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On a necessary condition for an entire function with the increasing second quotients of Taylor coefficients to belong to the Laguerre-Pólya class

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A B S T R A C T

For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k, a_k > 0$, we show that f does not belong to the Laguerre-Pólya class if the quotients $\frac{a_{n-1}^2}{a_{n-2}a_n}$ are increasing in n, and $c := \lim\limits_{n\to\infty} \frac{a_{n-1}^2}{a_{n-2}a_n}$ is smaller than an absolute constant q_∞ ($q_\infty \approx 3.2336$).

1. Introduction

The zero distribution of entire functions, its sections and tails have been studied many authors, see, for example, the remarkable survey of the topic in [21]. In this paper we investigate new necessary conditions under which some special entire functions have only real zeros. First, we need the definition of the famous Laguerre-Pólya class.

Definition 1. A real entire function f is said to be in the *Laguerre-Pólya class*, written $f \in \mathcal{L} - \mathcal{P}$, if it can be expressed in the form

$$f(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{x x_k^{-1}}, \tag{1}$$

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where $c, \alpha, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} x_k^{-2} < \infty$. As usual, the product on the right-hand side can be finite or empty (in the latter case the product equals 1).

This class is essential in the theory of entire functions due to the fact that the polynomials with only real zeros converge locally uniformly to these and only these functions. The following prominent theorem states an even stronger fact.

Theorem A. (E. Laguerre and G. Pólya, see, for example, [4, pp. 42–46]).

(i) Let $(P_n)_{n=1}^{\infty},\ P_n(0) = 1$, *be a sequence of complex polynomials having only real zeros which converges uniformly in the circle* $|z| \leq A, A > 0$. *Then this sequence converges locally uniformly to an entire function from the* $\mathcal{L} - \mathcal{P}$ *class.*

(ii) For any $f \in \mathcal{L} - \mathcal{P}$ *there is a sequence of complex polynomials with only real zeros which converges locally uniformly to* f.

In our research, we also need the following important subclass of the class $\mathcal{L} - \mathcal{P}$.

Definition 2. A real entire function f is said to be in the *Laguerre-Pólya class of type I*, written $f \in \mathcal{L} - \mathcal{P}I$, if it can be expressed in the following form

$$f(x) = cx^n e^{\beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right), \tag{2}$$

where $c \in \mathbb{R}, \beta \geq 0, x_k > 0$, n is a nonnegative integer, and $\sum_{k=1}^{\infty} x_k^{-1} < \infty$.

The famous theorem by E. Laguerre and G. Pólya (see, for example, [18, chapter VIII, §3]) states that the polynomials with only real nonpositive zeros converge locally uniformly to the function from the class $\mathcal{L} - \mathcal{P}I$. The following theorem states a stronger fact.

Theorem B. (E. Laguerre and G. Pólya, see, for example, [18, chapter VIII, §3]).

(i) Let $(P_n)_{n=1}^{\infty},\ P_n(0) = 1$, *be a sequence of complex polynomials having only real negative zeros which converges uniformly in the circle* $|z| \leq A, A > 0$. *Then this sequence converges locally uniformly to an entire function from the class* $\mathcal{L} - \mathcal{P}I$.

(ii) For any $f \in \mathcal{L} - \mathcal{P}I$ *there is a sequence of complex polynomials with only real nonpositive zeros which converges locally uniformly to* f.

For various properties and characterizations of the Laguerre-Pólya class and the Laguerre-Pólya class of type I, see [23, p. 100], [24] or [20, Kapitel II].

Note that for a real entire function (not identically zero) of order less than 2 having only real zeros is equivalent to belonging to the Laguerre-Pólya class. The situation is different when an entire function is of order 2. For example, the function $f_1(x) = e^{-x^2}$ belongs to the Laguerre-Pólya class, but the function $f_2(x) = e^{x^2}$ does not.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with positive coefficients. We define the quotients p_n and q_n:

$$p_n = p_n(f) := \frac{a_{n-1}}{a_n},\ n \geq 1; \tag{3}$$

$$q_n = q_n(f) := \frac{p_n}{p_{n-1}} = \frac{a_{n-1}^2}{a_{n-2}a_n},\ n \geq 2.$$

The following formulas can be verified by straightforward calculations.

$$a_n = \frac{a_0}{p_1 p_2 \ldots p_n},\ n \geq 1\ ; \tag{4}$$

$$a_n = \frac{a_1}{q_2^{n-1} q_3^{n-2} \dots q_{n-1}^2 q_n} \left(\frac{a_1}{a_0}\right)^{n-1}, \ n \geq 2.$$

Deciding whether a given entire function has only real zeros is a rather subtle problem. In 1926, J.I. Hutchinson found the following sufficient condition for an entire function with positive coefficients to have only real zeros.

Theorem C. (J.I. Hutchinson, [5]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k. Then $q_n(f) \geq 4$, for all $n \geq 2$, if and only if the following two conditions are fulfilled:*

(i) The zeros of $f(z)$ are all real, simple and negative, and

(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, $m < n$, formed by taking any number of consecutive terms of $f(z)$, are all real and non-positive.

