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On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind

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ABSTRACT

In the article, we prove that the double inequalities

$$\frac{1+(6p-7)r'}{p+(5p-6)r'}\frac{\pi\tanh^{-1}(r)}{2r} < \mathcal{K}(r) < \frac{1+(6q-7)r'}{q+(5q-6)r'}\frac{\pi\tanh^{-1}(r)}{2r},$$

$$\frac{qA(1,r)+(5q-6)G(1,r)}{A(1,r)+(6q-7)G(1,r)}L(1,r) < AGM(1,r) < \frac{pA(1,r)+(5p-6)G(1,r)}{A(1,r)+(6p-7)G(1,r)}L(1,r)$$

hold for all $r\in(0,1)$ if and only if $p\geq\pi/2=1.570796\cdots$ and $q\leq 89/69=1.289855\cdots$, where $\mathcal{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}dt$ is the complete elliptic integral of the first kind, $\tanh^{-1}(r)=\log[(1+r)/(1-r)]/2$ is the inverse hyperbolic tangent function, $r'=\sqrt{1-r^2}$, and $A(1,r)=(1+r)/2$, $G(1,r)=\sqrt{r}$, $L(1,r)=(r-1)/\log r$ and $AGM(1,r)$ are the arithmetic, geometric, logarithmic and Gaussian arithmetic-geometric means of 1 and r, respectively.

1. Introduction

For $r\in(0,1)$, Legendre's complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [13,14] of the first and second kinds are given by

$$\mathcal{K}(r)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-r^2\sin^2(t)}},$$

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$$\mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt,$$

respectively.

The Gaussian arithmetic-geometric mean $AGM(a,b)$ of two positive real numbers a and b is defined as the common limit of the sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$a_0=a,\quad b_0=b,\quad a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n}.$$

The Gaussian and Landen identities [7] show that

$$AGM(1,r)=\frac{\pi}{2\mathcal{K}(r')},\quad \mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right)=(1+r)\mathcal{K}(r) \tag{1.1}$$

for all $r\in(0,1)$, where and in what follows $r'=\sqrt{1-r^2}$.

It is well known that $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the particular cases of the Gaussian hypergeometric function [6,29,33,36,44,45,47]

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\quad(-1<x<1),$$

where $(a)_0=1$ for $a\neq 0$, $(a)_n=a(a+1)(a+2)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma function [25,26,52,53,55,58–60]. Indeed,

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n}, \tag{1.2}$$

$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}.$$

The complete elliptic integrals and Gaussian hypergeometric function have many important applications in mathematics, physics and engineering. For example, the modulus of the plane Grötzsch ring can be expressed in terms of the complete elliptic integral of the first kind, and the complete elliptic integral of the second kind gives the formula of the perimeter of an ellipse. Moreover, Ramanujan modular equation and continued fraction in number theory are both related to the Gaussian hypergeometric function $F(a,b;c;x)$.

Recently, the bounds for the complete elliptic integrals have attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $\mathcal{K}(r)$, $\mathcal{E}(r)$ and $F(a,b;c;x)$ can be found in the literature [3,4,8–12,16–24,27,28,32,35,37–43,46,48,50,51,54,56,57].

Carlson and Vuorinen [15], Vamanamurhty and Vuorinen [34], Qiu and Vamanamurthy [31] and Alzer [1] proved that the double inequalities

$$\frac{\log r'}{r'-1}<\mathcal{K}(r)<\frac{\pi\log r'}{2(r'-1)},$$

$$\left[1+\left(\frac{\pi}{4\log 2}-1\right)r'^2\right]\log\frac{4}{r'}<\mathcal{K}(r)<\left(1+\frac{1}{4}r'^2\right)\log\frac{4}{r'} \tag{1.3}$$

hold for all $r>0$.

Anderson, Vamanamurhty and Vuorinen [5] proved that the double inequality

$$\frac{\pi}{2}\left(\frac{\tanh^{-1}(r)}{r}\right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2}\frac{\tanh^{-1}(r)}{r} \tag{1.4}$$

holds for all $r \in (0,1)$, where $\tanh^{-1}(r) = \log[(1+r)/(1-r)]/2$ is the inverse hyperbolic tangent function.

