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Some weak flocking models and its application to target tracking

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ABSTRACT

In this paper, we introduce a new concept named the weak flocking behavior. That is, if the complex system has the weak flocking property, the agents do not need have the same velocities to keep together when $t \to \infty$. This shows the biggest difference between weak flocking behavior and the flocking behavior. The advantage of this concept is that its practical significance—the agents don't need always keep in step in lots of the complex systems but they still keep together. Then, we propose a weak flocking model with two agents and study two different mechanisms for this model. Moreover, we also consider a system with N agents. Under the weak link mechanism, the system with N agents can keep weak flocking.

1. Introduction

Clustered-Control in complex system and targets tracking have the wide application in industries and military fields [5,1,13]. How can these be realized automatically? Inspired by the research about the behavior of the similar biological group such as the flocking of birds migration, schools of fish and herds of wolves [17,18,2], many mathematical models have been proposed to reveal the inner mechanism of these animals' flocking behavior [15,6]. It is pointed out that, in 2007 Smale and Cucker introduced a classical flocking model which is called CS model [5,3,4]. This model describes how agents interact with each other by following the simple rule such as [5,3,4]:

$$\frac{\mathrm{d}\mathbf{x}_i(t)}{\mathrm{d}t} = \mathbf{v}_i(t), \frac{\mathrm{d}\mathbf{v}_i(t)}{\mathrm{d}t} = \alpha \sum_{j=1}^{N} \phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)), \tag{1}$$

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where α denotes a positive constant, and ϕ_{ij} quantifies the pairwise influence of agent "j" on the alignment of agent "i" as a function of their distance. The motion characters of each agent "i" is described by the position $\mathbf{x}_i(t) \in \mathbf{R}^n$ and the velocity $\mathbf{v}_i(t) \in \mathbf{R}^n$. More precisely, in the CS model, it has

$$\phi_{ij}(t) = \frac{\phi(\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|)}{N}, \qquad \phi(\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|) = \frac{1}{(1 + \|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|^2)^{\beta}},$$

where ϕ denotes given above or, in general, is a strictly positive decreasing function, and β stands for a parameter. This influence function possesses the symmetric property, that is, the agent "i" and the agent "j" have the same influence on each other ($\phi_{ij} = \phi_{ji}$).

In [5,3,4], Smale and Cucker gave a classical definition of flocking in mathematics. For any agents "i" and "j" in the complex system. For time $t > 0$, the positions and velocities are given as follows [5,3,4]:

$$\sup_{t>0} d_X(t) = \|\mathbf{x}_j(t) - \mathbf{x}_i(t)\| < \infty,$$

$$\lim_{t\to\infty} d_V(t) = \lim_{t\to\infty} \|\mathbf{v}_j(t) - \mathbf{v}_i(t)\| = 0.$$

According to the definition of flocking given by Smale and Cucker, there are two principles: i) bounded distance–individuals stay at bounded distance from each other; ii) alignment–they all move in the same direction and their velocities will become the same. Based on the mathematics definition of Smale and Cucker [5,3,4], present work can largely be categorized into ecology, robotics, control and economy and so on [14,8,7,19]. Up to now, many studies are mainly on what models can keep the system with the flocking behavior. For example, professor Shen proposed a hierarchical leadership flocking (HL) model [16]. Professor Li improved the HL model. They introduced an overall leader in their model where any other agents are led by the overall leader directly or in directly [11,12,10]. Both Shen and Li considered the free-will in their models. While their research was based on how the system can keep flocking. Moreover, the norm of difference between any two agents' free-will functions were required to convergence to zero. Also, in [9], a flocking model involving was built. However, this paper [9] only gave the condition to keep the system be flocking. It means that the impact from the out force to the system is very weak and even will be ignored at last. Meanwhile, in the numerical simulation experiment of the paper [9], an interesting phenomenon was found. That is to say, even though the impact from the out force always exist, the agents in the system can still keep together. However the paper [9] did not give the strict theoretical proof. So in this paper, we define a weak flocking behavior. It is defined as: if the complex system has the weak flocking behavior, it only to keep a property—the distance between any two agents has least upper bound. The research of this paper is benefit by the free-will. Through the free-will, we can control the system keep weak flocking. The significance of the weak flock are as follows: 1) The flock defined by the biologist does not require the velocity all always be same; 2) In the weak flocking system, the distance between any time is bounded. While in the real biology or the application of the flock to military, control, ecology and robotics fields, we cannot wait for the time $t \to \infty$ to realise the flock. We can realise our target easily by using weak flocking model.

In chart 2, we propose the mathematical definition of the weak flocking behavior. Also, we build a weak flocking model which contains two agents and strictly show that this model will be weak flocking if it satisfies the Theorem 2.1. In chart 3, we apply the weak flocking model to military field. Such as missile tracking, torpedo intercepting and so on. In chart 4, we extend the model (15), (16) to a complex system with N agents. Through the Theorem 4.1, we show that the distance of any two agents will be bounded, even the velocities always are different.

2. Weak flocking and weak flocking models with two agents

2.1. The definition of weak flocking

A complex system has N agents. The position and velocity of agents "i" and "j" are defined as $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ and $(\mathbf{x}_j(t), \mathbf{v}_j(t))$. For any $i \in \mathbb{N}$, $\mathbf{x}_i(t) \in \mathbf{R}^d$ and $\mathbf{v}_i(t) \in \mathbf{R}^d$. We define

$$\begin{aligned} d_X(t) &= \max_{i,j\in\mathbb{N}} \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|, \\ d_V(t) &= \max_{i,j\in\mathbb{N}} \|\mathbf{v}_i(t) - \mathbf{v}_j(t)\|. \end{aligned} \tag{2}$$

If the positions and velocities of the agents satisfy

$$\begin{aligned} &\sup_{t>0} d_X(t) < M_1, \\ &\lim_{t\to\infty} d_V(t) < M_2. \end{aligned} \tag{3}$$

Here, constants $M_1 > 0$ and $M_2 > 0$ Then, the complex system has weak flocking behavior.

2.2. Weak flocking model with two agents

Consider a complex system with two agents. This model has two agents "1" and "2". For any time t, the position and velocity of "1" are denoted by $\mathbf{x}_1(t) \in \mathbf{R}^n$, and $\mathbf{v}_1(t) \in \mathbf{R}^n$, satisfy

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_1}{\mathrm{d}t} &= \mathbf{v}_1(t), \\ \frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} &= \alpha b_{12}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)(\mathbf{v}_2(t) - \mathbf{v}_1(t)) + \mathbf{g}_1(t); \end{aligned} \tag{4}$$

At any time t, the position of "2" is $\mathbf{x}_2(t) \in \mathbf{R}^n$, and the velocity of "2" are $\mathbf{v}_2(t) \in \mathbf{R}^n$, satisfy

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_2}{\mathrm{d}t} &= \mathbf{v}_2(t), \\ \frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}t} &= \alpha b_{21}(\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|)(\mathbf{v}_1(t) - \mathbf{v}_2(t)) + \mathbf{g}_2(t). \end{aligned} \tag{5}$$

Here, $\alpha(\alpha > 0)$ measures the interaction strength, $b_{12}(t) = b_{21}(t) = \frac{1}{2}\phi(\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|)$, the influence function $\phi(r) = \frac{1}{(1+|r|^2)^\beta}$, parameter $\beta > 0$ and $\mathbf{g}_1(t)$, $\mathbf{g}_2(t)$ are free-will function.

Theorem 2.1. *The positions and velocities of the agents "1" and "2" are denoted by $(\mathbf{x}_1, \mathbf{v}_1)$, $(\mathbf{x}_2, \mathbf{v}_2)$ and satisfy the system (3), (4). If the influence function $\phi(r)$ satisfies $\int_0^\infty \phi(r)\mathrm{d}r = \infty$, and the free-will function satisfies $\| \int_0^\infty (\mathbf{g}_1(t) - \mathbf{g}_2(t))\mathrm{d}t\| < \infty$ and $\int_0^\infty \|(\mathbf{g}_1(t) - \mathbf{g}_2(t))\|\mathrm{d}t = \infty$, then the system (3), (4) has weak flocking behavior.*

Proof. For $\mathbf{x}_j(t) \in \mathbf{R}^n$ and $\mathbf{v}_j(t) \in R^n$, let $\mathbf{x}_j(t) = (x_j^1(t), x_j^2(t), \cdots, x_j^n(t))$, $\mathbf{v}_j(t) = (v_j^1(t), v_j^2(t), \cdots, v_j^n(t))$ $(j = 1, 2)$.

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_j(t)}{\mathrm{d}t} &= (\frac{\mathrm{d}x_j^1}{\mathrm{d}t}, \frac{\mathrm{d}x_j^2}{\mathrm{d}t}, \cdots \frac{\mathrm{d}_j^n}{\mathrm{d}t}) = (v_j^1(t), v_j^2(t), \cdots, v_j^n(t)), \\ \frac{\mathrm{d}\mathbf{v}_j(t)}{\mathrm{d}t} &= \alpha b_{21}(\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|)(v_i^1(t) - v_j^1(t), v_i^2(t) - v_j^2(t), \cdots, v_i^n(t) - v_j^n(t)) \end{aligned}$$

$$+ (g_j^1(t), g_j^2(t), \cdots, g_N^1(t)).$$

First we want to proof

$$\lim_{t\to\infty} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\| < \infty.$$

Let $\overline{\mathbf{v}}(t) = \mathbf{v}_1(t) - \mathbf{v}_2(t)$, $\overline{v}^i(t) = v_1^i(t) - v_2^i(t)$. Using the reduction to absurdity to prove that $\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|$ is bounded.

It supposes that $\lim\limits_{t\to\infty} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\| = \infty$ or $\lim\limits_{t\to\infty} \sup \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\| = \infty$.

