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Semiclassical states for Dirac-Klein-Gordon system with critical growth

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A B S T R A C T

In this paper, we study the following critical Dirac-Klein-Gordon system in $\mathbb{R}^3$:

$$\begin{cases} i\varepsilon \sum\limits_{k=1}^{3} \alpha_k \partial_k u - a\beta u + V(x)u - \lambda\phi\beta u = P(x)f(|u|)u + Q(x)|u|u, \\ -\varepsilon^2 \Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $a > 0$ is a constant. We prove the existence and concentration of solutions under suitable assumptions on the potential $V(x), P(x)$ and $Q(x)$. We also show the semiclassical solutions ω_ε with maximum points x_ε concentrating at a special set $\mathcal{H}_P$ characterized by $V(x), P(x)$ and $Q(x)$, and for any sequence $x_\varepsilon \to x_0 \in \mathcal{H}_P, v_\varepsilon(x) := \omega_\varepsilon(\varepsilon x + x_\varepsilon)$ converges in $H^1(\mathbb{R}^3, \mathbb{C}^4)$ to a least energy solution u of

$$\begin{cases} i \sum\limits_{k=1}^{3} \alpha_k \partial_k u - a\beta u + V(x_0)u - \lambda\phi\beta u = P(x_0)f(|u|)u + Q(x_0)|u|u, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u. \end{cases}$$

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1. Introduction and main results

The nonlinear Dirac equation

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + G_\psi(x,\psi), \tag{1.1}$$

has been widely used to build relativistic models of extended particles by means of nonlinear Dirac fields, where ψ represents the wave function of the state of an electron, c denotes the speed of light, $m > 0$, the mass of the electron, $\hbar$ is Planck's constant and $\partial_k = \frac{\partial}{\partial x_k}$, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 Pauli-Dirac matrices (in 2×2 blocks):

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is not difficult to check that β and α_k satisfy the following anticommutation relations

$$\begin{cases} \alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}I_4, \\ \alpha_k\beta + \beta\alpha_k = 0, \\ \beta^2 = I_4. \end{cases}$$

Different functions G model various types of self-coupling (see [27]). Such equations arise when one seeks for standing wave solutions of the nonlinear Dirac equation which describes the self-interaction in quantum electrodynamics and has been used as effective theories in atomic, and gravitational physical (see [31]). A standing wave solution of equation (1.1) is a solution of the form

$$\psi(t,x) = e^{-\frac{i\xi t}{\hbar}}u(x), \quad \xi \in \mathbb{R}, \ t \in \mathbb{R}, \ u : \mathbb{R}^3 \to \mathbb{C}^4,$$

and $u(x)$ solves the equation

$$-i\varepsilon\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + V(x)u = F_u(x,u) \tag{1.2}$$

with $\varepsilon = \hbar, a = mc > 0, V(x) = \frac{M(x)-\xi}{c}$ and $F(x,u) = \frac{G(x,u)}{c}$, where G satisfies that $G(x, e^{i\theta}\psi) = G(x,\psi)$. In the past decades, there are many works dedicated to study the Dirac equation (1.2) with the potential and the nonlinearity under several various hypotheses, see [2,4,10,15,17,25,32,34] and the references therein for the existence and multiplicity of solutions, and see [8,9,14,33] and the references therein for the concentration of semiclassical solutions.

If the Dirac field $\psi \in \mathbb{C}^4$ of equation (1.1) interacts with a scalar field $\phi \in \mathbb{R}$, then one lead to study the following Dirac-Klein-Gordon system involving an external self-coupling:

$$\begin{cases} i\frac{\hbar}{c}\partial_t\psi + i\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc\beta\psi - \lambda\phi\beta\psi = h(x,\psi), \\ \frac{\hbar^2}{c^2}\partial_t^2\phi - \hbar^2\Delta\phi + M\phi = 4\pi\lambda(\beta\psi)\cdot\psi \end{cases} \tag{1.3}$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}^3$, where $\alpha_k, \partial_k, \beta, c, \hbar, m$ as above and $\lambda > 0$ is coupling constant, M is the mass of the meson. System (1.3) arises in mathematical models of particle physics, especially in nonlinear topics. This system is inspired by approximate descriptions of the external force involve only functions of fields. The nonlinear self-coupling $h(x,\psi)$, which describes a self-interaction in Quantum electrodynamics, gives a closer description of many particles found in the real world. Various nonlinearities are considered to be possible basis models for unified filed theories, see [18,19,22] and references therein.

System (1.3) has been studied for a long time with null external self-coupling, i.e., $h \equiv 0$, and there are some results concerning the Cauchy problem, see [3,5,6,16,24,28]). The first result on the global existence and uniqueness of solutions of (1.3) (in one space dimension) was obtained by J.M. Chadam in [5] under suitable assumptions. Later, J.M. Chadam and R.T. Glasset [6] obtained the existence of a global solution in three space dimensionals. In [3], N. Bournaveas obtained low regularity solutions of the Dirac-Klein-Gordon system by using classical Strichartz-type time-space estimates. In [16], Esteban et al. obtained infinite many solutions by the variational arguments.

Likewise, a standing wave solution of system (1.3) is a solution of the form

$$\begin{cases} \psi(t,x) = u(x)e^{-\frac{i\xi t}{\hbar}}, \quad \xi\in\mathbb{R},\ t\in\mathbb{R},\ u:\mathbb{R}^3\to\mathbb{C}^4, \\ \phi = \phi(x), \end{cases}$$

and $u(x)$ solves the equation

$$\begin{cases} i\varepsilon\sum_{k=1}^{3}\alpha_k\partial_k u - a\beta u + \omega u - \lambda\phi\beta u = h(x,u), \\ -\varepsilon^2\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u, \end{cases} \tag{1.4}$$

with $\varepsilon = \hbar, a = mc > 0, \omega = \frac{\xi}{c}$, where h satisfies that $h(x, e^{i\theta}\psi) = e^{i\theta}h(x,\psi)$. A solution u is referred to as a bound state of (1.4) if $u \to 0$ as $|x| \to \infty$. When ε is sufficiently small, bound states of (1.4) are called semiclassical states, and an important feature of semiclassical states is their concentration as $\varepsilon \to 0$. To the best of our knowledge, there are few results concerning the concentration phenomenon of solutions to (1.4) except for work [13]. In [13], the authors developed cutting-off technique to obtain the existence and concentration of semiclassical solutions which seems the first one to consider the concentration behavior of semiclassical solutions of the Dirac-Klein-Gordon system with general nonlinearity.

Motivated by the references mentioned above, in this paper we consider the following critical Dirac-Klein-Gordon system in $\mathbb{R}^3$:

$$\begin{cases} i\varepsilon\sum_{k=1}^{3}\alpha_k\partial_k u - a\beta u + V(x)u - \lambda\phi\beta u = P(x)f(|u|)u + Q(x)|u|u, \\ -\varepsilon^2\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u, \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter, $a > 0$ is a constant, $V, P, Q \in C(\mathbb{R}^3, \mathbb{R})$ are three bounded functions, $V(x)$ has negative global minimum and $P(x), Q(x)$ have positive global maximum. Note that 3 is the critical exponent since the relevant Sobolev embedding is $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4) \hookrightarrow L^3(\mathbb{R}^3, \mathbb{C}^4)$. Thus, the term $|u|u$ has critical growth, while $f(|u|)u$ is assumed to be subcritical.

The aim of this paper is to focus on studying how the behavior of three potentials affect the existence and concentration of semiclassical solutions of system (1.5). Note that, in view of the presence of critical exponent term, this makes it more difficult to study the problem. It is natural to ask how about the concentration behavior of solutions of system (1.5) as $\varepsilon \to 0^+$? As far as we know such a critical problem was not considered before. There are some difficulties in such a problem. The first one is that (1.5) involves three different potentials which make our problem more complicated than that of [10]. This brings a competition between the potentials V, P and Q: each would try to attract ground states to their minimum and maximum points, respectively. Moreover, this makes the concentration sets more complex and we have to overcome many difficulties, as we shall see in the following section. The second one is the presence of the nonlocal term ϕ in (1.5), and this prevent us to use the Mountain-Pass reduction technique as [1]. In order to overcome this obstacle, we take advantage of the cut-off arguments developed by Ding and Xu in [13]. Roughly speaking, an accurate uniformly boundedness estimate on $(C)_c$-sequences of the associate energy functional Φ_ε enables us to introduce a new functional $\tilde{\Phi}_\varepsilon$ by virtue of the cut-off technique so that $\tilde{\Phi}_\varepsilon$ has the same least energy solutions as Φ_ε and can be dealt with more easily the influence of these nonlocal term for each $\lambda > 0$ small.

In this paper, we will give an answer to the above question. First, we obtain a least energy solution via cutting off technique for each $\varepsilon > 0$ small enough. Next, we study the concentration behavior of these solutions as $\varepsilon \to 0$. We determine a concrete set related to the potentials V, P and Q as the concentration position of these solutions. Roughly speaking, the least energy solutions concentrate at such points x where $V(x)$ is small or both $P(x)$ and $Q(x)$ are large. For a special case, we show that, as $\varepsilon \to 0$, these least energy solutions concentrate around such points which are both the minima points of the potential V and the maximum points of the potential P and Q. At last, we establish the exponential decay estimate of these solutions.

For notational convenience, writing $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$, we reread system (1.5) as

$$\begin{cases} i\varepsilon\alpha \cdot \nabla u - a\beta u + V(x)u - \lambda\phi\beta u = P(x)f(|u|)u + Q(x)|u|u, \\ -\varepsilon^2 \Delta\phi + M\phi = 4\pi\lambda(\beta u) \cdot u. \end{cases}$$

To state our main results, we need the following assumptions on the nonlinear self-coupling:

(f_1) $f(0) = 0, f \in C^1(\mathbb{R}, [0, \infty)), f'(t) > 0$ for $t > 0$, and there exist $p \in (2, 3), c_1 > 0$ such that

$$f(t) \leq c_1(1 + t^{p-2}) \text{ for all } t \geq 0;$$

(f_2) there exist $\sigma > 2$ and $c_0 > 0$ such that $F(t) \geq c_0 t^\sigma$ for all $t > 0$, where $F(t) = \int_0^t f(s)sds$.

Clearly, the power function $f(t) = t^{q-2}$ for $t \geq 0$ satisfies assumptions (f_1)-(f_2), where $2 < q \leq p$. Set

$$\mathcal{R}_\sigma := \left(\frac{S^{\frac{3}{2}} c_0^{\frac{2}{\sigma-2}}}{6\gamma} \right)^{\sigma-2}$$

where S denotes the Sobolev embedding constant: $S|u|_6^2 \leq |\nabla u|_2^2$ for all $u \in H^1(\mathbb{R}^3, \mathbb{R})$ and γ is the least energy of the following subcritical equation (which exists, see [8] for a similar argument)

$$i\alpha \cdot \nabla u - a\beta u = |u|^{\sigma-2}u$$

and c_0 is defined in assumption (f_2).

Remark 1.1. Note that, (f_1) implies that for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$f(t) \leq \epsilon + C_\epsilon t^{p-2} \quad \text{and} \quad F(t) \leq \epsilon t^2 + C_\epsilon t^p, \ \ \forall t \geq 0, \tag{1.6}$$

and

$$F(t) > 0 \quad \text{and} \quad \hat{F}(t) := f(t)t^2 - 2F(t) > 0, \ \ \forall t > 0. \tag{1.7}$$

We need some notations to help us to determine the concentration set of solutions. Set

$$V_{\min} := \min_{x\in\mathbb{R}^3} V(x), \ \ V_{\max} := \sup_{x\in\mathbb{R}^3} V(x), \ \ \mathcal{V} := \{x \in \mathbb{R}^3 : V(x) = V_{\min}\}, \ \ V_\infty := \liminf_{|x|\to\infty} V(x), \tag{1.8}$$

$$P_{\min} := \inf_{x\in\mathbb{R}^3} P(x), \ \ P_{\max} := \max_{x\in\mathbb{R}^3} P(x), \ \ \mathcal{P} := \{x \in \mathbb{R}^3 : P(x) = P_{\max}\}, \ P_\infty := \limsup_{|x|\to\infty} P(x), \tag{1.9}$$

$$Q_{\min} := \inf_{x\in\mathbb{R}^3} Q(x), \ Q_{\max} := \max_{x\in\mathbb{R}^3} Q(x), \ \ \mathcal{Q} := \{x \in \mathbb{R}^3 : Q(x) = Q_{\max}\}, \ Q_\infty := \limsup_{|x|\to\infty} Q(x), \tag{1.10}$$

$$V_{\mathcal{Q}} := \min_{x\in\mathcal{Q}} V(x), \qquad P_{\mathcal{Q}} := \max_{x\in\mathcal{Q}} P(x). \tag{1.11}$$

Moreover, we assume that $V(x) \leq 0, V_{\min} \in (-a, 0], P_{\min} > 0, Q_{\min} > 0$, and

(A_1) $P_{\mathcal{Q}} > P_\infty$ and there exists $x_P \in C_{\mathcal{P}}$ such that $V(x_P) \leq V(x)$ for $|x| \geq R$ with $R > 0$ sufficiently large, where $\mathcal{C}_P := \{x \in \mathcal{Q} : P(x) = P_{\mathcal{Q}}\}$.

