J. Math. Anal. Appl. 469 (2019) 480–492

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On Young's inequality

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ARTICLE INFO

Article history:
Received 14 March 2018
Available online 26 June 2018
Submitted by K. Driver

Keywords:
Young's inequality
Trigonometric sums
Turán-type inequalities
Harmonic functions
Absolutely monotonic

ABSTRACT

We present some inequalities for trigonometric sums. Among others, we prove the following refinements of the classical Young inequality.

(1) Let $m \geq 3$ be an odd integer, then for all $n \geq m-1$,

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} \geq \sum_{k=1}^{m} \frac{(-1)^k}{k}.$$

The sign of equality holds if and only if $n = m$ and $\theta = \pi$. The special case $m = 3$ is due to Brown and Koumandos (1997).

(2) For all even integers $n \geq 2$ and real numbers $r \in (0, 1]$ and $\theta \in [0, \pi]$ we have

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} r^k \geq -\frac{5}{48}(5+\sqrt{5}) = -0.75375\ldots.$$

The sign of equality holds if and only if $n = 4$, $r = 1$ and $\theta = 4\pi/5$. We apply this result to prove the absolute monotonicity of a function which is defined in terms of the log-function.

1. Introduction and statement of main results

In 1912, W.H. Young [11] published interesting inequalities for the cosine polynomial

$$C_n(\theta) = \sum_{k=1}^{n} \frac{\cos(k\theta)}{k}.$$

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https://doi.org/10.1016/j.jmaa.2018.06.061
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Among others, he proved that for all integers $n \geq 2$ and real numbers $\theta \in [0, \pi]$ we have

$$C_n(\theta) > -1. \tag{1.1}$$

This inequality is one of the classical results in the theory of trigonometric polynomials. It has attracted the attention of many researchers who presented numerous extensions, variants, related results and applications to geometric function theory and other branches. For more information on this subject we refer to Askey [1], Askey and Gasper [2], Koumandos [7], and Milovanović et al. [8].

We discovered only recently that there is an error in Young's original proof of (1.1). He first showed that C_n takes its minimum precisely at $\theta = \pi$ if n is odd, and at $\theta = \pi - \pi/(n+1)$ if n is even. Then, he noted that

$$C_n(\pi) > -1 \quad \text{for odd } n.$$

For even n, he proved that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) > -1$$

using an incorrect claim, namely, that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) = \sum_{k=1}^{n/2} (-1)^k u_k v_k \tag{1.2}$$

with

$$u_k = \frac{1}{k} - \frac{1}{n+1-k} \quad \text{and} \quad v_k = \cos\Big(\frac{k\pi}{n+1}\Big). \tag{1.3}$$

From

$$u_1 > u_2 > \cdots > u_{n/2} > 0 \quad \text{and} \quad v_1 > v_2 > \cdots > v_{n/2} > 0, \tag{1.4}$$

he concluded that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) \geq -u_1 v_1 = -\Big(1 - \frac{1}{n}\Big)\cos\Big(\frac{\pi}{n+1}\Big) > -1. \tag{1.5}$$

Indeed, formula (1.2) only holds when we replace u_k by

$$u_k^* = \frac{1}{k} + \frac{1}{n+1-k}\,.$$

Although (1.4) remains valid, (1.5) becomes

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) \geq -u_1^* v_1 = -\Big(1 + \frac{1}{n}\Big)\cos\Big(\frac{\pi}{n+1}\Big) > -1,$$

but the inequality

$$-u_1^* v_1 > -1$$

is only valid for $n = 2$. It is false for $n = 4, 6, 8, ...$ This means that Young's proof is still erroneous if we just correct the sign error in (1.3).

Nevertheless, (1.1) is true. If we replace ">" by "≥", then the inequality holds for all $n \geq 1$, $\theta \in [0, \pi]$ and the constant lower bound -1 is sharp. In 1997, Brown and Koumandos [5] provided the following remarkable refinement of this result.