Obviously $\overline{\mathbf{v}}(t)$ is continue, for $\lim\limits_{t\to\infty} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\| = \infty$, at least there exists one $\overline{v}^i(t)$ satisfies one of the two following conditions:

$$i).\ \lim_{t\to\infty} \overline{v}^i(t) = \infty, \tag{6}$$

$$ii).\ \lim_{t\to\infty} \overline{v}^i(t) = -\infty. \tag{7}$$

As $\lim\limits_{t\to\infty} \sup \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\| = \infty$, it is easy to deduce that there at least exists a velocity satisfying $\overline{v}^i(t)$

$$iii).\ \lim_{t\to\infty} \overline{v}^i(t) \neq \infty, \lim_{t\to\infty} \overline{v}^i(t) \neq -\infty, \lim_{t\to\infty} \sup |\overline{v}^i(t)| = \infty. \tag{8}$$

It supposes that i). $\lim\limits_{t\to\infty} \overline{v}^i(t) = \infty$ is right. Then it can find a time t_1 and a constant K, such that for any time $t > t_1$, we have $\overline{v}^i(t) \geq K > 0$. From the formula (4.3) and (4.4) known, for $t > t_1$, it can deduce that $\overline{x}^i(t) = x_1^i(t) - x_2^i(t)$ is increasing and $\lim\limits_{t\to\infty} \overline{x}^i(t) = \infty$. Furthermore $\lim\limits_{t\to\infty} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| = \infty$. For formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}^i(t) = -2\alpha b_{12}(\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|)\overline{v}^i(t) + g_1^i(t) - g_2^i(t) \tag{9}$$

the integral of this formula on $[t_1, t)$ is

$$\overline{v}^i(t) - \overline{v}^i(t_1) = -2\alpha \int_{t_1}^{t} \frac{1}{(1 + \|\mathbf{x}_2(s) - \mathbf{x}_1(s)\|^2)^\beta} \overline{v}^i(s)\mathrm{d}s + \int_{t_1}^{t} (g_1^i(s) - g_2^i(s))\mathrm{d}s.$$

As $\lim\limits_{t\to\infty} \|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2 \to \infty$ there are two conditions:

1). It exists a time $t_0 > 0$, when $t \in (t_0, \infty)$, $\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2$ is increasing. So $\frac{1}{(1+\|\mathbf{x}_2(s)-\mathbf{x}_1(s)\|^2)^\beta}$ is monotonous. Using mean value theorem of integrals, it exists a $\xi \in (t_1, t)$, such as

$$\begin{aligned}
&\int_{t_1}^{t} \frac{1}{(1 + \|\mathbf{x}_2(s) - \mathbf{x}_1(s)\|^2)^\beta} \overline{v}^i(s)\mathrm{d}s \\
=&\frac{x_2^i(t) - x_1^i(t)}{(1 + \|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2)^\beta} - \frac{{}_2^i(\xi) -{}_1^i(\xi)}{(1 + \|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2)^\beta} \\
&+ \frac{x_2^i(\xi) - x_1^i(\xi)}{(1 + \|\mathbf{x}_2(t_1) - \mathbf{x}_1(t_1)\|^2)^\beta} - \frac{x_2^i(t_1) -{}_1^i(t_1)}{(1 + \|\mathbf{x}_2(t_1) - \mathbf{x}_1(t_1)\|^2)^\beta}.
\end{aligned}$$

From the above analysis, it can easily deduce that for any $\xi \in (t_1, t)$,

$$-\frac{x_2^i(\xi)-x_1^i(\xi)}{(1+\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2)^\beta}+\frac{x_2^i(\xi)-x_1^i(\xi)}{(1+\|\mathbf{x}_2(t_1)-\mathbf{x}_1(t_1)\|^2)^\beta}>0,$$

is right. And $\frac{x_2^i(t)-{}^i_1(t)}{(1+\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2)^\beta}$ satisfies one the following two conditions:

$$\lim_{t\to\infty}\frac{x_2^i(t)-x_1^i(t)}{(1+\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2)^\beta}=\infty,$$

or

$$0\le\lim_{t\to\infty}\frac{x_2^i(t)-x_1^i(t)}{(1+\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2)^\beta}<\infty.$$

So there exists a constant M_0, such that

$$M_0\le\lim_{t\to\infty}\int_{t_1}^{t}\frac{1}{(1+\|\mathbf{x}_2(s)-\mathbf{x}_1(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s.$$

As $\lim\limits_{t\to\infty}|\int_{t_1}^t(g_1^i(s)-g_2^i(s))\mathrm{d}s|\le M<\infty$, one of the following conditions must correct:

$$-2\alpha\lim_{t\to\infty}\int_{t_1}^{t}\frac{1}{(1+\|\mathbf{x}_2(s)-\mathbf{x}_1(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s+\lim_{t\to\infty}\int_{t_1}^{t}(g_1^i(s)-g_2^i(s))\mathrm{d}s=-\infty$$

or

$$0<-2\alpha\lim_{t\to\infty}\int_{t_1}^{t}\frac{1}{(1+\|\mathbf{x}_2(s)-\mathbf{x}_1(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s+\lim_{t\to\infty}\int_{t_1}^{t}(g_1^i(s)-g_2^i(s))\mathrm{d}s<\infty.$$

While $\lim\limits_{t\to\infty}\overline{v}^i(t)=\infty$, it emerges a contradiction, so the condition i) is false.

2). There exists a time $t_0>0$, when $t\in(t_0,\infty)$, $\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2$ dose not have monotonous, but $\lim\limits_{t\to\infty}\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2\to\infty$. For any enough big time $t_n>t_0$, there exist $t_0<t_1<t_2<\cdots<t_n$, such that when $t\in(t_{i-1},t_i)(i=1,2,\cdots,n)$, $\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2$ has monotonous, then it exists $\xi_i\in(t_{i-1},t_i)$, such as:

$$\begin{aligned}
&\int_{t_0}^{t_n}\frac{1}{(1+\|\mathbf{x}_2(s)-\mathbf{x}_1(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s\\
&=\sum_{i=1,\cdots n}\int_{t_{i-1}}^{t_i}\frac{1}{(1+\|\mathbf{x}_2(s)-\mathbf{x}_1(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s\\
&=\sum_{i=1,\cdots n}\{\frac{x_2^i(t_i)-x_1^i(t_i)}{(1+\|\mathbf{x}_2(t_i)-\mathbf{x}_1(t_i)\|^2)^\beta}-\frac{x_2^i(\xi_i)-x_1^i(\xi_i)}{(1+\|\mathbf{x}_2(t_i)-\mathbf{x}_1(t_i)\|^2)^\beta}\\
&\quad+\frac{x_2^i(\xi_i)-x_1^i(\xi_i)}{(1+\|\mathbf{x}_2(t_{i-1})-\mathbf{x}_1(t_{i-1})\|^2)^\beta}-\frac{x_2^i(t_{i-1})-x_1^i(t_{i-1})}{(1+\|\mathbf{x}_2(t_{i-1})-\mathbf{x}_1(t_{i-1})\|^2)^\beta}\}\\
&=\frac{x_2^i(t_n)-x_1^i(t_n)}{(1+\|\mathbf{x}_2(t_n)-\mathbf{x}_1(t_n)\|^2)^\beta}+\sum_{i=1,\cdots n}\{-\frac{x_2^i(\xi_i)-x_1^i(\xi_i)}{(1+\|\mathbf{x}_2(t_i)-\mathbf{x}_1(t_i)\|^2)^\beta}\\
&\quad+\frac{x_2^i(\xi_i)-x_1^i(\xi_i)}{(1+\|\mathbf{x}_2(t_{i-1})-\mathbf{x}_1(t_{i-1})\|^2)^\beta}\}-\frac{x_2^i(t_0)-x_1^i(t_0)}{(1+\|\mathbf{x}_2(t_0)-\mathbf{x}_1(t_0)\|^2)^\beta}.
\end{aligned}$$

There is a time t_{k-1}, when $t \in (t_{k-1}, t_k)$, the function $\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2$ is decreasing. Also we can find a time $t_{k+1} > t_k$, such that

$$
\begin{aligned}
&|-\frac{1}{(1+\|\mathbf{x}_2(t_k)-\mathbf{x}_1(t_k)\|^2)^\beta}+\frac{1}{(1+\|\mathbf{x}_2(t_{k-1})-\mathbf{x}_1(t_{k-1})\|^2)^\beta}| \\
<&|-\frac{1}{(1+\|\mathbf{x}_2(t_{k+1})-\mathbf{x}_1(t_{k+1})\|^2)^\beta}+\frac{1}{(1+\|\mathbf{x}_2(t_k)-\mathbf{x}_1(t_k)\|^2)^\beta}|.
\end{aligned}
$$

I). For $t \in (t_k, t_{k+1})$, we consider this condition: when $t \in (t_k, t_{k+1})$, $\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2$ is increasing. There exist time t_0 and t_n, such that $x_2^i(t)-x_1^i(t)$ is increasing for $t \in (t_0, t_n)$. For $\xi_k \in (t_{k-1}, t_k)$ and $\xi_{k+1} \in (\mathbf{x}_k, \mathbf{x}_{k+1})$, obviously $\xi_k < \xi_{k+1}$, furthermore,

$$
\begin{aligned}
&\frac{x_2^i(\xi_k)-x_1^i(\xi_k)}{(1+\|\mathbf{x}_2(t_k)-\mathbf{x}_1(t_k)\|^2)^\beta}-\frac{x_2^i(\xi_k)-x_1^i(\xi_k)}{(1+\|\mathbf{x}_2(t_{k-1})-\mathbf{x}_1(t_{k-1})\|^2)^\beta} \\
<&-\frac{x_2^i(\xi_{k+1})-x_1^i(\xi_{k+1})}{(1+\|\mathbf{x}_2(t_{k+1})-\mathbf{x}_1(t_{k+1})\|^2)^\beta}+\frac{x_2^i(\xi_{k+1})-x_1^i(\xi_{k+1})}{(1+\|\mathbf{x}_2(t_k)-\mathbf{x}_1(t_k)\|^2)^\beta}.
\end{aligned}
$$

II). For the interval $t \in (t_k, t_{k+1})$, consider when $t \in (t_k, t_{k+1})$, $\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2$ is not monotonous. $t \in (t_k, t_{k+1})$ can be divided finite intervals, such that $\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2$ is monotonous on every interval. Similar to the above discussion can get

$$
\begin{aligned}
&\frac{x_2^i(\xi_k)-x_1^i(\xi_k)}{(1+\|\mathbf{x}_2(t_k)-\mathbf{x}_1(t_k)\|^2)^\beta}-\frac{x_2^i(\xi_k)-x_1^i(\xi_k)}{(1+\|\mathbf{x}_2(t_{k-1})-\mathbf{x}_1(t_{k-1})\|^2)^\beta} \\
<&-\frac{x_2^i(\xi_{k+1})-_1^i(\xi_{k+1})}{(1+\|\mathbf{x}_2(t_{k+1})-\mathbf{x}_1(t_{k+1})\|^2)^\beta}+\frac{x_2^i(\xi_{k+1})-x_1^i(\xi_{k+1})}{(1+\|\mathbf{x}_2(t_k)-\mathbf{x}_1(t_k)\|^2)^\beta}.
\end{aligned}
$$

In conclusion,

$$
\sum_{i=0,1,\cdots,n}-\frac{x_2^i(\xi_i)-x_1^i(\xi_i)}{(1+\|\mathbf{x}_2(t_i)-\mathbf{x}_1(t_i)\|^2)^\beta}+\frac{x_2^i(\xi_i)-x_1^i(\xi_i)}{(1+\|\mathbf{x}_2(t_{i-1})-\mathbf{x}_1(t_{i-1})\|^2)^\beta}>0,
$$

similar with the discussion from 1), it can easily get the contradiction, so i) is false.

ii). $\lim_{t\to\infty}(v_1^i(t) - v_2^i(t)) = -\infty$. From similar analysis with i), it also gets the contradiction, so ii) is false.

iii). $\lim_{t\to\infty}(v_1^i(t) - v_2^i(t)) \neq \infty$, $\lim_{t\to\infty}(v_1^i(t) - v_2^i(t)) \neq -\infty$ and $\lim_{t\to\infty} \sup |v_1^i(t) - v_2^i(t)| = \infty$. Now can find a time t_2, such that the function $\overline{\mathbf{v}}(t)$ is increasing on (t_2, t_3), and $|\overline{v}^i(t_3)| > |\overline{v}^i(t_2)| + 2M$.