For some extensions of Hutchinson's results see, for example, [3, §4].

We also use the well-known notion of a complex zero decreasing sequence. For a real polynomial P we denote by $Z_c(P)$ the number of nonreal zeros of P counting multiplicities.

Definition 3. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is said to be a complex zero decreasing sequence (we write $(\gamma_k)_{k=0}^{\infty} \in \mathcal{CZDS}$), if

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k z^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k z^k\right), \tag{5}$$

for any real polynomial $\sum_{k=0}^{n} a_k z^k$.

The existence of nontrivial $\mathcal{CZDS}$ sequences is a consequence of the following remarkable theorem proved by Laguerre and extended by Pólya.

Theorem D. (G. Pólya, see [22] or [23, pp. 314–321]). *Let f be an entire function from the Laguerre-Pólya class having only negative zeros. Then $(f(k))_{k=0}^{\infty} \in \mathcal{CZDS}$.*

As it follows from the theorem above,

$$\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{CZDS}, \ a \geq 1, \quad \left(\frac{1}{k!}\right)_{k=0}^{\infty} \in \mathcal{CZDS}. \tag{6}$$

The entire function $g_a(z) = \sum_{j=0}^{\infty} z^j a^{-j^2}$, $a > 1$, a so-called *partial theta-function*, was investigated in the paper [6]. Simple calculations show that $q_n(g_a) = a^2$ for all n. Since $\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{CZDS}$, for $a \geq 1$, we conclude that for every $n \geq 2$ there exists a constant $c_n > 1$ such that $S_n(z, g_a) := \sum_{j=0}^{n} z^j a^{-j^2} \in \mathcal{L} - \mathcal{P} \ \Leftrightarrow \ a^2 \geq c_n$.

The survey [26] by S.O. Warnaar contains the history of investigation of the partial theta-function and its interesting properties.

Theorem E. (O. Katkova, T. Lobova, A. Vishnyakova, [6]). *There exists a constant q_∞ ($q_\infty \approx 3.23363666\dots$) such that:*

(1) $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow \ a^2 \geq q_\infty$;

(2) $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow$ *there exists* $x_0 \in (-a^3, -a)$ *such that* $g_a(x_0) \leq 0$;

(3) *for a given* $n \geq 2$ *we have* $S_n(z, g_a) \in \mathcal{L} - \mathcal{P} \ \Leftrightarrow$ *there exists* $x_n \in (-a^3, -a)$ *such that* $S_n(x_n, g_a) \leq 0$;

(4) $4 = c_2 > c_4 > c_6 > \dots$ *and* $\lim_{n\to\infty} c_{2n} = q_\infty$;

(5) $3 = c_3 < c_5 < c_7 < \dots$ *and* $\lim_{n\to\infty} c_{2n+1} = q_\infty$.

There is a series of works by V.P. Kostov dedicated to the interesting properties of zeros of the partial theta-function and its derivative (see [8], [9], [10], [11], [12], [13], [14], [15] and [16]). For example, in [9] V.P. Kostov studied the so called spectrum of the partial theta function, i.e. the set of values of $a > 1$ for which the function g_a has a multiple real zero.

Theorem F. (V.P. Kostov, [9]).

(1) *The spectrum* Γ *of the partial theta-function consists of countably many values of* a *denoted by* $\widetilde{a}_1 > \widetilde{a}_2 > \dots > \widetilde{a}_k > \dots > 1$, $\lim_{j\to\infty} \widetilde{a}_j = 1$.
(2) *For* $\widetilde{a}_k \in \Gamma$ *the function* $g_{\widetilde{a}_k}$ *has exactly one multiple real zero which is of multiplicity 2 and is the rightmost of its real zeros.*
(3) *For* $a \in (\widetilde{a}_{k+1}, \widetilde{a}_k)$ *the function* g_a *has exactly* k *complex conjugate pairs of zeros (counted with multiplicities).*

A wonderful paper [17] among the other results explains the role of the constant q_∞ in the study of the set of entire functions with positive coefficients having all Taylor truncations with only real zeros.

Theorem G. (V.P. Kostov, B. Shapiro, [17]). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function with positive coefficients and* $S_n(z) = \sum_{j=0}^{n} a_j z^j$ *be its sections. Suppose that there exists* $N \in \mathbb{N}$*, such that for all* $n \geq N$ *the sections* $S_n(z) = \sum_{j=0}^{n} a_j z^j$ *belong to the Laguerre-Pólya class. Then* $\liminf_{n\to\infty} q_n(f) \geq q_\infty$.

In [7], some entire functions with a convergent sequence of second quotients of coefficients are investigated. The main question of [7] is whether a function and its Taylor sections belong to the Laguerre-Pólya class. In [2] and [1], some important special functions with increasing sequence of second quotients of Taylor coefficients are studied.

In the previous paper [19], we have studied the entire functions with positive Taylor coefficients such that $q_n(f)$ are decreasing in n.

