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Complex symmetric operators and isotropic vectors on Banach spaces ☆

Muneo Chō [a], Injo Hur [b,*], Ji Eun Lee [c]

[a] Department of Mathematics, Kanagawa University, Hiratsuka, 259-1293, Japan
[b] Department of Mathematics Education, Chonnam National University, 77 Yongbong-ro, Buk-gu, Gwangju 61186, Republic of Korea
[c] Department of Mathematics and Statistics, Sejong University, Seoul, 05006, Republic of Korea

A R T I C L E I N F O

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In this paper, we generalize the concepts of isotropic vectors and complex symmetric operators from Hilbert spaces to Banach spaces via their dual spaces. With this extension we show the existence of isotropic vectors on Banach spaces whose dimension is at least two and the relation between the simplicity of an eigenvalue and the non-existence of its isotropic eigenvectors.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ with its inner product $\langle\cdot,\cdot\rangle$ which is linear on the first and antilinear on the second. A conjugation C on $\mathcal{H}$ is an antilinear isometric involution on $\mathcal{H}$. In other words, for any vectors x and y in $\mathcal{H}$, the equality

$$\langle Cx, Cy\rangle = \langle y, x\rangle \tag{1}$$

holds. In particular, $C^2 = I$ where I is the identity operator on $\mathcal{H}$. We call an operator $T \in \mathcal{L}(\mathcal{H})$ *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$.

A vector $x \in \mathcal{H}$ is called *isotropic* (with respect to C) if $\langle Cx, x\rangle = 0$. For example, when $\mathcal{H} = \mathbb{C}^2$ and C is the canonical conjugation on $\mathbb{C}^2$, i.e., $C(x, y) = (\overline{x}, \overline{y})$, any vector of the form $(a, \pm ia)$ with any complex

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* Corresponding author.
E-mail addresses: chiyom01@kanagawa-u.ac.jp (M. Chō), injohur@jnu.ac.kr (I. Hur), jieunlee7@sejoung.ac.kr, jieun7@ewhain.net (J.E. Lee).

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number a becomes isotropic with respect to the canonical conjugation C. It seems that the existence of these isotropic vectors gives some difficulty in understanding the structure of conjugations. See [4] for more examples and details.

On these isotropic vectors on $\mathcal{H}$, Garcia et al. [4,5] showed two well-known results: the existence of isotropic vectors in any subspace of $\mathcal{H}$ whose dimension is at least two (which is Lemma 4.11 in [4] or Lemma 2.2 in this paper) and the relation between the simplicity of an eigenvalue λ of T and the non-existence of its isotropic eigenvectors for λ when T is complex symmetric (which is Theorem 4.12 in [4]). Here an eigenvalue λ is called *simple* if its algebraic multiplicity is 1, or equivalently $\dim\ker(\lambda I - T) = 1$.

Recently, Chō and Tanahashi [2] extended the concept of conjugations to a complex Banach space $\mathcal{X}$ (with its norm $\|\cdot\|$) as antilinear involutions whose operator norms are at most 1. More precisely, any operator $C : \mathcal{X} \to \mathcal{X}$ is called a conjugation on $\mathcal{X}$, if C satisfies

$$C^2 = I, \quad \|C\| \leq 1, \quad C(x + y) = Cx + Cy, \quad C(\lambda x) = \overline{\lambda}Cx, \tag{2}$$

where x and y are in $\mathcal{X}$ and λ is a complex number. Note that (2) implies that $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$. They showed that the definition above is the same as the usual one for conjugations in a complex Hilbert space $\mathcal{H}$.

With these extended conjugations C on $\mathcal{X}$, in this paper, we would like to give affirmative answers to two results related to isotropic vectors in [4], but on $\mathcal{X}$. More precisely, after generalizing isotropic vectors to $\mathcal{X}$, we show in Theorem 2.4 that they always exist if $\dim\mathcal{X} \geq 2$. In the proof of this we use linear functionals in the dual space of $\mathcal{X}$, denoted by $\mathcal{X}^*$. This gives us strong idea that, instead of an inner product, linear functionals can be employed on $\mathcal{X}$.

Based on this impression, Proposition 3.1 gives an equivalent condition for the C-symmetry of linear operators on $\mathcal{H}$ via its dual space $\mathcal{H}^*$. (For the definition of the C-symmetry, see Section 3.) This equivalent condition is applied to define complex symmetric operators or more generally C-symmetric operators on $\mathcal{X}$. Since the papers to deal with the C-symmetry on $\mathcal{X}$ are very rare (for example we knew only [2] where *normal* and C-symmetric operators are considered), we believe that our equivalent definition of C-symmetric operators via $\mathcal{X}^*$ sheds lights on understanding their true meaning and dealing with them on $\mathcal{X}$. After the new definition of the C-symmetry on $\mathcal{X}$, we discuss several properties of these C-symmetric operators and Theorem 3.11. The latter is the extension of Theorem 4.12 in [4].

2. Isotropic vectors in Banach spaces

Due to the lack of an inner product, we cannot directly use the original definition of isotropic vectors in a complex Hilbert space $\mathcal{H}$ in order to extend them to a complex Banach space $\mathcal{X}$. Instead observe that

$$\langle Cx, x\rangle = 0 \Longleftrightarrow Cx \perp x \Longleftrightarrow Cx \perp \mathcal{M}_x,$$

where $\mathcal{M}_x$ is the one-dimensional subspace generated by x.

As a generalization of orthogonality to $\mathcal{X}$, we adapt the orthogonality in the Birkhoff-James sense on [2,1,8,9]. For given two elements x and y in $\mathcal{X}$, x is called *orthogonal to y in the Birkhoff-James sense*, in short $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{C} \tag{3}$$

or, equivalently

$$\|x + m\| \geq \|x\| \quad \text{for all } m \in \mathcal{M}_y,$$

where M_y is the one-dimensional subspace generated by y. Note that, by the Hahn-Banach theorem, $x \perp_B y$ if and only if there is a norm-one linear functional f in $\mathcal{X}^*$ such that $f(x) = \|x\|$ and $f(y) = 0$. For more details, see [8] and the references therein.