It constructors a function

$$
V(t) = \overline{v}^{i^2}(t) - 2\int_0^t \overline{v}^i(s)(g_1^i(s) - g_2^i(s))\mathrm{d}s,
$$

then,

$$
\begin{aligned}
V'(t) =&2\overline{v}^i(t)\overline{v}'^i(t) - 2\overline{v}(t)(g_1^i(t) - g_2^i(t)) \\
=&-2\overline{v}^{i^2}(t)(b_{12}(t)+b_{21}(t)) + 2\overline{v}^i(t)(g_1^i(t)-g_2^i(t)) - 2\overline{v}^i(t)(g_1^i(t)-g_2^i(t)) \\
=&-4b_{12}(t)\overline{v}^{i^2}(t) \\
\le&0.
\end{aligned}
$$

As $V'(t) < 0$, then

$$V(t_2) \leq V(t_3),$$
$$\overline{v}^{i\,2}(t_3) - 2\int_0^{t_3} \overline{v}^i(s)(g_1^i(s) - g_2^i(s))\mathrm{d}s \leq \overline{v}^{i\,2}(t_2) - 2\int_0^{t_2} \overline{v}^i(s)(g_1^i(s) - g_2^i(s))\mathrm{d}s,$$

that is,

$$\overline{v}^{i\,2}(t_3) \leq \overline{v}^{i\,2}(t_2) - 2\int_{t_2}^{t_3} \overline{v}^i(s)(g_1^i(s) - g_2^i(s))\mathrm{d}s. \tag{10}$$

Using mean value theorem of integrals, there is a constant $\xi \in (t_2, t_3)$, such as

$$\int_{t_2}^{t_3} \overline{v}^i(s)(g_1^i(s) - g_2^i(s))\mathrm{d}s = \overline{v}^i(t_3)\int_{\xi}^{t_3} (g_1^i(s) - g_2^i(s))\mathrm{d}s$$
$$+ \overline{v}^i(t_2)\int_{t_2}^{\xi} (g_1^i(s) - g_2^i(s))\mathrm{d}s.$$

Into the formula (4.2.13),

$$\overline{v}^{i\,2}(t_3) \leq \overline{v}^{i\,2}(t_2) - 2(\overline{v}^i(t_3)\int_{\xi}^{t_2} (g_1^i(s) - g_2^i(s))\mathrm{d}s + \overline{v}^i(t_2)\int_{t_1}^{\xi} (g_1^i(s) - g_2^i(s))\mathrm{d}s).$$

As $|\int_{t_1}^{\infty} (g_1^i(s) - g_2^i(s))\mathrm{d}s| < M$, so,

$$\overline{v}^{i\,2}(t_3) \leq \overline{v}^{i\,2}(t_2) + 2M\overline{v}^i(t_3) + 2M\overline{v}^i(t_2),$$

furthermore,

$$(\overline{v}^i(t_3) - M)^2 \leq (\overline{v}^i(t_2) + M)^2. \tag{11}$$

From the condition iii), it is easy to know $(\overline{v}^i(t_3) - M)^2 > (\overline{v}^i(t_2) + M)^2$, it emerges a contradiction, so iii) is false.

From the discussion above, it can deduce that for all $t > 0$, the formula $\|v_1^i(t) - v_2^i(t)\| < \infty$ is right. So for any $\overline{v}^i(t)$, (here $i \in \mathbb{N}$), it has $\|\overline{v}^i(t)\| < \infty$. Then it can deduce $\|\overline{v}(t)\| < \infty$.

Following it needs to proof that for all $t \geq 0$, the formula $\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| < \infty$ is right.

For: $\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}^i(t) = -2\alpha b_{12}(\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|)\overline{v}^i(t) + g_1^i(t) - g_2^i(t)$. Using integration by parts, there is a constant C,

$$\overline{v}^i(t) = e^{-\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}(\int_0^t (e^{\alpha\int_0^h b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}(g_1^i(h) - g_2^i(h)))\mathrm{d}h + C)$$

For the formula $\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|$, has:

i). For any time $t > 0$, $\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq d^* < \infty$, it has $\int_0^t b_{12}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)\mathrm{d}s \geq d^* t$, as $\lim\limits_{t\to\infty} \int_0^t b_{12}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)\mathrm{d}s = \infty$.

ii). If $\lim\limits_{t\to\infty} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| = \infty$, for $\int_0^\infty b_{12}(s)\mathrm{d}s = \infty$, it has $\lim\limits_{t\to\infty} \int_0^t b_{12}(\|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|)\mathrm{d}s = \infty$.

So $\lim\limits_{t\to\infty} Ce^{-\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s} = 0$. Consider

$$
\begin{aligned}
f(t) =& e^{-\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}\Big(\int\limits_0^t e^{\alpha\int_0^h b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}(g_1^i(h) - g_2^i(h))\mathrm{d}h\Big) \\
=& \frac{\int_0^t e^{\alpha\int_0^h b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}(g_1^i(h) - g_2^i(h))\mathrm{d}h}{e^{\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}}.
\end{aligned}
$$

Using L'Hospital's rule,

$$
\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{t\to\infty} \frac{e^{\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}(g_1^i(t) - g_2^i(t))}{b_{12}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)e^{\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)\mathrm{d}s}} \\
&= \lim_{t\to\infty} \frac{g_1^i(t) - g_2^i(t)}{b_{12}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)}.
\end{aligned}
$$

If $\lim\limits_{t\to\infty} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| = \infty$, then it has

$$
\lim_{t\to\infty}(f(t) + Ce^{-\alpha\int_0^t b_{12}(\|\mathbf{x}_1(s)-\mathbf{x}_2(s)\|)ds}) = \lim_{t\to\infty} v^i(t) = \infty,
$$

from the formula above $\lim\limits_{t\to\infty} \|\overline{\mathbf{v}}(t)\| = \infty$.

While $\lim\limits_{t\to\infty} \|\overline{\mathbf{v}}(t)\| < \infty$, it emerges a contradiction. So it must have a constant d, for any time $t > 0$, it has $\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq d < \infty$.

In conclusion, we prove that this system (3), (4) has a weak flocking behavior. □

In this chart, we consider a system with two agents and analyze the weak behavior. First, we give the concept of weak flocking behavior and construct a model (4), (5), and study their weak flocking behavior in mathematics. In Theorem 2.1, it gives a condition (free-will) to keep the system with a weak flocking behavior. Although here it only gives a special system with two agents, it could still have wide applications. For example, in China's regional economics, the position $\mathbf{x}_i(t)$ $(i = 1, 2)$ represents the region's economy, the velocity $\mathbf{v}_i(t)$ $(i = 1, 2)$ represents the regional economic growth, free-will stands for the local government's economic policy. Here the two regions represent Guizhou province and Guangdong province. In three decades of reform and opening up, the economic development of Guangdong province is very fast, while the economic development in Guizhou is relatively slow. On national level, it will introduce some global economic policy to promote economic development of Guizhou province. So that it can realize the common development of regional economy in this two province. While every province has its own situation and characteristics, these may become some factors which could promote and restrict the development of the economy. These factors are not affected by national policy and economic factors of other provinces. So these factors can be consider as free-will in the model. From the study on the system (4), (5), it could realize the common development of the two provinces' economy by adjust the free-will. For example, there are lots of high quality pollution-free fruits and vegetables in Guizhou. But Due to the traffic inconvenience products are unsalable. We could improve the traffic conditions in Guizhou and process the fruits and vegetables to extend their sales cycle. Doing these will promote the economic development of Guizhou province. Also this model can apply on military field, such as submarines and missile tracking and interception. Taking missile tracking

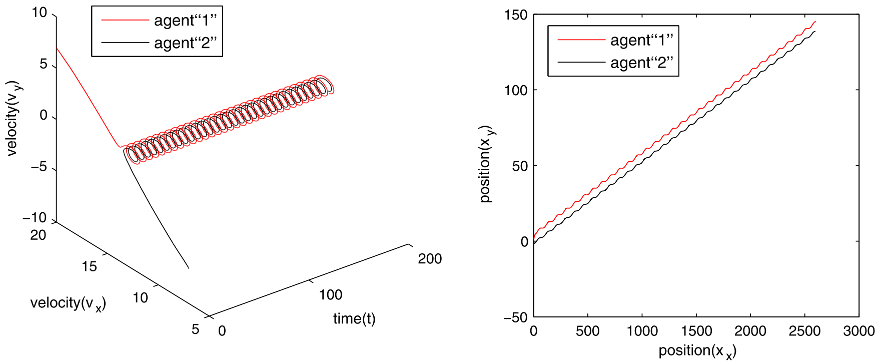

Fig. 1. Agent "1" and "2" form a system with weak flocking behavior. Parameter $\alpha = 0.5$, $\beta = 1/3$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

for example, the tracking signal will be affected by weather geography enemy deliberately to the factors of jamming signal. From the study of this model, it could make the intervention to the missile (adjust the free-will function) to track the enemy missile.

2.3. Example

In a two-dimensional plane system, there are two agents in a system. The position and velocity of agent "1" are denoted by $(\mathbf{x}_1(t), \mathbf{v}_1(t))$. Its free-will function is $\mathbf{g}_1(t) = (\sin t, \sin t)$, then,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_1(t)}{\mathrm{d}t} =& \mathbf{v}_1(t), \\ \frac{\mathrm{d}\mathbf{v}_1(t)}{\mathrm{d}t} =& \alpha \frac{1}{(1 + \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2)^\beta} (\mathbf{v}_2(t) - \mathbf{v}_1(t)) + (\cos t, -\sin t); \end{aligned} \tag{12}$$

The position and velocity of agent "2" are denoted by $(\mathbf{x}_2(t), \mathbf{v}_2(t))$. Its free-will function is $\mathbf{g}_2(t) = (\cos t, \sin t)$. Then,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_2(t)}{\mathrm{d}t} =& \mathbf{v}_2(t), \\ \frac{\mathrm{d}\mathbf{v}_2(t)}{\mathrm{d}t} =& \alpha \frac{1}{(1 + \|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2)^\beta} (\mathbf{v}_1(t) - \mathbf{v}_2(t)) + (\sin t, \cos t). \end{aligned} \tag{13}$$

Here $\beta = \frac{1}{3}$, and $\| \int_0^\infty (\mathbf{g}_1(t) - \mathbf{g}_2(t))\mathrm{d}t\| < \infty$, it takes notice of $\int_0^\infty \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|\mathrm{d}t = \infty$.