We set

$$\mathcal{H}_P = \{x \in \mathcal{C}_P : V(x) \leq V(x_P)\} \cup \{x \in \mathcal{Q} \setminus \mathcal{C}_P : V(x) < V(x_P)\} \cup \{x \notin \mathcal{Q} : P(x) > P_{\mathcal{Q}} \text{ or } V(x) < V(x_P)\}.$$

Remark 1.2.

(1) Obviously, $x_P \in \mathcal{H}_P$ and then $\mathcal{H}_P$ is non-empty and bounded by (1.8)-(1.11).

(2) If $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, we can set $V(x_P) = \min_{x\in\mathcal{P}\cap\mathcal{Q}} V(x)$ and

$$\mathcal{H}_P = \{x \in \mathcal{P} \cap \mathcal{Q} : V(x) = V(x_P)\} \cup \{x \notin \mathcal{P} \cap \mathcal{Q} : V(x) < V(x_P)\}.$$

In particular, if $P(x) \equiv Q(x)$ or $Q(x)$ is a constant function, we can let $V(x_P) = \min_{x\in\mathcal{P}} V(x)$ and

$$\mathcal{H}_P = \{x \in \mathcal{P} : V(x) = V(x_P)\} \cup \{x \notin \mathcal{P} : V(x) < V(x_P)\},$$

which is just the case in [10].

(3) If $\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} \neq \emptyset$, we can set $V(x_P) = \min_{x\in\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}} V(x)$ and

$$\mathcal{H}_P = \{x \in \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} : V(x) = V(x_P)\},$$

which implies that $\mathcal{H}_P = \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}$.

Define

$$\alpha_0 := (\frac{a}{P_\infty})^2(\frac{Q_{\max}}{a})^{2(\sigma-2)}, \ \alpha_1 := \left(\frac{a - |V|_\infty}{a}\right)^{12-5\sigma}(\frac{a}{P_\infty})^2(\frac{Q_{\max}}{a})^{2(\sigma-2)}.$$

Now we state our main results as follows.

Theorem 1.1. *Assume that* (f_1)-(f_2) *and* (A_1) *hold, and if* $2 < \sigma \le \frac{12}{5}$ *with* $\alpha_0 \le \mathcal{R}_\sigma$ *or* $\frac{12}{5} < \sigma < 3$ *with* $\alpha_1 \le \mathcal{R}_\sigma$. *Then there exists* $\lambda_0 > 0$ *such that given* $\lambda \in (0, \lambda_0]$, *for any* $\varepsilon > 0$ *small:*

(*i*) *The system* (1.5) *has a least energy solution* ω_ε.

(*ii*) $|\omega_\varepsilon|$ *possesses a maximum point* x_ε *such that, up to a subsequence,* $x_\varepsilon \to x_0$ *as* $\varepsilon \to 0$, $\lim\limits_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{H}_\mathcal{P}) = 0$, *and* $v_\varepsilon(x) := \omega_\varepsilon(\varepsilon x + x_\varepsilon)$ *converges in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$ *to a least energy solution of*

$$\begin{cases} i\alpha \cdot \nabla u - a\beta u + V(x_0)u - \lambda\phi\beta u = P(x_0)f(|u|)u + Q(x_0)|u|u, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u. \end{cases}$$

In particular if $\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q} \neq \emptyset$, *then* $\lim\limits_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}) = 0$, *and up to a subsequence,* v_ε *converges in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$ *to a least energy solution of*

$$\begin{cases} i\alpha \cdot \nabla u - a\beta u + V_{\min}u - \lambda\phi\beta u = P_{\max}f(|u|)u + Q_{\max}|u|u, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u. \end{cases}$$

(*iii*) *There are positive constants* C_1, C_2 *independent of* ε *such that*

$$|\omega_\varepsilon(x)| \le C_1 e^{-\frac{C_2}{\varepsilon}|x - x_\varepsilon|}, \ \forall x \in \mathbb{R}^3.$$

This paper is organized as follows. In section 2, we present some preliminary notions on the Dirac operator, introduce the modified functional by cutting approach and give some basic results which will be used later. In section 3, we prove the existence of least energy solutions of system (1.5) for small $\varepsilon > 0$. In section 4, we study the concentration phenomenon and convergence of least energy solutions. In section 5, we prove the exponential decay of solutions. Finally, we give the proof of Theorem 1.1.

Notation. Throughout this paper, we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^3$, $B_R(x)$ denotes the ball of radius R centered at x;
- $|\cdot|_q$ denotes the usual norm of the space $L^q(\mathbb{R}^3, \mathbb{C}^4), 1 \le q \le \infty$;
- $o_n(1)(o_\varepsilon(1))$ denotes $o_n(1)(o_\varepsilon(1)) \to 0$ as $n \to \infty(\varepsilon \to 0)$;
- $u \cdot v$ denotes the scalar product in $\mathbb{C}^4$ of u and v, i.e., $u \cdot v = \sum\limits_{i=1}^{4} u_i \bar{v}_i$;
- C or $C_i(i = 1, 2, \cdots)$ are some positive constants may change from line to line.

2. The functional-analytic setting and preliminary results

2.1. The functional-analytic setting

For convenience, let

$$H_0 := i\alpha \cdot \nabla - a\beta$$

denotes the Dirac operator. It is well known that H_0 is a selfadjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $D(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$, then by [2, Lemma 3.3(b)], we know that

$$\sigma(H_0) = \sigma_c(H_0) = \mathbb{R} \setminus (-a, a)$$

where $\sigma(H_0)$ and $\sigma_c(H_0)$ denote the spectrum and continuous spectrum of operator H_0. Thus the space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

so that H_0 is negative definite on L^- and positive definite on L^+. Let $|H_0|$ denote the absolute value of H_0 and $|H_0|^{\frac{1}{2}}$ denote its square root. Define $E := D(|H_0|^{\frac{1}{2}}) = H^{\frac{1}{2}}$ be the Hilbert space with the inner product

$$(u, v) = \Re(|H_0|^{\frac{1}{2}} u, |H_0|^{\frac{1}{2}} v)_{L^2}$$

and the induced norm $\|u\| = (u, u)^{\frac{1}{2}}$, where $\Re$ stands for the real part of a complex number. Since $\sigma(H_0) = \mathbb{R} \setminus (-a, a)$, one has

$$a|u|_2^2 \leq \|u\|^2, \text{ for all } u \in E. \tag{2.1}$$

Note that this norm is equivalent to the usual $H^{\frac{1}{2}}$-norm, hence E embeds continuously into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \in [2, 3]$ and compactly into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \in [1, 3)$. That is, there exists constant $\tau_q > 0$ such that

$$|u|_q \leq \tau_q \|u\|, \text{ for all } u \in E. \tag{2.2}$$

Moreover, it is clear that E possesses the following decomposition

$$E = E^- \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm$$

which is decomposition orthogonal with respect to inner products $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$. Furthermore, from [14, Proposition 2.1], this decomposition induces of E also a natural decomposition of $L^q(\mathbb{R}^3, \mathbb{C}^4)$, hence there is $c_q > 0$ such that

$$c_q |u^\pm|_q^q \leq |u|_q^q \text{ for all } u \in E. \tag{2.3}$$

Recall that by the Lax-Milgram theorem, we know that for every $u \in H^1(\mathbb{R}^3, \mathbb{C}^4)$, there exists a unique $\phi_u \in H^1(\mathbb{R}^3, \mathbb{R})$ such that $-\Delta \phi_u + M\phi_u = 4\pi\lambda(\beta u) \cdot u$ and ϕ_u can be repressed by

$$\phi_u(x) = \lambda \int_{\mathbb{R}^3} \frac{[(\beta u)u](y)}{|x-y|} e^{-\sqrt{M}|x-y|} dy.$$

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite the system (1.5) as the following equivalent system

$$\begin{cases} i\alpha \cdot \nabla u - a\beta u + V(\varepsilon x)u - \lambda\phi\beta u = P(\varepsilon x) f(|u|)u + Q(\varepsilon x)|u|u, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta u) \cdot u. \end{cases} \tag{2.4}$$

If u is a solution of the system (2.4), then $\omega(x) := u(\frac{x}{\varepsilon})$ is a solution of the system (1.5). Thus, to study the system (1.5), it suffices to study the system (2.4).

On E, we define the energy functional Φ_ε corresponding to system (2.4) by

$$\Phi_\varepsilon(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx - N_\lambda(u) - \Psi_\varepsilon(u),$$

for $u = u^+ + u^- \in E$, where

$$N_\lambda(u) = \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u \cdot (\beta u)u dx = \frac{\lambda^2}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{[(\beta u)u](x)[(\beta u)u](y)}{|x-y|} e^{-\sqrt{M}|x-y|}dxdy,$$

and

$$\Psi_\varepsilon(u) = \int_{\mathbb{R}^3} P(\varepsilon x)F(|u|)dx + \frac{1}{3}\int_{\mathbb{R}^3} Q(\varepsilon x)|u|^3 dx.$$

It follows by standard arguments that $\Phi_\varepsilon \in C^2(E,\mathbb{R})$. Also, for any $u, v \in E$, one has

$$\Phi_\varepsilon'(u)v = (u^+ - u^-, v) + \Re\int_{\mathbb{R}^3} V(\varepsilon x)u \cdot v dx - N_\lambda'(u)v - \Psi_\varepsilon'(u)v,$$

where

$$N_\lambda'(u)v = \lambda\int_{\mathbb{R}^3} \phi_u \cdot \Re(\beta u)v dx.$$

Moreover, it is proved that critical points of Φ_ε are weak solutions of system (2.4) in [13, Lemma 2.1].

2.2. Technical results

In this subsection, we shall introduce some preliminary lemmas. Firstly, we give some properties of ϕ_u, the proof can be found in [13], so we omit it here.

Lemma 2.1. *For any $u, v \in H^1(\mathbb{R}^3, \mathbb{C}^4)$, one has*

(i) *$\phi_u : E \to H^1(\mathbb{R}^3, \mathbb{R})$ is continuous and maps bounded sets into bounded sets;*
(ii) *$\|\phi_u\|_{H^1} \le 4\pi\lambda S^{-\frac{1}{2}}|u|_3 \cdot |u|_2 \le C\lambda\|u\|^2$ or $\|\phi_u\|_{H^1} \le 4\pi\lambda S^{-\frac{1}{2}}|u|^2_{\frac{12}{5}} \le C\lambda\|u\|^2$;*
(iii) *N_λ is non-negative and weakly sequentially lower semi-continuous. Moreover, N_λ vanishes only when $(\beta u)u = 0$ a.e. in $\mathbb{R}^3$.*
(iv) *there hold*

$$|N_\lambda(u)| \le S^{-1}\lambda^2|u|^4_{\frac{12}{5}} \le C\lambda^2\|u\|^4,$$
$$|N_\lambda'(u)v| \le 4\pi S^{-1}\lambda^2|u|_3^2 \cdot |u|_2 \cdot |v|_2 \le C\lambda^2\|u\|^3\|v\|,$$
$$|N_\lambda''(u)[v,v]| \le C\lambda^2\|u\|^2\|v\|^2,$$

where

$$\|v\|_{H^1} = \left(\int_{\mathbb{R}^3} |\nabla v|^2 + Mv^2 dx\right)^{\frac{1}{2}}, \quad v \in H^1(\mathbb{R}^3, \mathbb{R}),$$

and

$$N_\lambda''(u)[v,v] = 2\lambda^2 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{e^{-\sqrt{M}|x-y|}}{|x-y|}\Big(\Re[(\beta u)v](x)\Re[(\beta u)v](y)\Big)dxdy + \lambda\Re\int_{\mathbb{R}^3} \phi_u \cdot (\beta v)v dx.$$

In what following, set $\mathbb{R}^+ := [0, \infty)$ and define for any $u \in E^+ \setminus \{0\}$,

$$E_u := E^- \oplus \mathbb{R}^+ u.$$

The following Lemma implies that the functional Φ_ε possesses the link structure.

Lemma 2.2. *The functional Φ_ε possess the following properties:*

(i) There exist $r > 0$ and $\rho > 0$, both independent of ε, such that $\Phi_\varepsilon|_{B_r^+} \geq 0$ and $\Phi_\varepsilon|_{S_r^+} \geq \rho$, where $B_r^+ := \{u \in E^+ : \|u\| \leq r\}, S_r^+ := \{u \in E^+ : \|u\| = r\}$;
(ii) For any $e \in E^+ \setminus \{0\}$, there exist $R = R_e >$ and $C = C_e > 0$, both independent of ε, such that, for all $\varepsilon > 0$, there hold $\Phi_\varepsilon(u) < 0$ for all $u \in E_e \setminus B_R$ and $\max \Phi_\varepsilon(E_e) \leq C$.

