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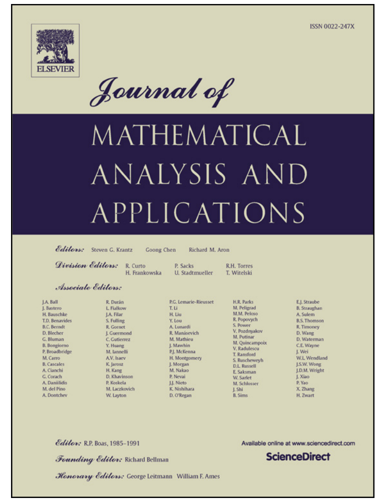

Conditioning of copulas: Transformations, invariance and measures of concordance

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Conditioning of copulas: Transformations, invariance and measures of concordance

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Abstract

In the present paper we study the problem of how to transform a copula for an arbitrary distribution function into a copula for its conditional distribution function where conditioning is meant with respect to a tail event in which the observations lie below some threshold. To this end, we consider conditioning of copulas as a map which transforms every copula into another one. Besides the general case, which refers to conditioning in all coordinates, we also pay attention to the special case of univariate conditioning, which refers to conditioning in a single coordinate. We investigate the behaviour of conditioning under composition and with respect to certain transformations of copulas, and we show that invariance of a copula under conditioning is equivalent to invariance of a copula under univariate conditioning in each coordinate. Finally, we apply conditioning of copulas to Sklar's Theorem and to measures of concordance.

Keywords: Copula, Conditioning, Invariance, Measures of concordance, Sklar's Theorem, Transformations

1. Introduction

In the present paper we study the problem of how to transform a copula for an arbitrary distribution function into a copula for its conditional distribution function where conditioning is meant with respect to a tail event in which the observations lie below some threshold, and hence with respect to a Borel set with positive measure. Besides the general case, which refers to conditioning in all coordinates, we also pay attention to the special case of univariate conditioning, which refers to conditioning in a single coordinate. In contrast to the literature, we do not assume that the coordinates of the distribution function are continuous, such that the initial copula and the transformed copula may fail to be unique.

Conditioning (or *truncation*) of copulas (also called *conditional copulas*, *threshold copulas* or *tail dependence copulas*) is a well–studied object. This is mainly due to its potential in describing and studying conditional dependence between random variables under a tail event, but also due to various applications in finance and reliability theory: This includes [5, 23, 24] for studying tail dependence, [8] for modeling market contagion, [1, 3, 22, 25] for the construction of systemic risk measures like *conditional value at risk* (CoVaR), [4, 7] for modeling credit derivatives and stock returns and [12, 30] for modeling bivariate ageing. In this context, copulas which are invariant under conditioning are of particular interest (see, e.g., [2, 4, 9, 10, 18, 29, 31]) since for such copulas the values of copula–based measures of association (like measures of concordance) remain unchanged under conditioning. For a comprehensive overview of univariate conditioning including theoretical results but also further applications, we refer to [19].

Here we consider conditioning as a map which transforms every copula into another one, and we investigate its behaviour under composition but also with respect to permutations and reflections. In addition, we show that invariance of a copula under conditioning is equivalent to invariance of a copula under univariate conditioning in each coordinate. In the bivariate case this result was proved in [9] using the fact that the copulas which are invariant under conditioning belong to the

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closure of the Clayton family of copulas. Moreover, to solve our initial problem, given a copula for a distribution function, we show how the introduced transformation can be used as a tool to obtain copulas for all those conditional distribution functions that are conditioned with respect to the event that observations lie below some threshold, but also for all those conditional distribution functions where the conditioning event is somehow reflected. Finally, we present representations of the usual multivariate generalizations of Kendall's tau and Spearman's rho for the transformed copula in terms of the initial copula.

This paper is organized as follows: We first recapitulate essential definitions and results concerning copulas and transformations of copulas (Section 2). In Section 3 we then introduce conditioning of copulas as a transformation on copulas and apply it to Sklar's Theorem (Section 4). We further investigate the behaviour of conditioning under composition (Section 5) and with respect to permutations and reflections (Section 6), and we study invariance of copulas under conditioning (Section 7). Finally, we apply conditioning to the usual multivariate generalizations of Kendall's tau and Spearman's rho (Section 8).

2. Prerequisites

Let $\mathrm{I} := [0,1]$ and let $d \geq 2$ be an integer which will be kept fix throughout this paper. For $K \subseteq \{1,...,d\}$, we first consider the map $\boldsymbol{\eta}_K : \mathrm{I}^d \times \mathrm{I}^d \to \mathrm{I}^d$ given coordinatewise by

$$\big(\boldsymbol{\eta}_K(\mathbf{u},\mathbf{v})\big)_k := \begin{cases} u_k & k \in \{1,...,d\}\backslash K \\ v_k & k \in K \end{cases}$$

and we put $\boldsymbol{\eta}_k := \boldsymbol{\eta}_{\{k\}}$ for $k \in \{1,...,d\}$. We denote by $\mathbf{0}$ the vector with entries 0 and by $\mathbf{1}$ the vector with entries 1. A *copula* is a function $C : \mathrm{I}^d \to \mathrm{I}$ satisfying the following conditions:

(i) The inequality

$$\sum_{L \subseteq \{1,...,d\}} (-1)^{d-|L|}\, C(\boldsymbol{\eta}_L(\mathbf{u},\mathbf{v})) \geq 0$$

holds for all $\mathbf{u},\mathbf{v} \in \mathrm{I}^d$ such that $\mathbf{u} \leq \mathbf{v}$.

(ii) The identity $C(\boldsymbol{\eta}_i(\mathbf{u},\mathbf{0})) = 0$ holds for all $\mathbf{u} \in \mathrm{I}^d$ and all $i \in \{1,...,d\}$.

(iii) The identity $C(\boldsymbol{\eta}_i(\mathbf{1},\mathbf{u})) = u_i$ holds for all $\mathbf{u} \in \mathrm{I}^d$ and all $i \in \{1,...,d\}$.

Note that this definition of a copula is in accordance with the literature; see, e.g., [11, 28]. The collection $\mathcal{C}$ of all copulas is convex.

A map $\varphi : \mathcal{C} \to \mathcal{C}$ is said to be a *transformation* on $\mathcal{C}$. Let Φ denote the collection of all transformations on $\mathcal{C}$ and define the *composition* $\circ : \Phi \times \Phi \to \Phi$ by letting $(\varphi_1 \circ \varphi_2)(C) := \varphi_1(\varphi_2(C))$. The composition is associative and the transformation $\iota \in \Phi$ given by $\iota(C) := C$ satisfies $\iota \circ \varphi = \varphi = \varphi \circ \iota$ and is therefore called *identity* on $\mathcal{C}$. Thus, $(\Phi, \circ)$ is a semigroup with neutral element ι. For the successive composition of $n \in \mathbb{N}_0$ transformations $\varphi_m \in \Phi$, $m \in \{1,...,n\}$, we write

$$\bigcirc_{m=1}^{n} \varphi_m := \begin{cases} \iota & n = 0 \\ \varphi_n \circ \bigcirc_{m=1}^{n-1} \varphi_m & \text{otherwise} \end{cases}$$

and, for $N = \{1,...,n\}$ and a set of pairwise commuting $\varphi_m \in \Phi$, $m \in N$, we put $\bigcirc_{m \in N} \varphi_m := \bigcirc_{m=1}^{n} \varphi_m$.

We now introduce two elementary transformations: For $i,j,k \in \{1,...,d\}$ with $i \neq j$, we define the maps $\pi_{i,j}, \nu_k : \mathcal{C} \to \mathcal{C}$ by letting

$$\begin{aligned} (\pi_{i,j}(C))(\mathbf{u}) &:= C(\boldsymbol{\eta}_{\{i,j\}}(\mathbf{u}, u_j\,\mathbf{e}_i + u_i\,\mathbf{e}_j)) \\ (\nu_k(C))(\mathbf{u}) &:= C(\boldsymbol{\eta}_k(\mathbf{u},\mathbf{1})) - C(\boldsymbol{\eta}_k(\mathbf{u},\mathbf{1}-\mathbf{u})) \end{aligned}$$

where $\mathbf{e}_i$ denotes the i–th unit vector. $\pi_{i,j}$ is called a *transposition*, and ν_k is called a *partial reflection*. Both, $\pi_{i,j}$ and ν_k, are involutions. There exists a smallest subgroup $(\Gamma, \circ)$ of Φ containing all transpositions and all partial reflections. This group Γ is non–commutative and is a representation of the hyperoctahedral group with $d!\,2^d$ elements.

A transformation is called a *permutation* if it can be expressed as a finite composition of transpositions, and it is called a *reflection* if it can be expressed as a finite composition of partial reflections. We denote by Γ^π the set of all permutations and by Γ^ν the set of all reflections. Then Γ^π and Γ^ν are subgroups of Γ, and every transformation in Γ can be expressed as a composition of a permutation and a reflection. Since Γ^ν is commutative, for every reflection $\nu \in \Gamma^\nu$, there exists a unique $K \subseteq \{1, ..., d\}$ such that

$$\nu = \nu_K := \bigcirc_{k \in K} \nu_k$$

Due to its particular interest we emphasize the *total reflection* $\tau := \nu_{\{1,...,d\}}$. The total reflection is an involution and transforms every copula into its survival copula. We refer to [13] for further details on the group Γ.
The hyperoctahedral group also has a geometric representation; see [32, 33, 34]: Consider the collection of all functions from I^d into I^d equipped with the composition $\diamond$ and the identity $\tilde{\iota}$. Then there is a smallest group $(\tilde{\Gamma}, \diamond)$ containing the functions $\tilde{\pi}_{i,j}, \tilde{\nu}_k : \mathrm{I}^d \to \mathrm{I}^d$ with $i, j, k \in \{1, ..., d\}$ and $i \neq j$, given by

$$\begin{aligned} \tilde{\pi}_{i,j}(\mathbf{u}) &:= \boldsymbol{\eta}_{\{i,j\}}(\mathbf{u}, u_j\,\mathbf{e}_i + u_i\,\mathbf{e}_j) \\ \tilde{\nu}_k(\mathbf{u}) &:= \boldsymbol{\eta}_k(\mathbf{u}, \mathbf{1} - \mathbf{u}) \end{aligned}$$

Note that every $\tilde{\gamma} \in \tilde{\Gamma}$ can be expressed as a finite composition of $\tilde{\pi}_{i,j}$ and $\tilde{\nu}_k$ with $i, j, k \in \{1, ..., d\}$ and $i \neq j$. Since $\tilde{\pi}_{i,j}$ and $\tilde{\nu}_k$ are continuous, every $\tilde{\gamma} \in \tilde{\Gamma}$ is continuous as well.
The groups Γ and $\tilde{\Gamma}$ are related to each other by an isomorphism $T : (\Gamma, \circ) \to (\tilde{\Gamma}, \diamond)$ satisfying $T(\pi_{i,j}) = \tilde{\pi}_{i,j}$ and $T(\nu_k) = \tilde{\nu}_k$ for all $i, j, k \in \{1, ..., d\}$ with $i \neq j$. For a detailed discussion of the groups $(\Gamma, \circ)$ and $(\tilde{\Gamma}, \diamond)$, see [13, 14].