Through Fig. 1, it can notice that when the system (12), (13) satisfies the Theorem 2.1. The agents "1" and "2" will form a weak flocking system, which can indicate that the condition shown in theorem is reasonable. Considering the requirement of free-will in Theorem 2.1, $\| \int_0^\infty (\mathbf{g}_1(t) - \mathbf{g}_2(t))\mathrm{d}t\| < \infty$, it indicates that the function $\lim_{t\to\infty} \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\| \neq 0$. This proper brings great convenience in practical application.

Next it is another system with agents "1" and "2", the free-will of these two agents satisfy $\| \int_0^\infty (\mathbf{g}_1(t) - \mathbf{g}_2(t))\mathrm{d}t\| = \infty$. the position and velocity of agent "1" denoted by $(\mathbf{x}_1(t), \mathbf{v}_1(t))$, satisfy

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_1(t)}{\mathrm{d}t} =& \mathbf{v}_1(t), \\ \frac{\mathrm{d}\mathbf{v}_1(t)}{\mathrm{d}t} =& \alpha \frac{1}{(1 + \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2)^\beta} (\mathbf{v}_2(t) - \mathbf{v}_1(t)) + (1, -\sin t); \end{aligned} \tag{14}$$

the position and velocity of agent "2" denoted by $(\mathbf{x}_2(t), \mathbf{v}_2(t))$, satisfy

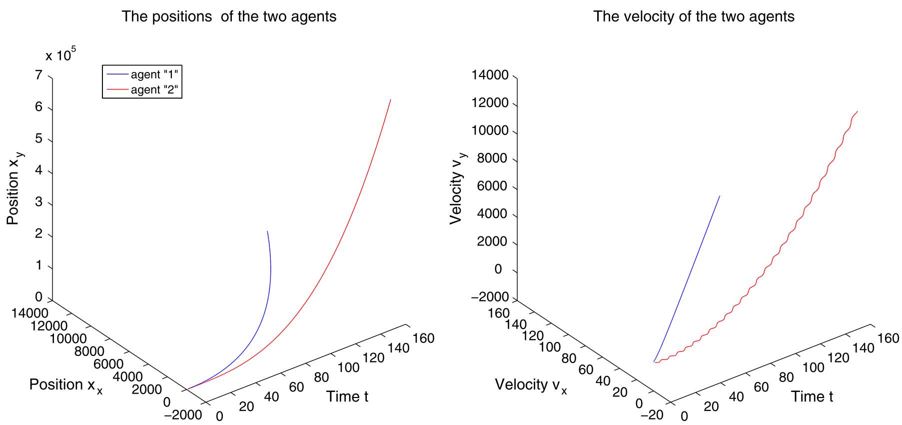

Fig. 2. Agent "1" and "2" form a system with weak flocking behavior. Parameter $\alpha = 0.5$, $\beta = 1/3$.

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_2(t)}{\mathrm{d}t} =& \mathbf{v}_2(t), \\ \frac{\mathrm{d}\mathbf{v}_2(t)}{\mathrm{d}t} =& \alpha \frac{1}{(1+\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2)^{\beta}}(\mathbf{v}_1(t)-\mathbf{v}_2(t)) + (\sin t, t); \end{aligned} \tag{15}$$

Here $\beta = \frac{1}{3}$, and $\| \int_0^\infty (\mathbf{g}_1(t) - \mathbf{g}_2(t))\mathrm{d}t\| = \infty$, not satisfy Theorem 2.1, notice that $\int_0^\infty \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|\mathrm{d}t = \infty$.

In system (14), (15), the free-will function requirements is loosed which means this system does not satisfy Theorem 2.1. Through Fig. 2, it is easy to deduce that the distance between the two agents is not bounded, and the weak flock won't occur. The results of numerical simulation from Fig. 1 and Fig. 2 shows that the condition of the free-will is very sharp. For the free-will does not satisfy the Theorem 2.1, the system will lose the weak flocking behavior.

3. Intelligent weak cluster complex system control application in missile tracking and defense

In Modern Warfare, missile is more and more obvious in the war. It is important to track and intercept enemy missiles and submarines. When the enemy missile or submarines are found, how to realise track and intercept them effectively and whether it could depict this process through rational mathematical models? It is a very interesting problem.

When we track the enemy missile in a war, its flight trajectory is without our interference before the tracking behavior is noticed. And after our tracking behavior is discovered, the enemy missile will make motor to get rid of the track. While how can we realise this tracking or automatic tracking? In current situation, the velocity of these two missiles cannot keep consistent. So the model given before cannot reflect it well. In this chart, it builds a model with weak flocking behavior which can fit the situation above. Although we cannot realise that the velocity of our missile match with the velocity of the enemy, we can control the free-will to realise tracking.

3.1. Missile tracking model

This model make up of a own missile "q" and an enemy missile "p". At time t, the position and velocity of "q" denote by $(\mathbf{x}_q(t), \mathbf{v}_q(t))$, the position and velocity of "p" are $(\mathbf{x}_p(t), \mathbf{v}_p(t))$.

For "p", $(\mathbf{x}_p(t), \mathbf{v}_p(t))$ satisfy

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_p(t)}{\mathrm{d}t} =& \mathbf{v}_p(t), \\ \frac{\mathrm{d}\mathbf{v}_p(t)}{\mathrm{d}t} =& \mathbf{g}_p(t); \end{aligned} \tag{16}$$

For "q" are $(\mathbf{x}_q(t), \mathbf{v}_q(t))$ satisfy

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_q(t)}{\mathrm{d}t} =& \mathbf{v}_q(t), \\ \frac{\mathrm{d}\mathbf{v}_q(t)}{\mathrm{d}t} =& \alpha b_{qp}(\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|)(\mathbf{v}_p(t) - \mathbf{v}_q(t)) + \mathbf{g}_q(t). \end{aligned} \tag{17}$$

The influence function $b_{qp}(\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|) = \frac{1}{(1+\|\mathbf{x}_p(t)-\mathbf{x}_q(t)\|^2)^\beta}$, parameters $\alpha > 0$, $\beta > 0$.

The motion trail of "q" is control by the enemy command center. Even though the enemy finds that its missile "q" is tracked "p". The enemy command will send the motor command to "q" in order to avoid being tracked. In math model, the motor of "q" reflects on the accelerated velocity $\mathbf{g}_p(t)$. Own missile's task is to track "P". But how to keep finish the track and intercept is very important. In this chart it builds the system (15), (16). The following Theorem 3.1 will give a method how to realise it in mathematical.

Theorem 3.1. *Let $(\mathbf{x}_p(t), \mathbf{v}_p(t))$ and $(\mathbf{x}_q(t), \mathbf{v}_q(t))$ of the solutions of system (15), (16). If the parameter $\beta < \frac{1}{2}$, and $\| \int_0^\infty (\mathbf{g}_p(t) - \mathbf{g}_q(t))\mathrm{d}t\| \le M < \infty$, then the system (15), (16) has the weak flocking behavior.*

Proof. As $\mathbf{x}_p(t) \in \mathbf{R}^n$, $\mathbf{v}_p(t) \in \mathbf{R}^n$ are n-dimension vectors. Denote $\mathbf{x}_p(t) = (x_p^1(t), x_p^2(t), \cdots, x_p^n(t))$, $\mathbf{v}_p(t) = (v_p^1(t), v_p^2(t), \cdots, v_p^n(t))$, then,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_p(t)}{\mathrm{d}t} &= \left(\frac{\mathrm{d}x_p^1}{\mathrm{d}t}, \frac{\mathrm{d}x_p^2}{\mathrm{d}t}, \cdots \frac{\mathrm{d}x_p^n}{\mathrm{d}t}\right) = (v_p^1(t), v_p^2(t), \cdots, v_p^n(t)) \\ \frac{\mathrm{d}\mathbf{v}_p(t)}{\mathrm{d}t} &= \mathbf{g}_p(t) = (g_p^1(t), g_p^2(t), \cdots, g_p^n(t)). \end{aligned}$$

$\mathbf{x}_q(t) \in R^n$, $\mathbf{v}_q(t) \in R^n$, denote $\mathbf{x}_q(t) = (x_q^1(t), x_q^2(t), \cdots, x_q^n(t))$ $\mathbf{v}_q(t) = (v_q^1(t), v_q^2(t), \cdots, v_q^n(t))$, then,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_q(t)}{\mathrm{d}t} =& \left(\frac{\mathrm{d}x_q^1}{\mathrm{d}t}, \frac{\mathrm{d}x_q^2}{\mathrm{d}t}, \cdots \frac{\mathrm{d}x_q^n}{\mathrm{d}t}\right) = (v_q^1(t), v_q^2(t), \cdots, v_q^n(t)), \\ \frac{\mathrm{d}\mathbf{v}_q(t)}{\mathrm{d}t} =& \alpha b_{pq}(\|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|)(v_p^1(t) - v_q^1(t), v_p^2(t) - v_q^2(t), \cdots, v_p^n(t) - v_q^n(t)) \\ &+ (g_q^1(t), g_q^2(t), \cdots, g_q^n(t)). \end{aligned}$$

First it wants to prove that

$$\|\mathbf{v}_p(t) - \mathbf{v}_q(t)\| < \infty.$$

Let $\overline{\mathbf{v}}(t) = \mathbf{v}_p(t) - \mathbf{v}_q(t)$, $\overline{v}^i(t) = v_p^i(t) - v_q^i(t)$. Using the reduction to absurdity to prove $\|\mathbf{v}_p(t) - \mathbf{v}_q(t)\|$ is bounded.

Suppose that $\lim\limits_{t\to\infty} \|\mathbf{v}_p(t) - \mathbf{v}_q(t)\| = \infty$ or $\lim\limits_{t\to\infty} \sup \|\mathbf{v}_p(t) - \mathbf{v}_q(t)\| = \infty$.