Based on our discussion, isotropic vectors on $\mathcal{X}$ are defined as follows:

Definition 2.1. A vector x in $\mathcal{X}$ is called isotropic, if $Cx \perp_B x$, or equivalently, for all $\lambda \in \mathbb{C}$,

$$\|Cx + \lambda x\| \geq \|Cx\|.$$

Since C is norm-preserving, $Cx \perp_B x$ implies that $x \perp_B Cx$ (which is notable since the orthogonality in the Birkhoff-James sense is not symmetric. See [1,8] for more details). Again, by the Hahn-Banach theorem, $x \perp_B Cx$ if and only if there is f in $\mathcal{X}^*$ such that $\|f\| = 1, f(x) = \|x\|$ and $f(Cx) = 0$. Keep in mind that the latter will be very useful later in order to show the orthogonality in the Birkhoff-James sense, or equivalently isotropic property.

From now on let us focus on the existence of isotropic vectors. To express the idea and to be self-contained, we insert their existence on $\mathcal{H}$ in [4].

Lemma 2.2. ([4, Lemma 4.11]) *If $C : \mathcal{H} \to \mathcal{H}$ is a conjugation, then every subspace whose dimension is at least two contains isotropic vectors for the bilinear form $\langle \cdot, C\cdot \rangle$.*

For example, if S is the unilateral shift on $\ell^2(\mathbb{N})$, then $T = S \oplus S^*$ is complex symmetric from [6]. Moreover, every eigenvector of T is isotropic because T has each point in $\mathbb{D}$ (open unit disk) as a simple eigenvalue from [4].

Proof of Lemma 2.2. We consider the span of two linearly independent vectors x_1 and x_2. If x_1 or x_2 is isotropic, then it is trivial. If neither x_1 nor x_2 is isotropic, then

$$y_1 = x_1 \text{ and } y_2 = x_2 - \frac{\langle x_2, Cx_1 \rangle}{\langle x_1, Cx_1 \rangle} x_1$$

are C-orthogonal and have the same span as x_1, x_2. In this case, either y_2 is isotropic (which is trivial) or neither y_1 nor y_2 is isotropic. If the latter happens, we may assume that y_1 and y_2 satisfy $\langle y_1, Cy_1 \rangle = \langle y_2, Cy_2 \rangle = 1$. Then the vectors $y_1 \pm iy_2$ are both isotropic and have the same as x_1 and x_2. □

In order to show our affirmative answer on Banach-space version of the lemma above, introduce the Gram-Schmidt process on $\mathcal{X}$ via linear functionals in [7].

Lemma 2.3. ([7, Proposition 2.1]) *Let x_1, x_2 be vectors of $\mathcal{X}$. Then the following are equivalent;*

(i) *$\{x_1, x_2\}$ is linearly independent.*
(ii) *There exist functionals f_1, f_2 in $\mathcal{X}^*$ such that $f_1(x_1) \neq 0$, $f_2(x_1) = 0$, and $f_2(x_2) \neq 0$.*

Proof. This result is an easy application of Hahn-Banach theorem. □

We now see the existence of isotropic vectors on $\mathcal{X}$ via a similar idea of Lemma 2.2.

Theorem 2.4. *If $C : \mathcal{X} \to \mathcal{X}$ is a conjugation, then every subspace whose dimension is at least two contains isotropic vectors for the conjugation C.*

Proof. We consider the span of two linearly independent vectors x_1 and x_2. If x_1 or x_2 is isotropic, then we are done. If neither x_1 nor x_2 is isotropic, then by Lemma 2.3, there exist functionals f_1, f_2 in $\mathcal{X}^*$ with $\|f_1\| = \|f_2\| = 1$ such that $f_1(Cx_1) \neq 0$, $f_2(Cx_1) = 0$ and $f_2(Cx_2) \neq 0$. Put

$$x_2^{(1)} = x_2 - \frac{\overline{f_1(Cx_2)}}{\overline{f_1(Cx_1)}} x_1, \quad y_1 := \frac{x_1}{\overline{f_1(Cx_1)}} \quad \text{and} \quad y_2 := \frac{x_2^{(1)}}{\overline{f_2(Cx_2^{(1)})}}.$$

Then $f_1(Cy_1) = 1$, $f_2(Cy_1) = 0$, $f_2(Cy_2) = 1$ and $f_1(Cy_2) = 0$. Indeed, by Gram-Schmidt process in [7],

$$\begin{aligned} f_1(Cy_2) &= \frac{f_1(Cx_2^{(1)})}{f_2(Cx_2^{(1)})} \\ &= \frac{f_1(Cx_2 - \frac{f_1(Cx_2)}{f_1(Cx_1)} Cx_1)}{f_2(Cx_2 - \frac{f_1(Cx_2)}{f_1(Cx_1)} Cx_1)} \\ &= \frac{f_1(Cx_2) - \frac{f_1(Cx_2)}{f_1(Cx_1)} f_1(Cx_1)}{f_2(Cx_2) - \frac{f_1(Cx_2)}{f_1(Cx_1)} f_2(Cx_1)} \\ &= \frac{f_1(Cx_2) - f_1(Cx_2)}{f_2(Cx_2)} = 0. \end{aligned}$$

Thus Cy_1 (resp. y_1) is Birkhoff-James orthogonal to Cy_2 (resp. y_2), and the linear span of y_1 and y_2 is the same one spanned by x_1 and x_2. In this case, either y_2 is isotropic (which is trivial), or neither y_1 nor y_2 is isotropic.

If the latter happens, observe that $f_1(y_1) \neq 0$, $f_1(y_2) = 0$, $f_2(y_1) = 0$ and $f_2(y_2) \neq 0$. This is because, for example if $f_1(y_1) = 0$ happens, then Cy_1 is orthogonal to y_1. Therefore y_1 becomes isotropic, which is impossible in this case. A similar argument also shows the other three cases.

Take $f := f_1 \pm i f_2$. Then

$$\begin{aligned} f(C(y_1 \mp iy_2)) &= f(Cy_1 \pm iCy_2) = (f_1 \pm if_2)(Cy_1 \pm iCy_2) \\ &= f_1(Cy_1) \pm if_1(Cy_2) \pm if_2(Cy_1) - f_2(Cy_2) \\ &= 1 - 1 = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(y_1 \mp iy_2) &= (f_1 \pm if_2)(y_1 \mp iy_2) \\ &= f_1(y_1) \mp if_1(y_2) \pm if_2(y_1) + f_2(y_2) \\ &= f_1(y_1) + f_2(y_2). \end{aligned}$$

The quantity $f_1(y_1) + f_2(y_2)$ can be chosen to be nonzero via selecting f_1 and f_2 such that $f_1(y_1) > 0$ and $f_2(y_2) > 0$ (for example by replacing f_1 by $e^{i\theta} f_1$ with a real number θ). All this means that the vectors $y_1 \mp iy_2$ are both isotropic. $\square$

Therefore, we have the following corollary.

