta)$ is obtained. At last, by using $F_a(X, Y)$, we write the value $h(\tau) = h((b + c\theta)/a)$ in terms of radicals.

If $n = 1$ or n is a square free positive integer with $n \equiv 3 \mod 4$, then the field $K = \mathbb{Q}(\sqrt{-n})$ has the discriminant $d_K \equiv 0 \pmod 4$. Hence, all $r\sqrt{-n} \in K$ and we get the value $h(r\sqrt{-n})$ in terms of radicals, immediately. □

Example 3.6. We can find the value of $h(\tau)$ as follows:

(1) $h(i/4) = 1/2\sqrt{2}$ and $h((8 + \sqrt{-1})/20) = -1/2\sqrt{2}$,

(2) $h(\frac{i}{2}) = -\frac{1}{4}\left(1 + \sqrt{2} - \sqrt{4 + 3\sqrt{2}}\right)\left(\sqrt{2} - 1\right)$.

Solution. (1) In Example 3.5, we can conclude that $\left\{h(\sqrt{-1}/4), h((8 + \sqrt{-1})/20)\right\} = \left\{\pm 1/2\sqrt{2}\right\}$. Since

$$h(\sqrt{-1}/4) \approx 0.3535533904$$

and

$$h((8 + \sqrt{-1})/20) \approx -0.3535533909 + 5.848119541 \cdot 10^{-13}\, i,$$

we get that $h(i/4) = 1/2\sqrt{2}$ and $h((8 + \sqrt{-1})/20) = -1/2\sqrt{2}$.

(2) We use the modular equation of level 2:

$$F_2(X, Y) = X^2 - Y(1 + 4X)(1 + 2Y).$$

Since $F_2(h(i/4), h(i/2)) = 0$, by solving the equation $F_2(1/2\sqrt{2}, x) = 0$, we get two zeros:

$$-\frac{1}{4}\left(1 + \sqrt{2} + \sqrt{4 + 3\sqrt{2}}\right)\left(\sqrt{2} - 1\right) \approx -0.5473017788,$$

$$-\frac{1}{4}\left(1 + \sqrt{2} - \sqrt{4 + 3\sqrt{2}}\right)\left(\sqrt{2} - 1\right) \approx 0.04730177875.$$

By calculating that $h(i/2) \approx 0.4730177873$, we get that

$$h\left(\frac{i}{2}\right) = -\frac{1}{4}\left(1 + \sqrt{2} - \sqrt{4 + 3\sqrt{2}}\right)\left(\sqrt{2} - 1\right).$$

□

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