For the function $\overline{\mathbf{v}}(t)$ is continue and $\lim\limits_{t\to\infty} \|\mathbf{v}_p(t) - \mathbf{v}_q(t)\| = \infty$, then these at least exist one component $\overline{v}^i(t)$ which satisfies one of the following two situations:

$$i). \lim_{t\to\infty} \overline{v}^i(t) = \infty, \tag{18}$$

$$ii). \lim_{t\to\infty} \overline{v}^i(t) = -\infty. \tag{19}$$

For $\lim_{t\to\infty} \sup \|\mathbf{v}_p(t) - \mathbf{v}_q(t)\| = \infty$, it also exist one component satisfy

$$iii). \lim_{t\to\infty} \overline{v}^i(t) \neq \infty, \lim_{t\to\infty} \overline{v}^i(t) \neq -\infty, \lim_{t\to\infty} \sup |\overline{v}^i(t)| = \infty. \tag{20}$$

Suppose that i). $\lim_{t\to\infty} \overline{v}^i(t) = \infty$ is right. Then it can find a time $t_1 > 0$ and K, such as for any time $t > t_1$, it has $\overline{v}^i(t) \geq K > 0$. From (15), (16), when $t > t_1$, the function $\overline{x}^i(t) = x_q^i(t) - x_p^i(t)$ is increasing and $\lim_{t\to\infty} \overline{x}^i(t) = \infty$. It easily deduces $\lim_{t\to\infty} \|\mathbf{x}_p(t) - \mathbf{x}_q(t)\| = \infty$. Consider the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}^i(t) = -\alpha b_{qp}(\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|)\overline{v}^i(t) + g_p^i(t) - g_q^i(t) \tag{21}$$

integer (20) on $[t_1, t)$, it has

$$\overline{v}^i(t) - \overline{v}^i(t_1) = -\alpha \int_{t_1}^{t} \frac{1}{(1 + \|\mathbf{x}_q(s) - \mathbf{x}_p(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s + \int_{t_1}^{t} (g_p^i(s) - g_q^i(s))\mathrm{d}s.$$

For the formula $\lim_{t\to\infty} \|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|^2 \to \infty$, there exist two situations:

1) It exist time $t_0 > 0$, when $t \in (t_0, \infty)$, $\|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|^2$ is increasing. Then $\frac{1}{(1+\|\mathbf{x}_q(s)-\mathbf{x}_p(s)\|^2)^\beta}$ is monotonous. By the integral mean value theorem, there exists $\xi \in (t_1, t)$, such that

$$\begin{aligned}
&\int_{t_1}^{t} \frac{1}{(1 + \|\mathbf{x}_q(s) - \mathbf{x}_p(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s \\
=&\frac{x_q^i(t) - x_p^i(t)}{(1 + \|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|^2)^\beta} - \frac{x_q^i(\xi) - x_p^i(\xi)}{(1 + \|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|^2)^\beta} \\
&+ \frac{x_q^i(\xi) - x_p^i(\xi)}{(1 + \|\mathbf{x}_q(t_1) - \mathbf{x}_p(t_1)\|^2)^\beta} - \frac{x_q^i(t_1) - x_p^i(t_1)}{(1 + \|\mathbf{x}_q(t_1) - \mathbf{x}_p(t_1)\|^2)^\beta}.
\end{aligned}$$

For any $\xi \in (t_1, t)$, it has

$$-\frac{x_q^i(\xi) - x_p^i(\xi)}{(1 + \|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|^2)^\beta} + \frac{x_p^i(\xi) - x_q^i(\xi)}{(1 + \|\mathbf{x}_q(t_1) - \mathbf{x}_p(t_1)\|^2)^\beta} > 0,$$

and the formula $\frac{x_q^i(t)-{}_p^i(t)}{(1+\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2)^\beta}$ must satisfy one of the following two conditions:

$$\lim_{t\to\infty} \frac{x_p^i(t) - x_q^i(t)}{(1 + \|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|^2)^\beta} = \infty$$

or

$$0 \leq \lim_{t\to\infty} \frac{x_q^i(t) - x_p^i(t)}{(1 + \|\mathbf{x}_q(t) - \mathbf{x}_p(t)\|^2)^\beta} < \infty.$$

So there exists a constant M_0, such as

$$M_0 \leq \lim_{t\to\infty} \int\limits_{t_1}^{t} \frac{1}{(1+\|\mathbf{x}_p(s)-\mathbf{x}_q(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s.$$

As $\lim\limits_{t\to\infty} |\int_{t_1}^{t}(g_p^i(s)-g_q^i(s))\mathrm{d}s| \leq M < \infty$, then one of the following conditions satisfies:

$$-\alpha \lim_{t\to\infty} \int\limits_{t_1}^{t} \frac{1}{(1+\|\mathbf{x}_q(s)-\mathbf{x}_p(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s + \lim_{t\to\infty}\int\limits_{t_1}^{t}(g_p^i(s)-g_q^i(s))\mathrm{d}s = -\infty$$

or

$$0 < -\alpha \lim_{t\to\infty} \int\limits_{t_1}^{t} \frac{1}{(1+\|\mathbf{x}_q(s)-\mathbf{x}_p(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s + \lim_{t\to\infty}\int\limits_{t_1}^{t}(g_p^i(s)-g_q^i(s))\mathrm{d}s < \infty.$$

While $\lim\limits_{t\to\infty}\overline{v}^i(t)=\infty$, it emerges a contradiction, so the condition i) is false.

2). It exists a time $t_0>0$, when $t\in(t_0,\infty)$, $\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2$ is not monotonous. But $\lim\limits_{t\to\infty}\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2\to\infty$. For any enough big time $t_n>t_0$, there exist time $t_0<t_1<t_2<\cdots<t_n$, such that on any interval $(t_{i-1},t_i)(i=1,2,\cdots,n)$, $\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2$ is monotonous. Then exists $\xi_i\in(t_{i-1},t_i)$, such as

$$\begin{aligned}
&\int\limits_{t_0}^{t_n} \frac{1}{(1+\|\mathbf{x}_p(s)-\mathbf{x}_q(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s\\
&=\sum_{i=1,\cdots n}\int\limits_{t_{i-1}}^{t_i} \frac{1}{(1+\|\mathbf{x}_q(s)-\mathbf{x}_p(s)\|^2)^\beta}\overline{v}^i(s)\mathrm{d}s\\
&=\sum_{i=1,\cdots n}\{\frac{x_q^i(t_i)-x_p^i(t_i)}{(1+\|\mathbf{x}_q(t_i)-\mathbf{x}_p(t_i)\|^2)^\beta}-\frac{x_q^i(\xi_i)-x_p^i(\xi_i)}{(1+\|\mathbf{x}_q(t_i)-\mathbf{x}_p(t_i)\|^2)^\beta}\\
&+\frac{x_q^i(\xi_i)-x_p^i(\xi_i)}{(1+\|\mathbf{x}_q(t_{i-1})-\mathbf{x}_p(t_{i-1})\|^2)^\beta}-\frac{x_q^i(t_{i-1})-x_p^i(t_{i-1})}{(1+\|\mathbf{x}_q(t_{i-1})-\mathbf{x}_p(t_{i-1})\|^2)^\beta}\}\\
&=\frac{x_q^i(t_n)-x_p^i(t_n)}{(1+\|\mathbf{x}_q(t_n)-\mathbf{x}_p(t_n)\|^2)^\beta}+\sum_{i=1,\cdots n}\{-\frac{x_q^i(\xi_i)-x_p^i(\xi_i)}{(1+\|\mathbf{x}_q(t_i)-\mathbf{x}_p(t_i)\|^2)^\beta}\\
&+\frac{x_q^i(\xi_i)-x_p^i(\xi_i)}{(1+\|\mathbf{x}_2(t_{i-1})-\mathbf{x}_p(t_{i-1})\|^2)^\beta}\}-\frac{x_q^i(t_0)-x_p^i(t_0)}{(1+\|\mathbf{x}_q(t_0)-\mathbf{x}_p(t_0)\|^2)^\beta}.
\end{aligned}$$

There exists time t_{k-1}, when $t\in(t_{k-1},t_k)$, $\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2$ is decreasing. Furthermore it can find the first time $t_{k+1}>t_k$, such as

$$\begin{aligned}
&|-\frac{1}{(1+\|\mathbf{x}_q(t_k)-\mathbf{x}_p(t_k)\|^2)^\beta}+\frac{1}{(1+\|\mathbf{x}_q(t_{k-1})-\mathbf{x}_p(t_{k-1})\|^2)^\beta}|\\
&<|-\frac{1}{(1+\|\mathbf{x}_q(t_{k+1})-\mathbf{x}_p(t_{k+1})\|^2)^\beta}+\frac{1}{(1+\|\mathbf{x}_q(t_k)-\mathbf{x}_p(t_k)\|^2)^\beta}|.
\end{aligned}$$

I). For the interval (t_k,t_{k+1}), we consider when $t\in(t_k,t_{k+1})$, $\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2$ is increasing. When $t\in(t_0,t_n)$, it has $x_q^i(t)-x_p^i(t)$ is increasing. For $\xi_k\in(t_{k-1},t_k)$ and $\xi_{k+1}\in(t_k,t_{k+1})$, easily $\xi_k<\xi_{k+1}$, furthermore,

$$\frac{x_q^i(\xi_k)-x_p^i(\xi_k)}{(1+\|\mathbf{x}_q(t_k)-\mathbf{x}_p(t_k)\|^2)^\beta}-\frac{x_q^i(\xi_k)-x_p^i(\xi_k)}{(1+\|\mathbf{x}_q(t_{k-1})-\mathbf{x}_p(t_{k-1})\|^2)^\beta}$$
$$<-\frac{x_q^i(\xi_{k+1})-x_p^i(\xi_{k+1})}{(1+\|\mathbf{x}_q(t_{k+1})-\mathbf{x}_p(t_{k+1})\|^2)^\beta}+\frac{x_q^i(\xi_{k+1})-x_p^i(\xi_{k+1})}{(1+\|\mathbf{x}_q(t_k)-\mathbf{x}_p(t_k)\|^2)^\beta}.$$

II). For the interval (t_k, t_{k+1}), we consider when $t\in(t_k,t_{k+1})$, $\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2$ is not monotonous. The interval (t_k,t_{k+1}) can be divided into finite subintervals. Such as $\|\mathbf{x}_q(t)-\mathbf{x}_p(t)\|^2$ is monotonous on every subinterval. Similar discussion from I) it has

$$\frac{x_q^i(\xi_k)-x_p^i(\xi_k)}{(1+\|\mathbf{x}_q(t_k)-\mathbf{x}_p(t_k)\|^2)^\beta}-\frac{x_q^i(\xi_k)-x_p^i(\xi_k)}{(1+\|\mathbf{x}_q(t_{k-1})-\mathbf{x}_p(t_{k-1})\|^2)^\beta}$$
$$<-\frac{x_q^i(\xi_{k+1})-x_p^i(\xi_{k+1})}{(1+\|\mathbf{x}_q(t_{k+1})-\mathbf{x}_p(t_{k+1})\|^2)^\beta}+\frac{x_q^i(\xi_{k+1})-x_p^i(\xi_{k+1})}{(1+\|\mathbf{x}_q(t_k)-\mathbf{x}_p(t_k)\|^2)^\beta}.$$

Above all

$$\sum_{i=0,1,\cdots,n}-\frac{x_q^i(\xi_i)-x_p^i(\xi_i)}{(1+\|\mathbf{x}_q(t_i)-\mathbf{x}_p(t_i)\|^2)^\beta}+\frac{x_q^i(\xi_i)-x_p^i(\xi_i)}{(1+\|\mathbf{x}_q(t_{i-1})-\mathbf{x}_p(t_{i-1})\|^2)^\beta}>0.$$

Similar analysis from 1) it is easily to get the contradiction. So i) is false.