3. Conditioning of copulas

In the present section we introduce conditioning of copulas as a transformation on copulas, and we provide some representations of the transformed copula in terms of the initial copula.

First, for $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$, we introduce the set $\mathcal{C}_\mathbf{a} := \{C \in \mathcal{C} \,|\, C(\mathbf{a}) > 0\}$. The following examples show that there exist copulas which are contained in $\mathcal{C}_\mathbf{a}$ for every $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$, and that there exists some $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ such that $\mathcal{C}_\mathbf{a} \neq \mathcal{C}$:

3.1 Examples.
(1) The *product copula* $\Pi : \mathrm{I}^d \to \mathrm{I}$ given by $\Pi(\mathbf{u}) := \prod_{i=1}^d u_i$ satisfies $\Pi \in \mathcal{C}_\mathbf{a}$ for every $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$.
(2) The *Fréchet–Hoeffding upper bound* $M : \mathrm{I}^d \to \mathrm{I}$ given by $M(\mathbf{u}) := \min\{u_i \,|\, i \in \{1, ..., d\}\}$ satisfies $M \in \mathcal{C}_\mathbf{a}$ for every $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$.
(3) The reflected Fréchet–Hoeffding upper bound $\nu_1(M)$ satisfies $\nu_1(M) \in \mathcal{C}_\mathbf{a}$ if, and only if, $a_1 + \min\{a_2, ..., a_d\} > 1$. Note that, for $d = 2$, $\nu_1(M)$ coincides with the *Fréchet–Hoeffding lower bound* W.

Moreover, for every $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ and every $\pi \in \Gamma^\pi$, $C \in \mathcal{C}_\mathbf{a}$ if, and only if, $\pi(C) \in \mathcal{C}_{(T(\pi))(\mathbf{a})}$.
The next result is evident:

3.2 Lemma.
(1) *The inclusions* $\mathcal{C}_\mathbf{a} \subseteq \mathcal{C}_\mathbf{b} \subseteq \mathcal{C}_\mathbf{1} = \mathcal{C}$ *hold for all* $\mathbf{a}, \mathbf{b} \in (\mathbf{0}, \mathbf{1}]$ *such that* $\mathbf{a} \leq \mathbf{b}$.
(2) *The identity* $\mathcal{C}_{\boldsymbol{\eta}_k(\mathbf{1}, a\,\mathbf{e}_k)} = \mathcal{C}$ *holds for every* $k \in \{1, ..., d\}$ *and every* $a \in (0, 1]$.

For $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$, $C \in \mathcal{C}_\mathbf{a}$ and $k \in \{1, ..., d\}$, we define the map $\delta^k_{\mathbf{a},C} : [0, a_k] \to \mathrm{I}$ by letting

$$\delta^k_{\mathbf{a},C}(s_k) := \frac{C\big(\boldsymbol{\eta}_k(\mathbf{a}, s_k \mathbf{e}_k)\big)}{C(\mathbf{a})}$$

and hence the map $\boldsymbol{\delta}_{\mathbf{a},C} : [\mathbf{0}, \mathbf{a}] \to \mathrm{I}^d$ by letting $\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}) := \big(\delta^1_{\mathbf{a},C}(s_1), ..., \delta^d_{\mathbf{a},C}(s_d)\big)'$. Note that $\delta^k_{\mathbf{a},C}$ is a normalized version of the univariate section from 0 to a_k in the k–th coordinate of the copula C and hence a continuous distribution function on $[0, a_k]$ satisfying $\delta^k_{\boldsymbol{\eta}_k(\mathbf{1},\mathbf{a}),C}(s_k) = s_k/a_k$. Also note that

$\boldsymbol{\delta}_{\mathbf{1},C}(\mathbf{s}) = \mathbf{s}$ and $\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{a}) = \mathbf{1}$. The maps $(\delta^k_{\mathbf{a},C})^{\leftarrow}, (\delta^k_{\mathbf{a},C})^{\rightarrow} : \mathrm{I} \to [0, a_k]$ given by

$$\begin{aligned}(\delta^k_{\mathbf{a},C})^{\leftarrow}(u_k) &:= \inf\{s_k \in [0, a_k] \,|\, \delta^k_{\mathbf{a},C}(s_k) = u_k\} \\ (\delta^k_{\mathbf{a},C})^{\rightarrow}(u_k) &:= \sup\{s_k \in [0, a_k] \,|\, \delta^k_{\mathbf{a},C}(s_k) = u_k\}\end{aligned}$$

are the lower respectively upper quantile functions of $\delta^k_{\mathbf{a},C}$ and we define the maps $(\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow}, (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} : \mathrm{I}^d \to [\mathbf{0}, \mathbf{a}]$ by letting

$$\begin{aligned}(\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow}(\mathbf{u}) &:= \big((\delta^1_{\mathbf{a},C})^{\leftarrow}(u_1), ..., (\delta^d_{\mathbf{a},C})^{\leftarrow}(u_d)\big)' \\ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}(\mathbf{u}) &:= \big((\delta^1_{\mathbf{a},C})^{\rightarrow}(u_1), ..., (\delta^d_{\mathbf{a},C})^{\rightarrow}(u_d)\big)'\end{aligned}$$

Then we have $(\delta^k_{\boldsymbol{\eta}_k(\mathbf{1},\mathbf{a}),C})^{\rightarrow}(u_k) = a_k u_k$ as well as $(\boldsymbol{\delta}_{\mathbf{1},C})^{\rightarrow}(\mathbf{u}) = \mathbf{u}$ and $(\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}(\mathbf{1}) = \mathbf{a}$. In addition, the identities $\chi_{[\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}),\mathbf{1}]}(\mathbf{u}) = \chi_{[\mathbf{0},(\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}(\mathbf{u})]}(\mathbf{s})$ and

$$(\boldsymbol{\delta}_{\mathbf{a},C} \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow})(\mathbf{u}) = \mathbf{u} = (\boldsymbol{\delta}_{\mathbf{a},C} \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow})(\mathbf{u})$$

and the inequality

$$((\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C})(\mathbf{s}) \leq \mathbf{s} \leq ((\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C})(\mathbf{s})$$

hold for all $\mathbf{s} \in [\mathbf{0}, \mathbf{a}]$ and all $\mathbf{u} \in \mathrm{I}^d$.

We base our definition of conditioning of copulas on Theorem 3.3 below. Note that the unique solution $C_{\mathbf{a}}$ of the functional equation in Theorem 3.3 is a copula for the conditional distribution function whenever C is a copula for the initial distribution function; see also Theorems 4.1 and 4.2.

3.3 Theorem. *For every $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ and every $C \in \mathcal{C}_{\mathbf{a}}$, there exists a unique copula $C_{\mathbf{a}} \in \mathcal{C}$ satisfying*

$$\big(C_{\mathbf{a}} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s}) = \frac{C(\mathbf{s})}{C(\mathbf{a})}$$

for all $\mathbf{s} \in [\mathbf{0}, \mathbf{a}]$, and the copula $C_{\mathbf{a}}$ satisfies

$$C_{\mathbf{a}} = \frac{C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow}}{C(\mathbf{a})} = \frac{C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}}{C(\mathbf{a})}$$

Moreover, $\big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s}) = C(\mathbf{s}) = \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s})$ holds for all $\mathbf{s} \in [\mathbf{0}, \mathbf{a}]$.

Proof. Consider $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$, $C \in \mathcal{C}_{\mathbf{a}}$ and define the map $F_{\mathbf{a},C} : \mathrm{I}^d \to \mathrm{I}$ by letting

$$F_{\mathbf{a},C}(\mathbf{u}) := \frac{C(\mathbf{u} \wedge \mathbf{a})}{C(\mathbf{a})}$$

where the minimum $\mathbf{u} \wedge \mathbf{a}$ is defined coordinatewise. Then $F_{\mathbf{a},C}$ is a continuous distribution function and the vector $\mathbf{F}_{\mathbf{a},C}$ of its univariate marginal distribution functions satisfies $\mathbf{F}_{\mathbf{a},C}(\mathbf{u}) = \boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{u} \wedge \mathbf{a})$ for all $\mathbf{u} \in \mathrm{I}^d$. Therefore, by Sklar's Theorem (see, e.g., [11, 28]), there exists a unique copula $C_{\mathbf{a}}$ satisfying $F_{\mathbf{a},C} = C_{\mathbf{a}} \circ \mathbf{F}_{\mathbf{a},C}$, and hence

$$\big(C_{\mathbf{a}} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{u} \wedge \mathbf{a}) = \big(C_{\mathbf{a}} \circ \mathbf{F}_{\mathbf{a},C}\big)(\mathbf{u}) = F_{\mathbf{a},C}(\mathbf{u}) = \frac{C(\mathbf{u} \wedge \mathbf{a})}{C(\mathbf{a})}$$

for all $\mathbf{u} \in \mathrm{I}^d$. Now, we prove the identities. Due to the properties of the lower and upper quantile function, we obtain

$$\frac{C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow}}{C(\mathbf{a})} = C_{\mathbf{a}} \circ \boldsymbol{\delta}_{\mathbf{a},C} \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow} = C_{\mathbf{a}} = C_{\mathbf{a}} \circ \boldsymbol{\delta}_{\mathbf{a},C} \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} = \frac{C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}}{C(\mathbf{a})}$$

Moreover, since $C_{\mathbf{a}}$ is increasing in each coordinate, we also obtain

$$\big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s}) \leq C(\mathbf{s}) \leq \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s}) = C(\mathbf{a})\,\big(C_{\mathbf{a}} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s}) = \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s})$$

for all $\mathbf{s} \in [\mathbf{0}, \mathbf{a}]$. □

3.4 Remark. A converse version of Theorem 3.3 is true as well: Consider $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ and a copula $C \in \mathcal{C}_{\mathbf{a}}$ satisfying $C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow} = C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}$. If

$$D := \frac{C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\leftarrow}}{C(\mathbf{a})} = \frac{C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}}{C(\mathbf{a})}$$

is a copula, then the identity

$$\big(D \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{s}) = \frac{C(\mathbf{s})}{C(\mathbf{a})}$$

holds for all $\mathbf{s} \in [\mathbf{0}, \mathbf{a}]$.