Proposition 1. *For all integers $n \geq 2$ and real numbers $x \in [0, \pi]$ we have*

$$C_n(\theta) \geq -\frac{5}{6}. \tag{1.6}$$

The constant lower bound is best possible.

In this article, we present an extension of (1.6). To that end, we need to know the ordering of the minimum values of C_n.

Theorem 1. *Let $n \geq 1$ be an integer and*

$$a_n = \min_{0 \leq \theta \leq \pi} C_n(\theta).$$

Then,

$$a_3 < a_5 < a_7 < a_4 < a_2 < a_9 < a_6 < a_8 \tag{1.7}$$

and

$$a_{2m+1} < a_{2m} < a_{2m+3} \quad for \quad m \geq 4. \tag{1.8}$$

As a consequence of Theorem 1 we obtain a generalization of (1.6). In fact, setting $m = 3$ in (1.9) below gives the Brown–Koumandos inequality.

Theorem 2. *Let $m \geq 3$ be an odd integer. For all integers $n \geq m - 1$ and real numbers $\theta \in [0, \pi]$ we have*

$$C_n(\theta) \geq C_m(\pi) = \sum_{k=1}^{m} \frac{(-1)^k}{k}. \tag{1.9}$$

The sign of equality holds if and only if $n = m$ and $\theta = \pi$.

From (1.7) and (1.8) we conclude that in (1.6) we can replace the lower bound $-5/6$ by a larger number if we assume that n is an even integer.

Theorem 3. *For all even integers $n \geq 2$ and real numbers $\theta \in [0, \pi]$ we have*

$$C_n(\theta) \geq C_4\Big(\frac{4\pi}{\pi}\Big) = -\frac{5}{48}(5 + \sqrt{5}) = -0.75375.... \tag{1.10}$$

The sign of equality holds if and only if $n = 4$ and $\theta = 4\pi/5$.

Moreover, an application of Theorem 1 leads to a companion of Theorem 2.

Theorem 4. *Let $m \geq 6$ be an even integer. For all even integers $n \geq m$ and real numbers $\theta \in [0, \pi]$ we have*

$$C_n(\theta) \geq C_m\Big(\pi - \frac{\pi}{m+1}\Big) = \sum_{k=1}^{m} \frac{(-1)^k}{k} \cos\Big(\frac{k\pi}{m+1}\Big).$$

The sign of equality holds if and only if $n = m$ and $\theta = \pi - \pi/(m+1)$.

In 1950, Turán published the elegant inequality

$$0 < P_{n-1}(x)P_{n+1}(x) - P_n^2(x) \quad (n = 1, 2, ...; \, -1 < x < 1),$$

where P_n denotes the nth Legendre polynomial. Since then, numerous papers appeared providing related results for other polynomials and special functions. We refer to Baricz et al. [3] and the references cited therein. Here, we show that with the help of Theorem 2 we are able to prove the following Turán-type inequalities.

Theorem 5. *For all integers $n \geq 2$ and real numbers θ we have*

$$-\frac{5}{12} \leq C_{n-1}(\theta)C_{n+1}(\theta) - C_n^2(\theta) \leq \frac{7}{12}. \tag{1.11}$$

Both bounds are best possible.

In 2015, Barnard et al. [4] proved that for odd integers $n \geq 1$ and real numbers $r \in (0, 1]$ and $\theta \in [0, \pi)$ we have

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} r^k > \sum_{k=1}^{n} \frac{(-1)^k}{k} r^k.$$

Since

$$\begin{aligned} -\sum_{k=1}^{n} \frac{(-1)^k}{k} r^k &= r + \sum_{j=1}^{[(n-1)/2]} \Big(\frac{r}{2j+1} - \frac{1}{2j}\Big) r^{2j} \\ &\leq r + \sum_{j=1}^{[(n-1)/2]} \Big(\frac{1}{2j+1} - \frac{1}{2j}\Big) r^{2j} = r - \sum_{j=1}^{[(n-1)/2]} \frac{r^{2j}}{2j(2j+1)} \\ &\leq r \, \leq \, 1, \end{aligned}$$

we obtain a counterpart of Young's inequality.