Corollary 2.5. *Let $T \in \mathcal{L}(\mathcal{X})$ be complex symmetric and let λ be an eigenvalue of T. If T has no isotropic eigenvectors for λ, then* $\dim \ker(T - \lambda) = 1$.

Here $\mathcal{L}(\mathcal{X})$ is the algebra of bounded linear operators on $\mathcal{X}$.

Proof. Suppose not, i.e., if $\dim\ker(T-\lambda) \geq 2$, then Theorem 2.4 indicates that the subspace $\ker(T-\lambda)$ contains an isotropic eigenvector which is impossible. □

In Section 3 this corollary (and its converse) will be discussed again, especially when T is C-symmetric in some sense on $\mathcal{X}$. It will also be seen how useful linear functionals are to deal with isotropic vectors and even to extend the C-symmetry to $\mathcal{X}$.

We now consider a natural dual element related to a given conjugation C. Let C be a conjugation on a complex Banach space $\mathcal{X}$. We define a dual conjugation C^* of C defined by

$$\big(C^*(f)\big)(x) := \overline{f(Cx)}. \tag{4}$$

Then C^* is a conjugation on the dual space $\mathcal{X}^*$ of $\mathcal{X}$. See [2] for more details. Then we have the following theorem:

Proposition 2.6. *Let x be a unit vector of a Banach space $\mathcal{X}$ and let C be a conjugation on $\mathcal{X}$. Let f be a linear functional on $\mathcal{X}$ such that $\|f\| = f(x) = \|x\| = 1$ and $f(Cx) = 0$. If $x \perp_B Cx$, then $f \perp_B C^*(f)$. In particular, x is isotropic.*

Proof. Let $\hat{x}$ be defined by $\hat{x}(g) = g(x)$ $(g \in \mathcal{X}^*)$, that is, Gelfand transformation of x. Then $\hat{x} \in \mathcal{X}^{**}$ and it satisfies

$$\|\hat{x}\| = \hat{x}(f) = f(x) = 1 \text{ and } \hat{x}(C^*(f)) = \overline{f(Cx)} = \overline{0} = 0.$$

Hence we have $f \perp_B C^*(f)$. □

Note that this natural dual conjugation C^* of C in (4) will be discussed and used in Section 3.

3. C-symmetric operators in Banach spaces

Let $\mathcal{L}(\mathcal{X})$ be the algebra of bounded linear operators on a complex Banach space $\mathcal{X}$. In this section, we would like to extend the concept of complex symmetric operators or more generally C-symmetric operators from a complex Hilbert space $\mathcal{H}$ to a complex Banach space $\mathcal{X}$. After that, we also want to see their properties. To do this, let us first recall that, in $\mathcal{H}$, the C-symmetry of a bounded linear operator T on $\mathcal{H}$ is expressed by $CT^*C = T$. In particular, with the aid of (1), for any vectors x and y in $\mathcal{H}$,

$$\langle CTy, x\rangle = \langle CTCCy, x\rangle = \langle T^*Cy, x\rangle = \langle Cy, Tx\rangle = \langle CTx, y\rangle. \tag{5}$$

However, due to the lack of an inner product, we cannot use the original condition to define complex symmetric operators on $\mathcal{X}$. Instead, we interpret (5) via linear functionals in the dual space $\mathcal{H}^*$ of $\mathcal{H}$ in order to generalize the notion of C-symmetric operators to $\mathcal{X}$.

Proposition 3.1. *For a bounded linear operator T on $\mathcal{H}$, $CT^*C = T$ if and only if for every pair of unit vectors, say x and y, there are two functionals f and g in $\mathcal{H}^*$ such that $||f|| = ||g|| = 1$, $f(x) = g(y) = 1$, $f(Cy) = g(Cx)$ and $f(CTy) = g(CTx)$.*

Proof. Choose two unit vectors x and y in $\mathcal{H}$. Then the necessity is trivially true by putting $f(\cdot) = \langle \cdot, x\rangle$ and $g(\cdot) = \langle \cdot, y\rangle$. For the sufficiency, observe that three conditions $||f|| = 1$, $||x|| = 1$ and $f(x) = 1$ imply that $f(\cdot) = \langle \cdot, x\rangle$ by Riesz representation theorem. (More precisely, by Riesz representation theorem $f(\cdot) = \langle \cdot, x_0\rangle$ with some $x_0 \in \mathcal{H}$. Moreover, x_0 should be exactly x, since the operator norm condition $||f|| = 1$ implies that x_0 is some unit vector and in this case the function value condition $f(x) = 1$ is true only when $x_0 = x$ among unit vectors.) Since $\langle CTx, y\rangle = \langle Cy, Tx\rangle = \langle T^*Cy, x\rangle = \langle Cx, CT^*Cy\rangle$, it follows that

$$\begin{aligned} f(CTy) = g(CTx) \quad &\Longleftrightarrow \quad \langle CTy, x\rangle = \langle CTx, y\rangle \\ &\Longleftrightarrow \quad \langle Cx, Ty\rangle = \langle Cx, CT^*Cy\rangle, \end{aligned}$$

which says that T is C-symmetric. □

Note that, in Proposition 3.1, we used the fact that, for given a unit vector $x \in \mathcal{H}$, there is a *unique* norm-one linear functional f with $f(x) = 1$, which is just $f(\cdot) = \langle \cdot, x\rangle$. Based on this observation, we would like to extend the C-symmetry of linear operators to Banach spaces and in particular to those satisfying the property above, that is, for each unit vector x, there is a unique norm-one functional f with $f(x) = 1$. It is then well-known that Phelps [10] and Taylor [11] characterize the condition for such a property on $\mathcal{X}$ as follows:

Theorem 3.2. ([10,11]) *If $\mathcal{X}^*$ is strictly convex, then every subspace, say $\mathcal{M}$, has a unique norm-preserving extension of continuous linear functionals on $\mathcal{M}$. The converse holds when $\mathcal{X}$ is reflexive.*

Recall that a normed space $\mathcal{X}$ is called *strictly convex*, if $x, y \in S$ and $x \neq y$ imply $\|\lambda x + (1-\lambda)y\| < 1$ for $0 < \lambda < 1$, where S is the unit sphere $\{x \in \mathcal{X} \mid \|x\| = 1\}$ in $\mathcal{X}$, i.e., S contains no line segments. See [10] for more details. It is worth to mention that, due to the theorem above, when $\mathcal{X}^*$ is strictly convex, then for given a unit vector x there exists the unique norm-one functional f in $\mathcal{X}^*$ satisfying $f(x) = 1$.