Theorem H. (T.H. Nguyen, A. Vishnyakova, [19]). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ *for all* k*, be an entire function. Suppose that* $q_n(f)$ *are decreasing in* n*, i.e.* $q_2 \geq q_3 \geq q_4 \geq \dots$, *and* $\lim\limits_{n\to\infty} q_n(f) = b \geq q_\infty$. *Then all the zeros of* f *are real and negative, in other words* $f \in \mathcal{L} - \mathcal{P}$.

It is easy to see that, if only the estimation of $q_n(f)$ from below is given and the assumption of monotonicity is omitted, then the constant 4 in $q_n(f) \geq 4$ is the smallest possible to conclude that $f \in \mathcal{L} - \mathcal{P}$.

In this paper, we study the case when $q_n(f)$ are increasing in n and obtained the following theorem.

Theorem 1.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ *for all* k*, be an entire function. Suppose that the quotients* $q_n(f)$ *are increasing in* n*, and* $\lim\limits_{n\to\infty} q_n(f) = c < q_\infty$. *Then the function* f *does not belong to the Laguerre-Pólya class.*

The theorem above provides the following necessary condition for an entire function with positive coefficients and with the increasing second quotients to belong to the Laguerre-Pólya class.

Corollary 1.2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ *for all* k*, be an entire function such that the quotients* $q_n(f)$ *are increasing in* n*. If* f *belongs to the Laguerre-Pólya class, then* $\lim\limits_{n\to\infty} q_n(f) \geq q_\infty$.

2. Proof of Theorem 1.1

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider a function $g(x) = a_0^{-1} f(a_0 a_1^{-1} x)$ instead of $f(x)$, due to the fact that such rescaling of f preserves its property of having real zeros and preserves the second quotients: $q_n(g) = q_n(f)$ for all n. During the proof we use notation p_n and q_n instead of $p_n(f)$ and $q_n(f)$. So, we can write

$$f(x) = 1 + x + \sum_{k=2}^{\infty} \frac{x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k}.$$

Let us introduce some more notations. For an entire function f, by $S_n(x, f)$ and $R_n(x, f)$ we denote the nth partial sum and the nth remainder of the series, i.e.

$$S_n(x, f) = \sum_{k=0}^{n} \frac{x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k},$$

and

$$R_n(x, f) = \sum_{k=n}^{\infty} \frac{x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k}.$$

We also consider a function

$$\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k}$$

instead of f.

Since the quotients q_n are increasing in n, and $\lim_{n\to\infty} q_n = c < q_\infty$, we conclude that $q_2 \le q_\infty < 4$. The following lemma shows that for $q_2 < 3$ we have $\varphi \notin \mathcal{L} - \mathcal{P}$.

Lemma 2.1. *Let $\varphi(z) = \sum_{k=0}^{\infty} (-1)^k a_k z^k$ be an entire function, $a_k > 0$ for all k, $a_0 = a_1 = 1$, and $q_n = q_n(\varphi)$ are increasing in n, i.e. $q_2 \le q_3 \le q_4 \le \dots$. If $\varphi \in \mathcal{L} - \mathcal{P}$, then $q_2(f) \ge 3$.*

Proof. Denote by $0 < z_1 \le z_2 \le z_3 \le \dots$ the real roots of φ. We observe that

$$0 \le \sum_{k=1}^{\infty} \frac{1}{z_k^2} = \left(\sum_{k=1}^{\infty} \frac{1}{z_k}\right)^2 - 2 \sum_{1\le i<j<\infty} \frac{1}{z_i z_j} = \left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0},$$

whence $q_2 \ge 2$.

According to the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\left(\frac{1}{z_1} + \frac{1}{z_2} + \ldots\right)\left(\frac{1}{z_1^3} + \frac{1}{z_2^3} + \ldots\right) \ge \left(\frac{1}{z_1^2} + \frac{1}{z_2^2} + \ldots\right)^2.$$

By Vieta's formulas, we have $\sigma_1 := \sum_{k=1}^{\infty} \frac{1}{z_k} = \frac{a_1}{a_0}$, $\sigma_2 = \sum_{1<i<j<\infty} \frac{1}{z_i z_j} = \frac{a_2}{a_0}$, and $\sigma_3 = \sum_{1<i<j<k<\infty} \frac{1}{z_i z_j z_k} = \frac{a_3}{a_0}$. We need further the following identities: $\sum_{k=1}^{\infty} \frac{1}{z_1^2} = \sigma_1^2 - 2\sigma_2$, and $\sum_{k=1}^{\infty} \frac{1}{z_1^3} = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$. Consequently, we have

$$\sigma_1(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) \ge (\sigma_1^2 - 2\sigma_2)^2,$$

or

$$\frac{a_1^2 a_2}{a_0^3} + 3\frac{a_1 a_3}{a_0^2} - 4\frac{a_2^2}{a_0^2} \geq 0.$$

Since $a_0 = a_1 = 1$ and $a_2 = \frac{1}{q_2}, a_3 = \frac{1}{q_2^2 q_3}$, we have:

$$q_3(q_2 - 4) + 3 \geq 0.$$

Since we have the conditions that $q_2 < 4$ and $q_2 \leq q_3$, we conclude that

$$q_2(q_2 - 4) + 3 \geq 0.$$

Therefore, we get that $q_2 \geq 3$. $\square$

Further, we assume that $3 \leq q_2 < q_\infty$.