Proposition 2. *For all odd integers $n \geq 1$ and real numbers $r \in (0, 1]$ and $\theta \in [0, \pi]$ we have*

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} r^k \geq -1. \tag{1.12}$$

The constant lower bound is best possible.

In view of this result it is natural to ask for a corresponding inequality which is valid for all even integers $n \geq 2$. An application of Theorem 3 leads to a companion of (1.12) and an extension of (1.10).

Theorem 6. *For all even integers $n \geq 2$ and real numbers $r \in (0, 1]$ and $\theta \in [0, \pi]$ we have*

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} r^k \geq -\frac{5}{48}(5 + \sqrt{5}). \tag{1.13}$$

The sign of equality holds if and only if $n = 4$, $r = 1$, $\theta = 4\pi/5$.

In the next section, we establish Theorems 1, 5 and 6. We conclude our paper with a few remarks and additional results. In Section 3, we apply Theorem 3 to prove the absolute monotonicity of a function defined in terms of the log-function, and we offer sharp upper and lower bounds for a trigonometric sum which is related to $C_n(\theta)$.

The numerical values given in the next section have been calculated via the computer program MAPLE 13.

2. Proofs

Proof of Theorem 1. As mentioned in Section 1 we have for all integers $m \geq 1$,

$$a_{2m-1} = C_{2m-1}(\pi) = \sum_{k=1}^{2m-1} \frac{(-1)^k}{k} \quad \text{and} \quad a_{2m} = C_{2m}\Big(\pi - \frac{\pi}{2m+1}\Big).$$

The numerical values

$$\begin{aligned} &a_2 = -0.75, && a_3 = -0.833333\cdots, \ a_4 = -0.753757\cdots, \ a_5 = -0.783333\cdots, \ a_6 = -0.744491\cdots, \\ &a_7 = -0.759523\cdots, && a_8 = -0.736555\cdots, \ a_9 = -0.745634\cdots, \ a_{11} = -0.736544\cdots, \end{aligned}$$

lead to (1.7) and (1.8) with $m = 4$.

We define

$$D_n(\theta) = C_n(\pi - \theta) = \sum_{k=1}^{n} (-1)^k \frac{\cos(k\theta)}{k}.$$

Let $m \geq 5$. By differentiation, we obtain

$$D'_{2m}(\theta) = \sum_{k=1}^{2m} (-1)^{k+1} \sin(k\theta) = -\frac{\sin(m\theta)\cos((2m+1)\theta/2)}{\cos(\theta/2)}.$$

Hence,

$$\begin{aligned} a_{2m} &= D_{2m}\Big(\frac{\pi}{2m+1}\Big) = D_{2m}(0) + \int_0^{\pi/(2m+1)} D'_{2m}(x)dx \\ &= \sum_{k=1}^{2m} \frac{(-1)^k}{k} - \frac{2}{2m+1} I_{2m} \end{aligned}$$

with

$$I_n = \int_0^{\pi/2} \frac{\sin(nx/(n+1))\cos(x)}{\cos(x/(n+1))} dx.$$

Then, (1.8) is equivalent to

$$\frac{4m^2+8m+5}{4(m+1)(2m+3)} < I_{2m} < \frac{1}{2}. \tag{2.1}$$

We obtain

$$\begin{aligned} I_{2m} &\geq \int_0^{\pi/2} \sin\Big(\frac{2mx}{2m+1}\Big)\cos(x)dx \\ &= \frac{1}{4m+1}\Big\{(2m+1)^2\cos\Big(\frac{\pi}{2(2m+1)}\Big) - 4m^2 - 2m\Big\} \\ &\geq \frac{1}{4m+1}\Big\{(2m+1)^2\Big(1-\frac{1}{2}\Big(\frac{\pi}{2(2m+1)}\Big)^2\Big) - 4m^2 - 2m\Big\}. \end{aligned}$$