The lower bound in (1.4) was improved by Alzer and Qiu [2], and Yang, Song and Chu [49] independently as follows:

$$\mathcal{K}(r) > \frac{\pi}{2}\left(\frac{\tanh^{-1}(r)}{r}\right)^{3/4} \tag{1.5}$$

for all $r \in (0,1)$.

The aim of this paper is to present new sharp upper and lower bounds for the complete elliptic integral $\mathcal{K}(r)$ of the first kind and the Gaussian arithmetic-geometric mean AGM.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1. *Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$, and*

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0,r)$ $(r > 0)$. Then the following statements are true:

(1) *If $S(r^-) \le 0$, then $S(t) < 0$ for all $t \in (0,r)$;*

(2) *If $S(r^-) > 0$, then there exists $t_0 \in (0,r)$ such that $S(t) < 0$ for $t \in (0,t_0)$ and $S(t) > 0$ for $t \in (t_0,r)$.*

Proof. We use mathematical induction to prove Lemma 2.1.

Let $m = 0$, then we clearly see that

$$S(0^+) = -a_0 < 0, \tag{2.1}$$

$$S'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} > 0 \tag{2.2}$$

for $t \in (0,r)$.

Inequality (2.2) implies that $S(t)$ is strictly increasing on $(0,r)$. If $S(r^-) \le 0$, then $S(t) < 0$ for all $t \in (0,r)$ follows from (2.1) and the monotonicity of $S(t)$ on the interval $(0,r)$. If $S(r^-) > 0$, then from (2.1) and the monotonicity of $S(t)$ on the interval $(0,r)$ we clearly see that exists $t_0 \in (0,r)$ such that $S(t) < 0$ for $t \in (0,t_0)$ and $S(t) > 0$ for $t \in (t_0,r)$. Therefore, Lemma 2.1 is true for $m = 0$.

Let $n \ge 0$ and Lemma 2.1 is true for $m = n$. Then we need to prove that Lemma 2.1 is also true for $m = n+1$. We clearly see that

$$S(0^+) = -a_0 \le 0 \tag{2.3}$$

if $m = n+1$.

We divide the proof into three cases.

Case 1 $S(r^-) \le 0$ and $S'(r^-) \le 0$. From $S'(r^-) \le 0$ and the induction hypothesis we know that $S'(t) < 0$ for $t \in (0, r)$ and $S(t)$ is strictly decreasing on $(0, r)$. Therefore, $S(t) < 0$ for all $t \in (0, r)$ follows from $S(r^-) \le 0$ and the monotonicity of the function $S(t)$ on $(0, r)$.

Case 2 $S(r^-) \le 0$ and $S'(r^-) > 0$. From $S'(r^-) > 0$ and the induction hypothesis we know that there exists $t_0 \in (0, r)$ such that $S'(t) < 0$ for $t \in (0, t_0)$ and $S'(t) > 0$ for $t \in (t_0, r)$, which implies that $S(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing on (t_0, r). Therefore, $S(t) < 0$ for all $t \in (0, r)$ follows from (2.3) and $S(r^-) \le 0$ together with the piecewise monotonicity of the function $S(t)$ on $(0, r)$.

Case 3 $S(r^-) > 0$. Then we assert that $S'(r^-) > 0$. Indeed, if $S'(r^-) \le 0$, then from the induction hypothesis and (2.3) we know that $S(t)$ is strictly decreasing on $(0, r)$ and $S(t) < 0$ for all $t \in (0, r)$, which contradicts with $S(r^-) > 0$.

It follows from $S'(r^-) > 0$ and the induction hypothesis we know that there exists $t^* \in (0, r)$ such that $S(t)$ is strictly decreasing on $(0, t^*)$ and strictly increasing on (t^*, r). Therefore, there exists $t_0 \in (0, r)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$ follows easily from (2.3) and $S(r^-) > 0$ together with the piecewise monotonicity of $S(t)$ on the interval $(0, r)$. □

Lemma 2.2. *Let $r \in (0, 1)$ and*

$$\frac{\sqrt{1-r^2}}{2r} \log\left(\frac{1+r}{1-r}\right) = \sum_{n=0}^{\infty} c_n r^{2n}. \tag{2.4}$$

Then

$$c_0 = 1, \quad c_n = \frac{\left(-\frac{1}{2}\right)_n}{(2n+1)n!} + \frac{2(n-1)}{2n+1} c_{n-1} \tag{2.5}$$

for all $n \ge 1$.