A typical example of such a complex Banach space in our mind is $L^p(\mathbb{T}, d\mu)$ with $p > 1$ but $p \neq 2$, where $\mathbb{T}$ is the unit circle and $d\mu$ is a finite measure on $\mathbb{T}$. More precisely, by Riesz representation theorem for $L^p(\mathbb{T})$ says that, for a given norm-one linear functional ϕ, there exists unique $g \in L^q(\mathbb{T}, d\theta)$ (essentially up to measure $d\mu$) with $1/p + 1/q = 1$, such that $||g||_q = ||\phi|| = 1$ and

$$\phi(f) = \int_{\mathbb{T}} fg\, d\mu, \quad f \in L^p(\mathbb{T}, d\mu).$$

It is also worth to mention the well-known James' theorem which says that $\mathcal{X}$ is a reflexive Banach space if and only if there exists $x \in \mathcal{X}$ with $||x|| \leq 1$ such that $||f|| = f(x)$. This means that every continuous linear functional on $\mathcal{X}$ attains its supremum on the closed unit ball.

Based on Proposition 3.1 and the observation above, we extend the C-symmetry of linear operators to complex Banach spaces $\mathcal{X}$ via their dual spaces as follows:

Definition 3.3. Let $T \in \mathcal{L}(\mathcal{X})$ and let C be a conjugation on $\mathcal{X}$. Then T is called C-symmetric in the sense of $\mathcal{X}^*$, if, for every pair of unit vectors x and y in $\mathcal{X}$, there exist two norm-one functionals f and g in $\mathcal{X}^*$ such that $f(x) = g(y) = 1$, $f(Cy) = g(Cx)$ and $f(CTy) = g(CTx)$.

Here $\mathcal{L}(\mathcal{X})$ is the set of all bounded linear operators on $\mathcal{X}$. The functionals are sometimes denoted by $f_{x,T}$ and $g_{y,T}$, especially when we emphasize the dependence of these functionals on x, y and T. However, in many cases when they are unambiguous, let us ignore these subscripts for convenience.

Remarks. 1. Even though the definition above seems a very natural extension of the C-symmetry of linear operators from $\mathcal{H}$ to $\mathcal{X}$ via its dual space, it would not be clear if the identity operator I of $\mathcal{X}$ would be C-symmetric in the sense of $\mathcal{X}^*$.

2. It is worthy mentioning that, if there exists $T \in \mathcal{L}(\mathcal{X})$ which is C-symmetric in the sense of $\mathcal{X}^*$, then so is the identity operator I. The functionals f and g can be chosen for both T and I at the same time (i.e., $f_{x,T} = f_{x,I}$ and $g_{y,T} = g_{y,I}$).

3. Even though T and S are C-symmetric in the sense of $\mathcal{X}^*$, we do not know if $f_{x,T} = f_{x,S}$ nor if $f(CTy) = g(CTx)$ implies $f(TCy) = g(TCx)$ in general.

4. On general complex Banach spaces, Definition 3.3 seems weaker. One of the reasons is that $\langle y, x\rangle = \langle Cx, Cy\rangle$ in $\mathcal{H}$, which can be re-written to

$$f_x(y) := \langle y, x\rangle = \langle Cx, Cy\rangle =: g_{Cy}(Cx)$$

via linear functionals in $\mathcal{X}^*$. This would, however, be hard to achieve on general Banach spaces.

To overcome the weakness of Definition 3.3 (which was addressed on 3 and 4 in Remarks above), in most cases we will assume that $\mathcal{X}^*$ is strictly convex. Then, due to Theorem 3.2, these functionals $f_{x,T}$ and $g_{y,T}$ are uniquely chosen independently of C-symmetric operators T in the sense of $\mathcal{X}^*$. Besides this, the following proposition expresses several useful properties when $\mathcal{X}^*$ is strictly convex:

Proposition 3.4. *Let $\mathcal{X}^*$ be strictly convex and let the identity operator I on $\mathcal{X}$ be C-symmetric in the sense of $\mathcal{X}^*$. Let $T \in \mathcal{L}(\mathcal{X})$. For two given unit vectors x and y denote by f and g two functionals satisfying $f(x) = g(y) = 1$ and $f(Cy) = g(Cx)$. Then the following statements hold:*

(i) *the (unique) norm-one functional whose function value at Cx is 1 is C^*f. (Note that $\|Cx\| = 1$ again.)*
(ii) *$f(y) = \overline{g(x)}$ and $f(y) = (C^*g)(Cx)$.*
(iii) *If T is C-symmetric in the sense of $\mathcal{X}^*$, then $f(TCy) = g(TCx)$.*
(iv) *T is C-symmetric in the sense of $\mathcal{X}^*$ if and only if CTC is also C-symmetric in the sense of $\mathcal{X}^*$.*

In any Hilbert space these are very easy to show. For example, (ii) is true due to $f(y) = \langle y, x\rangle = \overline{\langle x, y\rangle} = \overline{g(x)}$ and $f(y) = \langle y, x\rangle = \langle Cx, Cy\rangle = (C^*g)(Cx)$.