This implies

$$\begin{aligned} I_{2m} - \frac{4m^2+8m+5}{4(m+1)(2m+3)} &\geq \frac{(24-2\pi^2)m^2-(5\pi^2-32)m-(3\pi^2-14)}{8(m+1)(2m+3)(4m+1)} \\ &\geq \frac{4.2m^2-17.4m-15.7}{8(m+1)(2m+3)(4m+1)} \\ &> 0. \end{aligned} \tag{2.2}$$

Since

$$\begin{aligned} \frac{\sin(2mx/(2m+1))}{\cos(x/(2m+1))} &= \sin(x) - \cos(x)\frac{\sin(x/(2m+1))}{\cos(x/(2m+1))} \\ &< \sin(x) \end{aligned}$$

for $x \in (0, \pi/2)$, we get

$$I_{2m} < \int_0^{\pi/2} \sin(x)\cos(x)dx = \frac{1}{2}. \tag{2.3}$$

From (2.2) and (2.3) we conclude that (2.1) is valid. This completes the proof of Theorem 1. $\square$

Proof of Theorem 5. We define

$$G_n(\theta) = C_{n-1}(\theta)C_{n+1}(\theta) - C_n^2(\theta). \tag{2.4}$$

Since

$$G_n(\theta+2\pi) = G_n(\theta) \quad \text{and} \quad G_n(\pi+\theta) = G_n(\pi-\theta),$$

it suffices to prove (1.11) for $\theta \in [0, \pi]$.

Let $t = \cos(\theta) \in [-1, 1]$. We have

$$G_2(\theta) = \frac{7}{12} - \frac{1}{6}(t+1)w_1(t) \quad \text{and} \quad G_2(\theta) = -\frac{5}{12} + \frac{1}{6}(1-t)w_2(t)$$

with

$$w_1(t) = -2t^3 + 8t^2 - 8t + 5 \quad \text{and} \quad w_2(t) = -2t^3 + 4t^2 + 4t + 1.$$

Since w_1 and w_2 are positive on $[-1, 1]$, we conclude that (1.11) is valid for $n = 2$.

We have

$$G_3(\theta) = \frac{1}{72} w_3(t)$$

with

$$w_3(t) = 16t^6 + 48t^5 - 120t^4 - 24t^3 + 90t^2 - 18t - 9.$$

Applying Sturm's theorem (see [9, section 79]) gives that for $t \in [-1, 1]$ we obtain

$$42 - w_3(t) > 0 \quad \text{and} \quad 30 + w_3(t) > 0.$$

This leads to (1.11) with $n = 3$.

Next, we assume that $n \geq 4$. We have the representation

$$G_n(\theta) = \Big(\frac{\cos((n+1)\theta)}{n+1} - \frac{\cos(n\theta)}{n}\Big) C_n(\theta) - \frac{\cos(n\theta)\cos((n+1)\theta)}{n(n+1)}. \tag{2.5}$$

Using (2.5) and

$$\frac{\cos((n+1)\theta)}{n+1} - \frac{\cos(n\theta)}{n} = -\frac{2}{n+1}\sin(\theta/2)\sin((n+1/2)\theta) - \frac{\cos(n\theta)}{n(n+1)}$$

leads to

$$|G_n(\theta)| \leq \Big(\frac{2}{n+1}\sin(\theta/2) + \frac{1}{n(n+1)}\Big)|C_n(\theta)| + \frac{1}{n(n+1)}. \tag{2.6}$$

We consider two cases.

Case 1. $0 \leq \theta \leq 0.64$.

