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Generalizations of some conjectures of Chan on congruences for Appell–Lerch sums

Y.K. Qu [a], Y.J. Wang [b], Olivia X.M. Yao [b,*]

[a] Department of Mathematics, LuoYang Normal University, LuoYang 471022, PR China
[b] Department of Mathematics Jiangsu University, Jiangsu, Zhenjiang 212013, PR China

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In this paper, we prove several congruences for Appell–Lerch sums which generalize some conjectures given by Chan.

1. Introduction

This aim of this paper is to give extensions of several conjectures on congruences for Appell–Lerch sums due to Chan [6].

Throughout this paper, we assume $|q| < 1$ for convergence and let

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

and

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty.$$

Recall that an Appell–Lerch sum is a series of the form

$$AL(x,q,z) = \frac{1}{(q, q/z, q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}} z^{n+1}}{1 - xzq^n},$$

* Corresponding author.
E-mail addresses: yongke1239@163.com (Y.K. Qu), jsdxwyj@163.com (Y.J. Wang), yaoxiangmei@163.com (O.X.M. Yao).

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where x and z are nonzero complex numbers such that neither z nor xz is an integral power of q. Appell–Lerch sums were first studied by Appell [2–4] and then by Lerch [8]. In recent years, there has been much interest on Appell–Lerch sums and their connections to mock theta functions.

In [6], Chan gave the following definition which is related to Appell–Lerch sum:

$$\sum_{n=0}^{\infty} a_{j,p}(n)q^n := \frac{1}{(q^j, q^{p-j}, q^p; q^p)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{pn(n+1)/2+jn+j}}{1-q^{pn+j}}. \tag{1.1}$$

Chan [6], Chan and Mao [7] also proved some congruences for $a_{j,p}(n)$. At the end of his paper [6], Chan posed several conjectures on congruences for $a_{j,p}(n)$ and some of them can be stated as follows:

Conjecture 1.1. *For $n \geq 0$,*

$$a_{1,6}(2n) \equiv 0 \pmod 2, \tag{1.2}$$

$$a_{1,10}(2n) \equiv 0 \pmod 2, \tag{1.3}$$

$$a_{3,10}(2n) \equiv 0 \pmod 2, \tag{1.4}$$

$$a_{1,6}(6n+3) \equiv 0 \pmod 3. \tag{1.5}$$

The objective of this paper is to give extensions of Congruences (1.2)–(1.5).

Theorem 1.2. *Let k, j be positive integers with $1 \leq j \leq k-1$. If j is odd, then for $n \geq 0$,*

$$a_{j,2k}(2n) \equiv 0 \pmod 2. \tag{1.6}$$

It is easy to see that congruence (1.6) implies (1.2)–(1.4).

Moreover, we will establish the following generating function for $a_{1,6}(6n+3)$ which yields (1.5).

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty} a_{1,6}(6n+3)q^n = 3\frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^5}{(q;q)_\infty^6 (q^6;q^6)_\infty}. \tag{1.7}$$

From (1.7), we will prove some congruences modulo 9 for $a_{1,6}(n)$. The following theorem is a refinement of congruence (1.5).

Theorem 1.4. *For $n \geq 0$,*

$$a_{1,6}(18n+9) \equiv a_{1,6}(18n+15) \equiv 0 \pmod 9, \tag{1.8}$$

$$a_{1,6}(54n+21) \equiv a_{1,6}(54n+39) \equiv 0 \pmod 9 \tag{1.9}$$

and

$$a_{1,6}(54n+3) \equiv \begin{cases} 3 \pmod 9, & \text{if } n = \dfrac{k(3k+1)}{2} \text{ with } k \equiv 0 \pmod 2, \\ 6 \pmod 9, & \text{if } n = \dfrac{k(3k+1)}{2} \text{ with } k \equiv 1 \pmod 2, \\ 0 \pmod 9, & \text{otherwise.} \end{cases} \tag{1.10}$$

From (1.10), we can deduce the following corollary:

Corollary 1.5. *Let $p \geq 5$ be a prime and choose r, $1 \leq r \leq p-1$, such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,*

$$a_{1,6}(54(pn+r)+3) \equiv 0 \pmod{9}.$$

2. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first prove the following lemma.