In order to prove Theorem 1.1, we need some more Lemmas.

Lemma 2.2. *Let $\varphi(x) = 1 - x + \sum_{k=2}^\infty \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k}$ be an entire function. Suppose that $q_2 \geq 2$, q_k are increasing in k, i.e. $q_2 \leq q_3 \leq q_4 \dots$, and $\lim_{n\to\infty} q_n = c < q_\infty$. Then for any $x \in [0, q_2]$ we have $\varphi(x) > 0$, i.e. there are no real roots of φ in the segment $[0, q_2]$.*

Proof. For $x \in [0, 1]$ we have

$$1 \geq x > \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \frac{x^4}{q_2^3 q_3^2 q_4} > \cdots,$$

whence

$$\varphi(x) > 0 \quad \text{for all} \quad x \in [0, 1]. \tag{7}$$

Suppose that $x \in (1, q_2]$. Then we obtain

$$1 < x \geq \frac{x^2}{q_2} > \frac{x^3}{q_2^2 q_3} > \cdots > \frac{x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k} > \cdots \tag{8}$$

For an arbitrary $m \in \mathbb{N}$ we have

$$\varphi(x) = S_{2m+1}(x, \varphi) + R_{2m+2}(x, \varphi),$$

where

$$S_{2m+1}(x, \varphi) := 1 - x + \sum_{k=2}^{2m+1} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k},$$

and

$$R_{2m+2}(x, \varphi) := \sum_{k=2m+2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \dots q_{k-1}^2 q_k}.$$

By (8) and the Leibniz criterion for alternating series, we obtain $R_{2m+2}(x,\varphi)>0$ for all $x\in(1,q_2]$, or

$$\varphi(x) > S_{2m+1}(x,\varphi) \quad \text{for all} \quad x \in (1,q_2], m \in \mathbb{N}. \tag{9}$$

It remains to prove that there exists $m\in\mathbb{N}$ such that $S_{2m+1}(x,\varphi)>0$ for all $x\in(1,q_2]$. We have

$$S_{2m+1}(x,\varphi) = (1-x) + \left(\frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3}\right) + \left(\frac{x^4}{q_2^3 q_3^2 q_4} - \frac{x^5}{q_2^4 q_3^3 q_4^2 q_5}\right) + \ldots + \left(\frac{x^{2m}}{q_2^{2m-1} q_3^{2m-2} \cdot \ldots \cdot q_{2m-1}^2 q_{2m}} - \frac{x^{2m+1}}{q_2^{2m} q_3^{2m-1} \cdot \ldots \cdot q_{2m}^2 q_{2m+1}}\right). \tag{10}$$

Under our assumptions, q_k are increasing in k, and $\lim\limits_{n\to\infty} q_n = c$. We prove that for any fixed $k=1,2,\ldots,m$ and $x\in(1,q_2]$ the following inequality holds:

$$\frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k}^2 q_{2k+1}} \geq \frac{x^{2k}}{c^{2k-1} \cdot c^{2k-2} \cdot \ldots \cdot c} - \frac{x^{2k+1}}{c^{2k} \cdot c^{2k-1} \cdot \ldots \cdot c^2 \cdot c}.$$

For $x\in(1,q_2]$ and $k=1,2,\ldots,m$, we define the following function

$$F(q_2,q_3,\ldots,q_{2k},q_{2k+1}) := \frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdot \ldots \cdot q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdot \ldots \cdot q_{2k}^2 q_{2k+1}}.$$

We can observe that

$$\frac{\partial F(q_2,q_3,\ldots,q_{2k},q_{2k+1})}{\partial q_2} = -\frac{(2k-1)x^{2k}}{q_2^{2k} q_3^{2k-2} \ldots q_{2k}} + \frac{2k x^{2k+1}}{q_2^{2k+1} q_3^{2k-1} \ldots q_{2k}^2 q_{2k+1}} < 0$$
$$\Leftrightarrow x < \left(1 - \frac{1}{2k}\right) q_2 q_3 \ldots q_{2k} q_{2k+1}.$$

Thus, under our assumptions, the function $F(q_2,q_3,\ldots,q_{2k},q_{2k+1})$ is decreasing in q_2. Since $q_2\le q_3$,

$$F(q_2,q_3,q_4,\ldots,q_{2k},q_{2k+1}) \geq F(q_3,q_3,q_4,\ldots,q_{2k},q_{2k+1}) = \frac{x^{2k}}{q_3^{4k-3} q_4^{2k-3} \ldots q_{2k}} - \frac{x^{2k+1}}{q_3^{4k-1} q_4^{2k-2} \ldots q_{2k+1}}.$$