Then,

$$|G_n(\theta)| \leq \Big(\frac{2}{n+1}\sin(0.32) + \frac{1}{n(n+1)}\Big)\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n(n+1)} = y_n, \quad \text{say.}$$

Since $(y_n)_{n\geq 1}$ is decreasing with $y_4 = 0.4163...$, we obtain

$$|G_n(\theta)| < \frac{5}{12} = 0.4166....$$

Case 2. $0.64 \leq \theta \leq \pi$.

We set

$$c_0 = \frac{1}{2}\Big(\frac{1}{\sin(\theta/2)} - 1\Big), \quad c_k = -\cos(k\theta) \quad (1 \leq k \leq n), \quad \sigma_k = \sum_{j=0}^{k} c_j \quad (0 \leq k \leq n).$$

Then,

$$\sigma_0 > 0 \quad \text{and} \quad \sigma_k = \frac{1-\sin((k+1/2)\theta)}{2\sin(\theta/2)} \ge 0 \quad (1 \le k \le n).$$

Let

$$b_0 = b_1 = b_2 = b_3 = \frac{1}{4}, \quad b_k = \frac{1}{k} \quad (4 \le k \le n), \quad b_{n+1} = 0.$$

Summation by parts gives

$$0 \le \sum_{k=0}^{n} \sigma_k (b_k - b_{k+1}) = \sum_{k=0}^{n} b_k c_k = \frac{1}{8}\Big(\frac{1}{\sin(\theta/2)} - 1\Big) - \frac{1}{4}\sum_{k=1}^{3}\cos(k\theta) - \sum_{k=4}^{n}\frac{\cos(k\theta)}{k}.$$

Thus,

$$C_n(\theta) \le \sum_{k=1}^{3}\Big(\frac{1}{k} - \frac{1}{4}\Big)\cos(k\theta) + \frac{1}{8}\Big(\frac{1}{\sin(\theta/2)} - 1\Big) = J(\theta), \quad \text{say}.$$

Since

$$-J'(\theta) = \frac{1}{2}\sin(\theta)\big(\cos^2(\theta) + (1+\cos(\theta))^2\big) + \frac{\cos(\theta/2)}{16\sin^2(\theta/2)},$$

we conclude that J is decreasing on $(0, \pi]$. Now, we distinguish two subcases.

Case 2.1. $0.64 \le \theta \le 2$.

We use Theorem 2 (with $m = 5$) and the monotonicity of J. This gives

$$-0.783... = -\frac{47}{60} \le C_n(\theta) \le J(0.64) = 0.917....$$

Thus,

$$|C_n(\theta)| \le 0.92. \tag{2.7}$$

Applying (2.6) and (2.7) yields

$$|G_n(\theta)| \le \Big(\frac{2}{5}\sin(1) + \frac{1}{20}\Big)0.92 + \frac{1}{20} = 0.405... < \frac{5}{12}.$$

Case 2.2. $2 \le \theta \le \pi$.

Since

$$-0.783... = -\frac{47}{60} \le C_n(\theta) \le J(2) = -0.371...,$$

we find

$$|C_n(\theta)| \le \frac{47}{60}. \tag{2.8}$$

From (2.6) and (2.8) we obtain

$$|G_n(\theta)| \le \Big(\frac{2}{5}\sin(\pi/2) + \frac{1}{20}\Big)\frac{47}{60} + \frac{1}{20} = 0.402... < \frac{5}{12}.$$

This completes the proof of (1.11). Moreover, since $G_2(0) = -5/12$ and $G_2(\pi) = 7/12$, we conclude that the given bounds are sharp. □

Proof of Theorem 6. Let $n \geq 2$ be an even integer and

$$F(z) = \sum_{k=1}^{n} \frac{z^k}{k}.$$

Then,

$$u(x, y) = \Re F(x + iy)$$

is harmonic on $M = \{(x, y) \in \mathbb{R}^2 | x = r\cos(\theta), y = r\sin(\theta), 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}$, that is, we have

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0.$$

It follows that u takes its minimum on the boundary of M.