For $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$, we define the map $\vartheta_{\mathbf{a}} : \mathcal{C}_{\mathbf{a}} \to \mathcal{C}$ by letting

$$\vartheta_{\mathbf{a}}(C) := C_{\mathbf{a}}$$

where $C_{\mathbf{a}}$ is the unique copula satisfying the functional equation in Theorem 3.3. The transformation $\vartheta_{\mathbf{a}}$ is called the *(multivariate) conditioning* with respect to $\mathbf{a}$. In the sequel, we shall suppress the adjective *multivariate*. According to the literature, in the special case where $\mathbf{a} = \boldsymbol{\eta}_k(\mathbf{1}, a\,\mathbf{e}_k)$ holds for some $k \in \{1, ..., d\}$ and some $a \in (0, 1]$, the transformation $\vartheta_{\mathbf{a}} = \vartheta_{\boldsymbol{\eta}_k(\mathbf{1}, a\,\mathbf{e}_k)}$ is called the *univariate conditioning* with respect to k and a. We adopt this linguistic distinction and hence consider univariate conditioning as a special case of conditioning.

Due to Theorem 3.3, in the following, we restrict ourselves to representations concerning the upper quantile function; results concerning the lower quantile function (or any quantile function) can be obtained analogously. The following result is evident from Theorem 3.3; here and in the sequel we denote by Q^C the probability measure associated with the copula C, and by $(Q^C)_{\boldsymbol{\delta}_{\mathbf{a},C}}$ the pushforward of Q^C under the measurable map $\boldsymbol{\delta}_{\mathbf{a},C}$:

3.5 Corollary. *For every* $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ *and every* $C \in \mathcal{C}_{\mathbf{a}}$*, the copula* $\vartheta_{\mathbf{a}}(C)$ *satisfies*

$$\big(\vartheta_{\mathbf{a}}(C)\big)(\mathbf{u}) = \frac{1}{C(\mathbf{a})}\, \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}\big)(\mathbf{u}) = \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}\big(\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})\big)\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s})$$

for all $\mathbf{u} \in \mathrm{I}^d$ *and*

$$Q^{\vartheta_{\mathbf{a}}(C)} = \frac{1}{C(\mathbf{a})}\, \big(Q^C\big)_{\boldsymbol{\delta}_{\mathbf{a},C}}$$

In particular, $\vartheta_{\mathbf{1}}(C) = C$ *for every* $C \in \mathcal{C}$.

Because of the previous result, the copula $\vartheta_{\mathbf{a}}(C)$ is determined by the restriction of C to the interval $[\mathbf{0}, \mathbf{a}]$.

4. Applications to conditional distribution functions

In this section, we present a version of Sklar's Theorem for all those conditional distribution functions that are conditioned with respect to the event that observations lie below some threshold. In a second step we generalize this result and we present a version of Sklar's Theorem also for all those conditional distribution functions where the conditioning event is somehow reflected. In contrast to the literature, we do not assume that the coordinates of the distribution function are continuous, such that the initial copula and the transformed copula may fail to be unique; we refer to [6] for more information on copulas for non–continuous distribution functions.

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a random vector $\mathbf{X} : \Omega \to \mathbb{R}^d$. We denote by F the distribution function of $\mathbf{X}$ and by $\mathbf{F} := (F_1, ..., F_d)'$ the vector of its univariate marginal distribution functions, and we consider a fixed copula C satisfying

$$F = C \circ \mathbf{F}$$

which exists by Sklar's Theorem (see, e.g., [11, 28]). If the marginal distribution functions are continuous, then C is unique, but this will not be required here and in the sequel.

For $\mathbf{q} \in \mathbb{R}^d$ for which $P\big[\{\mathbf{X} \le \mathbf{q}\}\big] > 0$, we further define the map $F_{\mathbf{q}} : \mathbb{R}^d \to \mathrm{I}$ by letting

$$F_{\mathbf{q}}(\mathbf{x}) := P[\{\mathbf{X} \le \mathbf{x}\} \,|\, \{\mathbf{X} \le \mathbf{q}\}]$$

Then $\mathbf{F}(\mathbf{q}) \in (\mathbf{0}, \mathbf{1}]$ and $F_{\mathbf{q}}$ is the distribution function of the conditional distribution of $\mathbf{X}$ under the event $\mathbf{X} \le \mathbf{q}$. We denote by $\mathbf{F}_{\mathbf{q}}$ the vector of univariate margins of $F_{\mathbf{q}}$.

We show that the copula $\vartheta_{\mathbf{F}(\mathbf{q})}(C)$ is a copula for the conditional distribution function $F_{\mathbf{q}}$ whenever C is a copula for the distribution function F. While this result is well–known under certain assumptions (see, e.g., [4, 18]), no such assumptions will be made in the sequel.

4.1 Theorem. *Consider $\mathbf{q} \in \mathbb{R}^d$ for which $C \in \mathcal{C}_{\mathbf{F}(\mathbf{q})}$. Then*

$$\vartheta_{\mathbf{F}(\mathbf{q})}(C)$$

is a copula for $F_{\mathbf{q}}$. If the margins of F are continuous, then the copulas C and $\vartheta_{\mathbf{F}(\mathbf{q})}(C)$ are unique.

Proof. $F_{\mathbf{q}}$ and the vector of its univariate marginal distribution functions $\mathbf{F}_{\mathbf{q}}$ satisfy

$$F_{\mathbf{q}}(\mathbf{x}) = \frac{C\big(\mathbf{F}(\mathbf{x}) \wedge \mathbf{F}(\mathbf{q})\big)}{C(\mathbf{F}(\mathbf{q}))} \qquad \text{and} \qquad \mathbf{F}_{\mathbf{q}}(\mathbf{x}) = \boldsymbol{\delta}_{\mathbf{F}(\mathbf{q}),C}\big(\mathbf{F}(\mathbf{x}) \wedge \mathbf{F}(\mathbf{q})\big)$$

for all $\mathbf{x} \in \mathbb{R}^d$, where the minimum $\mathbf{F}(\mathbf{x}) \wedge \mathbf{F}(\mathbf{q})$ is defined coordinatewise. We hence obtain

$$F_{\mathbf{q}}(\mathbf{x}) = \frac{C\big(\mathbf{F}(\mathbf{x}) \wedge \mathbf{F}(\mathbf{q})\big)}{C(\mathbf{F}(\mathbf{q}))} = \big(\vartheta_{\mathbf{F}(\mathbf{q})}(C) \circ \boldsymbol{\delta}_{\mathbf{F}(\mathbf{q}),C}\big)\big(\mathbf{F}(\mathbf{x}) \wedge \mathbf{F}(\mathbf{q})\big) = \big(\vartheta_{\mathbf{F}(\mathbf{q})}(C)\big)\big(\mathbf{F}_{\mathbf{q}}(\mathbf{x})\big)$$

for all $\mathbf{x} \in \mathbb{R}^d$. Thus, $\vartheta_{\mathbf{F}(\mathbf{q})}(C)$ is a copula for $F_{\mathbf{q}}$. □

In the following we demonstrate how the conditioning $\vartheta_{\mathbf{a}}$ can be used as a tool to obtain copulas also for those conditional distribution functions where the conditioning event is somehow reflected. To this end, for $\mathbf{q} \in \mathbb{R}^d$ and $K \subseteq \{1, ..., d\}$ for which $P\big[\bigcap_{k\in K}\{X_k > q_k\} \cap \bigcap_{k\in\{1,...,d\}\setminus K}\{X_k \le q_k\}\big] > 0$, we define the map $F_{\mathbf{q},K} : \mathbb{R}^d \to \mathrm{I}$ by letting

$$F_{\mathbf{q},K}(\mathbf{x}) := P\left[\{\mathbf{X} \le \mathbf{x}\} \,\middle|\, \bigcap_{k\in K}\{X_k > q_k\} \cap \bigcap_{k\in\{1,...,d\}\setminus K}\{X_k \le q_k\}\right]$$

Then $\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}), \mathbf{1} - \mathbf{F}(\mathbf{q})) \in (\mathbf{0}, \mathbf{1}]$ and $F_{\mathbf{q},K}$ is the distribution function of the conditional distribution of $\mathbf{X}$ under the event $X_k > q_k$ for all $k \in K$ and $X_k \le q_k$ for all $k \in \{1, ..., d\}\setminus K$. We denote by $\mathbf{F}_{\mathbf{q},K} := (F_{\mathbf{q},K,1}, ..., F_{\mathbf{q},K,d})'$ the vector of univariate margins of $F_{\mathbf{q},K}$.

We next show that $(\nu_K \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}),\mathbf{1}-\mathbf{F}(\mathbf{q}))} \circ \nu_K)(C)$ (where $\nu_K(C)$ is the reflection of C with respect to the coordinates $k \in K$) is a copula for the conditional distribution function $F_{\mathbf{q},K}$ whenever C is a copula for F; in the case of univariate conditioning of the first coordinate compare also [17, 21]. Again, note that, in contrast to the literature, we here do not require any assumption neither on the distribution function F nor on the copula C, i.e. the presented copula $(\nu_K \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}),\mathbf{1}-\mathbf{F}(\mathbf{q}))} \circ \nu_K)(C)$ for the conditional distribution function $F_{\mathbf{q},K}$ may fail to be unique.

4.2 Theorem. *Consider $\mathbf{q} \in \mathbb{R}^d$ and $K \subseteq \{1, ..., d\}$ for which $\nu_K(C) \in \mathcal{C}_{\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}),\mathbf{1}-\mathbf{F}(\mathbf{q}))}$. Then*

$$\big(\nu_K \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}),\mathbf{1}-\mathbf{F}(\mathbf{q}))} \circ \nu_K\big)(C)$$

is a copula for $F_{\mathbf{q},K}$. If the margins of F are continuous, then the copulas C and $(\nu_K \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}),\mathbf{1}-\mathbf{F}(\mathbf{q}))} \circ \nu_K)(C)$ are unique.