Proof. (i) For any $u \in E^+$, it follows from (1.6) and Lemma 2.1(iv), for $\epsilon > 0$ small enough, that

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx - N_\lambda(u) - \Psi_\varepsilon(u) \\ &\geq \frac{a - |V|_\infty}{2a}\|u\|^2 - C\lambda^2\|u\|^4 - \epsilon|u|_2^2 - C_\epsilon|u|_p^p \\ &\geq \frac{a - |V|_\infty}{4a}\|u\|^2 - C\lambda^2\|u\|^4 - C\|u\|^p,\end{aligned}$$

which, jointly with $p > 2$, yields (i).

(ii) Take $e \in E^+ \setminus \{0\}$, by (2.3) and Lemma 2.1$(iii)$, for $u = se + v \in E_e$, one has

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}\|se\|^2 - \frac{1}{2}\|v\|^2 + \frac{1}{2}\int\limits_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx - N_\lambda(u) - \Psi_\varepsilon(u) \\ &\leq \frac{a + |V|_\infty}{2a}s^2\|e\|^2 - \frac{a - |V|_\infty}{2a}\|v\|^2 - \frac{Q_{\min}c_3}{3}s^3|e|_3^3,\end{aligned}$$

and so complete the proof of (ii). $\square$

Recall that a sequence $\{u_n\} \subset E$ is called to be a $(PS)_c$-sequence for functional $\Phi \in C^1(E, \mathbb{R})$ if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$, and is called to be $(C)_c$-sequence for Φ if $\Phi(u_n) \to c$ and $(1+\|u_n\|)\Phi'(u_n) \to 0$. It is clear that if $\{u_n\}$ is a $(PS)_c$-sequence with $\{\|u_n\|\}$ bounded, then it is also a $(C)_c$-sequence. Below we are going to study $(C)_c$-sequences for Φ_ε.

Lemma 2.3. *For every pair of constants $c_1, c_2 > 0$, there exist constants $\lambda_1 > 0$ and $\Lambda = \Lambda(c_1, c_2) > 0$ such that, for any $\lambda \in (0, \lambda_1]$ and $u \in E$ with*

$$|\Phi_\varepsilon(u)| \leq c_1 \quad \textit{and} \quad \|u\| \cdot \|\Phi_\varepsilon'(u)\| \leq c_2,$$

we have

$$\|u\| \leq \Lambda.$$

Proof. Without loss of generality, we may assume that $\|u\| \geq 1$. It follows from (1.7) and Lemma 2.1(iii) that

$$c_1 + c_2 \geq \Phi_\varepsilon(u) - \frac{1}{2}\Phi'_\varepsilon(u)u = N_\lambda(u) + \int_{\mathbb{R}^3} P(\varepsilon x)\hat{F}(|u|)dx + \frac{1}{6}\int_{\mathbb{R}^3} Q(\varepsilon x)|u|^3 dx \geq \frac{Q_{\min}}{6}|u|_3^3 \tag{2.5}$$

that is,

$$|u|_3 \leq \Big(\frac{6(c_1+c_2)}{Q_{\min}}\Big)^{\frac{1}{3}} := C_1. \tag{2.6}$$

Therefore, (1.6), (2.6) and Lemma 2.1(iv) imply that

$$\begin{aligned}
c_2 \geq \Phi'_\varepsilon(u)(u^+ - u^-) &= \|u\|^2 + \Re\int_{\mathbb{R}^3} V(\varepsilon x)u\cdot(u^+ - u^-)dx - N'_\lambda(u)(u^+ - u^-) \\
&\quad - \Re\int_{\mathbb{R}^3} P(\varepsilon x)f(|u|)u\cdot(u^+ - u^-)dx - \Re\int_{\mathbb{R}^3} Q(\varepsilon x)|u|u\cdot(u^+ - u^-)dx \\
&\geq \frac{a - |V|_\infty}{a}\|u\|^2 - 4\pi S^{-1}\lambda^2|u|_3^2\cdot|u|_2\cdot|u^+ - u^-|_2 - \epsilon|u|_2^2 - C_\epsilon|u|_p^p - C_2|u|_3^3 \\
&\geq \frac{a - |V|_\infty}{2a}\|u\|^2 - \frac{4\pi S^{-1}C_1^2}{a}\lambda^2\|u\|^2 - C_3|u|_p^p - C_2C_1^3 \\
&\geq \frac{a - |V|_\infty}{2a}\|u\|^2 - \frac{4\pi S^{-1}C_1^2}{a}\lambda^2\|u\|^2 - C_4|u|_3^{p-1}\cdot|u|_{\frac{3}{4-p}} - C_2C_1^3 \\
&\geq \frac{a - |V|_\infty}{2a}\|u\|^2 - \frac{4\pi S^{-1}C_1^2}{a}\lambda^2\|u\|^2 - C_5\|u\| - C_2C_1^3
\end{aligned}$$

where we have used the fact that $\frac{3}{4-p} \in (2,3)$ and $E \hookrightarrow L^{\frac{3}{4-p}}(\mathbb{R}^3, \mathbb{C}^4)$. Thus, we obtain that

$$\frac{a - |V|_\infty}{2a}\|u\|^2 \leq C_6\|u\| + \frac{4\pi S^{-1}C_1^2}{a}\lambda^2\|u\|^2, \tag{2.7}$$

which implies that there exist $\lambda_1 > 0$ and $\Lambda = \Lambda(c_1, c_2) > 0$ such that, for $\lambda \in (0, \lambda_1]$

$$\|u\| \leq \Lambda.$$

This completes the proof. $\square$

Lemma 2.3 has an immediate consequence which implies the boundedness of a $(C)_c$-sequence:

Corollary 2.1. *Let $\{u_n^\varepsilon\}$ is the corresponding $(C)_{c_\varepsilon}$-sequence for Φ_ε. If there exists $C > 0$ such that $|c_\varepsilon| \leq C$ for all ε, then we have (up to a subsequence)*

$$\|u_n^\varepsilon\| \leq \Lambda,$$

where Λ found in Lemma 2.3 and the pair $c_1 = C$ and $c_2 = 1$.

2.3. Cut-off arguments

The functional Φ_ε contains the non-local term N_λ, which is not convex on E. Thus, Mountain-Pass reduction technique could not be applied to functional Φ_ε. In order to overcome this difficulty, we will adopt the cut-off the non-local term argument of [13] to find critical point, and eventually shown to be a least energy solution of the original system.

Now fix $\Lambda > 0$ to be the constant found in Lemma 2.3 and the pair of the constants $c_1 = C_{e_0}$ and $c_2 = 1$, where C_{e_0} is the constant in Lemma 2.2 with $e_0 \in E^+ \setminus \{0\}$ being fixed. Denote $T = (\Lambda + 1)^2$ and let $\eta : [0, \infty) \to [0, 1]$ be a smooth function with $\eta(t) = 1$ if $0 \le t \le T, \eta(t) = 0$ if $t \ge T + 1$, $\max |\eta'(t)| \le 2$ and $|\eta''(t)| \le 2$. Define $\Gamma_\lambda : E \to \mathbb{R}$ as $\Gamma_\lambda(u) = \eta(\|u\|^2)N_\lambda(u)$. Then we have $\Gamma_\lambda \in C^2(E, \mathbb{R})$ and Γ_λ vanishes for all u with $\|u\| \ge \sqrt{T+1}$.

Consider the modified functional

$$\tilde{\Phi}_\varepsilon(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx - \Gamma_\lambda(u) - \Psi_\varepsilon(u).$$

By definition, $\tilde{\Phi}_\varepsilon|_{B_T} = \Phi_\varepsilon$, where $B_T := \{u \in E : \|u\| \le \sqrt{T}\}$. Clearly, $0 \le \Gamma_\lambda(u) \le N_\lambda(u)$ and for any $u, v \in E$

$$\Gamma'_\lambda(u)v = 2\eta'(\|u\|^2)N_\lambda(u)(u, v) + \eta(\|u\|^2)N'_\lambda(u)v.$$

Similar to Lemma 2.3, we have the following boundedness Lemma (with Λ being taken in Lemma 2.3 and large if necessary).

Lemma 2.4. *There exists $\lambda_2 > 0$ such that, for each $\lambda \in (0, \lambda_2]$, if $u \in E$ satisfies*

$$0 \le \tilde{\Phi}_\varepsilon(u) \le C_{e_0} \quad \text{and} \quad \|u\| \cdot \|\tilde{\Phi}'_\varepsilon(u)\| \le 1,$$

then we have $\|u\| \le \Lambda + 1$, and consequently $\tilde{\Phi}_\varepsilon(u) = \Phi_\varepsilon(u)$.

Proof. We repeat the arguments of Lemma 2.3. If $\|u\|^2 \ge T+1$ then $\Gamma_\lambda(u) = 0$. So, as proved in Lemma 2.3, we obtain that $\|u\| \le \tilde{C}$ for some $\tilde{C} > 0$ and get $\|u\| \le \tilde{C} \le \Lambda + 1$, a contradiction. Thus we assume that $\|u\|^2 \le T + 1$. Then, using Lemma 2.1(*iv*), there is $\lambda_2 > 0$ (such as $\lambda_2 = \frac{1}{(T+1)^{\frac{3}{2}}}$) such that, for any $\lambda \in (0, \lambda_2]$

$$|\eta'(\|u\|^2)\|u\|^2 N_\lambda(u)| \le r_0\lambda^2\|u\|^6 \le r_0\lambda^2(T + 1)^3 \le r_0, \tag{2.8}$$

where $r_0 > 0$ independent of T. Similar to (2.5), we get

$$C_{e_0} + 1 \ge (\eta(\|u\|^2) + \eta'(\|u\|^2)\|u\|^2)N_\lambda(u) + \int_{\mathbb{R}^3} P(\varepsilon x)\hat{F}(|u|)dx + \frac{1}{6}\int_{\mathbb{R}^3} Q(\varepsilon x)|u|^3 dx \ge \frac{Q_{\min}}{6}|u|_3^3$$

which, jointly with (2.8), yields

$$|u|_3^3 \le \frac{6(C_{e_0} + 1 + r_0)}{Q_{\min}} := \tilde{C}_1.$$

Similar to (2.7), for any $\lambda \in (0, \lambda_2]$, we get

$$\begin{aligned}\frac{a - |V|_\infty}{2a}\|u\|^2 &\le 2\eta'(\|u\|^2)(u, u^+ - u^-)N_\lambda(u) + \eta(\|u\|^2)N'_\lambda(u)(u^+ - u^-) + C\|u\| + C\tilde{C}_1 \\ &\le C\lambda^2\|u\|^6 + C\lambda^4\|u\|^4 + C\tilde{C}_1 \\ &\le R_0\end{aligned}$$

which implies that

$$\|u\| \leq \sqrt{\frac{2aR_0}{a-|V|_\infty}} \leq \Lambda + 1,$$

where $R_0 > 0$ is a constant independent of T. The proof is complete. □

Under Lemma 2.4, to prove our main results, it suffices to study $\tilde{\Phi}_\varepsilon$ and get its critical points with critical value in $[0, C_{e_0}]$. This will be done via a series of arguments. Firstly, a similar argument of Lemma 2.2 yields

Lemma 2.5. *$\tilde{\Phi}_\varepsilon$ possesses the linking structure, and the constants found in Lemma 2.2 are independent of ε.*

Define the following minimax value (see [26,30])

$$c_\varepsilon := \inf_{u\in E^+\setminus\{0\}} \max_{w\in E_u} \tilde{\Phi}_\varepsilon(w).$$

As a consequence of Lemma 2.4 and Lemma 2.5, one has

Lemma 2.6. *$\rho \leq c_\varepsilon \leq C_{e_0}$. Moreover, if c_ε is critical point values for $\tilde{\Phi}_\varepsilon$, then it is also critical values for Φ_ε.*

In order to get more information on c_ε, motivated by [1], we consider, for a fixed $u \in E^+$, the map $\phi_u : E^- \to \mathbb{R}$ defined by

$$\phi_u(v) = \tilde{\Phi}_\varepsilon(u+v).$$

Observe that, for any $v, w \in E^-$,

$$\begin{aligned}\phi_u''(v)[w,w] &= -\|w\|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)|w|^2dx - \Gamma_\lambda''(u+v)[w,w] - \Psi_\varepsilon''(u+v)[w,w] \\ &\leq -\frac{a-|V|_\infty}{a}\|w\|^2 - \Gamma_\lambda''(u+v)[w,w] - \Psi_\varepsilon''(u+v)[w,w],\end{aligned}$$

where

$$\begin{aligned}\Psi_\varepsilon''(u+v)[w,w] = &\int_{\mathbb{R}^3} P(\varepsilon x)\left[\frac{\big(f'(u+v)}{|u+v|}\Re(u+v,w)\big)^2 + f(|u+v|)|w|^2\right]dx \\ &+ \int_{\mathbb{R}^3} Q(\varepsilon x)\left[\frac{\big(\Re(u+v,w)\big)^2}{|u+v|} + |u+v||w|^2\right]dx > 0\end{aligned}$$

and

$$\begin{aligned}\Gamma_\lambda''(u+v)[w,w] = &(4\eta''(\|u+v\|^2)|(u+v,w)|^2 + 2\eta'(\|u+v\|^2)\|w\|^2)N_\lambda(u+v) \\ &+ 4\eta'(\|u+v\|^2)(u+v,w)N_\lambda'(u+v)w + \eta(\|u+v\|^2)N_\lambda''(u+v)[w,w].\end{aligned}$$