Further we have

$$\frac{\partial F(q_3,q_3,q_4,\ldots,q_{2k},q_{2k+1})}{\partial q_3} = -\frac{(4k-3)x^{2k}}{q_3^{4k-2} q_4^{2k-3} \ldots q_{2k}} + \frac{(4k-1)x^{2k+1}}{q_3^{4k} q_4^{2k-2} \ldots q_{2k+1}} < 0$$
$$\Leftrightarrow x < \frac{4k-3}{4k-1} q_3^2 q_4 \ldots q_{2k+1}.$$

Hence, under our assumptions, $F(q_3,q_3,q_4,\ldots,q_{2k},q_{2k+1})$ is decreasing in q_3, and since $q_3\le q_4$ we obtain

$$F(q_3,q_3,q_4\ldots,q_{2k},q_{2k+1}) \geq F(q_4,q_4,q_4,q_5,\ldots,q_{2k},q_{2k+1}).$$

Analogously, by the same computations, we obtain the following chain of inequalities

$$F(q_2, q_3, q_4, \dots, q_{2k}, q_{2k+1}) \geq F(q_3, q_3, q_4, \dots, q_{2k}, q_{2k+1}) \geq$$
$$F(q_4, q_4, q_4, q_5, \dots, q_{2k}, q_{2k+1}) \geq \ldots \geq F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1}).$$

Further, we have

$$\frac{\partial F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1})}{\partial q_{2k+1}} = -\frac{(2k^2-k)x^{2k}}{q_{2k+1}^{2k^2-k+1}} + \frac{(2k^2+k)x^{2k+1}}{q_{2k+1}^{2k^2+k+1}} < 0$$
$$\Leftrightarrow x < \frac{2k^2-k}{2k^2+k} q_{2k+1}^{2k}.$$

Thus, $F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1})$ is decreasing in q_{2k+1}, and since q_k are increasing in k, and $\lim\limits_{n\to\infty} q_n = c$, we conclude that

$$F(q_{2k+1}, q_{2k+1}, \dots, q_{2k+1}, q_{2k+1}) \geq F(c, c, \dots, c, c) = \frac{x^{2k}}{c^{k(2k-1)}} - \frac{x^{2k+1}}{c^{k(2k+1)}}.$$

Substituting the last inequality in (10) for every $x \in (1, q_2]$ and $k = 1, 2, \dots, m$, we get

$$S_{2m+1}(x, \varphi) \geq (1-x) + \left(\frac{x^2}{c} - \frac{x^3}{c^3}\right) + \left(\frac{x^4}{c^6} - \frac{x^5}{c^{10}}\right) + \ldots \tag{11}$$
$$+ \left(\frac{x^{2m}}{c^{m(2m-1)}} - \frac{x^{2m+1}}{c^{m(2m+1)}}\right) = \sum_{k=0}^{2m+1} \frac{(-1)^k x^k}{\sqrt{c}^{k(k-1)}} = S_{2m+1}(\sqrt{c}x, g_{\sqrt{c}}),$$

where g_a is the partial theta-function and $S_{2m+1}(y, g_a)$ are its $(2m+1)$-th section at the point y. Since, by our assumptions, $(\sqrt{c})^2 < q_\infty$, using the statement (5) of Theorem E, we obtain that there exists $m \in \mathbb{N}$ such that $S_{2m+1}(y, g_{\sqrt{c}}) \notin \mathcal{L}-\mathcal{P}$. Let us choose and fix such m. By the statement (3) of Theorem E, we obtain that for every x such that $\sqrt{c} < \sqrt{c}x < (\sqrt{c})^3$ we have $S_{2m+1}(\sqrt{c}x, g_{\sqrt{c}}) > 0$. It means that for every $x : 1 < x < c$ we have $S_{2m+1}(\sqrt{c}x, g_{\sqrt{c}}) > 0$, and, using (11) and (9),

$$\varphi(x) > S_{2m+1}(x, \varphi) > 0 \quad \text{for all} \quad x \in (1, q_2) \subset (1, c).$$

It remains to prove that $\varphi(q_2) > 0$. We have

$$\varphi(q_2) = \left(1 - q_2 + q_2 - \frac{q_2}{q_3}\right) + \left(\frac{q_2}{q_3^2 q_4} - \frac{q_2}{q_3^3 q_4^2 q_5}\right)$$
$$+ \left(\frac{q_2}{q_3^4 q_4^3 q_5^2 q_6} - \frac{q_2}{q_3^5 q_4^4 q_5^3 q_6^2 q_7}\right) + \ldots > 0$$

by our assumptions on q_j. □

Lemma 2.3. *Let $P(z) = 1 - z + \frac{z^2}{a} - \frac{z^3}{a^2 b} + \frac{z^4}{a^3 b^2 c}$ be a polynomial, $3 \leq a < 4$, and $a \leq b \leq c$. Then*