Let $x \in [0, 1]$. Then,

$$u(x, 0) = \sum_{k=1}^{n} \frac{x^k}{k} \geq 0.$$

Let $v(x) = u(-x, 0)$. Since

$$v'(x) = \frac{x^n - 1}{x + 1} \leq 0,$$

we obtain

$$\begin{aligned} v(x) \ \geq \ v(1) &= -\sum_{k=1}^{[n/2]} \frac{1}{2k(2k-1)} \\ &> -\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \\ &= -\log 2 \\ &> c\,, \end{aligned}$$

where $c = -5(5 + \sqrt{5})/48$. Applying Theorem 3 gives for $\theta \in [0, \pi]$,

$$u(\cos(\theta), \sin(\theta)) = \sum_{k=1}^{n} \frac{\cos(k\theta)}{k} \geq c\,.$$

Thus, we have $u(x, y) \geq c$ for $(x, y) \in \partial M$. This implies that (1.13) is valid for all even integers $n \geq 2$ and real numbers $r \in (0, 1]$, $\theta \in [0, \pi]$. Moreover, equality holds if and only if $n = 4$, $r = 1$, $\theta = 4\pi/5$. □

3. Remarks and additional results

Remark 1. Let $\omega_I(f)$ be the oscillation of a function f on an interval I. From Theorem 5 we obtain that if G_n denotes the function defined in (2.4), then we have for all $n \geq 2$,

$$\omega_{\mathbb{R}}(G_n) \leq 1.$$

Since $\omega_{\mathbb{R}}(G_2) = 1$, we conclude that the upper bound 1 is best possible.

Remark 2. The identity

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} r^k = \sum_{k=1}^{n} \frac{\cos(k(\pi-\theta))}{k} (-r)^k$$

reveals that the inequalities (1.12) and (1.13) are valid not only for $r \in (0,1]$, but for $r \in [-1,1]$.

Remark 3. A function $f : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is called absolutely monotonic if f has derivatives of all orders and satisfies

$$f^{(n)}(x) \geq 0 \quad \text{for} \quad n = 0, 1, 2, \dots \quad \text{and} \quad x \in I.$$

These functions have interesting applications in probability theory and other fields. The main properties of absolutely monotonic functions can be found in Widder [10, chapter 4].

An application of Young's inequality and (1.10) leads to the following result.

Theorem 7. *If $a \geq 5(5+\sqrt{5})/48 = 0.75375\dots$, $b \geq 1$ and $c \in [-1,1]$, then the function*

$$r \;\mapsto\; \left(\frac{a+br}{1-r^2} - \frac{1}{2(1-r)} \log(1 - 2cr + r^2) \right)$$

is absolutely monotonic on $[0,1)$.

Proof. Let $\theta = \arccos(c) \in [0,\pi]$ and

$$d_0 = 0, \quad d_k = d_k(\theta) = \frac{\cos(k\theta)}{k} \quad (k \geq 1).$$

We define for $r \in [0,1)$,

$$U_c(r) = -\frac{1}{2} \log(1 - 2cr + r^2).$$

Using the representation

$$U_{\cos(\theta)}(r) = \sum_{k=0}^{\infty} d_k(\theta) r^k,$$

see Gould [6, p. 7], gives

$$\begin{aligned}\frac{1}{1-r}\,U_c(r) &= \sum_{k=0}^{\infty} r^k \sum_{k=0}^{\infty} d_k r^k \\ &= \sum_{k=0}^{\infty}\sum_{\nu=0}^{k} d_\nu r^k \\ &= \sum_{n=0}^{\infty}\sum_{\nu=0}^{2n} d_\nu r^{2n} + \sum_{n=0}^{\infty}\sum_{\nu=0}^{2n+1} d_\nu r^{2n+1}.\end{aligned}$$