For condition ii) $\lim\limits_{t\to\infty}(v_p^i(t)-v_q^i(t))=-\infty$. Similar analysis from i) it can get the contradiction. So condition ii) is false.

For condition iii) $\lim\limits_{t\to\infty}(v_p^i(t)-v_q^i(t))\neq\infty$, $\lim\limits_{t\to\infty}(v_p^i(t)-v_q^i(t))\neq-\infty$ and $\lim\limits_{t\to\infty}\sup|v_p^i(t)-v_q^i(t)|=\infty$. It can find time t_2, such as function $\overline{\mathbf{v}}(t)$ is increasing on (t_2,t_3), and $|\overline{v}^i(t_3)|>|\overline{v}^i(t_2)|+2M$.

Construct a function

$$V(t)={\overline{v}^i}^2(t)-2\int_0^t\overline{v}^i(s)(g_p^i(s)-g_q^i(s))\mathrm{d}s.$$

Then it has,

$$\begin{aligned}V'(t)=&2\overline{v}^i(t){\overline{v}'}^i(t)-2\overline{v}(t)(g_p^i(t)-g_q^i(t))\\=&-2b_{qp}(t){\overline{v}^i}^2(t)+2\overline{v}^i(t)(g_p^i(t)-g_q^i(t))-2\overline{v}^i(t)(g_p^i(t)-g_q^i(t))\\=&-2b_{qp}(t){\overline{v}^i}^2(t)\\\le&0.\end{aligned}$$

From $V'(t)<0$, it has

$$\begin{gathered}V(t_2)\le V(t_3),\\{\overline{v}^i}^2(t_3)-2\int_0^{t_3}\overline{v}^i(s)(g_p^i(s)-g_q^i(s))\mathrm{d}s\le{\overline{v}^i}^2(t_2)-2\int_0^{t_2}\overline{v}^i(s)(g_p^i(s)-g_q^i(s))\mathrm{d}s.\end{gathered}$$

Furthermore,

$${\overline{v}^i}^2(t_3)\le{\overline{v}^i}^2(t_2)-2\int_{t_2}^{t_3}\overline{v}^i(s)(g_p^i(s)-g_q^i(s))\mathrm{d}s. \tag{22}$$

From the second mean value theorem of calculus, there exists constant $\xi \in (t_2, t_3)$, such as

$$\begin{aligned}\int_{t_2}^{t_3} \overline{v}^i(s)(g_p^i(s) - g_q^i(s))\mathrm{d}s =& \overline{v}^i(t_3) \int_{\xi}^{t_3} (g_p^i(s) - g_q^i(s))\mathrm{d}s \\ &+ \overline{v}^i(t_2) \int_{t_2}^{\xi} (g_p^i(s) - g_q^i(s))\mathrm{d}s,\end{aligned}$$

substitute into (21),

$$\overline{v}^{i^2}(t_3) \leq \overline{v}^{i^2}(t_2) - 2(\overline{v}^i(t_3) \int_{\xi}^{t_2} (g_p^i(s) - g_q^i(s))\mathrm{d}s + \overline{v}^i(t_2) \int_{t_1}^{\xi} (g_p^i(s) - g_q^i(s))\mathrm{d}s).$$

For $| \int_{t_1}^{\infty} (g_p^i(s) - g_q^i(s))\mathrm{d}s| < M$, it has

$$\overline{v}^{i^2}(t_3) \leq \overline{v}^{i^2}(t_2) + 2M\overline{v}^i(t_3) + 2M\overline{v}^i(t_2).$$

Furthermore,

$$(\overline{v}^i(t_3) - M)^2 \leq (\overline{v}^i(t_2) + M)^2. \tag{23}$$

From the situation iii), it has $(\overline{v}^i(t_3) - M)^2 > (\overline{v}^i(t_2) + M)^2$, it is a contradiction, so iii) is false.

To sum up all, it has prove that for all $t > 0$, $|v_p^i(t) - v_q^i(t)| < \infty$ is right. So for any $\overline{v}^i(t)$, here $i \in \mathbb{N}$, it has $|\overline{v}^i(t)| < \infty$, furthermore $\|\overline{\mathbf{v}}(t)\| < \infty$.

Next we want to prove that $\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\| < \infty$ for all $t \geq 0$. Consider the following formula:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}^i(t) = -\alpha b_{qp}(\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|)\overline{v}^i(t) + g_p^i(t) - g_q^i(t). \tag{24}$$

Using the part integral, there exists a constant C.

$$\overline{v}^i(t) = e^{-\alpha \int_0^t b_{qp}(\|\mathbf{x}_p(s) - \mathbf{x}_q(s)\|)\mathrm{d}s} (\int_0^t (e^{\alpha \int_0^h b_{qp}(\|\mathbf{x}_p(s) - \mathbf{x}_q(s)\|)\mathrm{d}s} (g_p^i(h) - g_q^i(h)))\mathrm{d}h + C)$$

For $\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|$, it has

i). For any time $t > 0$, if $\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\| \leq d^* < \infty$, we have $\int_0^t b_{pq}(\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|)\mathrm{d}s \geq d^* t$, mean $\lim_{t\to\infty} \int_0^t b_{qp}(\|\mathbf{x}_p(t) - \mathbf{x}_q(t)\|)\mathrm{d}s = \infty$.

ii). As $\int_0^{\infty} b_{qp}(s)\mathrm{d}s = \infty$, if $\lim_{t\to\infty} \|\mathbf{x}_p(t) - \mathbf{x}_q(t)\| = \infty$, then it has $\lim_{t\to\infty} \int_0^t b_{qp}(\|\mathbf{x}_p(s) - \mathbf{x}_q(s)\|)\mathrm{d}s = \infty$.

From the two above situations, it can deduce $\lim_{t\to\infty} Ce^{-\alpha \int_0^t b_{qp}(\|\mathbf{x}_p(s) - \mathbf{x}_q(s)\|)\mathrm{d}s} = 0$. Consider

$$\begin{aligned}f(t) =& e^{-\alpha \int_0^t b_{qp}(\|\mathbf{x}_p(s) - \mathbf{x}_q(s)\|)\mathrm{d}s} (\int_0^t e^{\alpha \int_0^h b_{qp}(\|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|)\mathrm{d}s} (g_1^i(h) - g_2^i(h))\mathrm{d}h) \\ =& \frac{\int_0^t e^{\alpha \int_0^h b_{qp}(\|\mathbf{x}_p(s) - \mathbf{x}_q(s)\|)\mathrm{d}s} (g_p^i(h) - g_q^i(h))\mathrm{d}h}{e^{\alpha \int_0^t b_{qp}(\|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|)\mathrm{d}s}}.\end{aligned}$$

Using L'Hospital's rule, it has

$$\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{t\to\infty} \frac{e^{\alpha\int_0^t b_{qp}(\|\mathbf{x}_p(s)-\mathbf{x}_q(s)\|)\mathrm{d}s}(g_p^i(t)-g_q^i(t))}{b_{pq}(\|\mathbf{x}_p(t)-\mathbf{x}_q(t)\|)e^{\alpha\int_0^t b_{pq}(\|\mathbf{x}_p(s)-\mathbf{x}_q(s)\|)\mathrm{d}s}} \\ &= \lim_{t\to\infty} \frac{g_p^i(t)-g_q^i(t)}{b_{qp}(\|\mathbf{x}_p(t)-\mathbf{x}_q(t)\|)}.\end{aligned}$$

If $\lim\limits_{t\to\infty} \|\mathbf{x}_p(t)-\mathbf{x}_q(t)\| = \infty$, then it has

$$\lim_{t\to\infty}(f(t)+Ce^{-\alpha\int_0^t b_{pq}(\|\mathbf{x}_p(s)-\mathbf{x}_q(s)\|)ds}) = \lim_{t\to\infty} v^i(t) = \infty.$$

By the above formula can launch $\lim\limits_{t\to\infty}\|\overline{\mathbf{v}}(t)\| = \infty$. While $\lim\limits_{t\to\infty}\|\overline{\mathbf{v}}(t)\| < \infty$, it is a contradiction. So it must exist a constant d, so that $\|\mathbf{x}_p(t)-\mathbf{x}_q(t)\| \le d < \infty$ for any time $t > 0$.

To sum up in conclusion, it has prove that the system has a weak flocking proper. □

The way to prove Theorem 2.1 and Theorem 3.1 are similar. While the model (3), (4) and model (15), (16) are inherently difference. In model (3), (4) the velocities of the two agents are bounded. It can describe the formation flight of Pterosaurs uav. When two Pterosaurs uavs finish the automatic formation, during their flight process there are all kinds of interference from external environment. If it applies the model (3), (4) to the formation flight of Pterosaurs uav, it can calculate what influence from external environment can keep the Pterosaurs uavs flying together. If the external disturbance is too strong, the ground command center only need make appropriate instructions to one of the uavs which satisfy our model requirement $\|\int_0^\infty(\mathbf{g}_1(t)-\mathbf{g}_2(t))dt\| < \infty$. This will improve operational efficiency, saving the cost of resources and improve the degree of automation.

It can describe another military combat mode—on enemy missile torpedo or warplane intercept by using model (15), (16). Obviously, the enemy missile torpedo and warplane will try to avoid the track and intercept, and the accelerated velocity of the enemy missile torpedo or warplane is depended on its own control. If we want to track with them, we should adjust the velocity to keep efficient trace and intercept. The model (15), (16) can describe it very well. Using this model it can realise the track and intercept.