Combining Lemma 2.1(iv) yields

$$|\Gamma_\lambda''(u+v)[w,w]| \leq C\lambda^2\|w\|^2 \leq \frac{a-|V|_\infty}{2a}\|w\|^2$$

for $\lambda \in (0, \lambda_3]$, where λ_3 is suitably chosen. Therefore, for each $\lambda \in (0, \lambda_3]$, we deduce

$$\phi_u''(v)[w,w] \leq -\frac{a-|V|_\infty}{2a}\|w\|^2.$$

Additionally, we find

$$\phi_u(v) \leq \frac{a-|V|_\infty}{2a}\|u\|^2 - \frac{a+|V|_\infty}{2a}\|v\|^2.$$

Therefore, there is a unique $h_\varepsilon : E^+ \to E^-$ such that

$$\phi_u(h_\varepsilon(u)) = \max_{v\in E^-} \phi_u(v).$$

It is clear that, for all $v \in E^-$

$$0 = \phi_u'(h_\varepsilon(u))v = -(h_\varepsilon(u), v) + \Re\int_{\mathbb{R}^3} V(\varepsilon x)(u + h_\varepsilon(u))v dx - \Gamma_\lambda'(u + h_\varepsilon(u))v - \Psi_\varepsilon'(u + h_\varepsilon(u))v$$

and

$$v \neq h_\varepsilon(u) \Leftrightarrow \tilde{\Phi}_\varepsilon(u+v) < \tilde{\Phi}_\varepsilon(u + h_\varepsilon(u)).$$

Define $I_\varepsilon : E^+ \to \mathbb{R}$ by

$$I_\varepsilon(u) = \tilde{\Phi}_\varepsilon(u + h_\varepsilon(u))$$

and let

$$\mathcal{N}_\varepsilon := \{u \in E^+ \setminus \{0\} : I_\varepsilon'(u)u = 0\}.$$

By a similar argument (see [1,11,13]), one has

Lemma 2.7. *For any $u \in E^+ \setminus \{0\}$, there is a unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathcal{N}_\varepsilon$ and $t_\varepsilon \leq T_u$ for some constant $T_u > 0$ (independent of ε). Moreover, $c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$.*

Next we estimate the regularities of critical points of $\tilde{\Phi}_\varepsilon$. Let $\mathscr{K}_\varepsilon := \{u \in E : \tilde{\Phi}_\varepsilon'(u) = 0\}$ be the critical set of $\tilde{\Phi}_\varepsilon$. It is easy to see that if $\mathscr{K}_\varepsilon \setminus \{0\} \neq \emptyset$ then

$$c_\varepsilon = \inf_{u\in\mathscr{K}_\varepsilon\setminus\{0\}} \tilde{\Phi}_\varepsilon(u)$$

(see an argument of [11]). Using the same argument as in [21] or [12, Lemma 3.19], one obtains the following Lemma.

Lemma 2.8. *Let u be a weak solution to the system (1.5). Then $u \in \cap_{q\geq 2} W^{1,q}_{loc} \cap L^\infty(\mathbb{R}^3, \mathbb{C}^4)$.*

2.4. The autonomous problem

In order to prove our main results, we will make use of the autonomous problem. Precisely, for any $\mu \in (-a, 0], \nu \in [P_\infty, P_{\max}], \tau \in [Q_{\min}, Q_{\max}]$, we consider the following constant coefficient system

$$\begin{cases} i\alpha\cdot\nabla u - a\beta u + \mu u - \lambda\phi\beta u = \nu f(|u|)u + \tau|u|u, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u. \end{cases} \tag{2.9}$$

As before, we consider the modified functional

$$\tilde{\mathcal{J}}_{\mu\nu\tau}(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{\mu}{2}\int_{\mathbb{R}^3} |u|^2 dx - \Gamma_\lambda(u) - H_{\nu\tau}(u),$$

where

$$H_{\nu\tau}(u) = \nu \int_{\mathbb{R}^3} F(|u|)dx + \frac{\tau}{3}\int_{\mathbb{R}^3} |u|^3 dx.$$

And define

$$\begin{aligned}
&\mathscr{J}_{\mu\nu\tau} : E^+ \to E^-,\ \tilde{\mathcal{J}}_{\mu\nu\tau}(u + \mathscr{J}_{\mu\nu\tau}(u)) = \max_{v\in E^-} \tilde{\mathcal{J}}_{\mu\nu\tau}(u+v),\\
&J_{\mu\nu\tau} : E^+ \to \mathbb{R},\ J_{\mu\nu\tau}(u) = \tilde{\mathcal{J}}_{\mu\nu\tau}(u + \mathscr{J}_{\mu\nu\tau}(u)),\\
&\mathscr{M}_{\mu\nu\tau} := \{u \in E^+ \setminus \{0\} : J'_{\mu\nu\tau}(u)u = 0\}.
\end{aligned}$$

Similar to Lemma 2.7, for each $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $tu \in \mathscr{M}_{\mu\nu\tau}$ and

$$\gamma_{\mu\nu\tau} = \inf_{u\in\mathscr{M}_{\mu\nu\tau}} J_{\mu\nu\tau}(u) = \inf_{e\in E^+\setminus\{0\}} \max_{u\in E_e} \tilde{\mathcal{J}}_{\mu\nu\tau}(u).$$

Lemma 2.9. *For any $\mu \in (-a, 0], \nu \in [P_\infty, P_{\max}], \tau \in [Q_{\min}, Q_{\max}]$. If $2 < \sigma \le \frac{12}{5}$ with $\alpha_0 \le \mathcal{R}_\sigma$ or $\frac{12}{5} < \sigma < 3$ with $\alpha_1 \le \mathcal{R}_\sigma$. Then there holds*

$$0 < \gamma_{\mu\nu\tau} < \frac{S^{\frac{3}{2}}}{6\tau^2}\Big(\frac{a+\mu}{a}\Big)^3.$$

Moreover, system (2.9) has a least energy solution u such that $\tilde{\mathcal{J}}_{\mu\nu\tau}(u) = \gamma_{\mu\nu\tau}$.

Proof. Note that, by the min-max scheme, we deduce

$$\gamma_{\mu\nu\tau} < \gamma_{\mu\nu}(\sigma) \tag{2.10}$$

where $\gamma_{\mu\nu}(\sigma)$ is the least energy of the following equation

$$i\alpha \cdot \nabla u - a\beta u + \mu u = c_0\nu|u|^{\sigma-2}u.$$

(2.10), jointly with [10, Lemma 4.6], that is

$$\gamma_{\mu\nu}(\sigma) \le (a+\mu)^{\frac{2(3-\sigma)}{\sigma-2}}(c_0\nu)^{\frac{-2}{\sigma-2}}\gamma$$

we get

$$\gamma_{\mu\nu\tau} < (a+\mu)^{\frac{2(3-\sigma)}{\sigma-2}}(c_0\nu)^{\frac{-2}{\sigma-2}}\gamma \le \frac{S^{\frac{3}{2}}}{6\tau^2}\Big(\frac{a+\mu}{a}\Big)^3.$$

Next, we show $\gamma_{\mu\nu\tau}$ is attained. Let $\{u_n\}$ be a $(C)_{\gamma_{\mu\nu\tau}}$-sequence. By the statements in Lemma 2.4, $\{u_n\}$ is bounded in E. Next, we claim that there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ and $R, \delta > 0$ such that

$$\int_{\mathbb{R}^3} |u_n|^2 dx \ge \delta,\ n \in \mathbb{N}. \tag{2.11}$$

Otherwise, by Lion's concentration principle [23], one has

$$u_n \to 0 \text{ in } L^r(\mathbb{R}^3, \mathbb{C}^4) \text{ for } 2 < r < 3.$$

Thus, it follows from (1.6) that

$$\int_{\mathbb{R}^3} F(|u_n|)dx \to 0, \quad \int_{\mathbb{R}^3} f(|u_n|)|u_n|^2 dx \to 0 \text{ as } n \to \infty. \tag{2.12}$$

Moreover, by Lemma 2.1(iv), we can obtain

$$\Gamma_\lambda(u_n) \le S^{-1}\lambda^2 |u_n|^4_{\frac{12}{5}} \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Note that

$$\begin{aligned}\tilde{\mathcal{J}}_{\mu\nu\tau}(u_n) &= \tilde{\mathcal{J}}_{\mu\nu\tau}(u_n) - \frac{1}{2}\tilde{\mathcal{J}}'_{\mu\nu\tau}(u_n)u_n \\ &= N_\lambda(u_n) + \nu \int_{\mathbb{R}^3} \hat{F}(|u_n|)dx + \frac{\tau}{6}\int_{\mathbb{R}^3} |u_n|^3 dx.\end{aligned}$$

Therefore, (2.12)-(2.13) imply that

$$\int_{\mathbb{R}^3} |u_n|^3 dx = \frac{6\gamma_{\mu\nu\tau}}{\tau} + o_n(1). \tag{2.14}$$

Similarly, we also have

$$\|u_n\|^2 + \Re\int_{\mathbb{R}^3} \mu u_n(u_n^+ - u_n^-)dx = \Re\int_{\mathbb{R}^3} \tau |u_n| u_n(u_n^+ - u_n^-)dx + o_n(1).$$

This, jointly with the fact $S^{\frac{1}{2}}|u|_3^2 \le \|u\|^2$ (see [2]), we get

$$\begin{aligned}\frac{a+\mu}{a}\|u_n\|^2 &\le \tau^{\frac{2}{3}}|u_n|_3|u_n^+ - u_n^-|_3\Big(\int_{\mathbb{R}^3} |u_n|^3 dx\Big)^{\frac{1}{3}} + o_n(1) \\ &\le \tau^{\frac{2}{3}}S^{-\frac{1}{2}}\|u_n\|\|u_n^+ - u_n^-\|\Big(\int_{\mathbb{R}^3} |u_n|^3 dx\Big)^{\frac{1}{3}} + o_n(1) \\ &\le \tau^{\frac{2}{3}}S^{-\frac{1}{2}}\|u_n\|^2(6\gamma_{\mu\nu\tau})^{\frac{1}{3}} + o_n(1),\end{aligned}$$

which implies

$$\gamma_{\mu\nu\tau} \ge \frac{S^{\frac{3}{2}}}{6\tau^2}\Big(\frac{a+\mu}{a}\Big)^3$$

a contradiction. Let $v_n(x) = u_n(x + y_n)$, then $\{v_n\}$ is bounded in E by the boundedness of $\{u_n\}$ and, up to a subsequence, we assume that $v_n \rightharpoonup v$ in E. By (2.11), we see that $v \neq 0$ and it is easy to check that $\tilde{\mathcal{J}}'_{\gamma_{\mu\nu\tau}}(v) = 0, \tilde{\mathcal{J}}_{\gamma_{\mu\nu\tau}}(v) = \gamma_{\mu\nu\tau}$. This completes the proof. □

Lemma 2.10. *Let $u \in \mathscr{M}_{\mu\nu\tau}$ be such that $J_{\mu\nu\tau}(u) = \gamma_{\mu\nu\tau}$. Then*

$$\max_{w\in E_u} \tilde{\mathcal{J}}_{\mu\nu\tau}(w) = J_{\mu\nu\tau}(u).$$

Proof. Clearly, since $u + \mathscr{J}_b(u) \in E_u$,

$$J_{\mu\nu\tau}(u) = \tilde{\mathcal{J}}_{\mu\nu\tau}(u + \mathscr{J}_{\mu\nu\tau}(u)) \leq \max_{w \in E_u} \tilde{\mathcal{J}}_{\mu\nu\tau}(u).$$

Moreover, for any $w = v + su \in E_u$,

$$\max_{w \in E_u} \tilde{\mathcal{J}}_{\mu\nu\tau}(w) \leq \max_{s \geq 0} \tilde{\mathcal{J}}_{\mu\nu\tau}(su + \mathscr{J}(su)) = \max_{s \geq 0} J_{\mu\nu\tau}(su) = J_{\mu\nu\tau}(u).$$

Therefore, $\max_{w \in E_u} \tilde{\mathcal{J}}_{\mu\nu\tau}(w) = J_{\mu\nu\tau}(u)$. $\square$

The following Lemma describes a comparison between the least energy values for different parameters μ, ν and τ, which will plays an important role in proving the existence result in Section 3.