Proof. It follows from (2.4) that

$$\begin{aligned} \frac{1}{\sqrt{1-r^2}} - \frac{r}{2\sqrt{1-r^2}} \log\left(\frac{1+r}{1-r}\right) &= \left[\frac{\sqrt{1-r^2}}{2} \log\left(\frac{1+r}{1-r}\right)\right]' \\ &= \left[\sum_{n=0}^{\infty} c_n r^{2n+1}\right]' = \sum_{n=0}^{\infty} (2n+1) c_n r^{2n}, \\ r^2 \left[\frac{\sqrt{1-r^2}}{2r} \log\left(\frac{1+r}{1-r}\right)\right] &= \sqrt{1-r^2} - (1-r^2) \sum_{n=0}^{\infty} (2n+1) c_n r^{2n}, \\ r^2 \sum_{n=0}^{\infty} c_n r^{2n} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n}{n!} r^{2n} - (1-r^2) \sum_{n=0}^{\infty} (2n+1) c_n r^{2n}, \\ \sum_{n=1}^{\infty} c_{n-1} r^{2n} &= 1 - c_0 + \sum_{n=1}^{\infty} \left[\frac{\left(-\frac{1}{2}\right)_n}{n!} - (2n+1) c_n + (2n-1) c_{n-1}\right] r^{2n}. \end{aligned} \tag{2.6}$$

Therefore, Lemma 2.2 follows easily from (2.6). □

Lemma 2.3. *Let $p \in \mathbb{R}$, $p_0 = \pi/2$, $r \in (0, 1)$, and $f(p, r)$ and $g(p, r)$ be defined by*

$$f(p, r) = \frac{2[p - (5p-6)r'][p + (5p-6)r']}{\pi} \mathcal{K}(r), \tag{2.7}$$

$$g(p,r)=\frac{[p-(5p-6)r'][1+(6p-7)r']}{2r}\log\left(\frac{1+r}{1-r}\right), \tag{2.8}$$

respectively. Then

$$\lim_{r\to1^-}[f(p_0,r)-g(p_0,r)]=\frac{\pi\log 2}{2}.$$

Proof. It follows from (1.3) that

$$\mathcal{K}(r)\sim\log\frac{4}{r'}=\log 4-\frac{1}{2}\log(1-r^2)\quad(r\to1^-). \tag{2.9}$$

Note that

$$\frac{1}{2}\log\left(\frac{1+r}{1-r}\right)=\log\left(\frac{1+r}{r'}\right)=\log\left(\frac{1+r}{4}\right)+\log\frac{4}{r'}. \tag{2.10}$$

From (2.7)–(2.10) we get

$$\begin{aligned}&\lim_{r\to1^-}\frac{f(p_0,r)-g(p_0,r)}{p_0-(5p_0-6)r'} \qquad (2.11)\\ &=\lim_{r\to1^-}\left[\frac{p_0+(5p_0-6)r'}{p_0}\log\frac{4}{r'}-\frac{1+(6p_0-7)r'}{r}\left(\log\left(\frac{1+r}{4}\right)+\log\frac{4}{r'}\right)\right]\\ &=\lim_{r\to1^-}\left[-\frac{\frac{p_0r'}{1+r}+6p_0^2-(7+5r)p_0+6r}{p_0r}r'\log\frac{4}{r'}-\frac{(6p_0-7)r'+1}{r}\log\frac{1+r}{4}\right]\\ &=\log 2.\end{aligned}$$

Therefore, Lemma 2.3 follows easily from (2.11). □

Lemma 2.4. *Let $n\geq0$, $W_n=\Gamma(n+1/2)/[\Gamma(1/2)\Gamma(n+1)]$, $r\in(0,1)$, c_n, $f(p,r)$ and $g(p,r)$ be respectively defined by (2.5), (2.7) and (2.8), and $a_n(p)$ and $b_n(p)$ be defined by*

$$a_n(p)=\frac{p^2n^2+12(2p-3)(p-1)n-3(2p-3)(p-1)}{\left(n-\frac{1}{2}\right)^2}W_n^2, \tag{2.12}$$

$$b_0=-12(p-1)(2p-3),\quad b_n=\frac{2np-143p+60p^2+84}{4n^2-1}+6(p-1)^2c_n\quad(n\geq1), \tag{2.13}$$

respectively. Then

$$f(p,r)=\sum_{n=0}^{\infty}a_n(p)r^{2n},\quad g(p,r)=\sum_{n=0}^{\infty}b_n(p)r^{2n}.$$