Lemma 2.1. *We have*

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n} \equiv \frac{(q;q)_\infty^4}{2(q^2;q^2)_\infty^2} \pmod{4}. \tag{2.1}$$

Proof. It is easy to check that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n} &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n} + \sum_{n=-\infty}^{-1} \frac{q^{n(n+1)/2}}{1+q^n} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n} + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{1+q^{-n}} \\ &= \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n}. \end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} + \sum_{n=1}^{\infty} \frac{(-1)^{-n} q^{n(n-1)/2}}{1+q^{-n}} \\ &= \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} \\ &\equiv \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n} \pmod{4}. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n} \equiv \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} \pmod{4}. \tag{2.4}$$

From Entry 12.2.2 on page 264 in Andrews and Berndt's book [1],

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-cq^n} = \frac{(q;q)_\infty^2}{(c;q)_\infty (q/c;q)_\infty}. \tag{2.5}$$

Setting $c=-1$ in (2.5), we get

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} = \frac{(q;q)_\infty^2}{2(-q;q)_\infty^2} = \frac{(q;q)_\infty^4}{2(q^2;q^2)_\infty^2}. \tag{2.6}$$

Congruence (2.1) follows from (2.4) and (2.6).

Now, we are ready to prove Theorem 1.2.

It follows from (7.2) in [6] that

$$4\sum_{n=0}^{\infty} a_{j,2k}(n)q^n = \frac{(-q^j,-q^{2k-j};q^{2k})_\infty^2 (q^{2k};q^{2k})_\infty^5}{(q^j,q^{2k-j};q^{2k})_\infty^2 (q^{4k};q^{4k})_\infty^4} - 2\frac{(q^{2k};q^{2k})_\infty}{(q^{4k};q^{4k})_\infty^2} \sum_{n=-\infty}^{\infty} \frac{q^{kn(n+1)}}{1+q^{2kn}}. \tag{2.7}$$

In view of (2.1) and (2.7),

$$4\sum_{n=0}^{\infty} a_{j,2k}(n)q^n \equiv \frac{(-q^j,-q^{2k-j};q^{2k})_\infty^2 (q^{2k};q^{2k})_\infty^5}{(q^j,q^{2k-j};q^{2k})_\infty^2 (q^{4k};q^{4k})_\infty^4} - \frac{(q^{2k};q^{2k})_\infty^5}{(q^{4k};q^{4k})_\infty^4} \pmod 8. \tag{2.8}$$

We can rewrite (2.8) as

$$4\sum_{n=0}^{\infty} a_{j,2k}(n)q^n \equiv \frac{(q^{2k};q^{2k})_\infty^5}{(q^{4k};q^{4k})_\infty^4} \frac{((-q^j,-q^{2k-j};q^{2k})_\infty^2 - (q^j,q^{2k-j};q^{2k})_\infty^2)}{(q^j,q^{2k-j};q^{2k})_\infty^2} \pmod 8. \tag{2.9}$$

By Entry 30 (vi) on page 46 in [5],

$$\begin{aligned} &(-q^j,-q^{2k-j},q^{2k};q^{2k})_\infty^2 - (q^j,q^{2k-j},q^{2k};q^{2k})_\infty^2 \\ &\qquad = 4q^j(-q^{2k-2j},-q^{2k+2j};q^{4k})_\infty (q^{8k};q^{8k})_\infty^2. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\sum_{n=0}^{\infty} a_{j,2k}(n)q^n \equiv q^j \frac{(q^{2k};q^{2k})_\infty^3}{(q^{4k};q^{4k})_\infty^4} \frac{(-q^{2k-2j},-q^{2k+2j};q^{4k})_\infty (q^{8k};q^{8k})_\infty^2}{(q^j,q^{2k-j};q^{2k})_\infty^2} \pmod 2. \tag{2.11}$$

By the binomial theorem,

$$(q^j,q^{2k-j};q^{2k})_\infty^2 \equiv (q^{2j},q^{4k-2j};q^{4k})_\infty \pmod 2. \tag{2.12}$$

Thanks to (2.11) and (2.12),

$$\sum_{n=0}^{\infty} a_{j,2k}(n)q^n \equiv q^j \frac{(q^{2k};q^{2k})_\infty^3}{(q^{4k};q^{4k})_\infty^4} \frac{(-q^{2k-2j},-q^{2k+2j};q^{4k})_\infty (q^{8k};q^{8k})_\infty^2}{(q^{2j},q^{4k-2j};q^{4k})_\infty} \pmod 2. \tag{2.13}$$

Congruence (1.6) follows from (2.13) and the fact that j is an odd integer. This completes the proof.