Lemma 2.11. *Let $\mu_j \in (-a, 0], \nu_j \in [P_\infty, P_{\max}]$ and $\tau_j \in [Q_{\min}, Q_{\max}], j = 1, 2,$ with $\mu_1 \leq \mu_2, \nu_1 \geq \nu_2$ and $\tau_1 \geq \tau_2$. Then $\gamma_{\mu_1\nu_1\tau_1} \leq \gamma_{\mu_2\nu_2\tau_2}$. In particular, if one of inequalities is strict, then $\gamma_{\mu_1\nu_1\tau_1} < \gamma_{\mu_2\nu_2\tau_2}$.*

Proof. Let u be a least energy solution of $\tilde{\mathcal{J}}_{\mu_2\nu_2\tau_2}$ and set $e = u^+$. Then

$$\gamma_{\mu_2\nu_2\tau_2} = \tilde{\mathcal{J}}_{\mu_2\nu_2\tau_2}(u) = \max_{w \in E_e} \tilde{\mathcal{J}}_{\mu_2\nu_2\tau_2}(w).$$

Let $u_0 \in E_e$ be such that $\tilde{\mathcal{J}}_{\mu_1\nu_1\tau_1}(u_0) = \max_{w \in E_e} \tilde{\mathcal{J}}_{\mu_1\nu_1\tau_1}(w)$. One has

$$\begin{aligned}
\gamma_{\mu_2\nu_2\tau_2} &= \tilde{\mathcal{J}}_{\mu_2\nu_2\tau_2}(u) \geq \tilde{\mathcal{J}}_{\mu_2\nu_2\tau_2}(u_0) \\
&= \tilde{\mathcal{J}}_{\mu_1\nu_1\tau_1}(u_0) + \frac{\mu_2 - \mu_1}{2} \int_{\mathbb{R}^3} |u_0|^2 dx + (\nu_1 - \nu_2) \int_{\mathbb{R}^3} F(|u_0|) dx + \frac{\tau_1 - \tau_2}{3} \int_{\mathbb{R}^3} |u_0|^3 dx \\
&\geq \gamma_{\mu_1\nu_1\tau_1}.
\end{aligned}$$

Thus, we complete the proof. $\square$

3. Existence of least energy solutions

In the section, we will prove the existence of least energy solutions to system (2.4). Observing that given any $x_{\mathcal{P}} \in \mathcal{C}_{\mathcal{P}}$, we set $\tilde{V}(x) = V(x + x_{\mathcal{P}}), \tilde{P}(x) = P(x + x_{\mathcal{P}})$ and $\tilde{Q}(x) = Q(x + x_{\mathcal{P}})$. Clearly, if $\tilde{u}(x)$ is a solution of

$$\begin{cases} i\alpha \cdot \nabla \tilde{u} - a\beta\tilde{u} + \tilde{V}(\varepsilon x)\tilde{u} - \lambda\phi\beta\tilde{u} = \tilde{P}(\varepsilon x) f(|\tilde{u}|)\tilde{u} + \tilde{Q}(\varepsilon x)|\tilde{u}|\tilde{u}, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta\tilde{u}) \cdot \tilde{u}, \end{cases}$$

then $u(x) = \tilde{u}(x - x_{\mathcal{P}})$ solves (2.4). Thus, without loss of generality, we may assume that

$$x_{\mathcal{P}} = 0 \in \mathcal{C}_{\mathcal{P}},$$

so

$$Q(0) = Q_{\max}, P(0) = P_{\mathcal{Q}} \text{ and } v_0 := V(0) \leq V(x) \text{ for all } |x| \geq R. \tag{3.1}$$

Lemma 3.1. $\limsup_{\varepsilon \to 0} c_\varepsilon \leq \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}$.

Proof. Denote $V_\varepsilon^\mu(x) = \max\{\mu, V(\varepsilon x)\}$, $P_\varepsilon^\nu(x) = \min\{\nu, P(\varepsilon x)\}$ and $Q_\varepsilon^\tau(x) = \min\{\tau, Q(\varepsilon x)\}$, where $\mu \in (-a, 0]$ and ν, τ are two positive constants. Define the auxiliary functional as follows:

$$\tilde{\Phi}_\varepsilon^{\mu\nu\tau}(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon^\mu(x)|u|^2 dx - \Gamma_\lambda(u) - \int_{\mathbb{R}^3} P_\varepsilon^\nu(x)F(|u|)dx - \frac{1}{3}\int_{\mathbb{R}^3} Q_\varepsilon^\tau(x)|u|^3 dx,$$

which implies that $\tilde{\mathcal{J}}_{\mu\nu\tau}(u) \le \tilde{\Phi}_\varepsilon^{\mu\nu\tau}(u)$, and thus $\gamma_{\mu\nu\tau} \le c_\varepsilon^{\mu\nu\tau}$, where $c_\varepsilon^{\mu\nu\tau}$ is the least energy of $\tilde{\Phi}_\varepsilon^{\mu\nu\tau}$. By definition, one has $V_\varepsilon^{V_{\min}}(x) \to V(0) = v_0$, $P_\varepsilon^{P_{\max}}(x) \to P(0) = P_{\mathcal{Q}}$, $Q_\varepsilon^{Q_{\max}}(x) \to Q(0) = Q_{\max}$ on bounded sets of x as $\varepsilon \to 0$. Set $V_\varepsilon^0(x) = V(\varepsilon x) - v_0$, $P_\varepsilon^0(x) = P_{\mathcal{Q}} - P(\varepsilon x)$ and $Q_\varepsilon^0(x) = Q_{\max} - Q(\varepsilon x)$. Then

$$\tilde{\Phi}_\varepsilon(u) = \tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(u) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon^0(x)|u|^2 dx + \int_{\mathbb{R}^3} P_\varepsilon^0(x)F(|u|)dx + \frac{1}{3}\int_{\mathbb{R}^3} Q_\varepsilon^0(x)|u|^3 dx. \tag{3.2}$$

Let u be a least energy solution of $\tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}$ by Lemma 2.9, that is, $\tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(u) = \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}$ and let $e = u^+$. Clearly, $e \in \mathscr{M}_{v_0 P_{\mathcal{Q}} Q_{\max}}$, $\mathscr{J}_{v_0 P_{\mathcal{Q}} Q_{\max}}(e) = u^-$ and $J_{v_0 P_{\mathcal{Q}} Q_{\max}}(e) = \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}$. There is a unique $t_\varepsilon > 0$ such that $t_\varepsilon e \in \mathcal{N}_\varepsilon$ and one has

$$c_\varepsilon \le I_\varepsilon(t_\varepsilon e). \tag{3.3}$$

By Lemma 2.7, t_ε is bounded. Hence, without loss of generality we can assume $t_\varepsilon \to t_0$ as $\varepsilon \to 0$.

Let $\ell(t) = \tilde{\Phi}_\varepsilon(w_\varepsilon + tv_\varepsilon)$, one has $\ell(1) = \tilde{\Phi}_\varepsilon(u_\varepsilon), \ell(0) = \tilde{\Phi}_\varepsilon(w_\varepsilon)$ and $\ell'(0) = 0$, where

$$u_\varepsilon = t_\varepsilon e + \mathscr{J}_{v_0 P_{\mathcal{Q}} Q_{\max}}(t_\varepsilon e), \quad w_\varepsilon = t_\varepsilon e + h_\varepsilon(t_\varepsilon e), \quad v_\varepsilon = u_\varepsilon - w_\varepsilon.$$

Thus, $\ell(1) - \ell(0) = \int_0^1 (1-t)\ell''(t)dt$. This implies that

$$\begin{aligned} \tilde{\Phi}_\varepsilon(w_\varepsilon) - \tilde{\Phi}_\varepsilon(u_\varepsilon) &= -\int_0^1 (1-t)\tilde{\Phi}_\varepsilon''(w_\varepsilon + tv_\varepsilon)[v_\varepsilon, v_\varepsilon]dt \\ &= \frac{1}{2}\|v_\varepsilon\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)|v_\varepsilon|^2 dx + \int_0^1 (1-t)\Gamma_\lambda''(w_\varepsilon + tv_\varepsilon)[v_\varepsilon, v_\varepsilon]dt \\ &\quad + \int_0^1 (1-t)\Psi_\varepsilon''(w_\varepsilon + tv_\varepsilon)[v_\varepsilon, v_\varepsilon]dt. \end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned} \tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(w_\varepsilon) - \tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(u_\varepsilon) &= -\frac{1}{2}\|v_\varepsilon\|^2 + \frac{v_0}{2}\int_{\mathbb{R}^3} |v_\varepsilon|^2 dx \\ &- \int_0^1 (1-t)\Gamma_\lambda''(u_\varepsilon - tv_\varepsilon)[v_\varepsilon, v_\varepsilon]dt - \int_0^1 (1-t)H_{P_{\mathcal{Q}} Q_{\max}}''(u_\varepsilon - tv_\varepsilon)[v_\varepsilon, v_\varepsilon]dt. \end{aligned} \tag{3.5}$$

Thus, (3.4) and (3.5) imply that

$$
\begin{aligned}
&\tilde{\Phi}_\varepsilon(w_\varepsilon) - \tilde{\Phi}_\varepsilon(u_\varepsilon) - \big(\tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(w_\varepsilon) - \tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(u_\varepsilon)\big) \\
&= \|v_\varepsilon\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} (V(\varepsilon x) + v_0)|u_\varepsilon|^2 dx + \int_0^1 \Gamma_\lambda''(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt \\
&+ \int_0^1 t H''_{P_{\mathcal{Q}} Q_{\max}}(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt + \int_0^1 (1-t)\Psi_\varepsilon''(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt.
\end{aligned} \tag{3.6}
$$

On the other hand,

$$
\begin{aligned}
\tilde{\Phi}_\varepsilon(w_\varepsilon) - \tilde{\Phi}_\varepsilon(u_\varepsilon) = \tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(w_\varepsilon) - \tilde{\mathcal{J}}_{v_0 P_{\mathcal{Q}} Q_{\max}}(u_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon^0(x)(|w_\varepsilon|^2 - |u_\varepsilon|^2) dx \\
+ \int_{\mathbb{R}^3} P_\varepsilon^0(x)(F(|w_\varepsilon|) - F(|u_\varepsilon|)) dx + \frac{1}{3}\int_{\mathbb{R}^3} Q_\varepsilon^0(x)(|w_\varepsilon|^3 - |u_\varepsilon|^3) dx.
\end{aligned} \tag{3.7}
$$

By a direct computation, we deduce

$$
\int_{\mathbb{R}^3} V_\varepsilon^0(x)(|w_\varepsilon|^2 - |u_\varepsilon|^2) dx = \int_{\mathbb{R}^3} V_\varepsilon^0(x)|v_\varepsilon|^2 dx - 2\Re\int_{\mathbb{R}^3} V_\varepsilon^0(x) u_\varepsilon \cdot v_\varepsilon dx \tag{3.8}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} P_\varepsilon^0(x)(F(|w_\varepsilon|) - F(|u_\varepsilon|)) dx + \frac{1}{3}\int_{\mathbb{R}^3} Q_\varepsilon^0(x)(|w_\varepsilon|^3 - |u_\varepsilon|^3) dx \\
&= -\Re\Big[\int_{\mathbb{R}^3} P_\varepsilon^0(x) f(|u_\varepsilon|) u_\varepsilon \cdot v_\varepsilon dx + \int_{\mathbb{R}^3} Q_\varepsilon^0(x)|u_\varepsilon| u_\varepsilon \cdot v_\varepsilon dx\Big] \\
&+ \int_0^1 (1-t) H''_{P_{\mathcal{Q}} Q_{\max}}(u_\varepsilon - t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt - \int_0^1 (1-t)\Psi_\varepsilon''(u_\varepsilon - t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt.
\end{aligned} \tag{3.9}
$$

It follows from (3.6)-(3.9) that

$$
\begin{aligned}
&\|v_\varepsilon\|^2 - \int_{\mathbb{R}^3} V(\varepsilon x)|v_\varepsilon|^2 dx + \int_0^1 \Gamma_\lambda''(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt + \int_0^1 \Psi_\varepsilon''(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon] dt \\
&= -\Re\Big[\int_{\mathbb{R}^3} V_\varepsilon^0(x) u_\varepsilon \cdot v_\varepsilon dx + \int_{\mathbb{R}^3} P_\varepsilon^0(x) f(|u_\varepsilon|) u_\varepsilon \cdot v_\varepsilon dx + \int_{\mathbb{R}^3} Q_\varepsilon^0(x)|u_\varepsilon| u_\varepsilon \cdot v_\varepsilon dx\Big].
\end{aligned}
$$

Note that

$$
\Psi_\varepsilon''(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon] \geq 0 \quad \text{and} \quad |\Gamma_\lambda''(w_\varepsilon + t v_\varepsilon)[v_\varepsilon, v_\varepsilon]| \leq \frac{a - |V|_\infty}{2a}\|v_\varepsilon\|^2
$$

we deduce that

$$
\begin{aligned}
&\frac{a-|V|_\infty}{2a}\|v_\varepsilon\|^2 \le -\Re\Big[\int\limits_{\mathbb{R}^3} V_\varepsilon^0(x)u_\varepsilon\cdot v_\varepsilon dx + \int\limits_{\mathbb{R}^3} P_\varepsilon^0(x)f(|u_\varepsilon|)u_\varepsilon\cdot v_\varepsilon dx + \int\limits_{\mathbb{R}^3} Q_\varepsilon^0(x)|u_\varepsilon|u_\varepsilon\cdot v_\varepsilon dx\Big]\\
&\le \int\limits_{\mathbb{R}^3} |V_\varepsilon^0(x)||u_\varepsilon|\cdot|v_\varepsilon|dx + \int\limits_{\mathbb{R}^3} |P_\varepsilon^0(x)|f(|u_\varepsilon|)|u_\varepsilon|\cdot|v_\varepsilon|dx + \int\limits_{\mathbb{R}^3} |Q_\varepsilon^0(x)||u_\varepsilon|^2\cdot|v_\varepsilon|dx\\
&\le C_1\int\limits_{\mathbb{R}^3}\big(|V_\varepsilon^0(x)|+|P_\varepsilon^0(x)|\big)|u_\varepsilon|\cdot|v_\varepsilon|dx + C_2\int\limits_{\mathbb{R}^3}\big(|P_\varepsilon^0(x)|+|Q_\varepsilon^0(x)|\big)|u_\varepsilon|^2\cdot|v_\varepsilon|dx\\
&\le C_3\Big(\int\limits_{\mathbb{R}^3}\big(|V_\varepsilon^0(x)|+|P_\varepsilon^0(x)|\big)^2|u_\varepsilon|^2dx\Big)^{\frac12}|v_\varepsilon|_2 + C_4\Big(\int\limits_{\mathbb{R}^3}\big(|P_\varepsilon^0(x)|+|Q_\varepsilon^0(x)|\big)^{\frac32}|u_\varepsilon|^3dx\Big)^{\frac23}|v_\varepsilon|_3.
\end{aligned}
\tag{3.10}
$$