Proof. It follows from (1.2), Lemma 2.2, (2.5), (2.7), (2.8), (2.12) and (2.13) that

$$\begin{aligned}f(p,r)&=(5p-6)^2\sum_{n=1}^{\infty}\left(\frac{\Gamma(n-1/2)}{\Gamma(1/2)}\right)^2\frac{n^2r^{2n}}{(n!)^2}-12(p-1)(2p-3)\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+1/2)}{\Gamma(1/2)}\right)^2\frac{r^{2n}}{(n!)^2}\\ &=\sum_{n=0}^{\infty}\frac{p^2n^2+12(2p-3)(p-1)n-3(2p-3)(p-1)}{(n-1/2)^2}W_n^2r^{2n}=\sum_{n=0}^{\infty}a_n(p)r^{2n},\end{aligned}$$

$$\begin{aligned}
g(p,r) &= \left[(6p-7)(5p-6)r^2 - 6(p-1)(5p-7) + 6(p-1)^2\sqrt{1-r^2}\right]\frac{\log\left(\frac{1+r}{1-r}\right)}{2r} \\
&= (6p-7)(5p-6)\sum_{n=1}^{\infty}\frac{r^{2n}}{2n-1} - 6(p-1)(5p-7)\sum_{n=0}^{\infty}\frac{r^{2n}}{2n+1} + 6(p-1)^2\sum_{n=0}^{\infty}c_n r^{2n} \\
&= -12p(p-1)(2p-3) + \sum_{n=1}^{\infty}\left[\frac{2np-143p+60p^2+84}{4n^2-1} + 6(p-1)^2 c_n\right]r^{2n} \\
&= \sum_{n=0}^{\infty} b_n(p) r^{2n}. \quad \square
\end{aligned}$$

Lemma 2.5. *Let $n \ge 0$, and $a_n(p)$ and $b_n(p)$ be respectively defined by (2.12) and (2.13), and $d_n(p) = a_n(p) - b_n(p)$. Then $d_0(89/69) = d_0(\pi/2) = d_1(89/69) = d_1(\pi/2) = d_2(89/69) = 0$, $d_2(\pi/2) < 0$, $d_n(89/69) < 0$ and $d_n(\pi/2) > 0$ for all for all $n \ge 3$.*

Proof. From (2.5), (2.12) and (2.13) we have

$$c_1 = -\frac{1}{6}, \quad c_2 = -\frac{11}{120}, \quad c_3 = -\frac{103}{1680},$$

$$d_0(p) = d_1(p) = 0, \tag{2.14}$$

$$d_n(p) = \frac{p^2n^2 + 12(2p-3)(p-1)n - 3(2p-3)(p-1)}{\left(n-\frac{1}{2}\right)^2}W_n^2 \tag{2.15}$$

$$-\frac{2np-143p+60p^2+84}{4n^2-1} - 6(p-1)^2 c_n \quad (n \ge 2),$$

$$d_2(p) = -\frac{1}{240}(69p-89)(2p-3), \quad d_3(p) = -\frac{391}{2240}p^2 + \frac{269}{448}p - \frac{1087}{2240},$$

$$d_2(89/69) = 0, \quad d_3(89/69) = -2.6618\cdots < 0, \tag{2.16}$$

$$d_2(\pi/2) = -\frac{1}{240}\left(\frac{69\pi}{2} - 89\right)(\pi-3) < 0, \quad d_3(\pi/2) = 0.02721\cdots > 0. \tag{2.17}$$

From (2.14), (2.16) and (2.17) we clearly see that we only need to prove that $d_n(89/69) < 0$ and $d_n(\pi/2) > 0$ for all $n \ge 4$.