Proof. Define $\mathbf{a} := \mathbf{F}(\mathbf{q})$. Then $\boldsymbol{\eta}_K(\mathbf{a}, \mathbf{1} - \mathbf{a}) \in (\mathbf{0}, \mathbf{1}]$. Note that, by [15, Theorem 2.2], we first have

$$F_{\mathbf{q},K}(\mathbf{x}) \;=\; \frac{P[\bigcap_{k\in K}\{q_k < X_k \le x_k\} \cap \bigcap_{k\notin K}\{X_k \le x_k \wedge q_k\}]}{P[\bigcap_{k\in K}\{q_k < X_k\} \cap \bigcap_{k\notin K}\{X_k \le q_k\}]}$$

$$
\begin{aligned}
&= \frac{P_{\mathbf{X}}\left[\prod_{k\in K}(q_k,x_k]\times\prod_{k\notin K}(-\infty,x_k\wedge q_k]\right]}{P_{\mathbf{X}}\left[\prod_{k\in K}(q_k,\infty)\times\prod_{k\notin K}(-\infty,q_k]\right]}\\
&= \frac{Q^C\left[\prod_{k\in K}\big(a_k,F_k(x_k)\big]\times\prod_{k\notin K}\big[0,F_k(x_k)\wedge a_k\big]\right]}{Q^C\left[\prod_{k\in K}(a_k,1]\times\prod_{k\notin K}[0,a_k]\right]}\\
&= \frac{(Q^{\nu_K(C)})_{T(\nu_K)}\left[\prod_{k\in K}\big(a_k,F_k(x_k)\big]\times\prod_{k\notin K}\big[0,F_k(x_k)\wedge a_k\big]\right]}{(Q^{\nu_K(C)})_{T(\nu_K)}\left[\prod_{k\in K}(a_k,1]\times\prod_{k\notin K}[0,a_k]\right]}\\
&= \frac{Q^{\nu_K(C)}\left[\prod_{k\in K}\big(1-F_k(x_k),1-a_k\big]\times\prod_{k\notin K}\big[0,F_k(x_k)\wedge a_k\big]\right]}{Q^{\nu_K(C)}\left[\prod_{k\in K}[0,1-a_k]\times\prod_{k\notin K}[0,a_k]\right]}\\
&= \frac{Q^{\nu_K(C)}\left[\prod_{k\in K}\big(1-F_k(x_k),1-a_k\big]\times\prod_{k\notin K}\big[0,F_k(x_k)\wedge a_k\big]\right]}{(\nu_K(C))(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}))}
\end{aligned}
$$

for all $\mathbf{x}\in\mathbb{R}^d$ and hence

$$
\begin{aligned}
1-F_{\mathbf{q},K,k}(x_k) &= 1-\frac{Q^{\nu_K(C)}\left[\prod_{l\in K\setminus\{k\}}\big[0,1-a_l\big]\times\big(1-F_k(x_k),1-a_k\big]\times\prod_{l\notin K}\big[0,a_l\big]\right]}{Q^{\nu_K(C)}\left[\prod_{l\in K}[0,1-a_l]\times\prod_{l\notin K}[0,a_l]\right]}\\
&= \frac{Q^{\nu_K(C)}\left[\prod_{l\in K\setminus\{k\}}\big[0,1-a_l\big]\times\big[0,1-F_k(x_k)\big]\times\prod_{l\notin K}\big[0,a_l\big]\right]}{Q^{\nu_K(C)}\left[\prod_{l\in K}[0,1-a_l]\times\prod_{l\notin K}[0,a_l]\right]}\\
&= \frac{(\nu_K(C))\big(\boldsymbol{\eta}_k\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),(1-F_k(x_k))\,\mathbf{e}_k\big)\big)}{(\nu_K(C))(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}))}\\
&= \delta^k_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),\nu_K(C)}(1-F_k(x_k))
\end{aligned}
$$

for all $x_k\in\mathbb{R}$ and all $k\in K$, as well as

$$
\begin{aligned}
F_{\mathbf{q},K,m}(x_m) &= \frac{Q^{\nu_K(C)}\left[\prod_{l\in K}\big[0,1-a_l\big]\times\big[0,F_m(x_m)\wedge a_m\big]\times\prod_{l\in\{1,\dots,d\}\setminus(K\cup\{m\})}\big[0,a_l\big]\right]}{(\nu_K(C))(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}))}\\
&= \frac{(\nu_K(C))\big(\boldsymbol{\eta}_m\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),(F_m(x_m)\wedge a_m)\,\mathbf{e}_m\big)\big)}{(\nu_K(C))(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}))}\\
&= \delta^m_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),\nu_K(C)}(F_m(x_m)\wedge a_m)
\end{aligned}
$$

for all $x_m\in\mathbb{R}$ and all $m\in\{1,...,d\}\setminus K$. Thus,

$$
\big(T(\nu_K)\circ\mathbf{F}_{\mathbf{q},K}\big)(\mathbf{x})=\boldsymbol{\eta}_K\big(\mathbf{F}_{\mathbf{q},K}(\mathbf{x}),\mathbf{1}-\mathbf{F}_{\mathbf{q},K}(\mathbf{x})\big)=\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),\nu_K(C)}\big(\boldsymbol{\eta}_K\big(\mathbf{F}(\mathbf{x})\wedge\mathbf{a},\mathbf{1}-\mathbf{F}(\mathbf{x})\big)\big)
$$

for all $\mathbf{x}\in\mathbb{R}^d$. By [13, Theorem 4.1], we finally obtain

$$
\begin{aligned}
&\Big(\big(\nu_K\circ\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})}\circ\nu_K\big)(C)\Big)(\mathbf{F}_{\mathbf{q},K}(\mathbf{x}))\\
&= \sum_{L\subseteq K}(-1)^{|K|-|L|}\,\Big(\big(\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})}\circ\nu_K\big)(C)\Big)\Big(\boldsymbol{\eta}_L\big(\boldsymbol{\eta}_K\big(\mathbf{F}_{\mathbf{q},K}(\mathbf{x}),\mathbf{1}-\mathbf{F}_{\mathbf{q},K}(\mathbf{x})\big),\mathbf{1}\big)\Big)\\
&= \sum_{L\subseteq K}(-1)^{|K|-|L|}\,\Big(\big(\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})}\circ\nu_K\big)(C)\Big)\Big(\boldsymbol{\eta}_L\Big(\boldsymbol{\eta}_K\big(\mathbf{F}_{\mathbf{q},K}(\mathbf{x}),\mathbf{1}-\mathbf{F}_{\mathbf{q},K}(\mathbf{x})\big),\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),\nu_K(C)}\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})\big)\Big)\Big)\\
&= \sum_{L\subseteq K}(-1)^{|K|-|L|}\,\Big(\big(\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})}\circ\nu_K\big)(C)\circ\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),\nu_K(C)}\Big)\Big(\boldsymbol{\eta}_L\Big(\boldsymbol{\eta}_K\big(\mathbf{F}(\mathbf{x})\wedge\mathbf{a},\mathbf{1}-\mathbf{F}(\mathbf{x})\big),\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})\Big)\Big)\\
&= \sum_{L\subseteq K}(-1)^{|K|-|L|}\,\Big(\big(\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a})}\circ\nu_K\big)(C)\circ\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}),\nu_K(C)}\Big)\Big(\boldsymbol{\eta}_L\Big(\boldsymbol{\eta}_K\big(\mathbf{F}(\mathbf{x})\wedge\mathbf{a},\mathbf{1}-\mathbf{F}(\mathbf{x})\big),\mathbf{1}-\mathbf{a}\Big)\Big)\\
&= \sum_{L\subseteq K}(-1)^{|K|-|L|}\,\frac{(\nu_K(C))\Big(\boldsymbol{\eta}_L\Big(\boldsymbol{\eta}_K\big(\mathbf{F}(\mathbf{x})\wedge\mathbf{a},\mathbf{1}-\mathbf{F}(\mathbf{x})\big),\mathbf{1}-\mathbf{a}\Big)\Big)}{(\nu_K(C))(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}))}\\
&= \frac{Q^{\nu_K(C)}\left[\prod_{k\in K}\big(1-F_k(x_k),1-a_k\big]\times\prod_{k\notin K}\big[0,F_k(x_k)\wedge a_k\big]\right]}{(\nu_K(C))(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}-\mathbf{a}))}\\
&= F_{\mathbf{q},K}(\mathbf{x})
\end{aligned}
$$

for all $\mathbf{x} \in \mathbb{R}^d$. Thus, $(\nu_K \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{F}(\mathbf{q}),\mathbf{1}-\mathbf{F}(\mathbf{q}))} \circ \nu_K)(C)$ is a copula for $F_{\mathbf{q},K}$. $\square$

4.3 Remark.
(1) For $K = \emptyset$, Theorem 4.2 presents a copula for the conditional distribution function that is conditioned with respect to the event that the observations lie below some threshold $\mathbf{q}$. It thus coincides with Theorem 4.1.
(2) For $K = \{1, ..., d\}$, Theorem 4.2 presents a copula for the conditional distribution function that is conditioned with respect to the event that the observations lie above some threshold $\mathbf{q}$.
(3) We note that, similar to Theorem 4.2, conditionings of copulas and reflections of copulas can also be used as a tool to obtain copulas for conditional survival functions.

5. The class Θ

In the present section we study some properties of the class

$$\Theta := \{\vartheta_{\mathbf{a}} \,|\, \mathbf{a} \in (\mathbf{0}, \mathbf{1}]\}$$

of all conditionings. First, we investigate the behaviour of conditioning under composition; see also [4, Lemma 4.2 (ii)] who showed a similar result in the bivariate case under the assumption that the copula under consideration is strictly increasing in each coordinate:

5.1 Lemma. *Consider $\mathbf{a}, \mathbf{b} \in (\mathbf{0}, \mathbf{1}]$. The identity*

$$\big(\vartheta_{\mathbf{a}} \circ \vartheta_{\mathbf{b}}\big)(C) = \vartheta_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})}(C)$$

holds for all $C \in \mathcal{C}_{\mathbf{b}}$ such that $\vartheta_{\mathbf{b}}(C) \in \mathcal{C}_{\mathbf{a}}$. In particular, $\vartheta_{\mathbf{a}} \circ \vartheta_{\mathbf{1}} = \vartheta_{\mathbf{a}} = \vartheta_{\mathbf{1}} \circ \vartheta_{\mathbf{a}}$.

Proof. First, note that $(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}) \in (\mathbf{0}, \mathbf{b}]$ for all $C \in \mathcal{C}_{\mathbf{b}}$, and hence $\mathbf{s} \in [\mathbf{0}, \mathbf{b}]$ such that $\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s}) \in [\mathbf{0}, \mathbf{a}]$ if, and only if, $\mathbf{s} \in [\mathbf{0}, (\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})]$. Now, consider $C \in \mathcal{C}$, and observe that $C \in \mathcal{C}_{\mathbf{b}}$ such that $\vartheta_{\mathbf{b}}(C) \in \mathcal{C}_{\mathbf{a}}$ if, and only if, $C \in \mathcal{C}_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})}$. In this case, by Corollary 3.5 and Theorem 3.3, we have

$$\begin{aligned}
(\delta^k_{\mathbf{a},\vartheta_{\mathbf{b}}(C)} \circ \delta^k_{\mathbf{b},C})(s_k) &= \frac{1}{(\vartheta_{\mathbf{b}}(C))(\mathbf{a})} \, \big(\vartheta_{\mathbf{b}}(C)\big)\big(\boldsymbol{\eta}_k\big(\mathbf{a}, \delta^k_{\mathbf{b},C}(s_k)\, \mathbf{e}_k\big)\big) \\
&= \frac{C(\mathbf{b})}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, \frac{1}{C(\mathbf{b})} \, \big(C \circ (\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\big)\big(\boldsymbol{\eta}_k\big(\mathbf{a}, \delta^k_{\mathbf{b},C}(s_k)\, \mathbf{e}_k\big)\big) \\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, \big(C \circ (\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\big)\big(\boldsymbol{\eta}_k\big((\boldsymbol{\delta}_{\mathbf{b},C} \circ (\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow})(\mathbf{a}), \delta^k_{\mathbf{b},C}(s_k)\, \mathbf{e}_k\big)\big) \\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, \big(C \circ (\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{b},C}\big)\big(\boldsymbol{\eta}_k\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}), s_k\, \mathbf{e}_k\big)\big) \\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, C\big(\boldsymbol{\eta}_k\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}), s_k\, \mathbf{e}_k\big)\big) \\
&= \delta^k_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}),C}(s_k)
\end{aligned}$$