4. Intelligent weak cluster complex system with a leader

In this chapter It considers a model which its agents are not all connected. There are $N+1$ agents in the model. The leader remarks with "0". Any other agents's behavior has no influence on it. There N followers in the model. The position and velocity of the leader are denoted by $(\mathbf{x}_0(t), \mathbf{v}_0(t))$. The position and velocity of the follower "i" are denoted by $(\mathbf{x}_i(t), \mathbf{v}_i(t))$, and its free-will function is $\mathbf{f}_i(t)$.

For the leader "0", $(\mathbf{x}_0(t), \mathbf{v}_0(t))$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_0(t)}{\mathrm{d}t} &= \mathbf{v}_0(t), \\ \frac{\mathrm{d}\mathbf{v}_0(t)}{\mathrm{d}t} &= \mathbf{f}_0(t);\end{aligned} \tag{25}$$

For the follower "i"$(i \in \mathbb{N})$, $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_i(t)}{\mathrm{d}t} &= \mathbf{v}_i(t), \\ \frac{\mathrm{d}\mathbf{v}_i(t)}{\mathrm{d}t} &= \alpha a_{i,i-1}(\|\mathbf{x}_i(t)-\mathbf{x}_{i-1}(t)\|)(\mathbf{v}_{i-1}(t)-\mathbf{v}_i(t)) + \mathbf{f}_i(t).\end{aligned} \tag{26}$$

Theorem 4.1. *At time t, let the velocity and position of the leader "0" and the follower "i" are $(\mathbf{x}_0(t), \mathbf{v}_0(t))$ and $(\mathbf{x}_i(t), \mathbf{v}_i(t))$. They satisfy system (24), (25). If the parameter $\beta < \frac{1}{2}$, and there exists a constant $M_0 > 0$, $M_i > 0$ ($i \in \mathbb{N}$), such that the free-will functions satisfy $\| \int_0^\infty \mathbf{f}_0(t)\mathrm{d}t\| \leq M_0 < \infty$, $\| \int_0^\infty (\mathbf{f}_1(t) - \mathbf{f}_0(t))\mathrm{d}t\| \leq M_1 < \infty$, and $\| \int_0^\infty (\mathbf{f}_i(t) - \mathbf{f}_{i-1}(t))\mathrm{d}t\| \leq M_i < \infty$ ($i \in \mathbb{N}$), then system (24), (25) has weak flocking proper.*

Proof. Using mathematical induction to prove this theorem.

First, when $i = 1$, the system (24), (25) become into the system (15), (16). From the Theorem 3.1, it is easy to deduce the system with weak flocking proper. So for any time t, it can find two constants $A_1 > 0$, $K_1 > 0$, such as $\|\mathbf{x}_1(t) - \mathbf{x}_0(t)\| < A_1$ and $\|\mathbf{v}_1(t) - \mathbf{v}_0(t)\| < K_1$. As

$$\lim_{t\to\infty} \|\mathbf{v}_0(t) - \mathbf{v}_0(0)\| = \lim_{t\to\infty} \| \int_0^t \mathbf{f}_0(t)\mathrm{d}t\| \leq M_0.$$

So there exists a constant $K_0 > 0$, such as $\|\mathbf{v}_0(t)\| < K_0$, for any time t. Furthermore

$$\begin{aligned}\|\mathbf{v}_1(t)\| &\leq \|\mathbf{v}_0(t)\| + \|\mathbf{v}_1(t) - \mathbf{v}_0(t)\| \\ &\leq K_0 + K_1.\end{aligned}$$

To sum up, it can deduce that there exists a constant $K_1^* = K_0 + K_1$, such as $\|\mathbf{v}_1(t)\| < K_1^*$.

When $i = 2$, it wants to study $\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|$ and $\|\mathbf{v}_2(t) - \mathbf{v}_1(t)\|$. Firstly, it considers $\overline{\mathbf{v}}_{21}(t) = \mathbf{v}_2(t) - \mathbf{v}_1(t)$, for any time $t > 0$, it has

$$\begin{aligned}\frac{\mathrm{d}\overline{\mathbf{v}}_{21}(t)}{\mathrm{d}t} =& -\alpha a_{21}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)(\mathbf{v}_2(t) - \mathbf{v}_1(t)) + \mathbf{f}_2(t) \\ & - [a_{10}(\|\mathbf{x}_1(t) - \mathbf{x}_0(t)\|)(\mathbf{v}_0(t) - \mathbf{v}_1(t)) + \mathbf{f}_1(t)].\end{aligned} \tag{27}$$

Let $\mathbf{h}_2(t) = \mathbf{f}_2(t) - [a_{10}(\|\mathbf{x}_1(t) - \mathbf{x}_0(t)\|)(\mathbf{v}_0(t) - \mathbf{v}_1(t)) + \mathbf{f}_1(t)]$, the formula (26) can be turned into

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathbf{v}}_{21}(t)) &= -\alpha a_{21}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)(\mathbf{v}_2(t) - \mathbf{v}_1(t)) + \mathbf{h}_2(t) \\ &= -\alpha a_{21}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)\overline{\mathbf{v}}_{21}(t) + \mathbf{h}_2(t).\end{aligned} \tag{28}$$

As $\| \int_0^\infty \mathbf{f}_0(t)\mathrm{d}t\| < M_0$ and $\| \int_0^\infty (\mathbf{f}_1(t) - \mathbf{f}_0(t))\mathrm{d}t\| < M_1$, then

$$\begin{aligned}\| \int_0^\infty \mathbf{f}_1(t)\mathrm{d}t\| &\leq \| \int_0^\infty (\mathbf{f}_1(t) - \mathbf{f}_0(t))\mathrm{d}t\| + \| \int_0^\infty \mathbf{f}_0(t)\mathrm{d}t\| \\ &< M_0 + M_1.\end{aligned}$$

The same procedure may be easily adapted to obtain $\| \int_0^\infty \mathbf{f}_2(t)\mathrm{d}t\| < M_0 + M_1 + M_2$. From $\mathbf{h}_2(t) = \mathbf{f}_2(t) - v_1'(t)$ it can deduce

$$\begin{aligned}\| \int_0^\infty \mathbf{h}_2(t)\mathrm{d}t\| &= \| \int_0^\infty \mathbf{f}_2(t) - v_1'(t)\mathrm{d}t\| \\ &= \| \int_0^\infty \mathbf{f}_2(t)\mathrm{d}t - \int_0^\infty v_1'(t)\mathrm{d}t\|\end{aligned}$$

$$\le \| \int_0^\infty \mathbf{f}_2(t)\mathrm{d}t\| + \| \int_0^\infty v_1'(t)\mathrm{d}t\|$$
$$\le M_0 + M_1 + M_2 + K_1^* + \|v_1(0)\|.$$

Let $M_2^* = M_0 + M_1 + M_2 + K_1^* + \|\mathbf{v}_1(0)\|$, then

$$\| \int_0^\infty \mathbf{h}_2(t)\| < M_2^*.$$

The displacement and velocity difference of the agent "1" and "2" $(\overline{\mathbf{x}}_{21}(t), \overline{\mathbf{v}}_{21}(t))$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathbf{x}}_{21}(t)) = \overline{\mathbf{v}}_{21}(t), \tag{29}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathbf{v}}_{21}(t)) = -\alpha a_{21}(\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|)\overline{\mathbf{v}}_{21}(t) + \mathbf{h}_2(t). \tag{30}$$

Similar analysis from Theorem 3.1, there are two constants A_2 and K_2, for any time $t > 0$, it can deduce that $\|\overline{\mathbf{x}}_{21}(t)\| < A_2$, $\|\overline{\mathbf{v}}_{21}(t)\| < K_2$.

Suppose when $N = k$, system has the weak flocking behavior.

For $N = k + 1$, it wants to prove the system also has the weak flocking behavior.

When $N = k$ the system has the weak flocking behavior. Denote by $\overline{\mathbf{x}}_{k+1,k}(t) = \mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)$, $\overline{\mathbf{v}}_{k+1,k}(t) = \mathbf{v}_{k+1}(t) - \mathbf{v}_k(t)$, it is easy to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{v}}_{k+1,k}(t) = \alpha a_{k+1,k}(\mathbf{v}_k(t) - \mathbf{v}_{k+1}(t)) - \alpha a_{k,k-1}(\mathbf{v}_{k-1}(t) - \mathbf{v}_k(t)) + \mathbf{g}_{k+1}(t) - \mathbf{g}_k(t).$$

For there exists two constants $M_0 > 0$, $K_{i,i-1} > 0$, such that $\|\mathbf{v}_0(t)\| \le M_0$ and $\|\overline{\mathbf{v}}_{i,i-1}(t)\| \le K_{i,i-1}$, for any time $t > 0$. For any time $t > 0$, it could easily deduce that there is a constant M_i, such as $\|\mathbf{v}_i(t)\| < M_i$. Let $\overline{M}_k = \max_{i \in 1,2,\cdots,k}\{M_0, \cdots, M_{i,i-1}\}$. Integral the formula above, then

$$\|\mathbf{v}_k(t)\| = \|\mathbf{v}_k(0) - \int_0^t (\alpha a_{k,k-1}(\mathbf{v}_{k-1}(s) - \mathbf{v}_k(s)) + \mathbf{g}_k(s))\mathrm{d}s\| < \overline{M}_k.$$

Let $M^* = \mathbf{v}_k(0) + \overline{M}_k$, then

$$\| \int_0^t (\alpha a_{k,k-1}(\mathbf{v}_{k-1}(s) - \mathbf{v}_k(s)) + \mathbf{g}_k(s))\mathrm{d}s\| < M^*.$$

Here it is easy to know $\int_0^t \mathbf{g}_{k+1}(s)\mathrm{d}s$ for all $t > 0$. Similar proof method from above, it is easily to know $\|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)\| < \infty$, and $\|\mathbf{v}_{k+1}(t) - \mathbf{v}_k(t)\| < \infty$ for all $t > 0$. As the number of the agents in the system id finite. By the triangle inequality, it can deduce that for any time $t > 0$, the position and velocity of the leader "0" and follower "i" $(\mathbf{x}_0(t), \mathbf{v}_0(t))$, $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ satisfy

$$\|\mathbf{x}_0(t) - \mathbf{x}_i(t)\| < \infty,$$
$$\|\mathbf{v}_0(t) - \mathbf{v}_i(t)\| < \infty.$$

Then the system (24), (25) has the weak flocking behavior. □

In this chapter, it describes a complex system with simply connected directed graph proper. Theorem 4.1 gives the initial conditions to keep the weak flocking behavior and strictly proved this conclusion. This model can be widely application to various domains.