Let

$$\xi_n(p) = 4p^2n^4 - 2(49p^2-120p+72)n^3 - (277p^2-660p+396)n^2 \tag{2.18}$$

$$-2(32p^2-75p+45)n + 3(19p^2-45p+27),$$

$$\eta_n(p) = 8p^2n^5 + (216p^2-480p+288)n^4 + (1940p^2-4680p+2808)n^3 \tag{2.19}$$

$$+(3045p^2-7320p+4392)n^2 + (302p^2-720p+432)n - 3(202p^2-485p+291),$$

$$u_n(p) = (n+1)(2n+3)d_{n+1}(p) - 2n(n+1)d_n(p), \quad v_n(p) = u_{n+1}(p) - \frac{n+1/2}{n+1}u_n(p). \tag{2.20}$$

Then from (2.15) and (2.18)–(2.20) we get

$$u_n(p) = \frac{\xi_n(p)}{(n+1)(2n-1)^2}W_n^2 + 3(p-1)^2W_n - \frac{(n+1)(2np-60p^2+141p-84)}{(2n-1)(2n+1)},$$

$$v_n(p) = \frac{\eta_n(p)}{4(n+2)(n+1)^2(2n-1)^2}W_n^2 \tag{2.21}$$

$$-\frac{4pn^2+4(6p-7)(5p-6)n+(420p^2-995p+588)}{2(2n+3)(2n+1)(2n-1)}.$$

We first prove that $d_n(89/69)<0$ for $n\geq 4$. From the inequality $\Gamma(n+1/2)/\Gamma(n+1)<1/\sqrt{n+1/4}$ given in [30] and $\Gamma^2(1/2)=\pi>28/9$ we get

$$W_n^2<\frac{1}{\pi(n+1/4)}<\frac{9}{28(n+1/4)} \tag{2.22}$$

for all $n\geq 1$.

It is not difficult to verify that

$$\eta_n(89/69)>0 \tag{2.23}$$

for all $n\geq 1$.

It follows from (2.19) and (2.21)–(2.23) that

$$\begin{aligned} v_n(89/69) &< \frac{\eta_n(89/69)}{4(n+2)(n+1)^2(2n-1)^2}\times\frac{9}{28(n+1/4)} \\ &\quad -\frac{8188n^2+2108n+5331}{3174(2n+3)(2n+1)(2n-1)} \\ &= -\frac{\lambda_n}{44436(n+2)(2n+1)(2n+3)(4n+1)(n+1)^2(2n-1)^2} \end{aligned} \tag{2.24}$$

for all $n\geq 1$, where

$$\begin{aligned} \lambda_n &= 156640n^7+541136n^6+1290560n^5+1343744n^4 \\ &\quad -812518n^3-1445425n^2-465320n+42648>0 \end{aligned} \tag{2.25}$$

for all $n\geq 1$.

From (2.20), (2.24), (2.25) and $u_1(89/69)=0$ we clearly see that

$$u_n(89/69)<0 \tag{2.26}$$

for all $n\geq 2$.

Therefore, $d_n(89/69)<0$ for $n\geq 4$ follows easily from (2.20) and (2.26).

Next, we prove that $d_n(\pi/2)>0$ for all $n\geq 4$. Let $p_0=\pi/2$, then (2.21) and (2.22) lead to

$$\begin{aligned} v_n(p_0) &< \frac{\eta_n(p_0)}{4(n+2)(n+1)^2(2n-1)^2}\times\frac{1}{2p_0(n+1/4)} \\ &\quad -\frac{4p_0n^2+4(6p_0-7)(5p_0-6)n+(420p_0^2-995p_0+588)}{2(2n+3)(2n+1)(2n-1)} \\ &= -\frac{\mu_n(p_0)}{2p_0(2n+3)(2n+1)(4n+1)(n+2)(n+1)^2(2n-1)^2} \end{aligned} \tag{2.27}$$

for all $n\geq 1$, where

$$\begin{aligned} \mu_n(p_0) &= 60(2n+7)(2n-1)(4n+1)(n+2)(n+1)^2p_0^3 \\ &\quad -(3080n^6+25868n^5+66994n^4+62837n^3+8569n^2-13465n-3808)p_0^2 \end{aligned} \tag{2.28}$$