Let

$$V_{a,b,c}(r) = \frac{a}{1-r^2} + \frac{br}{1-r^2} + \frac{U_c(r)}{1-r}.$$

Then,

$$\begin{aligned}V_{a,b,c}(r) &= \sum_{n=0}^{\infty}\Big(\sum_{\nu=0}^{2n} d_\nu + a\Big) r^{2n} + \sum_{n=0}^{\infty}\Big(\sum_{\nu=0}^{2n+1} d_\nu + b\Big) r^{2n+1} \\ &= \sum_{n=0}^{\infty} d_n^* r^n.\end{aligned}$$

From Young's inequality and (1.10) we conclude that $d_n^* \geq 0$ for $n \geq 0$. It follows that

$$\frac{d^n}{dr^n} V_{a,b,c}(r) \geq 0 \quad \text{for} \quad n = 0, 1, 2, \ldots \quad \text{and} \quad r \in [0, 1).$$

This means that $V_{a,b,c}$ is absolutely monotonic on $[0, 1)$. □

Remark 4. The sine counterpart of Young's inequality states that for all $n \geq 1$ and $\theta \in [0, \pi]$ we have

$$\sum_{k=1}^{n} \frac{\sin(k\theta)}{k} \geq 0. \tag{3.1}$$

This result is known in the literature as the Fejér–Jackson inequality. A proof as well as historical comments on (3.1) are given in Milovanović et al. [8, chapter 4]. We conclude this paper with a companion of (1.1) and (3.1).

Theorem 8. *For all integers $n \geq 2$ and real numbers θ we have*

$$-\frac{1}{2} \leq -\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k} \leq \sum_{k=1}^{n-1}\frac{\cos(k\theta)\sin((n-k)\theta)}{k(n-k)} \leq \frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k} \leq \frac{1}{2}. \tag{3.2}$$

All bounds are best possible.

Proof. We denote the interior sum in (3.2) by $H_n(\theta)$. By differentiation we obtain

$$H_n'(\theta) = \sum_{k=1}^{n-1}\Big(\frac{\cos(k\theta)\cos((n-k)\theta)}{k} - \frac{\sin(k\theta)\sin((n-k)\theta)}{n-k}\Big)$$
$$= \frac{1}{2}\sum_{k=1}^{n-1}\Big\{\Big(\frac{1}{k}-\frac{1}{n-k}\Big)\cos((2k-n)\theta) + \Big(\frac{1}{k}+\frac{1}{n-k}\Big)\cos(n\theta)\Big\} = \cos(n\theta)\sum_{k=1}^{n-1}\frac{1}{k}.$$

This leads to the representation

$$H_n(\theta) = \frac{\sin(n\theta)}{n}\sum_{k=1}^{n-1}\frac{1}{k}.$$

It follows that

$$|H_n(\theta)| \le \frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k} = z_n, \quad \text{say}.$$

Since

$$z_2 = z_3 = \frac{1}{2} \quad \text{and} \quad z_n = z_{n+1} + \frac{1}{n(n+1)}\sum_{k=2}^{n-1}\frac{1}{k} > z_{n+1} \quad (n \ge 3),$$

we obtain

$$z_n \le 1/2 \quad \text{for} \quad n \ge 2.$$

This settles (3.2). Moreover, from

$$H_n\Big(\frac{3\pi}{2n}\Big) = -\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k}, \quad H_n\Big(\frac{\pi}{2n}\Big) = \frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k}$$

and

$$H_2\Big(\frac{3\pi}{4}\Big) = H_3\Big(\frac{\pi}{2}\Big) = -\frac{1}{2}, \quad H_2\Big(\frac{\pi}{4}\Big) = H_3\Big(\frac{\pi}{6}\Big) = \frac{1}{2}$$

we conclude that the given bounds are sharp. □

Acknowledgment

We thank the referee for helpful comments.

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