Since $t_\varepsilon \to t_0$ and e is exponentially decaying, we have for $q = 2, 3$,

$$\limsup_{r\to\infty}\int\limits_{|x|>r}|u_\varepsilon|^q dx = 0,$$

which implies that

$$
\begin{aligned}
\int\limits_{\mathbb{R}^3}\big(|V_\varepsilon^0(x)|+|P_\varepsilon^0(x)|\big)^2|u_\varepsilon|^2dx &= \Big(\int\limits_{|x|\le r} + \int\limits_{|x|>r}\Big)\big(|V_\varepsilon^0(x)|+|P_\varepsilon^0(x)|\big)^2|u_\varepsilon|^2dx\\
&\le \int\limits_{|x|\le r}\big(|V_\varepsilon^0(x)|+|P_\varepsilon^0(x)|\big)^2|u_\varepsilon|^2dx + C\int\limits_{|x|>r}|u_\varepsilon|^2dx\\
&= o_\varepsilon(1).
\end{aligned}
$$

Similarly,

$$\int\limits_{\mathbb{R}^3}\big(|P_\varepsilon^0(x)|+|Q_\varepsilon^0(x)|\big)^{\frac32}|u_\varepsilon|^3dx = o_\varepsilon(1).$$

Thus by (3.10) one has $\|v_\varepsilon\|^2 \to 0$, that is, $h_\varepsilon(t_\varepsilon e) \to \mathscr{J}_{v_0 P_Q Q_{\max}}(t_0 e)$. Consequently,

$$\int\limits_{\mathbb{R}^3} V_\varepsilon^0(x)|w_\varepsilon|^2dx \to 0 \quad\text{and}\quad \int\limits_{\mathbb{R}^3} P_\varepsilon^0(x)F(|w_\varepsilon|)dx + \frac13\int\limits_{\mathbb{R}^3} Q_\varepsilon^0(x)|w_\varepsilon|^3dx \to 0$$

as $\varepsilon \to 0$. This, together with (3.2), shows

$$\tilde\Phi_\varepsilon(w_\varepsilon) = \tilde{\mathcal{J}}_{v_0 P_Q Q_{\max}}(w_\varepsilon) + o_\varepsilon(1) = \tilde{\mathcal{J}}_{v_0 P_Q Q_{\max}}(u_\varepsilon) + o_\varepsilon(1),$$

that is

$$I_\varepsilon(t_\varepsilon e) = J_{v_0 P_Q Q_{\max}}(t_0 e) + o_\varepsilon(1)$$

as $\varepsilon \to 0$. Then, since

$$J_{v_0 P_Q Q_{\max}}(t_0 e) \le \max_{v\in E_e}\tilde{\mathcal{J}}_{v_0 P_Q Q_{\max}}(v) = J_{v_0 P_Q Q_{\max}}(e) = \gamma_{v_0 P_Q Q_{\max}},$$

we obtain by (3.3)

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \limsup_{\varepsilon\to 0} I_\varepsilon(t_\varepsilon e) = J_{v_0 P_Q Q_{\max}}(t_0 e) \le \gamma_{v_0 P_Q Q_{\max}}.$$

This completes the proof. □

Next we only truncate the functional $V(x)$ and $P(x)$ with $\mu = v_0$ and $\nu \in (P_\infty, P_Q)$ and consider the truncated energy functional

$$\tilde{\tilde{\Phi}}_\varepsilon^{v_0\nu}(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon^{v_0}(x)|u|^2 dx - \Gamma_\lambda(u) - \int_{\mathbb{R}^3} P_\varepsilon^\nu(x)F(|u|)dx - \frac{1}{3}\int_{\mathbb{R}^3} Q_\varepsilon(\varepsilon x)|u|^3 dx.$$

As before define correspondingly $\tilde{h}_\varepsilon^{v_0\nu} : E^+ \to E^-, \tilde{I}_\varepsilon^{v_0\nu} : E^+ \to \mathbb{R}, \tilde{\mathcal{N}}_\varepsilon^{v_0\nu}, \tilde{c}_\varepsilon^{v_0\nu}$ and so on.

We have an important lower bound for the least energy $\tilde{c}_\varepsilon^{v_0\nu}$.

Lemma 3.2. $\tilde{c}_\varepsilon^{v_0\nu} \ge \gamma_{v_0\nu Q_{\max}}$.

Proof. Since $V_\varepsilon^{v_0}(x) \ge v_0, P_\varepsilon^\nu(x) \le \nu, Q(\varepsilon x) \le Q_{\max}$, from the characterization of the value $\gamma_{v_0\nu Q_{\max}}$, we know that

$$\inf_{w\in E^+\setminus\{0\}} \max_{u\in \hat{E}(w)} \tilde{\tilde{\Phi}}_\varepsilon^{v_0\nu}(u) \ge \inf_{w\in E^+\setminus\{0\}} \max_{u\in \hat{E}(w)} \tilde{\mathcal{J}}_{v_0\nu Q_{\max}}(u)$$

which gives

$$\tilde{c}_\varepsilon^{v_0\nu} \ge \gamma_{v_0\nu Q_{\max}}.$$

This completes the proof. □

Lemma 3.3. *c_ε is attained at some non-trivial u_ε for small $\varepsilon > 0$.*

Proof. Let $\{w_n\} \subset \mathcal{N}_\varepsilon$ be a minimization sequence: $I_\varepsilon(w_n) \to c_\varepsilon$. By the Ekeland variational principle we can assume that $\{w_n\}$ is a $(PS)_{c_\varepsilon}$-sequence for I_ε on E^+. Then $u_n = w_n + h_\varepsilon(w_n)$ is a $(PS)_{c_\varepsilon}$-sequence for $\tilde{\Phi}_\varepsilon$ on E. It is clear that $\{u_n\}$ is bounded, hence is a $(C)_{c_\varepsilon}$-sequence. Assume that $u_n \rightharpoonup u_\varepsilon$ in E and then $\tilde{\Phi}'(u_\varepsilon) = 0$. If $u_\varepsilon \neq 0$, it is easy to check that $\tilde{\Phi}_\varepsilon(u_\varepsilon) = c_\varepsilon$. Next we show that $u_\varepsilon \neq 0$ for small $\varepsilon > 0$. Assume by contradiction that there exists a sequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} = 0$, then $u_n \rightharpoonup 0$ in E, and thus $u_n \to 0$ in $L^r_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $r \in [1, 3)$ and $u_n(x) \to 0$ a.e. in $x \in \mathbb{R}^3$.

By (A_1), choose $\nu \in (P_\infty, P_Q)$ and consider the auxiliary functional $\tilde{\tilde{\Phi}}_{\varepsilon_j}^{v_0\nu}$, where v_0 is defined in (3.1). Let $t_n > 0$ be such that $t_n w_n \in \tilde{\mathcal{N}}_{\varepsilon_j}^{v_0\nu}$. Then $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ for some $t_0 \ge 0$ as $n \to \infty$. Remark that $\tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n) \rightharpoonup 0$ in E and $\tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n) \to 0$ in $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [1, 3)$. By (A_1) again, the set $O_\varepsilon := \{x \in \mathbb{R}^3 : V(\varepsilon x) < v_0\}$ is bounded. Thus,

$$\int_{\mathbb{R}^3} \big(V_{\varepsilon_j}^{v_0}(x) - V(\varepsilon_j x)\big)|t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|^2 dx = \int_{O_{\varepsilon_j}} \big(v_0 - V(\varepsilon_j x)\big)|t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|^2 dx = o_n(1). \quad (3.11)$$

Similarly, since the set $\{x \in \mathbb{R}^3 : P(\varepsilon x) \ge \nu\}$ is bounded and f is subcritical growth, we have

$$\int_{\mathbb{R}^3} \big(P(\varepsilon_j x) - P_{\varepsilon_j}^\nu(x)\big)F(|t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|)dx = o_n(1). \quad (3.12)$$

Therefore, by (3.11)-(3.12) and $\tilde{\Phi}_{\varepsilon_j}(t_n w_n + h_{\varepsilon_j}^{v_0\nu}(t_n w_n)) \le I_{\varepsilon_j}(w_n)$, one has

$$
\begin{aligned}
\tilde{c}_{\varepsilon_j}^{v_0\nu} &\le \tilde{I}_{\varepsilon_j}^{v_0\nu}(t_n w_n) = \tilde{\tilde{\Phi}}_{\varepsilon_j}^{v_0\nu}(t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)) \\
&= \tilde{\Phi}_{\varepsilon_j}(t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)) + \frac{1}{2}\int\limits_{\mathbb{R}^3} \big(V_{\varepsilon_j}^{v_0}(x) - V(\varepsilon_j x)\big)|t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|^2 dx \\
&\quad + \int\limits_{\mathbb{R}^3} \big(P(\varepsilon_j x) - P_{\varepsilon_j}^{\nu}(x)\big) F(|t_{w_n} w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|) dx \\
&\le I_{\varepsilon_j}(w_n) + \frac{1}{2}\int\limits_{\mathbb{R}^3} \big(V_{\varepsilon_j}^{v_0}(x) - V(\varepsilon_j x)\big)|t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|^2 dx \\
&\quad + \int\limits_{\mathbb{R}^3} \big(P(\varepsilon_j x) - P_{\varepsilon_j}^{\nu}(x)\big) F(|t_n w_n + \tilde{h}_{\varepsilon_j}^{v_0\nu}(t_n w_n)|) dx \\
&= I_{\varepsilon_j}(w_n) + o_n(1),
\end{aligned}
$$

which implies that $\tilde{c}_{\varepsilon_j}^{v_0\nu} \le c_{\varepsilon_j}$ as $n \to \infty$. Note that $\tilde{c}_{\varepsilon_j}^{v_0\nu} \ge \gamma_{v_0\nu Q_{\max}}$ in view of Lemma 3.2. Hence, we get

$$
\gamma_{v_0\nu Q_{\max}} \le c_{\varepsilon_j}.
$$

In virtue of Lemma 3.1, letting $\varepsilon_j \to 0$ yields

$$
\gamma_{v_0\nu Q_{\max}} \le \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}.
$$

Applying Lemma 2.11 and the fact that $\nu < P_{\mathcal{Q}}$ yield a contradiction. Therefore, c_ε is attained at some u_ε for small $\varepsilon > 0$. $\square$

4. Concentration and convergence of least energy solutions

This section is devoted to the concentration behavior of the least energy solutions u_ε as $\varepsilon \to 0$. We will prove the following results.