$$\min_{0 \leq \theta \leq 2\pi} |P(ae^{i\theta})| \geq \frac{a}{b^2 c}.$$

Proof. By direct calculation, we have

$$|P(ae^{i\theta})|^2 = (1 - a\cos\theta + a\cos 2\theta - \frac{a}{b}\cos 3\theta + \frac{a}{b^2c}\cos 4\theta)^2 +$$
$$(-a\sin\theta + a\sin 2\theta - \frac{a}{b}\sin 3\theta + \frac{a}{b^2c}\sin 4\theta)^2$$
$$= 1 + 2a^2 + \frac{a^2}{b^2} + \frac{a^2}{b^4c^2} - 2a\cos\theta + 2a\cos 2\theta - 2\frac{a}{b}\cos 3\theta$$
$$+ 2\frac{a}{b^2c}\cos 4\theta - 2a^2\cos\theta + 2\frac{a^2}{b}\cos 2\theta - 2\frac{a^2}{b^2c}\cos 3\theta$$
$$- 2\frac{a^2}{b}\cos\theta + 2\frac{a^2}{b^2c}\cos 2\theta - 2\frac{a^2}{b^3c}\cos\theta.$$

Set $t := \cos\theta, t \in [-1, 1]$. Since $\cos 2\theta = 2t^2 - 1$, $\cos 3\theta = 4t^3 - 3t$, and $\cos 4\theta = 8t^4 - 8t^2 + 1$, we get

$$|P(ae^{i\theta})|^2 = \frac{16a}{b^2c}t^4 + \left(-\frac{8a}{b} - \frac{8a^2}{b^2c}\right)t^3 + \left(4a - \frac{16a}{b^2c} + \frac{4a^2}{b} + \frac{4a^2}{b^2c}\right)t^2 +$$
$$\left(-2a + \frac{6a}{b} - 2a^2 + \frac{6a^2}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^3c}\right)t$$
$$+ \left(1 + 2a^2 + \frac{a^2}{b^2} + \frac{a^2}{b^4c^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\right).$$

We want to show that $\min_{0\le\theta\le2\pi}|P(ae^{i\theta})|^2 \ge \frac{a^2}{b^4c^2}$, or to prove the inequality $\min_{0\le\theta\le2\pi}|P(ae^{i\theta})|^2 - \frac{a^2}{b^4c^2} \ge 0$. Using the last expression we see that the inequality we want to get is equivalent to the following: for all $t \in [-1, 1]$ the next inequality holds

$$\frac{16a}{b^2c}t^4 - \frac{8a}{b}(1 + \frac{a}{bc})t^3 + 4a\big(1 - \frac{4}{b^2c} + \frac{a}{b} + \frac{a}{b^2c}\big)t^2 - 2a\big(1 - \frac{3}{b} + a - \frac{3a}{b^2c} + \frac{a}{b}$$
$$+ \frac{a}{b^3c}\big)t + \big(1 + 2a^2 + \frac{a^2}{b^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\big) \ge 0.$$

Let $y := 2t$, $y \in [-2, 2]$. We rewrite the last inequality in the form

$$\frac{a}{b^2c}y^4 - \frac{a}{b}\left(1 + \frac{a}{bc}\right)y^3 + a\left(1 - \frac{4}{b^2c} + \frac{a}{b} + \frac{a}{b^2c}\right)y^2$$
$$-a\left(1 - \frac{3}{b} + a - \frac{3a}{b^2c} + \frac{a}{b} + \frac{a}{b^3c}\right)y +$$
$$\left(1 + 2a^2 + \frac{a^2}{b^2} - 2a + \frac{2a}{b^2c} - \frac{2a^2}{b} - \frac{2a^2}{b^2c}\right) \ge 0.$$

We note that the coefficient of y^4 is positive, and the coefficient of y^3 is negative. It is easy to show that the other coefficients are also sign-changing. For y^2: $1 - \frac{4}{b^2c} > 0$ since $b^2c > 4$, thus,

$$1 + \frac{a}{b} + \frac{a}{b^2c} - \frac{4}{b^2c} = (1 - \frac{4}{b^2c}) + \frac{a}{b} + \frac{a}{b^2c} > 0.$$

For y:

$$1 + a + \frac{a}{b} + \frac{a}{b^3c} - \frac{3}{b} - \frac{3a}{b^2c} = (1 + a - \frac{3}{b}) +$$
$$(\frac{a}{b} - \frac{3a}{b^2c}) + \frac{a}{b^3c} > 0.$$

Finally,

$$1 + 2a^2 + \frac{a^2}{b^2} - 2a - 2\frac{a^2}{b} - 2\frac{a^2}{b^2c} + 2\frac{a}{b^2c} =$$
$$(1 + a^2 - 2a) + (a^2 - 2\frac{a^2}{b}) + (\frac{a^2}{b^2} - 2\frac{a^2}{b^2c}) + 2\frac{a}{b^2c} > 0,$$

since $1 - 2a + a^2 \geq 0$; $a^2 - 2\frac{a^2}{b} > 0$ and $\frac{a^2}{b^2} - 2\frac{a^2}{b^2c} > 0$ by our assumptions.