Proof. With the computation $(C^*f)(Cx) = \overline{f(x)} = 1$ and $(C^*g)(Cy) = \overline{g(y)} = 1$, the strict convexity of $\mathcal{X}^*$ leads (i). For (ii), apply the C-symmetry of I in the sense of $\mathcal{X}^*$ on (two unit vectors) x and Cy. Then this and (i) show that

$$f(y) = f(C(Cy)) = (C^*g)(Cx) = \overline{g(C(Cx))} = \overline{g(x)}.$$

For (iii), the C-symmetry of T in the sense of $\mathcal{X}^*$ on Cx and Cy (with (i)) implies that

$$\overline{f(TCy)} = \overline{f(C(CTCy))} = (C^*f)(CTCy) = (C^*g)(CTCx) = \overline{g(TCx)},$$

which shows (iii). For the last (iv), due to the strict convexity of $\mathcal{X}^*$, it is enough to show that $f(C(CTC)y) = g(C(CTC)x)$, which is just (iii). $\square$

From now on let us see some properties of the C-symmetric operators in the sense of $\mathcal{X}^*$.

Proposition 3.5. *If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric in the sense of $\mathcal{X}^*$, then the following properties hold:*

(i) *λT and $T - \lambda I$ are C-symmetric in the sense of $\mathcal{X}^*$ for any complex number λ (and vice versa).*
(ii) *$T|_{\mathcal{M}}$ is C-symmetric in the sense of $\mathcal{X}^*$ for any nonzero subspace $\mathcal{M}$ of $\mathcal{X}$.*

Note that in the proposition above we do not assume that $\mathcal{X}^*$ is strictly convex.

Proof. Since these are straightforward, they are left to the readers. □

Theorem 3.6. *Let $\mathcal{X}^*$ be strictly convex and let $T, S \in \mathcal{L}(\mathcal{X})$ be C-symmetric in the sense of $\mathcal{X}^*$. Then the following statements hold.*

(i) *$T + S$ is C-symmetric in the sense of $\mathcal{X}^*$.*
(ii) *If $TS = ST$, then TS is C-symmetric in the sense of $\mathcal{X}^*$.*

Proof. (i) Choose two unit vectors x and y in $\mathcal{X}$. Since T and S are C-symmetric in the sense of $\mathcal{X}^*$, there are four norm-one functionals f_T, f_S, g_T and g_S such that $f_T(x) = f_S(x) = g_T(y) = g_S(y) = 1$, $f_T(Cy) = g_T(Cx)$, $f_T(CTy) = g_T(CTx)$, $f_S(Cy) = g_S(Cx)$ and $f_S(CSy) = g_S(CSx)$. Due to the strict convexity of $\mathcal{X}^*$, $f_T = f_S$ and $g_T = g_S$. Put $f = f_T\ (= f_S)$ and $g = g_T\ (= g_S)$ simply. Therefore, we have that

$$f(C(T+S)y) = f(CTy) + f(CSy) = g(CTx) + g(CSx) = g(C(T+S)x),$$

which means that $T + S$ is C-symmetric in the sense of $\mathcal{X}^*$.

(ii) To see that TS is C-symmetric in the sense of $\mathcal{X}^*$, for given two unit vectors x and y, we need to show that there exist two norm-one linear functionals f_{TS} and g_{TS} such that $f_{TS}(x) = g_{TS}(y) = 1$, $f_{TS}(Cy) = g_{TS}(Cx)$ and $f_{TS}(CTSy) = g_{TS}(CTSx)$.

Due to the C-symmetry in the sense of $\mathcal{X}^*$ of T and S, there are norm-one functionals f_T, f_S, g_T and g_S satisfying the following:

$$\begin{aligned} &f_T(x) = f_S(x) = g_T(y) = g_S(y) = 1, \\ &f_T(Cy) = g_T(Cx),\ f_S(Cy) = g_S(Cx), \\ &f_T(CTy) = g_T(CTx),\ f_S(CSy) = g_S(CSx). \end{aligned}$$

Since $\mathcal{X}^*$ is strictly convex, Theorem 3.2 reveals that $f_T = f_S$ and $g_T = g_S$. (This is because f_T and f_S are the same on the one-dimensional linear span of x and g_T and g_S are the same on that of y.) Due to the same reason, if such f_{TS} and g_{TS} exist, they should also be f and g, respectively. So we get rid of the subscripts and put them by f and g. They then satisfy

$$\begin{aligned} &f(x) = g(y) = 1,\ f(Cy) = g(Cx), \\ &f(CTy) = g(CTx),\ f(CSy) = g(CSx) \end{aligned}$$

and it suffices to show that $f(CTSy) = g(CTSx)$.

Let us now assume that $Tx \neq 0$ and $Sy \neq 0$. Since $Tx/\|Tx\|$ and $Sy/\|Sy\|$ are unit vectors, a similar argument above indicates that there are norm-one functionals h and k such that

$$h\Big(\frac{Tx}{\|Tx\|}\Big) = k\Big(\frac{Sy}{\|Sy\|}\Big) = 1, \quad h\Big(C\frac{Sy}{\|Sy\|}\Big) = k\Big(C\frac{Tx}{\|Tx\|}\Big), \tag{6}$$

$$h\big(CT\frac{Sy}{\|Sy\|}\big) = k\big(CT^2x/\|Tx\|\big) \text{ and } h\big(CS^2y/\|Sy\|\big) = k\big(CS\frac{Tx}{\|Tx\|}\big).$$

Similarly we apply T and S's C-symmetry on two other pairs of unit vectors, $\{x, Sy/\|Sy\|\}$ and $\{y, Tx/\|Tx\|\}$. Then, by the uniqueness of the norm-one functional of each unit vector, it implies that

$$\begin{aligned} f(x) = k(Sy/\|Sy\|) = 1, &\quad g(y) = h(Tx/\|Tx\|) = 1, \\ f(CSy/\|Sy\|) = k(Cx), &\quad g(CTx/\|Tx\|) = h(Cy), \\ f(CTSy/\|Sy\|) = k(CTx), &\quad g(CSTx/\|Tx\|) = h(CSy). \end{aligned} \tag{7}$$

Due to (6), (7) and $TS = ST$, we have that

$$f(CT(Sy)) = \|Sy\|k(CTx) = \|Tx\|h(CSy) = g(CS(Tx)) = g(CTSx), \tag{8}$$

as desired.