for all $s_k \in [0, b_k]$ such that $\delta^k_{\mathbf{b},C}(s_k) \in [0, a_k]$ and all $k \in \{1, ..., d\}$, and hence $(\boldsymbol{\delta}_{\mathbf{a},\vartheta_{\mathbf{b}}(C)} \circ \boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{s}) = \boldsymbol{\delta}_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}),C}(\mathbf{s})$ for all $\mathbf{s} \in [\mathbf{0}, \mathbf{b}]$ such that $\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s}) \in [\mathbf{0}, \mathbf{a}]$. Thus, by Corollary 3.5, we finally obtain

$$\begin{aligned}
\big((\vartheta_{\mathbf{a}} \circ \vartheta_{\mathbf{b}})(C)\big)(\mathbf{u}) &= \frac{1}{(\vartheta_{\mathbf{b}}(C))(\mathbf{a})} \, \Big(\vartheta_{\mathbf{b}}(C) \circ (\boldsymbol{\delta}_{\mathbf{a},\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\Big)(\mathbf{u}) \\
&= \frac{C(\mathbf{b})}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, \frac{1}{C(\mathbf{b})} \, \Big(C \circ (\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow} \circ (\boldsymbol{\delta}_{\mathbf{a},\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\Big)(\mathbf{u}) \\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, Q^C\big[\big\{\mathbf{s} \in [\mathbf{0}, \mathbf{b}] \,\big|\, \mathbf{s} \le \big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow} \circ (\boldsymbol{\delta}_{\mathbf{a},\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\big)(\mathbf{u})\big\}\big] \\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)} \, Q^C\big[\big\{\mathbf{s} \in [\mathbf{0}, \mathbf{b}] \,\big|\, \boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s}) \in [\mathbf{0}, \mathbf{a}], \boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s}) \le (\boldsymbol{\delta}_{\mathbf{a},\vartheta_{\mathbf{b}}(C)})^{\rightarrow}(\mathbf{u})\big\}\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)}\, Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{b}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\mathbf{a}], \big(\boldsymbol{\delta}_{\mathbf{a},\vartheta_{\mathbf{b}}(C)}\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{s})\le\mathbf{u}\big\}\big]\\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)}\, Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{b}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\mathbf{a}], \boldsymbol{\delta}_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}),C}(\mathbf{s})\le\mathbf{u}\big\}\big]\\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)}\, Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})]\,\big|\,\mathbf{s}\le(\boldsymbol{\delta}_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}),C})^{\rightarrow}(\mathbf{u})\big\}\big]\\
&= \frac{1}{C\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})\big)}\, \big(C\circ(\boldsymbol{\delta}_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a}),C})^{\rightarrow}\big)(\mathbf{u})\\
&= \big(\vartheta_{(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}(\mathbf{a})}(C)\big)(\mathbf{u})
\end{aligned}
$$

for all $\mathbf{u}\in\mathrm{I}^d$. This proves the identity. □

5.2 Remark. The class Θ fails to be closed under composition since the transformation on the right hand side of Lemma 5.1 depends on the argument C. In addition, Θ is non–commutative (see, e.g., Example 3 in [26]).

It turns out that, for the composition of two univariate conditionings of the same coordinate, the conditioning transformation on the right hand side of Lemma 5.1 is independent of the copula to be transformed. Therefore, for every $k\in\{1,...,d\}$, the class

$$\Theta_k := \{\vartheta_{\boldsymbol{\eta}_k(\mathbf{1},a\,\mathbf{e}_k)}\,|\,a\in(0,1]\}$$

of all univariate conditionings with respect to k has a particularly nice property; see also [19, 20, 26]:

5.3 Proposition.
(1) *The identity*

$$\vartheta_{\boldsymbol{\eta}_k(\mathbf{1},a\,\mathbf{e}_k)}\circ\vartheta_{\boldsymbol{\eta}_k(\mathbf{1},b\,\mathbf{e}_k)} = \vartheta_{\boldsymbol{\eta}_k(\mathbf{1},ab\,\mathbf{e}_k)}$$

holds for all $k\in\{1,...,d\}$ *and all* $a,b\in(0,1]$.
(2) *For every* $k\in\{1,...,d\}$, $(\Theta_k,\circ)$ *is a commutative subsemigroup of* Φ *including the identity of* Φ.

A converse version of Lemma 5.1 is given by the following result; see also [4, Lemma 4.2 (i)] who showed the result in the bivariate case under the assumption that the copula under consideration is strictly increasing in each coordinate:

5.4 Lemma. *The identity*

$$\big(\vartheta_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})}\circ\vartheta_{\mathbf{b}}\big)(C) = \vartheta_{\mathbf{a}}(C)$$

holds for all $\mathbf{b}\in(\mathbf{0},\mathbf{1}]$*, all* $\mathbf{a}\in(\mathbf{0},\mathbf{b}]$ *and all* $C\in\mathcal{C}_{\mathbf{b}}$ *such that* $\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})\in(\mathbf{0},\mathbf{1}]$ *and* $\vartheta_{\mathbf{b}}(C)\in\mathcal{C}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})}$.

Proof. Consider $\mathbf{b}\in(\mathbf{0},\mathbf{1}]$ and $\mathbf{a}\in(\mathbf{0},\mathbf{b}]$. First note that, for every $C\in\mathcal{C}$ with $\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})\in(\mathbf{0},\mathbf{1}]$, $C\in\mathcal{C}_{\mathbf{b}}$ such that $\vartheta_{\mathbf{b}}(C)\in\mathcal{C}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})}$ if, and only if, $C\in\mathcal{C}_{\mathbf{a}}$. In this case, we have

$$
\begin{aligned}
(\delta^k_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)}\circ\delta^k_{\mathbf{b},C})(s_k) &= \frac{1}{\big(\vartheta_{\mathbf{b}}(C)\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{a})}\,\big(\vartheta_{\mathbf{b}}(C)\big)\big(\boldsymbol{\eta}_k\big(\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\delta^k_{\mathbf{b},C}(s_k)\,\mathbf{e}_k\big)\big)\\
&= \frac{1}{\big(\vartheta_{\mathbf{b}}(C)\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{a})}\,\big(\vartheta_{\mathbf{b}}(C)\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)\big(\boldsymbol{\eta}_k(\mathbf{a},s_k\mathbf{e}_k)\big)\\
&= \frac{C(\mathbf{b})}{C(\mathbf{a})}\,\frac{1}{C(\mathbf{b})}\,C\big(\boldsymbol{\eta}_k(\mathbf{a},s_k\mathbf{e}_k)\big)\\
&= \frac{1}{C(\mathbf{a})}\,C\big(\boldsymbol{\eta}_k(\mathbf{a},s_k\mathbf{e}_k)\big)\\
&= \delta^k_{\mathbf{a},C}(s_k)
\end{aligned}
$$

for all $s_k\in[0,a_k]$ such that $\delta^k_{\mathbf{b},C}(s_k)\in[0,\delta^k_{\mathbf{b},C}(a_k)]$ and all $k\in\{1,...,d\}$, and hence $(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)}\circ\boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{s})=\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})$ for all $\mathbf{s}\in[\mathbf{0},\mathbf{a}]$ such that $\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})]$. Due to Theorem 3.3 we further obtain the (in–)equality

$$0 \le Q^C\big[\big\{\mathbf{s}\in(\mathbf{a},\mathbf{b}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})],\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\le(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}(\mathbf{u})\big\}\big]$$

$$
\begin{aligned}
&= Q^C\big[\big\{\mathbf{s}\in(\mathbf{a},\mathbf{b}]\,\big|\,\mathbf{s}\le((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ\boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{a}),\mathbf{s}\le\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\big)(\mathbf{u})\big\}\big]\\
&\le Q^C\big[\big(\mathbf{a},((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ\boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{a})\big]\big]\\
&\le Q^C\big[\big[\mathbf{0},((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ\boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{a})\big]\big]-Q^C\big[[\mathbf{0},\mathbf{a}]\big]\\
&= \big(C\circ(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{a})-C(\mathbf{a})\\
&= 0
\end{aligned}
$$

for all $\mathbf{u}\in\mathrm{I}^d$ and Corollary 3.5 hence yields

$$
\begin{aligned}
&\big((\vartheta_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})}\circ\vartheta_{\mathbf{b}})(C)\big)(\mathbf{u})\\
&= \frac{1}{\big(\vartheta_{\mathbf{b}}(C)\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{a})}\,\Big(\vartheta_{\mathbf{b}}(C)\circ(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\Big)(\mathbf{u})\\
&= \frac{C(\mathbf{b})}{C(\mathbf{a})}\,\frac{1}{C(\mathbf{b})}\,\Big(C\circ(\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\Big)(\mathbf{u})\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{b}]\,\big|\,\mathbf{s}\le\big((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}\big)(\mathbf{u})\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{b}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})],\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\le(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}(\mathbf{u})\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{a}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})],\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\le(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)})^{\rightarrow}(\mathbf{u})\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{a}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})],\big(\boldsymbol{\delta}_{\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a}),\vartheta_{\mathbf{b}}(C)}\circ\boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{s})\le\mathbf{u}\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{a}]\,\big|\,\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{s})\in[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{a})],\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})\le\mathbf{u}\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{a}]\,\big|\,\mathbf{s}\le((\boldsymbol{\delta}_{\mathbf{b},C})^{\rightarrow}\circ\boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{a}),\mathbf{s}\le(\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}(\mathbf{u})\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,Q^C\big[\big\{\mathbf{s}\in[\mathbf{0},\mathbf{a}]\,\big|\,\mathbf{s}\le(\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}(\mathbf{u})\big\}\big]\\
&= \frac{1}{C(\mathbf{a})}\,\big(C\circ(\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow}\big)(\mathbf{u})\\
&= \big(\vartheta_{\mathbf{a}}(C)\big)(\mathbf{u})
\end{aligned}
$$

for all $\mathbf{u}\in\mathrm{I}^d$. This proves the identity. □

6. The class $\boldsymbol{\Theta}$ and the group $\boldsymbol{\Gamma}$

In this section we investigate the behaviour of conditioning with respect to the transformations in the group Γ. We start with a quite helpful result concerning permutations $\pi\in\Gamma^\pi$ and reflections $\nu\in\Gamma^\nu$:

6.1 Lemma. *Consider $\mathbf{a}\in(\mathbf{0},\mathbf{1}]$.*

(1) *The identity*

$$\boldsymbol{\delta}_{(T(\pi))(\mathbf{a}),\pi(C)}\circ T(\pi)=T(\pi)\circ\boldsymbol{\delta}_{\mathbf{a},C}$$

holds for all $\pi\in\Gamma^\pi$ and all $C\in\mathcal{C}_{\mathbf{a}}$.