Consequently, the inequality we need holds for any $y \in [-2, 0]$, and we have to prove it for $y \in [0, 2]$. Multiplying our inequality by $\frac{b^2c}{a}$, we get

$$y^4 - (bc + a)y^3 + (b^2c + abc + a - 4)y^2 - (b^2c + ab^2c + abc + \frac{a}{b} - 3bc - 3a)y$$
$$+(\frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2) =: \psi(y),$$

and we want to prove that $\psi(y) \geq 0$ for all $y \in [0, 2]$.

Let $\chi(y) := \psi(y) - \frac{1}{b}(b - a)y$, whence $\chi(y) \leq \psi(y)$ for all $y \in [0, 2]$. It is sufficient to prove that $\chi(y) \geq 0$ for all $y \in [0, 2]$. We have

$$\chi(0) = \psi(0) = \frac{b^2c}{a} + 2ab^2c + ac - 2b^2c - 2abc - 2a + 2 \geq 0,$$

as it was previously shown. We also have $\chi(2) = \psi(2) - \frac{2}{c}(b - a) \geq 0$, since

$$\psi(2) = -2bc - 2\frac{a}{b} + \frac{b^2c}{a} + ac + 2 =$$
$$\frac{1}{b}\left(2(b - a) + \frac{b^2c}{a}(b - a) - bc(b - a)\right) =$$
$$\frac{1}{b}(b - a)\left(2 + \frac{bc}{a}(b - a)\right) \geq \frac{2}{b}(b - a) \geq 0.$$

Now we consider the following function:

$$\nu(y) := \frac{\partial^2\chi(y)}{\partial y^2} = \frac{\partial^2\psi(y)}{\partial y^2} = 12y^2 - 6(bc + a)y + 2(b^2c + abc + a - 4).$$

The vertex point of this parabola is $y_v = \frac{bc+a}{4} \geq 3$. Accordingly, we can observe that $\nu(y)$ decreases for $y \in [0, 2]$. We have

$$\nu(0) = 2(b^2c + abc + a - 4) > 0,$$

and

$$\nu(2) = 2abc + 2b^2c - 12bc - 10a + 40.$$

We want to show that $\nu(2)$ is positive. We have

$$abc + b^2c - 6bc - 5a + 20 = (20 - 5a) + (b^2c - 3bc) + (abc - 3bc) =$$
$$5(4 - a) + bc(c - 3) + bc(a - 3) > 0$$

due to our assumptions. We conclude that $\nu(y)$ is nonnegative for $y \in [0,2]$, and it follows that $\chi'(y)$ increases for $y \in [0,2]$.

We want to show that $\chi'(y) \leq 0$ for $y \in [0,2]$, and it is sufficient to show that $\chi'(2) \leq 0$. We have

$$\chi'(2) = \psi'(2) - \frac{b-a}{b} = 15 - 9bc - 5a + 3b^2c + 3abc - ab^2c =$$
$$5(3-a) + bc(-9 + 3b + 3a - ab) = 5(3-a) + bc(a-3)(3-b) \leq 0.$$

Thus, $\chi(y)$ decreases, $\chi(2) \geq 0$, so it is positive for $y \in [0,2]$. Since $\chi(y) \leq \psi(y)$, it follows that $\psi(y)$ is positive for $y \in [0,2]$. $\square$

The function φ can be presented in the following form:

$$\varphi(x) = S_4(x,\varphi) + R_5(x,\varphi),$$

where

$$S_4(x,\varphi) := (1 - x + \frac{x^2}{q_2} - \frac{x^3}{q_2^2q_3} + \frac{x^4}{q_2^3q_3^2q_4}),$$

and

$$R_5(x,\varphi) := \sum_{k=5}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\dots q_k}.$$

By Lemma 2.3 we have

$$\min_{0\leq\theta\leq 2\pi} |S_4(q_2e^{i\theta},\varphi)| \geq \frac{q_2}{q_3^2q_4}. \tag{12}$$

Now we need the estimation on $|R_5(q_2e^{i\theta},\varphi)|$ from above.

Lemma 2.4. *Let $R_5(z,\varphi) := \sum_{k=5}^{\infty} \frac{(-1)^k z^k}{q_2^{k-1}q_3^{k-2}\dots q_k}$, q_n be increasing in n, and let $\lim\limits_{n\to\infty} q_n(f) = c < q_\infty$. Then*

$$\max_{0\leq\theta\leq 2\pi} |R_5(q_2e^{i\theta},\varphi)| \leq \frac{q_2}{q_3^3q_4^3 - q_3^2}.$$

Proof. We have

$$|R_5(q_2e^{i\theta},\varphi)| \leq \sum_{k=5}^{\infty} \frac{q_2^k}{q_2^{k-1}q_3^{k-2}\dots q_k} = \sum_{k=5}^{\infty} \frac{q_2}{q_3^{k-2}\dots q_k} =$$
$$\frac{q_2}{q_3^3q_4^2q_5} + \frac{q_2}{q_3^4q_4^3q_5^2q_6} + \dots + \frac{q_2}{q_3^{k-2}\dots q_k} + \dots$$
$$\leq \frac{q_2}{q_3^3q_4^3}(1 + \frac{1}{q_3q_4^3} + \frac{1}{q_3^2q_4^7} + \dots + \frac{1}{q_3^{k-5}q_4^{\frac{k(k-5)}{2}}} + \dots) \leq$$
$$\frac{q_2}{q_3^3q_4^3} \cdot \frac{1}{1 - \frac{1}{q_3q_4^3}} = \frac{q_2}{q_3^3q_4^3 - q_3^2}. \quad \square$$