Theorem 4.1. *Let u_ε be a least energy solution of the system (2.4) given by Lemma 3.3, then for $\lambda > 0$ small, $|u_\varepsilon|$ possesses a maximum point y_ε such that, up to a subsequence, $\varepsilon y_\varepsilon \to x_0$ as $\varepsilon \to 0$, $\lim\limits_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathcal{H}_{\mathcal{P}}) = 0$ and $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$ converges in $H^1(\mathbb{R}^3, \mathbb{C}^4)$ to a least energy solution of*

$$
\begin{cases}
i\alpha \cdot \nabla u - a\beta u + V(x_0)u - \lambda\phi\beta u = P(x_0) f(|u|)u + Q(x_0)|u|u, \\
-\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u.
\end{cases}
$$

In particular, if $\mathcal{V} \cap \mathcal{P} \cap \mathcal{Q} \ne \emptyset$, then $\lim\limits_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathcal{V} \cap \mathcal{P} \cap \mathcal{Q}) = 0$, and up to a subsequence, v_ε converges in $H^1(\mathbb{R}^3, \mathbb{C}^4)$ to a least energy solution of

$$
\begin{cases}
i\alpha \cdot \nabla u - a\beta u + V_{\min} u - \lambda\phi\beta u = P_{\max} f(|u|)u + Q_{\max}|u|u, \\
-\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u.
\end{cases}
$$

Lemma 4.1. *There exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, there exist $\{y'_\varepsilon\} \subset \mathbb{R}^3$ and R', $\delta' > 0$ such that*

$$
\int\limits_{B_{R'}(y'_\varepsilon)} |u_\varepsilon|^2 dx \ge \delta'.
$$

Proof. Assume by contradiction that there exists a sequence $\varepsilon_j \to 0$ as $j \to \infty$, such that for any $R_1 > 0$,

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}^3} \int\limits_{B_{R_1}(y)} |u_{\varepsilon_j}|^2 dx = 0.$$

Thus, by Lion's concentration principle [23, Lemma1.1], we have

$$u_{\varepsilon_j} \to 0 \text{ in } L^r(\mathbb{R}^3, \mathbb{C}^4) \text{ for } 2 < r < 3,$$

which implies, from the boundedness of the potential function P and (1.6), that

$$\int\limits_{\mathbb{R}^3} P^{\nu}_{\varepsilon_j}(x)F(|u_{\varepsilon_j}|)dx \to 0, \int\limits_{\mathbb{R}^3} P^{\nu}_{\varepsilon_j}(x)f(|u_{\varepsilon_j}|)|u_{\varepsilon_j}|^2 dx \to 0 \text{ as } j \to \infty, \tag{4.1}$$

for any $\nu \in (P_\infty, P_\mathcal{Q})$, and

$$\Gamma_\lambda(u_{\varepsilon_j}) \le C\lambda^2 |u_{\varepsilon_j}|^4_{\frac{12}{5}} \to 0 \text{ as } j \to \infty. \tag{4.2}$$

It is not difficult to check that

$$\tilde{\tilde{\Phi}}^{v_0\nu}_{\varepsilon_j}(u_{\varepsilon_j}) = c_{\varepsilon_j} + o_j(1) \text{ and } (\tilde{\tilde{\Phi}}^{v_0\nu}_{\varepsilon_j})'(u_{\varepsilon_j})u_{\varepsilon_j} = o_j(1). \tag{4.3}$$

Note that

$$\begin{aligned}\tilde{\tilde{\Phi}}^{v_0\nu}_{\varepsilon_j}(u_{\varepsilon_j}) &= \tilde{\tilde{\Phi}}^{v_0\nu}_{\varepsilon_j}(u_{\varepsilon_j}) - \frac{1}{2}(\tilde{\tilde{\Phi}}^{v_0\nu}_{\varepsilon_j})'(u_{\varepsilon_j})u_{\varepsilon_j} + o_j(1) \\ &= \Gamma_\lambda(u_{\varepsilon_j}) + \int\limits_{\mathbb{R}^3} P^{\nu}_{\varepsilon_j}(x)\hat{F}(|u_{\varepsilon_j}|)dx + \frac{1}{6}\int\limits_{\mathbb{R}^3} Q(\varepsilon_j x)|u_{\varepsilon_j}|^3 dx + o_j(1).\end{aligned}$$

Thus, by (4.1)-(4.3), one has

$$\int\limits_{\mathbb{R}^3} Q(\varepsilon_j x)|u_{\varepsilon_j}|^3 dx = 6c_{\varepsilon_j} + o_j(1).$$

Moreover, we also have

$$\|u_{\varepsilon_j}\|^2 + \Re\int\limits_{\mathbb{R}^3} V^{v_0}_{\varepsilon_j}(x)u_{\varepsilon_j}(u^+_{\varepsilon_j} - u^-_{\varepsilon_j})dx = \Re\int\limits_{\mathbb{R}^3} Q(\varepsilon_j x)|u_{\varepsilon_j}|u_{\varepsilon_j}(u^+_{\varepsilon_j} - u^-_{\varepsilon_j})dx + o_j(1).$$

$$\begin{aligned}\frac{a+v_0}{a}\|u_{\varepsilon_j}\|^2 &\le Q^{\frac{2}{3}}_{\max}|u_{\varepsilon_j}|_3 \cdot |u^+_{\varepsilon_j} - u^-_{\varepsilon_j}|_3 \cdot \Big(\int\limits_{\mathbb{R}^3} Q(\varepsilon_j x)|u_{\varepsilon_j}|^3 dx\Big)^{\frac{1}{3}} + o_j(1) \\ &\le Q^{\frac{2}{3}}_{\max} S^{-\frac{1}{2}}\|u_{\varepsilon_j}\| \cdot \|u^+_{\varepsilon_j} - u^-_{\varepsilon_j}\| \cdot \Big(\int\limits_{\mathbb{R}^3} Q(\varepsilon_j x)|u_{\varepsilon_j}|^3 dx\Big)^{\frac{1}{3}} + o_j(1) \\ &\le Q^{\frac{2}{3}}_{\max} S^{-\frac{1}{2}}\|u_{\varepsilon_j}\|^2 (6c_{\varepsilon_j})^{\frac{1}{3}} + o_j(1),\end{aligned}$$

which implies

$$\liminf_{j\to\infty} c_{\varepsilon_j} \ge \frac{S^{\frac{3}{2}}}{6Q^2_{\max}}\Big(\frac{a+v_0}{a}\Big)^3. \tag{4.4}$$

Moreover, it follows from Lemma 2.11 and [10, Lemma 4.6] again that

$$\gamma_{v_0 P_Q Q_{\max}} < \gamma_{v_0 P_\infty Q_{\max}} < \gamma_{v_0 P_\infty}(\sigma) \le (a+v_0)^{\frac{2(3-\sigma)}{\sigma-2}} (c_0 P_\infty)^{\frac{-2}{\sigma-2}} \gamma \le \frac{S^{\frac{3}{2}}}{6Q_{\max}^2} (\frac{a+v_0}{a})^3$$

which is a contradiction with Lemma 3.1 and (4.4). □

Let $\{y_\varepsilon\} \subset \mathbb{R}^3$ is maximum point of $|u_\varepsilon|$, that is

$$|u_\varepsilon(y_\varepsilon)| = \max_{x\in\mathbb{R}^3} |u_\varepsilon(x)|, \ \varepsilon \in (0, \varepsilon^*).$$

We claim that there exists $\theta_0 > 0$ (independent of ε) such that

$$|u_\varepsilon(y_\varepsilon)| \ge \theta_0, \quad \text{uniformly for all } \varepsilon \in (0, \varepsilon^*).$$

Assume by contradiction that $|u_\varepsilon(y_\varepsilon)| \to 0$ as $\varepsilon \to 0$. We deduce from Lemma 4.1 that

$$0 < \delta' \le \int_{B_{R'}(y'_\varepsilon)} |u_\varepsilon|^2 dx \le C|u_\varepsilon(y_\varepsilon)|^2 \to 0 \text{ as } \varepsilon \to 0.$$

This is a contradiction. So it follows from the above claim that there exists $R > R' > 0$ and $\delta > 0$ such that

$$\int_{B_R(y_\varepsilon)} |u_\varepsilon|^2 dx \ge \delta.$$

Set $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, then v_ε satisfies

$$i\alpha \cdot \nabla v_\varepsilon - a\beta v_\varepsilon + \hat{V}_\varepsilon(x) v_\varepsilon - \lambda \phi_{v_\varepsilon} \beta v_\varepsilon = \hat{P}_\varepsilon(x) f(|v_\varepsilon|) v_\varepsilon + \hat{Q}_\varepsilon(x)|v_\varepsilon| v_\varepsilon, \tag{4.5}$$

with energy

$$\begin{aligned}
\mathcal{E}_\varepsilon(v_\varepsilon) &= \frac{1}{2}\|v_\varepsilon^+\|^2 - \frac{1}{2}\|v_\varepsilon^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \hat{V}_\varepsilon(x)|v_\varepsilon|^2 dx - \Gamma_\lambda(v_\varepsilon) \\
&\quad - \int_{\mathbb{R}^3} \hat{P}_\varepsilon(x) F(|v_\varepsilon|) dx - \frac{1}{3}\int_{\mathbb{R}^3} \hat{Q}_\varepsilon(x)|v_\varepsilon|^3 dx \\
&= \mathcal{E}_\varepsilon(v_\varepsilon) - \frac{1}{2}\mathcal{E}'_\varepsilon(v_\varepsilon) v_\varepsilon \\
&= \Gamma_\lambda(v_\varepsilon) + \int_{\mathbb{R}^3} \hat{P}_\varepsilon(x) \hat{F}(|v_\varepsilon|) dx + \frac{1}{6}\int_{\mathbb{R}^3} \hat{Q}_\varepsilon(x)|v_\varepsilon|^3 dx \\
&= \tilde{\Phi}_\varepsilon(u_\varepsilon) - \frac{1}{2}\tilde{\Phi}'_\varepsilon(u_\varepsilon) u_\varepsilon = \tilde{\Phi}_\varepsilon(u_\varepsilon) = c_\varepsilon,
\end{aligned}$$

where $\hat{V}_\varepsilon(x) = V(\varepsilon(x + y_\varepsilon)), \hat{P}_\varepsilon(x) = P(\varepsilon(x + y_\varepsilon))$ and $\hat{Q}_\varepsilon(x) = Q(\varepsilon(x + y_\varepsilon))$. We may assume $v_\varepsilon \rightharpoonup u$ in E, and $v_\varepsilon \to u$ in $L^r_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $r \in [1, 3)$ with $u \ne 0$.

By the boundedness of V, P and Q, without loss of generality, we may assume that $V(\varepsilon y_\varepsilon) \to V_0, P(\varepsilon y_\varepsilon) \to P_0$ and $Q(\varepsilon y_\varepsilon) \to Q_0$ as $\varepsilon \to 0$.

Lemma 4.2. *u is a least energy solution of*

$$i\alpha\cdot\nabla u - a\beta u + V_0 u - \lambda\phi_u\beta u = P_0 f(|u|)u + Q_0|u|u. \tag{4.6}$$

Proof. By (4.5), for any $\varphi\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^4)$, there holds that

$$0=\lim_{\varepsilon\to 0}\Re\int_{\mathbb{R}^3}\left(i\alpha\cdot\nabla v_\varepsilon - a\beta v_\varepsilon + \hat V_\varepsilon(x)v_\varepsilon - \lambda\phi_{v_\varepsilon}\beta v_\varepsilon - \hat P_\varepsilon(x)f(|v_\varepsilon|)v_\varepsilon - \hat Q_\varepsilon(x)|v_\varepsilon|v_\varepsilon\right)\cdot\varphi dx. \tag{4.7}$$

Since V, P, Q are all continuous and bounded, one has

$$\Re\int_{\mathbb{R}^3}\hat V_\varepsilon(x)v_\varepsilon\cdot\varphi dx\to\Re\int_{\mathbb{R}^3}V_0 u\cdot\varphi dx,\ \Re\int_{\mathbb{R}^3}\hat P_\varepsilon(x)f(|v_\varepsilon|)v_\varepsilon\cdot\varphi dx\to\Re\int_{\mathbb{R}^3}P_0 f(|u|)u\cdot\varphi dx,$$

and

$$\Re\int_{\mathbb{R}^3}\hat Q_\varepsilon(x)|v_\varepsilon|v_\varepsilon\cdot\varphi dx\to\Re\int_{\mathbb{R}^3}Q_0|u|u\cdot\varphi dx,$$

which combined with (4.7) imply that

$$i\alpha\cdot\nabla u - a\beta u + V_0 u - \lambda\phi_u\beta u = P_0 f(|u|)u + Q_0|u|u,$$

this is, u solves (4.6) with energy

$$\begin{aligned}
\tilde{\mathcal{J}}_{V_0P_0Q_0}(u) &= \frac12\|u^+\|^2-\frac12\|u^-\|^2+\frac12 V_0\int_{\mathbb{R}^3}|u|^2dx-\Gamma_\lambda(u)-P_0\int_{\mathbb{R}^3}F(|u|)dx-\frac{Q_0}{3}\int_{\mathbb{R}^3}|u|^3dx\\
&=\tilde{\mathcal{J}}_{V_0P_0Q_0}(u)-\frac12\tilde{\mathcal{J}}'_{V_0P_0Q_0}(u)u\\
&=\Gamma_\lambda(u)+P_0\int_{\mathbb{R}^3}\hat F(|u|)dx+\frac{Q_0}{6}\int_{\mathbb{R}^3}|u|^3dx\\
&\ge\gamma_{V_0P_0Q_0}.
\end{aligned}$$

By Fatou's Lemma and the proof of Lemma 3.1, we have

$$\begin{aligned}
\gamma_{V_0P_0Q_0} &\le \Gamma_\lambda(u)+P_0\int_{\mathbb{R}^3}\hat F(|u|)dx+\frac{Q_0}{6}\int_{\mathbb{R}^3}|u|^3dx\\
&\le\liminf_{\varepsilon\to0}\left[\Gamma_\lambda(v_\varepsilon)+\int_{\mathbb{R}^3}\hat P_\varepsilon(x)\hat F(|v_\varepsilon|)dx+\frac16\int_{\mathbb{R}^3}\hat Q_\varepsilon(x)|v_\varepsilon|^3dx\right]\\
&=\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(v_\varepsilon)\\
&\le\limsup_{\varepsilon\to0}\tilde\Phi_\varepsilon(u_\varepsilon)\\
&\le\gamma_{V_0P_0Q_0}.
\end{aligned}\tag{4.8}$$

Consequently,

$$\lim_{\varepsilon\to0}\mathcal{E}_\varepsilon(v_\varepsilon)=\lim_{\varepsilon\to0}c_\varepsilon=\tilde{\mathcal{J}}_{V_0P_0Q_0}(u)=\gamma_{V_0P_0Q_0}. \tag{4.9}$$

Thus, u is a least energy solution of the system (4.6). □

Lemma 4.3. $\{\varepsilon y_\varepsilon\}$ *is bounded.*

Proof. Suppose to the contrary that, up to a subsequence, $|\varepsilon y_\varepsilon| \to \infty$. Since $P(0) = P_{\mathcal{Q}}$ and $v_0 = V(0) \leq V(x)$ for all $|x| \geq R$, we deduce that $P_0 > P_{\mathcal{Q}}$ and $v_0 \leq V_0$. So it follows from Lemma 2.11 that $\gamma_{V_0 P_0 Q_0} > \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}$.