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$$+(3264n^6+32304n^5+91008n^4+94044n^3+21564n^2-15108n-5541)p_0$$
$$-9(2n+1)(2n+3)(32n^4+312n^3+488n^2+48n-97).$$

Let $\theta=\pi/2-3/2=p_0-3/2>0$, then (2.28) leads to

$$\begin{aligned}\mu_{n+2}(p_0)&=\mu_{n+2}(\theta+3/2)=60(2n+11)(2n+3)(4n+9)(n+4)(n+3)^2\theta^3\\&+(1240n^6+20332n^5+135366n^4+464881n^3+859813n^2+799287n+283806)\theta^2\\&+(504n^6+7728n^5+47226n^4+144366n^3+224289n^2+152676n+22761)\theta\\&+\frac{9}{4}(n+4)(24n^5+284n^4+1206n^3+2173n^2+1329n-153)>0\end{aligned}\tag{2.29}$$

for all $n\geq 1$.

It follows from (2.20) and (2.27)–(2.29) that

$$\begin{aligned}&v_n(p_0)=u_{n+1}(p_0)-\frac{n+1/2}{n+1}u_n(p_0)<0,\\&u_n(p_0)>\frac{n+1}{n+1/2}u_{n+1}(p_0)>\frac{(n+1)(n+2)}{(n+1/2)(n+3/2)}u_{n+2}(p_0)\\&>\cdots>\frac{(n+1)_m}{(n+1/2)_m}u_{n+m}(p_0)=\frac{u_{n+m}(p_0)}{W_{n+m}}W_n\end{aligned}\tag{2.30}$$

for all $n\geq 3$.

It follows from the well known asymptotic formula

$$W_n=\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}\sim\frac{1}{\sqrt{\pi(n+1/4+\epsilon_n)}}=\frac{1}{\sqrt{2p_0(n+1/4+\epsilon_n)}},\quad \epsilon_n\to 0$$

as $n\to\infty$ we have

$$\begin{aligned}\frac{u_n(p_0)}{W_n}&=3(p_0-1)^2+\frac{\xi_n(p_0)}{(n+1)(2n-1)^2}W_n\\&-\frac{(n+1)(2np_0-60p_0^2+141p_0-84)}{(2n-1)(2n+1)W_n}\\&\sim 3(p_0-1)^2+\frac{\frac{\xi_n(p_0)}{(n+1)(2n-1)^2}-\frac{2p_0(n+1/4+\epsilon_n)(n+1)(2np_0-60p_0^2+141p_0-84)}{(2n-1)(2n+1)}}{\sqrt{2p_0(n+1/4+\varepsilon_n)}}\\&=3(p_0-1)^2+\frac{\tau_n(p_0)}{2\sqrt{2p_0(n+1/4+\epsilon_n)}}\quad(n\to\infty),\end{aligned}\tag{2.31}$$

where

$$\begin{aligned}\tau_n(p_0)&=\frac{4[-120p_0^3+(385+4\epsilon_n)p_0^2-408p_0+144]n^4}{(n+1)(2n+1)(2n-1)^2}\\&-\frac{4[(210+120\epsilon_n)p_0^3-(821+288\epsilon_n)p_0^2+(1074+168\epsilon_n)p_0-468]n^3}{(n+1)(2n+1)(2n-1)^2}\\&-\frac{[(180+720\epsilon_n)p_0^3-(1225+1692\epsilon_n)p_0^2+(2172+1008\epsilon_n)p_0-1152]n^2}{(n+1)(2n+1)(2n-1)^2}\\&+\frac{[240p_0^3-(666+8\epsilon_n)p_0^2+576p_0-144]n}{(n+1)(2n+1)(2n-1)^2}\end{aligned}$$

$$+\frac{(60+240\epsilon_n)p_0^3-(255+564\epsilon_n)p_0^2+(354+336\epsilon_n)p_0-162}{(n+1)(2n+1)(2n-1)^2}.$$

We clearly see that

$$\lim_{n\to\infty}\tau_n(p_0)=-\frac{(3p_0-4)(40p_0^2-75p_0+36)}{2}. \tag{2.32}$$

From (2.30)–(2.32) we get

$$\frac{u_n(p_0)}{W_n}>\lim_{m\to\infty}\frac{u_{n+m}(p_0)}{W_{n+m}}=3(p_0-1)^2>0 \tag{2.33}$$

for all $n\geq 3$.