If $Tx = Sy = 0$, then the commutativity of T and S, $TS = ST$, leads $f(CTSy) = g(CTSx)$ trivially. Hence, let us assume that $Tx = 0$ but $Sy \neq 0$. (A similar proof works for the other case when $Sy = 0$ but $Tx \neq 0$.) Then the first equality on (7) shows that

$$f(CTSy) = \|Sy\|k(CTx) = 0,$$

which implies $f(CTSy) = g(CTSx)$. Therefore TS becomes C-symmetric in the sense of $\mathcal{X}^*$. $\square$

To see more intuitively what is going on, note that, when $\mathcal{X}$ is a Hilbert space, our four functionals are just $f(\cdot) = \langle \cdot, x\rangle$, $g(\cdot) = \langle \cdot, y\rangle$, $\|Tx\|h(\cdot) = \langle \cdot, Tx\rangle$ and $\|Sy\|k(\cdot) = \langle \cdot, Sy\rangle$. This means that (8) is nothing but

$$\langle CTSy, x\rangle = \langle CTx, Sy\rangle = \langle CSy, Tx\rangle = \langle CSTx, y\rangle = \langle CTSx, y\rangle,$$

which is clear due to the C-symmetry of T and S, i.e., $CTC = T^*$ and $CSC = S^*$ and the properties of any conjugation C such as $\langle Cx, Cy\rangle = \langle y, x\rangle$ and $C^2 = I$.

As an easy consequence of the theorem above, we have the following:

Corollary 3.7. *Let $\mathcal{X}^*$ be strictly convex. If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric in the sense of $\mathcal{X}^*$, then so is $p(T)$ for any polynomial $p(z)$.*

Proposition 3.8. *Let $\mathcal{X}^*$ be strictly convex and let $T \in \mathcal{L}(\mathcal{X})$ be C-symmetric in the sense of $\mathcal{X}^*$. If T^{-1} exists, then it is also C-symmetric in the sense of $\mathcal{X}^*$.*

It is worth to mention that this proof is similar to that of (ii) in Theorem 3.6.

Proof. Choose any two unit vectors x and y in $\mathcal{X}$. Due to the C-symmetry in the sense of $\mathcal{X}^*$ of T, there are two norm-one functionals f and g such that

$$f(x) = g(y) = 1, \quad f(Cy) = g(Cx), \quad f(CTy) = g(CTx).$$

Since $\mathcal{X}^*$ is strictly convex, it suffices to show that $f(CT^{-1}y) = g(CT^{-1}x)$ for the C-symmetry of T^{-1} in the sense of $\mathcal{X}^*$.

The invertibility of T says that both $T^{-1}x$ and $T^{-1}y$ should not be zero vectors. We now apply the similar argument in the proof of (ii) in Theorem 3.6 except that $\frac{T^{-1}x}{\|T^{-1}x\|}$ and $\frac{T^{-1}y}{\|T^{-1}y\|}$ are two chosen unit vectors. Denote by h and k two norm-one functionals which satisfy

$$\begin{aligned} h\left(\frac{T^{-1}x}{\|T^{-1}x\|}\right) &= k\left(\frac{T^{-1}y}{\|T^{-1}y\|}\right) = 1, \\ h\left(C\frac{T^{-1}y}{\|T^{-1}y\|}\right) &= k\left(C\frac{T^{-1}x}{\|T^{-1}x\|}\right), \\ h\left(C\frac{y}{\|T^{-1}y\|}\right) &= k\left(C\frac{x}{\|T^{-1}x\|}\right). \end{aligned} \tag{9}$$

By performing the C-symmetry in the sense of $\mathcal{X}^*$ of T on two other pairs of unit vectors $\{x, T^{-1}y/\|T^{-1}y\|\}$ and $\{y, T^{-1}x/\|T^{-1}x\|\}$, the strict convexity of $\mathcal{X}^*$ indicates that

$$\begin{aligned} f(x) = k(T^{-1}y/\|T^{-1}y\|) &= 1, \quad & g(y) = h(T^{-1}x/\|T^{-1}x\|) &= 1, \\ f(CT^{-1}y/\|T^{-1}y\|) &= k(Cx), \quad & g(CT^{-1}x/\|T^{-1}x\|) &= h(Cy), \\ f(Cy/\|T^{-1}y\|) &= k(CTx), \quad & g(Cx/\|T^{-1}x\|) &= h(CTy). \end{aligned} \tag{10}$$

Then (9) and (10) say that

$$f(C(T^{-1}y)) = \|T^{-1}y\|k(Cx) = \|T^{-1}x\|h(Cy) = g(C(T^{-1}x)),$$

which means that T^{-1} is also C-symmetric in the sense of $\mathcal{X}^*$. □

Proposition 3.9. *Let $\mathcal{X}^*$ be strictly convex. If the sequence $\{T_n\}$ of C-symmetric operators in the sense of $\mathcal{X}^*$ converges to $T \in \mathcal{L}(\mathcal{X})$ in the strong operator norm topology, then T is also C-symmetric in the sense of $\mathcal{X}^*$.*

Proof. With a similar argument in the proof of Theorem 3.6 it suffices to show that $f(CTy) = g(CTx)$ for given two unit vectors x and y. Due to the continuity of f and g and the assumption on the convergence above, it follows that

$$f(CTy) = \lim_{n\to\infty} f(CT_n y) = \lim_{n\to\infty} g(CT_n x) = g(CTx). \qquad \square$$

So far we have examined basic properties of C-symmetric operators in the sense of $\mathcal{X}^*$ and showed the closedness of the set of all such operators under the strong operator norm topology. Let us extend Theorem 4.12 in [4] on $\mathcal{H}$ which shows the relation between the simplicity of an eigenvalue and the non-existence of its isotropic eigenvectors. For this, we generalize so-called C-projections to $\mathcal{X}$.

Definition 3.10. A projection P is called a C-projection in the sense of $\mathcal{X}^*$ if it is C-symmetric in the sense of $\mathcal{X}^*$.

Let λ be an isolated eigenvalue of a bounded linear operator T on $\mathcal{X}$. Then the *Riesz idempotent* E of T with respect to λ is defined by

$$E := \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (z - T)^{-1} dz$$

where $\mathbb{D}$ is an open disk centered at λ with $\overline{\mathbb{D}} \cup \sigma(T) = \{\lambda\}$. It is then well-known from [3] that $E^2 = E, ET = TE$, $\sigma(T|_{\mathrm{ran}(E)}) = \{\lambda\}$ and $\ker(T - \lambda I) \subset \mathrm{ran}\,(E)$. When T is C-symmetric in the sense of $\mathcal{X}^*$, Theorem 3.6 and Proposition 3.9 imply that E is a C-projection in the sense of $\mathcal{X}^*$, i.e., for given two unit vectors x and y, there exist two norm-one linear functionals f and g satisfying $f(x) = g(y) = 1$, $f(Cy) = g(Cx)$ and $f(CEy) = g(CEx)$.