3. Proofs of Theorems 1.3 and 1.4

Setting $j = 1$ and $k = 3$ in (2.7), we deduce that

$$4\sum_{n=0}^{\infty} a_{1,6}(n)q^n = \left(\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2}\right)^2 \cdot \frac{(q^2;q^2)_\infty^6}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2} - 2\frac{(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{3n(n+1)}}{1+q^{6n}}. \tag{3.1}$$

Xia and Yao [10] that

$$\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2} = \frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty(q^{24};q^{24})_\infty} + 2q\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^8;q^8)_\infty(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^4(q^{12};q^{12})_\infty}; \tag{3.2}$$

see also [9]. Substituting (3.2) into (3.1) and expanding it, we get

$$4\sum_{n=0}^{\infty} a_{1,6}(n)q^n = \frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty^2} - 2\frac{(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{3n(n+1)}}{1+q^{6n}} + 4q\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty} + 4q^2\frac{(q^6;q^6)_\infty^3(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^4}. \tag{3.3}$$

Extracting those terms in which the power of q is congruent to 1 modulo 2 in (3.3), then dividing q and replacing q^2 by q, we deduce that

$$\sum_{n=0}^{\infty} a_{1,6}(2n+1)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \cdot \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \cdot \frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}. \tag{3.4}$$

From Corollary (ii) on page 49 in [5],

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} + q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}. \tag{3.5}$$

From Lemma 2.1 in [10],

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8(q^{18};q^{18})_\infty^3} + 2q\frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^7} + 4q^2\frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^6}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$\sum_{n=0}^{\infty} a_{1,6}(2n+1)q^n = \left(\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} + q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}\right)\cdot\left(\frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8(q^{18};q^{18})_\infty^3} + 2q\frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^7} + 4q^2\frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^6}\right)\cdot\frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}$$

$$=\frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^8}{(q^3;q^3)_\infty^7(q^{18};q^{18})_\infty^4}+3q\frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^5}{(q^3;q^3)_\infty^6(q^{18};q^{18})_\infty}$$

$$+6q^2\frac{(q^6;q^6)_\infty^2(q^9;q^9)_\infty^2(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty^5}+4q^3\frac{(q^6;q^6)_\infty(q^{18};q^{18})_\infty^5}{(q^3;q^3)_\infty^4(q^9;q^9)_\infty}. \tag{3.7}$$

Picking out those terms in which the power of q is congruent to 1 modulo 3 in (3.7), then dividing q and replacing q^3 by q, we arrive at (1.7). This completes the proof of Theorem 1.3.

Now, we turn to prove Theorem 1.4.

By the binomial theorem,

$$(q;q)_\infty^3 \equiv (q^3;q^3)_\infty \pmod 3. \tag{3.8}$$

Based on (1.7) and (3.8),

$$\sum_{n=0}^{\infty} a_{1,6}(6n+3)q^n = 3\frac{(q^2;q^2)_\infty^3}{(q^6;q^6)_\infty}\cdot\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^6}\cdot\frac{(q^3;q^3)_\infty^3}{(q^9;q^9)_\infty}\cdot(q^9;q^9)_\infty$$

$$\equiv 3(q^9;q^9)_\infty \pmod 9, \tag{3.9}$$

which yields (1.8) and (1.9).

From (3.9), we have

$$\sum_{n=0}^{\infty} a_{1,6}(54n+3)q^n \equiv 3(q;q)_\infty \pmod 9. \tag{3.10}$$

The following identity is commonly known as Euler's pentagonal number theorem and is worth highlighting here:

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\sum_{n=0}^{\infty} a_{1,6}(54n+3)q^n \equiv 3\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k+1)}{2}} \pmod 9. \tag{3.12}$$

Congruence (1.10) follows from (3.12). The proof of Theorem 1.4 is complete.

To conclude this section, we present a proof of Corollary 1.5.

Since r was chosen such that $24r+1$ is a quadratic nonresidue modulo p, then for any integer k,

$$24pn+24r+1 \neq (6k+1)^2.$$

This implies that $pn+r$ can not be represented as $\frac{k(3k+1)}{2}$ for any integer k. Based on (1.10),

$$a_{1,6}(54pn+54r+3) \equiv 0 \pmod 9.$$

This completes the proof of Corollary 1.5.

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