However, by (4.6) and Lemma 3.1, $c_\varepsilon \to \gamma_{V_0 P_0 Q_0} \leq \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}$, which is a contradiction. Therefore, $\{\varepsilon y_\varepsilon\}$ is bounded. □

After extracting a subsequence, we may assume $\varepsilon y_\varepsilon \to x_0$ as $\varepsilon \to 0$, then $V_0 = V(x_0), P_0 = P(x_0)$ and $Q_0 = Q(x_0)$.

Lemma 4.4. $\lim\limits_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathcal{H}_{\mathcal{P}}) = 0.$

Proof. It suffices to show that $x_0 \in \mathcal{H}_{\mathcal{P}}$. We argue by contradiction, if $x_0 \notin \mathcal{H}_{\mathcal{P}}$, then it is easy to check that $\gamma_{V(x_0)P(x_0)Q(x_0)} > \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}}$ by (A_1) and Lemma 2.11. Therefore, by Lemma 3.1, we have

$$\lim_{\varepsilon\to 0} c_\varepsilon = \gamma_{V(x_0)P(x_0)Q(x_0)} > \gamma_{v_0 P_{\mathcal{Q}} Q_{\max}} \geq \lim_{\varepsilon\to 0} c_\varepsilon,$$

which is absurd. □

Lemma 4.5. $v_\varepsilon \to u$ *in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Proof. By (4.8), it is easy to check that

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3} \hat{Q}_\varepsilon(x)|v_\varepsilon|^3 dx = Q_0 \int\limits_{\mathbb{R}^3} |u|^3 dx. \tag{4.10}$$

By the decay of u and $\hat{Q}_\varepsilon(x) \to Q_0$ on bounded sets of x as $\varepsilon \to 0$, one has

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^3} \hat{Q}_\varepsilon(x)|u|^3 dx = Q_0 \int\limits_{\mathbb{R}^3} |u|^3 dx. \tag{4.11}$$

It follows from (4.10)-(4.11) and the Brezis-Lieb lemma that $v_\varepsilon \to u$ in $L^3(\mathbb{R}^3, \mathbb{C}^4)$. Hence, using the interpolation inequality and the boundedness of v_ε in E yields $v_\varepsilon \to u$ in $L^t(\mathbb{R}^3, \mathbb{C}^4)$ for $t \in (2, 3]$. Denote $z_\varepsilon = v_\varepsilon - u$. The scalar product of (4.5) and (4.6) with z_ε, respectively, we get

$$(v_\varepsilon, z_\varepsilon) + \Re \int\limits_{\mathbb{R}^3} \hat{V}_\varepsilon(x) v_\varepsilon \cdot z_\varepsilon dx = o_\varepsilon(1), \tag{4.12}$$

and

$$(u, z_\varepsilon) + \Re V_0 \int\limits_{\mathbb{R}^3} u \cdot z_\varepsilon dx = o_\varepsilon(1). \tag{4.13}$$

Note that

$$\lim_{\varepsilon\to 0} \Re\Big(\int\limits_{\mathbb{R}^3} \hat{V}_\varepsilon(x) u \cdot z_\varepsilon dx - V_0 \int\limits_{\mathbb{R}^3} u \cdot z_\varepsilon dx\Big) = 0. \tag{4.14}$$

Hence, (4.12)-(4.14) imply that

$$\|z_\varepsilon\|^2 + \int_{\mathbb{R}^3} \hat{V}_\varepsilon(x)|z_\varepsilon|^2 dx = o_\varepsilon(1),$$

and then we get $v_\varepsilon \to u$ in E, and the arguments in [13, Lemma 4.3] show that $v_\varepsilon \to u$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$. This completes the proof. □

Proof of Theorem 4.1. By Lemma 4.1-Lemma 4.5 above, one can obtains the conclusions of Theorem 4.1. □

5. Decay estimate

In this section, we estimate the exponential decay properties of solutions. Let $\varepsilon_j \to 0$ and v_{ε_j} be a solution given by Theorem 4.1. For simplicity of notations, we denote v_{ε_j} and y_{ε_j} by v_j and y_j, respectively.

Let $\mathscr{L}_{\varepsilon_j}$ denote the set of all solutions of the following system

$$\begin{cases} i\alpha \cdot \nabla u - a\beta u + \hat{V}_{\varepsilon_j}(x)u - \lambda\phi\beta u = \hat{P}_{\varepsilon_j}(x)f(|u|)u + \hat{Q}_{\varepsilon_j}(x)|u|u, \\ -\Delta\phi + M\phi = 4\pi\lambda(\beta u)\cdot u. \end{cases} \tag{5.1}$$

For $u \in \mathscr{L}_{\varepsilon_j}$, similar to Lemma 2.8, we see that $u \in L^\infty(\mathbb{R}^3, \mathbb{C}^4)$. We rewrite (5.1) as

$$Du = a\beta u - \hat{V}_{\varepsilon_j}(x)u + \lambda\phi_u\beta u + \hat{P}_{\varepsilon_j}(x)f(|u|)u + \hat{Q}_{\varepsilon_j}(x)|u|u.$$

Acting the operator D on the two sides and noting that $D^2 = -\Delta$, we get

$$\begin{aligned} \Delta u = {} & \Big(a + \lambda\phi_u\Big)^2 u - \Big(\hat{P}_{\varepsilon_j}(x)f(|u|) + \hat{Q}_{\varepsilon_j}(x)|u| - \hat{V}_{\varepsilon_j}(x)\Big)^2 u \\ & - D(\lambda\phi_u)\beta u - D\Big(\hat{P}_{\varepsilon_j}(x)f(|u|) + \hat{Q}_{\varepsilon_j}(x)|u| - \hat{V}_{\varepsilon_j}(x)\Big)u. \end{aligned}$$

Now define

$$\mathrm{sgn}\ u = \begin{cases} \frac{\bar{u}}{|u|}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

By Kato's inequality [7], there holds

$$\Delta|u| \geq \Re[\Delta u(\mathrm{sgn}\ u)].$$

Note that

$$\Re\big[D\big(\hat{P}_{\varepsilon_j}(x)f(|u|) + \hat{Q}_{\varepsilon_j}(x)|u| - \hat{V}_{\varepsilon_j}(x)\big)u(\mathrm{sgn}\ u)\big] = 0,$$

we obtain

$$\Delta|u| \geq \Big(a + \lambda\phi_u\Big)^2|u| - \Big(\hat{P}_{\varepsilon_j}(x)f(|u|) + \hat{Q}_{\varepsilon_j}(x)|u| - \hat{V}_{\varepsilon_j}(x)\Big)^2|u| - |D(\lambda\phi_u)|\cdot|u|. \tag{5.2}$$

Choosing $\alpha \in [\frac{3}{2}, 2]$, it follows from Hölder inequality and $u \in L^\infty(\mathbb{R}^3, \mathbb{C}^4)$ that, for any $x \in \mathbb{R}^3$

$$\begin{aligned}
|\phi_u(x)| &= \lambda \Big| \int\limits_{\mathbb{R}^3} \frac{[(\beta u)u](y)}{|x-y|} dy \Big| \le \lambda \int\limits_{|x-y|\ge 1} \frac{|u(y)|^2}{|x-y|} dy + \lambda \int\limits_{|x-y|\le 1} \frac{|u(y)|^2}{|x-y|} dy \\
&\le \lambda \int\limits_{|x-y|\ge 1} |u(y)|^2 dy + \lambda |u|_\infty \int\limits_{|x-y|\le 1} \frac{|u(y)|}{|x-y|} dy \\
&\le \lambda \int\limits_{|x-y|\ge 1} |u(y)|^2 dy + \lambda |u|_\infty \Big(\int\limits_{|x-y|\le 1} \frac{1}{|x-y|^\alpha} dy \Big)^{\frac{1}{\alpha}} |u|_{\frac{\alpha}{\alpha-1}} \\
&\le \lambda C
\end{aligned}$$

for some $C > 0$, where we have used the fact that $\frac{\alpha}{\alpha-1} \in [2, 3]$. Similarly, we also have

$$|D(\phi_u)(x)| \le \lambda C, \text{ for any } x \in \mathbb{R}^3.$$

So, it follows from (5.2) that there exist constants $M > 0$ and $\lambda_4 \in (0, \min\{\lambda_1, \lambda_2, \lambda_3, \lambda^*\})$ such that, for any $\lambda \in (0, \lambda_4]$

$$\Delta|u| \ge -M|u|.$$

It then follows from the sub-solution estimate [20,29] that

$$|u(x)| \le C_0 \int\limits_{B_1(x)} |u(y)| dy \tag{5.3}$$

where C_0 independent of x and ε.

Lemma 5.1. *$|v_j(x)| \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$.*

Proof. Assume by contradiction that the conclusion of the Lemma does not hold. Then, it follows from (5.3) that there exist $\bar{r} > 0$ and $x_j \in \mathbb{R}^3$ with $|x_j| \to \infty$ such that

$$\bar{r} \le |v_j(x_j)| \le C_0 \int\limits_{B_1(x_j)} |v_j(x)| dx.$$

Since $v_j \to u$ in E, one obtains

$$\begin{aligned}
\bar{r} &\le C_0 \Big(\int\limits_{B_1(x_j)} |v_j(x)|^2 dx \Big)^{\frac{1}{2}} \\
&\le C_0 \Big(\int\limits_{\mathbb{R}^3} |v_j - u|^2 dx \Big)^{\frac{1}{2}} + C_0 \Big(\int\limits_{B_1(x_j)} |u(x)|^2 dx \Big)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $j \to \infty$, a contradiction. $\square$

Lemma 5.2. *There exists $C > 0$ such that*

$$|v_j(x)| \le Ce^{-\bar{\omega}|x|}, \quad \forall x \in \mathbb{R}^3$$

uniformly in $j \in \mathbb{N}$, where $\bar{\omega} = \frac{a-|V|_\infty}{2}$.

Proof. It follows from (5.2), Lemma 5.1 and the boundedness of ϕ_u and $D(\phi_u)$ that there exist $\lambda_0 \in (0, \lambda_4]$ and $R > 0$ such that, for any $\lambda \in (0, \lambda_0]$

$$\Delta|v_j| \ge \frac{(a-|V|_\infty)^2}{4}|v_j| = \bar{\omega}^2|v_j| \ \text{ for all } |x| \ge R\ , j \in \mathbb{N}.$$

Let $\Gamma(x) = \Gamma(x, 0)$ be a fundamental solution to $-\Delta + \bar{\omega}^2$ (see [29]). Using the uniform boundedness, one may choose Γ so that $|v_j(x)| \le \bar{\omega}^2\Gamma(y)$ holds on $|x| = R$, all $j \in \mathbb{N}$. Let $z_j = |v_j| - \bar{\omega}^2\Gamma$. Then

$$\Delta z_j = \Delta|v_j| - \bar{\omega}^2\Delta\Gamma \ge \bar{\omega}^2\big(|v_j| - \bar{\omega}^2\Gamma\big) = \bar{\omega}^2 z_j.$$

By the maximum principle we can conclude that $z_j(x) \le 0$ on $|x| \ge R$. It is well known that there is $C' > 0$ such that $\Gamma(x) \le C'e^{-\bar{\omega}|x|}$ on $|x| \ge 1$. We see that

$$|v_j(x)| \le Ce^{-\bar{\omega}|x|}$$

for all $x \in \mathbb{R}^3$ and all $j \in \mathbb{N}$. This completes the proof. □

Proof of Theorem 1.1. Define $\omega_j(x) := u_j(\frac{x}{\varepsilon_j})$, then ω_j is a least energy solution of system (1.5) and $x_j := \varepsilon_j y_j$ is a maximum point of $|\omega_j|$, and by Theorem 4.1, we know that the Theorem 1.1$(i), (ii)$ hold. Moreover, one has

$$|\omega_j(x)| = |u_j(\frac{x}{\varepsilon_j})| = |v_j(\frac{x}{\varepsilon_j} - y_j)| \le Ce^{-\bar{\omega}|\frac{x}{\varepsilon_j} - y_j|} = Ce^{-\frac{\bar{\omega}}{\varepsilon_j}|x-\varepsilon_j y_j|} = Ce^{-\frac{\bar{\omega}}{\varepsilon_j}|x-x_j|}.$$

Thus, the proof of Theorem 1.1 is completed. □

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