The following theorem is a generalization of Theorem 4.12 in [4] with the C-symmetry in the sense of $\mathcal{X}^*$:

Theorem 3.11. *Let $T \in \mathcal{L}(\mathcal{X})$ be C-symmetric in the sense of $\mathcal{X}^*$ and λ an isolated eigenvalue of T. If T has no isotropic eigenvectors for λ, then λ is simple. Moreover, the converse is true if $\mathcal{X}^*$ is strictly convex.*

Proof. Let us mention that the theorem above will be proven in the contrapositive way. Assume that λ is not a simple eigenvalue. Then there are two cases to discuss as follows. If $\dim\ker(T - \lambda) > 1$, then Theorem 2.4 says that T has an isotropic eigenvector corresponding to λ. If $\dim\ker(T - \lambda) = 1$, then choose two (generalized) eigenvectors x and y satisfying $(T - \lambda)x = 0$ and $(T - \lambda)y = x$. Put the condition $\|x\| = 1$ for convenience. Since T is C-symmetric in the sense of $\mathcal{X}^*$, we can choose two norm-one linear functionals f and g satisfying $f(x) = g(y/\|y\|) = 1$, $f(Cy) = \|y\|g(Cx)$ and $f(CTy) = \|y\|g(CTx)$. Then

$$f(Cx) = f(C(T - \lambda)y) = \|y\|g(C(T - \lambda)x) = 0.$$

Since we have found one functional f with $f(x) = 1$ and $f(Cx) = 0$, x becomes isotropic, which is impossible. Therefore, λ is simple.

For the converse, assume that $\mathcal{X}^*$ is strictly convex and λ is a simple eigenvalue with an isotropic eigenvector x, i.e., $Tx = \lambda x$. Since x is isotropic, there exists a unique norm-one functional f such that $f(x) = 1$ and $f(Cx) = 0$. Since the Riesz idempotent E with respect to λ is a C-projection in the sense of $\mathcal{X}^*$ onto the span of x, the C-symmetry of T and E in $\mathcal{X}^*$ and the strict convexity of $\mathcal{X}^*$ imply that, for given any unit vector z, there exist unique norm-one functionals f and g such that $f(x) = 1$, $g(z) = 1$, $f(Cz) = g(Cx)$, $f(CEz) = g(CEx)$ and $f(CTz) = g(CTx)$. Then (when $Ez = \alpha x$ with some number α)

$$f(Cz) = f(CEz) + f(C(I - E)z) = \alpha f(Cx) + g(C(I - E)x) = 0.$$

This means that $f(Cz) = 0$ for every $z \in \mathcal{X}$, so $f = 0$, which contradicts to the fact that $f(x) = 1$. Therefore, x should not be isotropic, as desired. $\square$

Note that, for given $T \in \mathcal{L}(\mathcal{X})$, there exists a natural element in $\mathcal{L}(\mathcal{X}^*)$, say T^* by

$$(T^* f)(x) := f(Tx) \tag{11}$$

where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. We now apply Definition 3.3 to the dual space $\mathcal{X}^*$ naturally as follows:

Definition 3.12. Let $S \in \mathcal{L}(\mathcal{X}^*)$ and let J be a conjugation on $\mathcal{X}^*$. Then S is called *J-symmetric in the sense of $\mathcal{X}$* (or $\mathcal{X}^{**}$), if, for every pair of norm-one functionals f and g in $\mathcal{X}^*$, there exist two unit vectors x and y satisfying $\hat{x}(f) = \hat{y}(g) = 1$, $\hat{x}(Jg) = \hat{y}(Jf)$ and $\hat{x}(SJg) = \hat{y}(SJf)$, where $\hat{x}$ and $\hat{y}$ are point evaluations (or Gelfand transformations) of x and y, respectively.

Be careful for the reverse order SJ in the definition above, compared to Definition 3.3. However, due to the same reason for (iii) in Proposition 3.4, the order of SJ can be reversed, when $\mathcal{X}^*$ is strictly convex.

Proposition 3.13. *Let $\mathcal{X}^*$ be strictly convex. If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric in the sense of $\mathcal{X}^*$, then T^* is C^*-symmetric in the sense of $\mathcal{X}$.*

Proof. Choose two norm-one functional f and g in $\mathcal{X}^*$. Since $\|f\| = \sup\{|f(x)| : x \in \mathcal{X}, \|x\| = 1\} = 1 = \|g\|$ and $\mathcal{X}$ is complete, there exist unit vectors x and y in $\mathcal{X}$ such that $f(x) = g(y) = 1$. Due to the C-symmetry of T in the sense of $\mathcal{X}^*$, there exist norm-one functionals f' and g' such that $f'(x) = g'(y) = 1$, $f'(Cy) = g'(Cx)$ and $f'(CTy) = g'(CTx)$. Since $\mathcal{X}^*$ is strictly convex, we have $f' = f$ and $g' = g$. Hence it holds that $f(Cy) = g(Cx)$ and $f(CTy) = g(CTx)$. Therefore, we have

$$\hat{y}(T^*C^*g) = (T^*C^*g)(y) = (C^*g)(Ty) = \overline{g(CTy)}$$
$$\hat{x}(T^*C^*f) = (T^*C^*f)(x) = (C^*f)(Tx) = \overline{f(CTx)}.$$

Similarly, we can show that $\hat{x}(C^*g) = \hat{y}(C^*f)$. Therefore T^* is C^*-symmetric in the sense of $\mathcal{X}$. $\square$

Let us recall that $\mathcal{X}_1 \otimes \mathcal{X}_2$ denotes the completion (endowed with a sensible uniform cross-norm) of the algebraic tensor product $\mathcal{X}_1 \otimes \mathcal{X}_2$ of $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are complex Banach spaces. For operators $T \in \mathcal{L}(\mathcal{X}_1)$ and $S \in \mathcal{L}(\mathcal{X}_2)$, we define the *tensor product* operator $T \otimes S$ on $\mathcal{L}(\mathcal{X}_1 \otimes \mathcal{X}_2)$ by

$$(T \otimes S)\big(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j\big) = \sum_{j=1}^{n} \alpha_j \big(Tx_j \otimes Sy_j\big).$$

Then it is well-known that $T \otimes S \in \mathcal{L}(\mathcal{X}_1 \otimes \mathcal{X}_2)$. The definition of $T \otimes S$ is extended from these finite linear combinations of simple tensors to the whole space.