(2) *The identity*

$$\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),\nu_L(C)}\circ T(\nu_L)=T(\nu_L)\circ\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}$$

holds for all $K\subseteq\{1,...,d\}$, all $L\subseteq K$ and all $C\in\mathcal{C}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})}$.

Proof. To prove (1), consider $C\in\mathcal{C}_{\mathbf{a}}$ and $i,j\in\{1,...,d\}$ such that $i\neq j$. Then

$$\delta^i_{(T(\pi_{i,j}))(\mathbf{a}),\pi_{i,j}(C)}(s_j)=\frac{\big(\pi_{i,j}(C)\big)\big(\boldsymbol{\eta}_i\big((T(\pi_{i,j}))(\mathbf{a}),s_j\,\mathbf{e}_i\big)\big)}{\big(\pi_{i,j}(C)\big)\big((T(\pi_{i,j}))(\mathbf{a})\big)}=\frac{C\big(\boldsymbol{\eta}_j(\mathbf{a},s_j\,\mathbf{e}_j)\big)}{C(\mathbf{a})}=\delta^j_{\mathbf{a},C}(s_j)$$

for all $s_j\in[0,a_j]$ and, analogously, we have $\delta^j_{(T(\pi_{i,j}))(\mathbf{a}),\pi_{i,j}(C)}(s_i)=\delta^i_{\mathbf{a},C}(s_i)$ for all $s_i\in[0,a_i]$ and $\delta^l_{(T(\pi_{i,j}))(\mathbf{a}),\pi_{i,j}(C)}=\delta^l_{\mathbf{a},C}$ for all $l\in\{1,...,d\}\backslash\{i,j\}$. Thus, $\boldsymbol{\delta}_{(T(\pi_{i,j}))(\mathbf{a}),\pi_{i,j}(C)}\circ T(\pi_{i,j})=T(\pi_{i,j})\circ\boldsymbol{\delta}_{\mathbf{a},C}$. The assertion hence follows from the fact that the set $\{\pi_{i,j}\,|\,i,j\in\{1,...,d\},i\neq j\}$ generates Γ^π.

To prove (2), consider $K \subseteq \{1,...,d\}$, $L \subseteq K$ and $C \in \mathcal{C}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})}$. Then, for every $l,k \in L$ such that $l \neq k$, we have

$$\begin{aligned}
\delta^l_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),\nu_l(C)}(1-s_l) &= \frac{\big(\nu_l(C)\big)\big(\boldsymbol{\eta}_l\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),(1-s_l)\,\mathbf{e}_l\big)\big)}{\big(\nu_l(C)\big)\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big)} \\
&= \frac{C\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big) - C\big(\boldsymbol{\eta}_l\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),s_l\,\mathbf{e}_l\big)\big)}{C\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big) - 0} \\
&= 1 - \delta^l_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}(s_l)
\end{aligned}$$

and

$$\delta^l_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),\nu_k(C)}(s_l) = \frac{\big(\nu_k(C)\big)\big(\boldsymbol{\eta}_l\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),s_l\,\mathbf{e}_l\big)\big)}{\big(\nu_k(C)\big)\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big)} = \frac{C\big(\boldsymbol{\eta}_l\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),s_l\,\mathbf{e}_l\big)\big) - 0}{C\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big) - 0} = \delta^l_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}(s_l)$$

for all $s_l \in [0,1]$, and (analogously) $\delta^m_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),\nu_l(C)} = \delta^m_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}$ for all $l \in L$ and all $m \in \{1,...,d\}\backslash L$. Thus, $\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),\nu_L(C)} \circ T(\nu_L) = T(\nu_L) \circ \boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}$. This proves (2). □

We first point out the behaviour of conditioning with respect to permutations. It turns out that (1) univariate conditionings of different coordinates are interchangeable using transpositions, and that (2) every univariate conditioning commutes with permutations of other coordinates (see also [19]).

6.2 Theorem. *Consider $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$. The identity*

$$\vartheta_{\mathbf{a}} = \pi \circ \vartheta_{(T(\pi))(\mathbf{a})} \circ \pi$$

holds for all $\pi \in \Gamma^\pi$. In particular, for $a \in (0,1]$,
(1) *the identity $\vartheta_{\boldsymbol{\eta}_i(\mathbf{1},a\,\mathbf{e}_i)} = \pi_{i,j} \circ \vartheta_{\boldsymbol{\eta}_j(\mathbf{1},a\,\mathbf{e}_j)} \circ \pi_{i,j}$ holds for all $i,j \in \{1,...,d\}$ such that $i \neq j$, and*
(2) *the identity $\vartheta_{\boldsymbol{\eta}_k(\mathbf{1},a\,\mathbf{e}_k)} = \pi_{i,j} \circ \vartheta_{\boldsymbol{\eta}_k(\mathbf{1},a\,\mathbf{e}_k)} \circ \pi_{i,j}$ holds for all $k \in \{1,...,d\}$ and all $i,j \in \{1,...,d\}\backslash\{k\}$ such that $i \neq j$.*

Proof. Consider $C \in \mathcal{C}_{\mathbf{a}}$ and $\pi \in \Gamma^\pi$. By [15, Theorem 2.2], Corollary 3.5 and Lemma 6.1, we obtain

$$\begin{aligned}
Q^{(\pi\circ\vartheta_{(T(\pi))(\mathbf{a})}\circ\pi)(C)} &= \big(Q^{(\vartheta_{(T(\pi))(\mathbf{a})}\circ\pi)(C)}\big)_{T(\pi)} \\
&= \frac{1}{\big(\pi(C)\big)\big((T(\pi))(\mathbf{a})\big)}\,\big(Q^{\pi(C)}\big)_{\boldsymbol{\delta}_{(T(\pi))(\mathbf{a}),\pi(C)}\circ T(\pi)} \\
&= \frac{1}{C(\mathbf{a})}\,\big(Q^{\pi(C)}\big)_{T(\pi)\circ\boldsymbol{\delta}_{\mathbf{a},C}} \\
&= \frac{1}{C(\mathbf{a})}\,\big(Q^{C}\big)_{\boldsymbol{\delta}_{\mathbf{a},C}} \\
&= Q^{\vartheta_{\mathbf{a}}(C)}
\end{aligned}$$

This proves the assertion. □

Now, we point out the behaviour of conditioning with respect to reflections. It turns out that every univariate conditioning also commutes with reflections of other coordinates; see also [19].

6.3 Theorem. *Consider $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$. The identity*

$$\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})} = \nu_L \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})} \circ \nu_L$$

holds for all $K \subseteq \{1,...,d\}$ and all $L \subseteq K$. In particular, for $a \in (0,1]$, the identity

$$\vartheta_{\boldsymbol{\eta}_k(\mathbf{1},a\,\mathbf{e}_k)} = \nu_l \circ \vartheta_{\boldsymbol{\eta}_k(\mathbf{1},a\,\mathbf{e}_k)} \circ \nu_l$$

holds for $k,l \in \{1,...,d\}$ such that $k \neq l$.

Proof. Consider $K \subseteq \{1, ..., d\}$, $L \subseteq K$ and $C \in \mathcal{C}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})}$. By [15, Theorem 2.2], Corollary 3.5 and Lemma 6.1, we obtain

$$\begin{aligned} Q^{(\nu_L \circ \vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})} \circ \nu_L)(C)} &= \Big(Q^{(\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})} \circ \nu_L)(C)}\Big)_{T(\nu_L)} \\ &= \frac{1}{\big(\nu_L(C)\big)\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big)} \,\big(Q^{\nu_L(C)}\big)_{\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),\nu_L(C)} \circ T(\nu_L)} \\ &= \frac{1}{C\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big)} \,\big(Q^{\nu_L(C)}\big)_{T(\nu_L) \circ \boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}} \\ &= \frac{1}{C\big(\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})\big)} \,\big(Q^{C}\big)_{\boldsymbol{\delta}_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1}),C}} \\ &= Q^{\vartheta_{\boldsymbol{\eta}_K(\mathbf{a},\mathbf{1})}(C)} \end{aligned}$$

This proves the assertion. □

7. The class Θ and invariance

Copulas which are invariant under conditioning have been studied frequently in literature (see, e.g., [2, 4, 9, 10, 18, 19, 29, 31]), and they are of particular interest since for such copulas the values of copula–based measures of association (like measures of concordance) remain unchanged under conditioning.

A copula $C \in \mathcal{C}$ is said to be *invariant under* Θ (or *invariant under conditioning*) if it satisfies $\vartheta_{\mathbf{a}}(C) = C$ for all $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ such that $C \in \mathcal{C}_{\mathbf{a}}$. The following examples show that the product copula Π and the Fréchet–Hoeffding upper bound M are invariant under Θ; see also [18]. We additionally show that every reflection of M is invariant under Θ as well; note that, for $d = 2$, this includes the Fréchet–Hoeffding lower bound W.

7.1 Examples.

(1) The product copula Π is invariant under Θ.
Indeed, for every $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$, the copula Π satisfies $\Pi \in \mathcal{C}_{\mathbf{a}}$ and $\big(\Pi \circ (\boldsymbol{\delta}_{\mathbf{a},\Pi})^{\rightarrow}\big)(\mathbf{u}) = \Pi(\mathbf{a})\,\Pi(\mathbf{u})$ and, by Corollary 3.5, we obtain

$$\big(\vartheta_{\mathbf{a}}(\Pi)\big)(\mathbf{u}) = \frac{\big(\Pi \circ (\boldsymbol{\delta}_{\mathbf{a},\Pi})^{\rightarrow}\big)(\mathbf{u})}{\Pi(\mathbf{a})} = \frac{\Pi(\mathbf{a})\,\Pi(\mathbf{u})}{\Pi(\mathbf{a})} = \Pi(\mathbf{u})$$

for all $\mathbf{u} \in \mathrm{I}^d$.

(2) For every $\nu \in \Gamma^{\nu}$, the reflected Fréchet–Hoeffding upper bound $\nu(M)$ is invariant under Θ. In particular,
- the Fréchet–Hoeffding upper bound $M = \iota(M)$ is invariant under Θ.
- for $d = 2$, the Fréchet–Hoeffding lower bound $W = \nu_1(M)$ is invariant under Θ.