Therefore, $d_n(\pi/2)>0$ for all $n\geq 4$ follows easily from (2.17) and (2.20) together with (2.33). □

3. Main result

Theorem 3.1. *Let $p,q>1$. Then the double inequality*

$$\frac{1+(6p-7)r'}{p+(5p-6)r'}\frac{\pi\tanh^{-1}(r)}{2r}<\mathcal{K}(r)<\frac{1+(6q-7)r'}{q+(5q-6)r'}\frac{\pi\tanh^{-1}(r)}{2r} \tag{3.1}$$

holds for all $r\in(0,1)$ if and only if $p\geq\pi/2=1.570796\cdots$ and $q\leq 89/69=1.289855\cdots$.

Proof. We first prove that the second inequality of (3.1) holds for all $r\in(0,1)$ if and only if $q\leq 89/69$.

If the second inequality of (3.1) holds for all $r\in(0,1)$, then

$$\lim_{r\to0^+}\frac{\mathcal{K}(r)-\frac{1+(6q-7)r'}{q+(5q-6)r'}\frac{\pi\tanh^{-1}(r)}{2r}}{r^4}=\frac{23\pi}{1920(q-1)}\left(q-\frac{89}{69}\right)\leq 0. \tag{3.2}$$

Therefore, $q\leq 89/69$ follows from (3.2) and $q>1$.

Let $n\geq 0$, $f(p,r)$, $g(p,r)$, $\{a_n(p)\}$, $\{b_n(p)\}$ and $\{d_n(p)\}$ be respectively defined by (2.7), (2.8), (2.12), (2.13) and Lemma 2.5, and $h(p,r)$ and $D(p,r)$ be defined by

$$h(p,r)=p^2-(5p-6)^2(1-r^2), \tag{3.3}$$

$$D(p,r)=\frac{2}{\pi}\mathcal{K}(r)-\frac{1+(6p-7)r'}{p+(5p-6)r'}\frac{\tanh^{-1}(r)}{r}, \tag{3.4}$$

respectively.

Then it follows from Lemma 2.4, (3.3) and (3.4) that

$$D(p,r)=\frac{f(p,r)-g(p,r)}{h(p,r)} \tag{3.5}$$

$$=\frac{\sum_{n=0}^{\infty}(a_n(p)-b_n(p))r^{2n}}{h(p,r)}=\frac{\sum_{n=0}^{\infty}d_n(p)r^{2n}}{h(p,r)},$$

$$h\left(\frac{89}{69},r\right)=\frac{961}{4761}r^2+\frac{2320}{1587}>0 \tag{3.6}$$

for all $r\in(0,1)$.

From Lemma 2.5, (3.5) and (3.6) we clearly see that

$$D\left(\frac{89}{69}, r\right) < 0 \tag{3.7}$$

for all $r \in (0, 1)$.

It is not difficult to verify that the function $p \to [1 + (6p - 7)r']/[p + (5p - 6)r']$ is strictly decreasing on $(1, \infty)$.

Therefore, the second inequality of (3.1) holds for all $q \leq 89/69$ and $r \in (0, 1)$ follows easily from (3.4) and (3.7) together with the monotonicity of the function $p \to [1 + (6p - 7)r']/[p + (5p - 6)r']$ on the interval $(1, \infty)$.

Next, we prove that the first inequality of (3.1) holds for all $r \in (0, 1)$ if and only if $p \geq \pi/2$.

If the first inequality of (3.1) holds for all $r \in (0, 1)$, then we clearly see that

$$\lim_{r \to 1^-} \frac{\mathcal{K}(r) - \frac{1+(6p-7)r'}{p+(5p-6)r'} \frac{\pi \tanh^{-1}(r)}{2r}}{\log \frac{4}{r'}} = 1 - \frac{\pi}{2p} \geq 0. \tag{3.8}$$

Therefore, $p \geq \pi/2$ follows from $p > 1$ and (3.8).