Let us check that $\frac{q_2}{q_3^2 q_4} > \frac{q_2}{q_3^3 q_4^3 - q_3^2}$, which is equivalent to $q_4 < q_3 q_4^3 - 1$. The last inequality obviously holds under our assumptions. Therefore, according to Rouché's theorem, the functions $S_4(z,\varphi)$ and $\varphi(z)$ have the same number of zeros inside the circle $\{z : |z| < q_2\}$ counting multiplicities.

It remains to prove that $S_4(z,\varphi)$ has zeros in the circle $\{z : |z| < q_2\}$. To do this we need the notion of apolar polynomials and the famous theorem by J.H. Grace.

Definition 4. (See, for example [25, Chapter 2, $ 3, p. 59]). Two complex polynomials $P(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$ and $Q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$ of degree n are called apolar if

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{13}$$

The following famous theorem due to J.H. Grace states that the complex zeros of two apolar polynomials cannot be separated by a straight line or by a circumference.

Theorem I. (J.H. Grace, see for example [25, Chapter 2, $ 3, Problem 145]). *Suppose P and Q are two apolar polynomials of degree $n \geq 1$. If all zeros of P lie in a circular region C, then Q has at least one zero in C. (A circular region is a closed or open half-plane, disk or exterior of a disk).*

Lemma 2.5. *Let $S_4(z,\varphi) = 1 - z + \frac{1}{q_2} z^2 - \frac{1}{q_2^2 q_3} z^3 + \frac{1}{q_2^3 q_3^2 q_4} z^4$ be a polynomial and $q_2 \geq 3$. Then $S_4(z,\varphi)$ has at least one root in the circle $\{z : |z| \leq q_2\}$.*

Proof. We have

$$S_4(z,\varphi) = \binom{4}{0} + \binom{4}{1}(-\frac{1}{4})z + \binom{4}{2}\frac{1}{6q_2}z^2 + \binom{4}{3}(-\frac{1}{4q_2^2 q_3})z^3 + \binom{4}{4}\frac{1}{q_2^3 q_3^2 q_4}z^4.$$

Let

$$Q(z) = \binom{4}{2} b_2 z^2 + \binom{4}{3} b_3 z^3 + \binom{4}{4} z^4.$$

Then the condition for $S_4(z,\varphi)$ and $Q(z)$ to be apolar is the following

$$\binom{4}{0} - \binom{4}{1}\left(-\frac{1}{4}\right) b_3 + \binom{4}{2}\frac{1}{6q_2} b_2 = 0.$$

We have $1 + b_3 + \frac{b_2}{q_2} = 0$. Further, we choose $b_3 = \frac{q_2-6}{2}$, and, by the apolarity condition, $b_2 = -q_2(1 + \frac{q_2-6}{2})$. So, we have

$$\begin{aligned} Q(z) &= -6q_2\left(1 + \frac{q_2-6}{2}\right) z^2 + 4\left(\frac{q_2-6}{2}\right) z^3 + z^4 \\ &= z^2\left(-3q_2(q_2-4) + 2(q_2-6)z + z^2\right). \end{aligned}$$

As we can see, the zeros of Q are $z_1 = 0, z_2 = 0, z_3 = q_2, z_4 = -3(q_2-4)$. To show that z_4 lies in the circle of radius q_2, we solve the inequality $|-3(q_2-4)| \leq q_2$. Hence, we obtain that if $q_2 \geq 3$, then all zeros of Q are in the circle $\{z : |z| \leq q_2\}$. Since all the zeros of Q are in the circle $\{z : |z| \leq q_2\}$, we obtain by the Grace theorem that $S_4(z,\varphi)$ has at least one zero in the circle $\{z : |z| \leq q_2\}$. $\square$

Thus, $S_4(z,\varphi)$ has at least one zero in the circle $\{z : |z| \le q_2\}$, and, by Lemma 2.3 applying to the $S_4(z,\varphi)$, $S_4(z,\varphi)$ does not have zeros on $\{z : |z| = q_2\}$. So, the polynomial $S_4(z,\varphi)$ has at least one zero in the open circle $\{z : |z| < q_2\}$. By Rouché's theorem, the functions $S_4(z,\varphi)$ and $\varphi(z)$ have the same number of zeros inside the circle $\{z : |z| < q_2\}$, whence φ has at least one zero in the open circle $\{z : |z| < q_2\}$. If φ is in the Laguerre-Pólya class, this zero must be real, and, since coefficients of φ are sign-changing, this zero belongs to $(0, q_2)$. But, by Lemma 2.2 we have $\varphi(x) > 0$ for all $x \in [0, q_2]$. This contradiction shows that $\varphi \notin \mathcal{L} - \mathcal{P}$.

Theorem 1.1 is proved.

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