4.1. Numerical simulation example

In 2-dimensional planar, consider a system (31), (32), (33), (34) with a leader and three followers. At time t, let the position and velocity of leader "0" is $(\mathbf{x}_0(t), \mathbf{v}_0(t))$, the position and velocity of the follower "$i = 1, 2, 3$" is $(\mathbf{x}_i(t), \mathbf{v}_i(t))$. The free-will function of "0" is $\mathbf{g}_0(t) = (\cos t, \sin t)$, the free-will function of "i" are $\mathbf{g}_1(t) = (\sin t, \cos t)$, $\mathbf{g}_2(t) = (-\cos t, e^{-t})$ and $\mathbf{g}_3(t) = (e^{-t}, \sin t)$. The position and velocity of the leader $(\mathbf{x}_0(t), \mathbf{v}_0(t))$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_0(t)}{\mathrm{d}t} =&\mathbf{v}_0(t),\\ \frac{\mathrm{d}\mathbf{v}_0(t)}{\mathrm{d}t} =&(\cos t, \sin t);\end{aligned} \tag{31}$$

The position and velocity of the follower "1" $(\mathbf{x}_1(t), \mathbf{v}_1(t))$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_1(t)}{\mathrm{d}t} =&\mathbf{v}_1(t),\\ \frac{\mathrm{d}\mathbf{v}_1(t)}{\mathrm{d}t} =&\alpha\frac{1}{2(1+\|\mathbf{x}_1(t)-\mathbf{x}_0(t)\|^2)^{\frac{1}{3}}}(\mathbf{v}_0(t)-\mathbf{v}_1(t)) + (\sin t, \cos t);\end{aligned} \tag{32}$$

The position and velocity of the follower "2" $(\mathbf{x}_2(t), \mathbf{v}_2(t))$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_2(t)}{\mathrm{d}t} =&\mathbf{v}_2(t),\\ \frac{\mathrm{d}\mathbf{v}_2(t)}{\mathrm{d}t} =&\alpha\frac{1}{2(1+\|\mathbf{x}_2(t)-\mathbf{x}_1(t)\|^2)^{\frac{1}{3}}}(\mathbf{v}_1(t)-\mathbf{v}_2(t)) + (-\cos t, e^{-2t});\end{aligned} \tag{33}$$

The position and velocity of the follower "3" $(\mathbf{x}_3(t), \mathbf{v}_3(t))$ satisfy

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_3(t)}{\mathrm{d}t} =&\mathbf{v}_3(t),\\ \frac{\mathrm{d}\mathbf{v}_3(t)}{\mathrm{d}t} =&\alpha\frac{1}{2(1+\|\mathbf{x}_3(t)-\mathbf{x}_2(t)\|^2)^{\frac{1}{3}}}(\mathbf{v}_2(t)-\mathbf{v}_3(t)) + (e^{-t}, \sin(t)).\end{aligned} \tag{34}$$

Here $\| \int_0^\infty (\mathbf{g}_i(t) - \mathbf{g}_{i-1}(t))dt\| < \infty$, satisfy Theorem 4.1. Notice that $\int_0^\infty \|\mathbf{g}_i(t) - \mathbf{g}_{i-1}(t))\|\mathrm{d}t = \infty$ for $(i = 1, 2, 3, 4)$.

It is easy to verify this system satisfies the Theorem 4.1. Through Fig. 3, it can deduce that this system has the weak flocking behavior.

Next for the system (35) (36) (37) (38), it gives another free-will function for the agents, such that $\| \int_0^\infty (\mathbf{g}_i(t) - \mathbf{g}_{i-1}(t))\mathrm{d}t\| = \infty$. Through the numerical simulation results, it reveals that if the free-will functions don't satisfy the Theorem 4.1, the system don't have the weak flocking behavior.

Here the free-will functions the system (35) (36) (37) (38) are changed. The free-will of the leader "0" is $\mathbf{h}_0(t) = (\cos t, \sin t)$, the free-will of the followers are $\mathbf{h}_1(t) = (\frac{1}{t}, \cos t)$, $\mathbf{h}_2(t) = (-\cos t, \frac{1}{t^{1/2}})$ and $\mathbf{h}_3(t) = (1, \sin t)$. Then the position and velocity of "0" $(\mathbf{x}_0(t), \mathbf{v}_0(t))$ satisfy

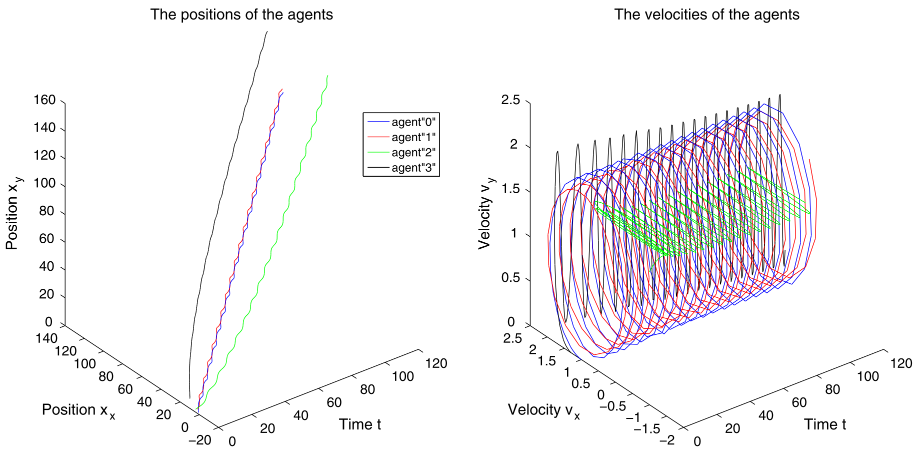

Fig. 3. The system contains a leader "0" and three followers "$i = 1, 2, 3$" with the weak flocking behavior. Parameter $\alpha = 0.5$, $\beta = 1/3$.

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}_0(t)}{\mathrm{d}t} =& \mathbf{v}_0(t), \\
\frac{\mathrm{d}\mathbf{v}_0(t)}{\mathrm{d}t} =& (\cos t, \sin t);
\end{aligned}
\tag{35}
$$

The position and velocity of "1" $(\mathbf{x}_1(t), \mathbf{v}_1(t))$ satisfy

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}_1(t)}{\mathrm{d}t} =& \mathbf{v}_1(t), \\
\frac{\mathrm{d}\mathbf{v}_1(t)}{\mathrm{d}t} =& \alpha \frac{1}{2(1 + \|\mathbf{x}_1(t) - \mathbf{x}_0(t)\|^2)^{\frac{1}{3}}} (\mathbf{v}_0(t) - \mathbf{v}_1(t)) + (\frac{1}{t}, \cos t);
\end{aligned}
\tag{36}
$$

The position and velocity of "2" $(\mathbf{x}_2(t), \mathbf{v}_2(t))$ satisfy

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}_2(t)}{\mathrm{d}t} =& \mathbf{v}_2(t), \\
\frac{\mathrm{d}\mathbf{v}_2(t)}{\mathrm{d}t} =& \alpha \frac{1}{2(1 + \|\mathbf{x}_2(t) - \mathbf{x}_1(t)\|^2)^{\frac{1}{3}}} (\mathbf{v}_1(t) - \mathbf{v}_2(t)) + (-\cos t, \frac{1}{t^{1/2}});
\end{aligned}
\tag{37}
$$

The position and velocity of "3" $(\mathbf{x}_3(t), \mathbf{v}_3(t))$ satisfy

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}_3(t)}{\mathrm{d}t} =& \mathbf{v}_3(t), \\
\frac{\mathrm{d}\mathbf{v}_3(t)}{\mathrm{d}t} =& \alpha \frac{1}{2(1 + \|\mathbf{x}_3(t) - \mathbf{x}_2(t)\|^2)^{\frac{1}{3}}} (\mathbf{v}_2(t) - \mathbf{v}_3(t)) + (1, \sin(t)).
\end{aligned}
\tag{38}
$$

Here $\| \int_0^\infty (\mathbf{f}_i(t) - \mathbf{f}_{i-1}(t))\mathrm{d}t\| < \infty$, satisfy the conditions of Theorem 4.1. Notice that $\int_0^\infty \|\mathbf{f}_i(t) - \mathbf{f}_{i-1}(t)\|\mathrm{d}t = \infty$. Here $\beta = \frac{1}{3}$, $\| \int_0^\infty (\mathbf{f}_1(t) - \mathbf{f}_2(t))\mathrm{d}t\| = \infty$, do not satisfy Theorem 4.1 notice that $\int_0^\infty \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|\mathrm{d}t = \infty$.

Obviously, the free-will functions do not satisfy Theorem 4.1 in this model. Through the Fig. 4, it is easy to see the agents are getting further and further apart and the position between the agents is not bounded. So this model does not have the weak flocking behavior.

5. Conclusions

This paper give a definition of weak flocking behavior in mathematics. One of the major contributions for this paper is that we use the different method to strictly prove: when the agents' velocities do not

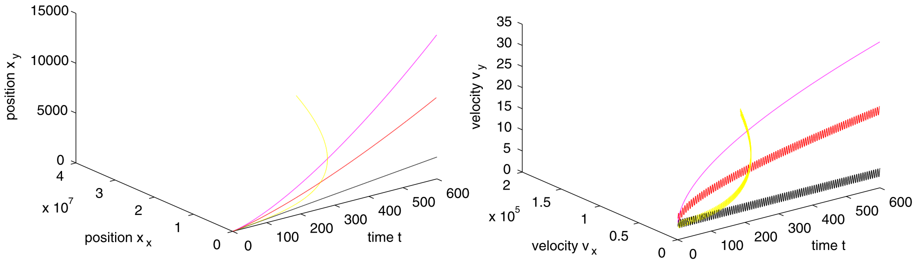

Fig. 4. The system has a leader "0" and three agents "$i = 1, 2, 3$". Parameters $\alpha = 0.5$, $\beta = 1/3$.

convergence the same at last, the distance of any two agents is bound at any time. In many classical paper to analysis the flocking behavior, the condition that the agents' velocities convergence to the same is the necessary condition to keep distance to keep bound. While in the realize world, keeping the velocities agents in complex system same is hard to realize. Our model has more wide applications, such as target tracking model. Also, this model still need to be improved. Such as, we only consider the agents will be boundary, in factory, whether we can consider that how the agents can avoid to collide to each others. In the further work, we will continue to study these problems.

In this paper, we consider a factor called free-will that can be viewed as noise or external interference. The weak flocking models proposed in this paper are relatively ideal models. In these models, we assume that all variables can be measured. Nevertheless, in a real application, the velocities and positions of the agents in the system are very complex to measure. Especially, the noise will bring uncertainty which will add the difficulty to measure the variables such as position and velocity. In our future work, we will consider to build the model in which the variables cannot be measured in sometimes. And we will carry out numerical simulations and then try to conduct theoretical analysis.

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