Let $p_0 = \pi/2$, then Lemma 2.3 leads to

$$\lim_{r \to 1^-} [f(p_0, r) - g(p_0, r)] = \frac{\pi \log 2}{2} > 0. \tag{3.9}$$

It follows from Lemma 2.1, Lemma 2.4, Lemma 2.5, (2.17) and (3.9) that there exists $r_0 \in (0, 1)$ such that

$$f(p_0, r) < g(p_0, r) \tag{3.10}$$

for $r \in (0, r_0)$ and

$$f(p_0, r) > g(p_0, r) \tag{3.11}$$

for $r \in (r_0, 1)$.

From (2.7), (2.8) and (3.3) we clearly see that

$$r_0 = \frac{2\sqrt{6(\pi^2 - 5\pi + 6)}}{5\pi - 12} = 0.531185\cdots$$

is the unique solution of the equation $h(p_0, r) = 0$ on the interval $(0, 1)$ such that

$$h(p_0, r) < 0 \tag{3.12}$$

for $r \in (0, r_0)$ and

$$h(p_0, r) > 0 \tag{3.13}$$

for $r \in (r_0, 1)$.

Equation (3.5) and inequalities (3.10)–(3.13) lead to the conclusion that

$$D(p_0, r) > 0 \tag{3.14}$$

for $r \in (0, r_0) \cup (r_0, 1)$.

From (3.5) and (3.14) together with the fact that r_0 is the one order common zero point of the function $h(p_0, r)$ and $f(p_0, r) - g(p_0, r)$ on the interval $(0, 1)$ we know that

$$D(p_0, r) > 0 \tag{3.15}$$

for $r \in (0, 1)$.

Therefore, the first inequality of (3.1) holds for all $r \in (0, 1)$ if and only if $p \geq \pi/2$ follows from (3.4) and (3.15) together with the monotonicity of the function $p \to [1 + (6p - 7)r']/[p + (5p - 6)r']$ on the interval $(1, \infty)$. $\square$

Let $A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$ be the arithmetic, geometric and logarithmic means of two distinct positive real numbers a and b, respectively. Then from (1.1) and Theorem 3.1 we get Theorem 3.2 immediately.

Theorem 3.2. *Let $p, q > 1$. Then the double inequality*

$$\frac{qA(1,r) + (5q-6)G(1,r)}{A(1,r) + (6q-7)G(1,r)} L(1,r) < AGM(1,r) < \frac{pA(1,r) + (5p-6)G(1,r)}{A(1,r) + (6p-7)G(1,r)} L(1,r)$$

holds for all $r \in (0, 1)$ if and only if $p \geq \pi/2$ and $q \leq 89/69$. In particular, one has

$$\frac{89A(1,r) + 31G(1,r)}{3[23A(1,r) + 17G(1,r)]} L(1,r) < AGM(1,r) < \frac{\pi A(1,r) + (5\pi - 12)G(1,r)}{2[A(1,r) + (3\pi - 7)G(1,r)]} L(1,r)$$

for all $r \in (0, 1)$.

Remark 3.3. Let $p = \pi/2$ and $q = 89/69$. Then Theorem 3.1 leads to the double inequality

$$\frac{2 + 2(3\pi - 7)r'}{\pi + (5\pi - 12)r'} \frac{\pi \tanh^{-1}(r)}{2r} < \mathcal{K}(r) < \frac{69 + 51r'}{89 + 31r'} \frac{\pi \tanh^{-1}(r)}{2r} \tag{3.16}$$

for all $r \in (0, 1)$.

Remark 3.4. We clearly see that

$$\lim_{r \to 1^-} \left(\frac{\tanh^{-1}(r)}{r} \right)^{1/4} = \infty, \quad \lim_{r \to 1^-} \frac{\pi + (5\pi - 12)r'}{2 + 2(3\pi - 7)r'} = \frac{\pi}{2}, \tag{3.17}$$

$$\frac{69 + 51r'}{89 + 31r'} < 1 \tag{3.18}$$

for all $r \in (0, 1)$.

Equation (3.17) implies that there exists $\delta \in (0, 1)$ such that the lower bound for $\mathcal{K}(r)$ given in (3.16) is better than that given in (1.5) for $r \in (\delta, 1)$, and inequality (3.17) implies that the upper bound for $\mathcal{K}(r)$ given in (3.16) is better than that given in (1.4) for all $r \in (0, 1)$.

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