Proposition 3.14. *Let $\mathcal{X}_1^*$ and $\mathcal{X}_2^*$ be strictly convex. If T, S are C_1-symmetric and C_2-symmetric in the sense of $\mathcal{X}_1^*$ and $\mathcal{X}_2^*$ respectively, then the following properties hold:*

(i) *$T \oplus S$ is $C_1 \oplus C_2$-symmetric in the sense of $\mathcal{X}_1^* \oplus \mathcal{X}_2^*$.*
(ii) *$T \otimes S$ is $C_1 \otimes C_2$-symmetric in the sense of $\mathcal{X}_1^* \otimes \mathcal{X}_2^*$.*

Proof. Let us see (i) first. It is clear that $C_1 \oplus C_2$ is a conjugation on $\mathcal{X}_1 \oplus \mathcal{X}_2$. Choose two unit vectors in $\mathcal{X}_1 \oplus \mathcal{X}_2$, say $x_1 \oplus y_1$ and $x_2 \oplus y_2$. In other words, $\|x_i\|^2 + \|y_i\|^2 = 1$ for $i = 1, 2$. Let us first assume that x_i and y_i are not zero vectors. Since T and S are C_1-symmetric and C_2-symmetric in the sense of $\mathcal{X}_1^*$ and $\mathcal{X}_2^*$ respectively, there are four norm-one functionals f_1, f_2, g_1 and g_2 such that

$$f_1(x_1/\|x_1\|) = f_2(x_2/\|x_2\|) = g_1(y_1/\|y_1\|) = g_2(y_2/\|y_2\|) = 1,$$
$$f_1(C_1x_2/\|x_2\|) = f_2(C_1x_1/\|x_1\|),\ g_1(C_2y_2/\|y_2\|) = g_1(C_2y_1/\|y_1\|),$$
$$f_1(C_1Tx_2/\|x_2\|) = f_2(C_1Tx_1/\|x_1\|),\ g_1(C_2Sy_2/\|y_2\|) = g_2(C_2Sy_1/\|y_1\|).$$

To deal with the remaining case, assume that one of x_i or y_i are the zero vector. (Since $x_i \oplus y_i$ is a unit vector, both x_i and y_i cannot be zero vector at the same time.) In this case we assume that the corresponding functional $\|x_i\|f_i$ or $\|y_i\|g_i$ is the trivial functional, i.e., if $y_1 = 0$, then $\|y_1\|g_1$ is interpreted as the zero functional.

Then $\|x_1\|f_1 \oplus \|y_1\|g_1$ and $\|x_2\|f_2 \oplus \|y_2\|g_2$ are such functionals we looked for. (Again, if y_1 is the zero vector, then $\|y_1\|g_1(z) = 0$ for all $z \in \mathcal{X}_2$, and all the computations below work for this case, too.) Indeed, for $i, j = 1, 2$ and $i \neq j$,

$$(\|x_i\|f_i \oplus \|y_i\|g_i)(x_i \oplus y_i) := \|x_i\|f_i(x_i) + \|y_i\|g_i(y_i) = \|x_i\|^2 + \|y_i\|^2 = 1,$$

$$\begin{aligned}(\|x_i\|f_i \oplus \|y_i\|g_i)\big((C_1 \oplus C_2)(x_j \oplus y_j)\big) &= (\|x_i\|f_i \oplus \|y_i\|g_i)(C_1x_j \oplus C_2y_j)\\ &= \|x_i\|f_i(C_1x_j) + \|y_i\|g_i(C_2y_j)\\ &= \|x_j\|f_j(C_1x_i) + \|y_j\|g_j(C_2y_i)\\ &= (\|x_j\|f_j \oplus \|y_j\|g_j)(C_1x_i \oplus C_2y_i)\\ &= (\|x_j\|f_j \oplus \|y_j\|g_j)\big((C_1 \oplus C_2)(x_i \oplus y_i)\big),\end{aligned}$$

and

$$\begin{aligned}&(\|x_i\|f_i \oplus \|y_i\|g_i)\big((C_1 \oplus C_2)(T \oplus S)(x_j \oplus y_j)\big)\\ &= (\|x_i\|f_i \oplus \|y_i\|g_i)(C_1Tx_j \oplus C_2Sy_j)\\ &= \|x_i\|f_i(C_1Tx_j) + \|y_i\|g_i(C_2Sy_j)\\ &= \|x_j\|f_j(C_1Tx_i) + \|y_j\|g_j(C_2Sy_i)\\ &= (\|x_j\|f_j \oplus \|y_j\|g_j)(C_1Tx_i \oplus C_2Sy_i)\\ &= (\|x_j\|f_j \oplus \|y_j\|g_j)\big((C_1 \oplus C_2)(T \oplus S)(x_i \oplus y_i)\big).\end{aligned}$$

Therefore $T \oplus S$ is $C_1 \oplus C_2$-symmetric in the sense of $\mathcal{X}_1^* \oplus \mathcal{X}_2^*$.

For (ii), it is clear that $C_1 \otimes C_2$ is a conjugation on $\mathcal{X}_1 \otimes \mathcal{X}_2$. Then choose two unit tensors, say $x_1 \otimes y_1$ and $x_2 \otimes y_2$. Since $\|x \otimes y\| = \|x\|\,\|y\|$, select four norm-one functionals f_1, f_2, g_1 and g_2 such that

$$\begin{gathered}f_1(x_1) = f_2(x_2) = g_1(y_1) = g_2(y_2) = 1,\\ f_1(C_1x_2) = f_2(C_1x_1),\ g_1(C_2y_2) = g_1(C_2y_1),\\ f_1(C_1Tx_2) = f_2(C_1Tx_1),\ g_1(C_2Sy_2) = g_2(C_2Sy_1).\end{gathered}$$

Then by a similar (but easier) computation for (i), $T \otimes S$ is $C_1 \otimes C_2$-symmetric in the sense of $\mathcal{X}_1^* \otimes \mathcal{X}_2^*$. □

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