Indeed, consider $K \subseteq \{1, ..., d\}$ and $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ such that $\nu_K(M) \in \mathcal{C}_{\mathbf{a}}$. First, by [15, Lemma 2.3], we have

$$\begin{aligned} \big(\nu_K(M)\big)(\mathbf{u}) &= \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}(\mathbf{s}) \,\mathrm{d}Q^{\nu_K(M)}(\mathbf{s}) \\ &= \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}(\boldsymbol{\eta}_K(\mathbf{s}, \mathbf{1} - \mathbf{s})) \,\mathrm{d}Q^{M}(\mathbf{s}) \\ &= \int_{\mathrm{I}} \chi_{[\boldsymbol{\eta}_K(\mathbf{0},\mathbf{1}-\mathbf{u}),\boldsymbol{\eta}_K(\mathbf{u},\mathbf{1})]}(s\mathbf{1}) \,\mathrm{d}\boldsymbol{\lambda}(s) \\ &= \big(\min\{u_k \,|\, k \notin K\} - \max\{1 - u_k \,|\, k \in K\}\big)^{+} \end{aligned}$$

Thus, for $l \in K$ and $s \in [\max\{1 - a_k \,|\, k \in K\}, \min\{a_k \,|\, k \notin K\}]$, we have

$$\delta^{l}_{\mathbf{a},\nu_K(M)}(1 - s) = \frac{\big(\nu_K(M)\big)\big(\boldsymbol{\eta}_l(\mathbf{a}, (1 - s)\,\mathbf{e}_l)\big)}{\big(\nu_K(M)\big)(\mathbf{a})} = \frac{\min\{a_k \,|\, k \notin K\} - s}{\big(\nu_K(M)\big)(\mathbf{a})}$$

Similarly, for $l \notin K$ and $s \in [\max\{1-a_k \,|\, k \in K\}, \min\{a_k \,|\, k \notin K\}]$, we have

$$\delta^l_{\mathbf{a},\nu_K(M)}(s) = \frac{\big(\nu_K(M)\big)\big(\boldsymbol{\eta}_l(\mathbf{a}, s\,\mathbf{e}_l)\big)}{\big(\nu_K(M)\big)(\mathbf{a})} = \frac{s - \max\{1-a_k \,|\, k \in K\}}{\big(\nu_K(M)\big)(\mathbf{a})}$$

Thus, by Corollary 3.5 and [15, Lemma 2.3], we obtain

$$\begin{aligned}
&\big(\vartheta_{\mathbf{a}}\big(\nu_K(M)\big)\big)(\mathbf{u}) \\
&= \frac{1}{\big(\nu_K(M)\big)(\mathbf{a})} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}\big(\boldsymbol{\delta}_{\mathbf{a},\nu_K(M)}(\mathbf{s})\big)\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^{\nu_K(M)}(\mathbf{s}) \\
&= \frac{1}{\big(\nu_K(M)\big)(\mathbf{a})} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}\big(\boldsymbol{\delta}_{\mathbf{a},\nu_K(M)}\big(\boldsymbol{\eta}_K(\mathbf{s}, \mathbf{1}-\mathbf{s})\big)\big)\, \chi_{[\mathbf{0},\mathbf{a}]}\big(\boldsymbol{\eta}_K(\mathbf{s}, \mathbf{1}-\mathbf{s})\big)\, \mathrm{d}Q^{M}(\mathbf{s}) \\
&= \frac{1}{\big(\nu_K(M)\big)(\mathbf{a})} \int_{\mathrm{I}} \chi_{[\mathbf{0},\mathbf{u}]}\big(\boldsymbol{\delta}_{\mathbf{a},\nu_K(M)}\big(\boldsymbol{\eta}_K(s\mathbf{1}, \mathbf{1}-s\mathbf{1})\big)\big)\, \chi_{[\mathbf{0},\mathbf{a}]}\big(\boldsymbol{\eta}_K(s\mathbf{1}, \mathbf{1}-s\mathbf{1})\big)\, \mathrm{d}\boldsymbol{\lambda}(s) \\
&= \frac{1}{\big(\nu_K(M)\big)(\mathbf{a})} \int_{\mathrm{I}} \chi_{[\min\{a_k \,|\, k\notin K\}-(\nu_K(M))(\mathbf{a})\, \min\{u_k \,|\, k\in K\},(\nu_K(M))(\mathbf{a})\, \min\{u_k \,|\, k\notin K\}+\max\{1-a_k \,|\, k\in K\}]}(s) \\
&\qquad\qquad \chi_{[\mathbf{0},\mathbf{a}]}\big(\boldsymbol{\eta}_K(s\mathbf{1}, \mathbf{1}-s\mathbf{1})\big)\, \mathrm{d}\boldsymbol{\lambda}(s) \\
&= \frac{1}{\big(\nu_K(M)\big)(\mathbf{a})} \Big((\nu_K(M))(\mathbf{a})\, \min\{u_k \,|\, k \notin K\} - \big(\nu_K(M)\big)(\mathbf{a}) + (\nu_K(M))(\mathbf{a})\, \min\{u_k \,|\, k \in K\}\Big)^+ \\
&= \Big(\min\{u_k \,|\, k \notin K\} - 1 + \min\{u_k \,|\, k \in K\}\Big)^+ \\
&= \Big(\min\{u_k \,|\, k \notin K\} - \max\{1-u_k \,|\, k \in K\}\Big)^+ \\
&= \big(\nu_K(M)\big)(\mathbf{u})
\end{aligned}$$

for all $\mathbf{u} \in \mathrm{I}^d$.

A copula $C \in \mathcal{C}$ is said to be *invariant under* Θ_k (or *invariant under univariate conditioning with respect to* k) if it satisfies $\vartheta_{\boldsymbol{\eta}_k(\mathbf{1}, a\,\mathbf{e}_k)}(C) = C$ for all $a \in (0,1]$.

In Theorem 7.2 we show that a copula C is invariant under Θ if, and only if, it is invariant under Θ_k for every $k \in \{1,...,d\}$; see also [9] who proved the result in the bivariate case using the fact that the copulas which are invariant under conditioning belong to the closure of the Clayton family of copulas:

7.2 Theorem. *For a copula $C \in \mathcal{C}$, the following are equivalent:*
(a) *C is invariant under Θ.*
(b) *C is invariant under Θ_k for every $k \in \{1,...,d\}$.*

Proof. It is evident that (a) implies (b). Assume now that (b) holds and consider $C \in \mathcal{C}$. We prove (a) by induction. To this end, for $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$ and for $k \in \{1,...,d\}$, define $\mathbf{a}^k := \boldsymbol{\eta}_{\{1,...,k\}}(\mathbf{1},\mathbf{a})$. Then, by assumption, $\vartheta_{\mathbf{a}^1}(C) = C$, $\mathbf{a}^k \le \mathbf{a}^{k-1}$ and hence $\mathcal{C}_{\mathbf{a}^k} \subseteq \mathcal{C}_{\mathbf{a}^{k-1}}$ for all $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$ and all $k \in \{2,...,d\}$. Further, let $m \in \{2,...,d\}$ and assume that the identity $\vartheta_{\mathbf{a}^k}(C) = C$ holds for all $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$ and all $k \in \{1,...,m-1\}$ such that $C \in \mathcal{C}_{\mathbf{a}^k}$. Then

$$\boldsymbol{\delta}_{\mathbf{a}^{m-1},C}(\mathbf{a}^m) = \boldsymbol{\eta}_m\left(\mathbf{1}, \frac{C(\mathbf{a}^m)}{C(\mathbf{a}^{m-1})}\,\mathbf{e}_m\right) \in (\mathbf{0},\mathbf{1}]$$

for all $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$ such that $C \in \mathcal{C}_{\mathbf{a}^m}$. Since $\mathcal{C}_{\boldsymbol{\delta}_{\mathbf{a}^{m-1},C}(\mathbf{a}^m)} = \mathcal{C}$, by Lemma 5.4 and the assumption, we hence obtain

$$\vartheta_{\mathbf{a}^m}(C) = (\vartheta_{\boldsymbol{\delta}_{\mathbf{a}^{m-1},C}(\mathbf{a}^m)} \circ \vartheta_{\mathbf{a}^{m-1}})(C) = \vartheta_{\boldsymbol{\delta}_{\mathbf{a}^{m-1},C}(\mathbf{a}^m)}(C) = C$$

for all $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$ such that $C \in \mathcal{C}_{\mathbf{a}^m}$. This proves (a). □

A copula $C \in \mathcal{C}$ is said to be *symmetric* if $\pi(C) = C$ holds for every $\pi \in \Gamma^\pi$. For symmetric copulas we obtain the following refinement of Theorem 7.2; see also [9]:

7.3 Corollary. *For a symmetric copula $C \in \mathcal{C}$, the following are equivalent:*

(a) *C is invariant under Θ.*
(b) *C is invariant under Θ_k for every $k \in \{1, ...d\}$.*
(c) *C is invariant under Θ_k for some $k \in \{1, ...d\}$.*

Proof. The assertion immediately follows applying Theorem 6.2 and Theorem 7.2. □

8. The class Θ and measures of concordance

In this final section we study conditioning of copulas in connection with the usual multivariate generalizations of Kendall's tau and Spearman's rho. These measures of concordance can be expressed in terms of the biconvex form $[.\,,.] : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ which is given by

$$[C, D] := \int_{\mathrm{I}^d} C(\mathbf{u})\, \mathrm{d}Q^D(\mathbf{u})$$

and was introduced in [15]. Our aim is to express Kendall's tau and Spearman's rho for the transformed copula $\vartheta_{\mathbf{a}}(C)$ in terms of the initial copula $C \in \mathcal{C}_{\mathbf{a}}$.

Let us first consider Kendall's tau:

The map $\kappa^{(\tau)} : \mathcal{C} \to \mathbb{R}$ given by

$$\kappa^{(\tau)}[C] := \frac{[C, C] - [\Pi, \Pi]}{[M, M] - [\Pi, \Pi]} = \frac{1}{2^{d-1} - 1} \Big(2^d\, [C, C] - 1\Big)$$

is called *Kendall's tau*. This definition is in accordance with the definition proposed in [27]. Since $[C, C] \geq 0 = [\nu_1(M), \nu_1(M)]$ (see [15]), it is evident that C minimizes Kendall's tau if, and only if, $[C, C] = 0$. We shall need the following lemma:

8.1 Lemma. *The identity*

$$[\vartheta_{\mathbf{a}}(C), \vartheta_{\mathbf{b}}(C)] = \frac{1}{C(\mathbf{a})\, C(\mathbf{b})} \int_{\mathrm{I}^d} \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{u})\, \chi_{[\mathbf{0},\mathbf{b}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u})$$

holds for all $\mathbf{a}, \mathbf{b} \in (\mathbf{0}, \mathbf{1}]$ and all $C \in \mathcal{C}_{\mathbf{a}} \cap \mathcal{C}_{\mathbf{b}}$. In particular, the identity

$$[\vartheta_{\mathbf{a}}(C), \vartheta_{\mathbf{a}}(C)] = \frac{1}{C(\mathbf{a})^2} \int_{\mathrm{I}^d} C(\mathbf{u})\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u})$$

holds for all $\mathbf{a} \in (\mathbf{0}, \mathbf{1}]$ and all $C \in \mathcal{C}_{\mathbf{a}}$, and $[C, C] = 0$ implies $[\vartheta_{\mathbf{a}}(C), \vartheta_{\mathbf{a}}(C)] = 0$.

Proof. Consider $\mathbf{a}, \mathbf{b} \in (\mathbf{0}, \mathbf{1}]$. By Corollary 3.5, we obtain

$$\begin{aligned}
[\vartheta_{\mathbf{a}}(C), \vartheta_{\mathbf{b}}(C)] &= \int_{\mathrm{I}^d} \big(\vartheta_{\mathbf{a}}(C)\big)(\mathbf{u})\, \mathrm{d}Q^{\vartheta_{\mathbf{b}}(C)}(\mathbf{u}) \\
&= \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}\big(\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})\big)\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s})\, \mathrm{d}Q^{\vartheta_{\mathbf{b}}(C)}(\mathbf{u}) \\
&= \frac{1}{C(\mathbf{a})\, C(\mathbf{b})} \int_{\mathrm{I}^d} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]}\big(\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})\big)\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s})\, \mathrm{d}\big(Q^C\big)_{\boldsymbol{\delta}_{\mathbf{b},C}}(\mathbf{u}) \\
&= \frac{1}{C(\mathbf{a})\, C(\mathbf{b})} \int_{\mathrm{I}^d} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\boldsymbol{\delta}_{\mathbf{b},C}(\mathbf{u})]}\big(\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})\big)\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \chi_{[\mathbf{0},\mathbf{b}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{u}) \\
&= \frac{1}{C(\mathbf{a})\, C(\mathbf{b})} \int_{\mathrm{I}^d} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},((\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{b},C})(\mathbf{u})]}(\mathbf{s})\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \chi_{[\mathbf{0},\mathbf{b}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{u}) \\
&= \frac{1}{C(\mathbf{a})\, C(\mathbf{b})} \int_{\mathrm{I}^d} \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{b},C}\big)(\mathbf{u})\, \chi_{[\mathbf{0},\mathbf{b}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u})
\end{aligned}$$

for all $C \in \mathcal{C}_{\mathbf{a}} \cap \mathcal{C}_{\mathbf{b}}$, and Theorem 3.3 hence yields

$$[\vartheta_{\mathbf{a}}(C), \vartheta_{\mathbf{a}}(C)] = \frac{1}{C(\mathbf{a})^2} \int_{\mathrm{I}^d} \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \circ \boldsymbol{\delta}_{\mathbf{a},C}\big)(\mathbf{u})\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u})$$

$$= \quad \frac{1}{C(\mathbf{a})^2} \int_{\mathrm{I}^d} C(\mathbf{u})\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u})$$

for all $C \in \mathcal{C}_{\mathbf{a}}$. This proves the assertion. □

The following result is immediate from Lemma 8.1; see also [29]:

8.2 Theorem. *Kendall's tau satisfies*

$$\kappa^{(\tau)}[\vartheta_{\mathbf{a}}(C)] = \frac{1}{2^{d-1}-1} \left(2^d \int_{\mathrm{I}^d} \frac{1}{C(\mathbf{a})^2}\, C(\mathbf{u})\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u}) - 1 \right)$$

for all $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$ *and all* $C \in \mathcal{C}_{\mathbf{a}}$. *In particular, if* C *minimizes Kendall's tau, then* $\vartheta_{\mathbf{a}}(C)$ *minimizes Kendall's tau as well.*

Let us now consider Spearman's rho:

The map $\kappa^{(\rho)} : \mathcal{C} \to \mathbb{R}$ given by

$$\kappa^{(\rho)}[C] := \frac{\frac{1}{2}\,[C,\Pi] + \frac{1}{2}[\tau(C),\Pi] - [\Pi,\Pi]}{[M,\Pi] - [\Pi,\Pi]} = \frac{d+1}{2^d-(d+1)} \Big(2^{d-1} \, \Big([C,\Pi] + [\tau(C),\Pi] \Big) - 1 \Big)$$

is called *Spearman's rho*. This definition is in accordance with that in [16, 27]. Unlike Kendall's tau, Spearman's rho evaluates not only the copula C but also its survival copula $\tau(C)$, which is obtained from C via the total reflection τ. We shall need the following lemma:

8.3 Lemma. *The identity*

$$\begin{aligned}
[\vartheta_{\mathbf{a}}(C), D] \quad &= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \big(C \circ (\boldsymbol{\delta}_{\mathbf{a},C})^{\rightarrow} \big)(\mathbf{u})\, \mathrm{d}Q^D(\mathbf{u}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \big(\tau(D) \big) \big(\mathbf{1} - \boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{u}) \big) \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\, \mathrm{d}Q^C(\mathbf{u})
\end{aligned}$$

holds for all $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$, *all* $C \in \mathcal{C}_{\mathbf{a}}$ *and all* $D \in \mathcal{C}$.

Proof. Consider $\mathbf{a} \in (\mathbf{0},\mathbf{1}]$, $C \in \mathcal{C}_{\mathbf{a}}$ and $D \in \mathcal{C}$. The first identity is a consequence of Corollary 3.5. Moreover, applying Corollary 3.5, Fubini's Theorem and [15, Theorem 2.2], we obtain

$$\begin{aligned}
[\vartheta_{\mathbf{a}}(C), D] \quad &= \quad \int_{\mathrm{I}^d} \big(\vartheta_{\mathbf{a}}(C) \big)(\mathbf{u})\, \mathrm{d}Q^D(\mathbf{u}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \int_{\mathrm{I}^d} \chi_{[\mathbf{0},\mathbf{u}]} \big(\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}) \big)\, \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s})\, \mathrm{d}Q^D(\mathbf{u}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \left(\int_{\mathrm{I}^d} \chi_{[\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}),\mathbf{1}]}(\mathbf{u})\, \mathrm{d}Q^D(\mathbf{u}) \right) \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} Q^D \big[[\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}), \mathbf{1}] \big] \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \big(Q^{\tau(D)} \big)_{T(\tau)} \big[[\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}), \mathbf{1}] \big] \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} Q^{\tau(D)} \big[[\mathbf{0}, \mathbf{1} - \boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s})] \big] \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s}) \\
&= \quad \frac{1}{C(\mathbf{a})} \int_{\mathrm{I}^d} \big(\tau(D) \big) \big(\mathbf{1} - \boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{s}) \big) \chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{s})\, \mathrm{d}Q^C(\mathbf{s})
\end{aligned}$$

This proves the assertion. □

The previous lemma yields the following result:

8.4 Theorem. *Spearman's rho satisfies*

$$\begin{aligned}\kappa^{(\rho)}[\vartheta_{\mathbf{a}}(C)] &= \frac{d+1}{2^d-(d+1)}\biggl(2^{d-1}\int_{\mathrm{I}^d}\frac{1}{(C(\mathbf{a}))^{d+1}}\prod_{k=1}^{d}\Bigl(C(\mathbf{a})-C\bigl(\boldsymbol{\eta}_k(\mathbf{a},u_k\mathbf{e}_k)\bigr)\Bigr)\chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\\ &\qquad +\frac{1}{((\tau(C))(\mathbf{a}))^{d+1}}\prod_{k=1}^{d}\Bigl((\tau(C))(\mathbf{a})-(\tau(C))\bigl(\boldsymbol{\eta}_k(\mathbf{a},(1-u_k)\,\mathbf{e}_k)\bigr)\Bigr)\chi_{[\mathbf{1}-\mathbf{a},\mathbf{1}]}(\mathbf{u})\,\mathrm{d}Q^C(\mathbf{u})-1\biggr)\end{aligned}$$

for all $\mathbf{a}\in(\mathbf{0},\mathbf{1}]$ *and all* $C\in\mathcal{C}_{\mathbf{a}}$.

Proof. Note that $\tau(\Pi)=\Pi$ and consider $\mathbf{a}\in(\mathbf{0},\mathbf{1}]$ and $C\in\mathcal{C}_{\mathbf{a}}$. By Lemma 8.3 and [15, Lemma 2.3], we obtain

$$\begin{aligned}&\left(\frac{2^d-(d+1)}{d+1}\,\kappa^{(\rho)}[\vartheta_{\mathbf{a}}(C)]+1\right)\frac{1}{2^{d-1}}\\ &= [C,\Pi]+[\tau(C),\Pi]\\ &= \int_{\mathrm{I}^d}\frac{\bigl(\tau(\Pi)\bigr)\bigl(\mathbf{1}-\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{u})\bigr)}{C(\mathbf{a})}\chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\,\mathrm{d}Q^C(\mathbf{u})+\int_{\mathrm{I}^d}\frac{\bigl(\tau(\Pi)\bigr)\bigl(\mathbf{1}-\boldsymbol{\delta}_{\mathbf{a},\tau(C)}(\mathbf{u})\bigr)}{(\tau(C))(\mathbf{a})}\chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\,\mathrm{d}Q^{\tau(C)}(\mathbf{u})\\ &= \int_{\mathrm{I}^d}\frac{\Pi\bigl(\mathbf{1}-\boldsymbol{\delta}_{\mathbf{a},C}(\mathbf{u})\bigr)}{C(\mathbf{a})}\chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})+\frac{\Pi\bigl(\mathbf{1}-\boldsymbol{\delta}_{\mathbf{a},\tau(C)}(\mathbf{1}-\mathbf{u})\bigr)}{(\tau(C))(\mathbf{a})}\chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{1}-\mathbf{u})\,\mathrm{d}Q^C(\mathbf{u})\\ &= \int_{\mathrm{I}^d}\frac{1}{(C(\mathbf{a}))^{d+1}}\prod_{k=1}^{d}\Bigl(C(\mathbf{a})-C\bigl(\boldsymbol{\eta}_k(\mathbf{a},u_k\mathbf{e}_k)\bigr)\Bigr)\chi_{[\mathbf{0},\mathbf{a}]}(\mathbf{u})\\ &\qquad +\frac{1}{((\tau(C))(\mathbf{a}))^{d+1}}\prod_{k=1}^{d}\Bigl((\tau(C))(\mathbf{a})-(\tau(C))\bigl(\boldsymbol{\eta}_k(\mathbf{a},(1-u_k)\,\mathbf{e}_k)\bigr)\Bigr)\chi_{[\mathbf{1}-\mathbf{a},\mathbf{1}]}(\mathbf{u})\,\mathrm{d}Q^C(\mathbf{u})\end{aligned}$$

This proves the assertion. □

9. Conclusion

We have introduced conditioning of copulas as a map which transforms every copula into another one, and we have shown, given a copula for a distribution function, how this transformation can be used as a tool to obtain copulas for all those conditional distribution functions that are conditioned with respect to a tail event. We have further investigated the behaviour of conditioning under composition and with respect to permutations and reflections, and we have shown that invariance of a copula under conditioning is equivalent to invariance of a copula under univariate conditioning in each coordinate. We have finally presented representations of the usual multivariate generalizations of Kendall's tau and Spearman's rho for the transformed copula in terms of the initial copula.

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