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Free boundary problems with nonlocal and local diffusions I: Global solution ☆

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A R T I C L E I N F O

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A B S T R A C T

We study a class of free boundary problems of ecological models with nonlocal and local diffusions, which are natural extensions of free boundary problems of reaction diffusion systems in there local diffusions are used to describe the population dispersal, with the free boundary representing the spreading front of the species. We prove that such kind of nonlocal and local diffusion problems has a unique global solution, and then show that a spreading-vanishing dichotomy holds. Moreover, criteria of spreading and vanishing, and long time behavior of solution when spreading happens are established for the classical Lotka-Volterra competition and prey-predator models. Compared with free boundary problems of reaction diffusion systems with local diffusions ([10,24,25]), with nonlocal diffusions ([9]) as well as with nonlocal and local diffusions ([14]) (one equation is Cauchy problem and the other one is free boundary problem), the present paper involves some new difficulties, which should be overcome by use of new techniques. This is part I of a two part series, where we prove the existence, uniqueness, regularity and estimates of global solution. The spreading-vanishing dichotomy, criteria of spreading and vanishing, and long-time behavior of solution when spreading happens will be studied in the separate part II ([17]).

1. Introduction

The spreading and vanishing of multiple species is an important content in understanding ecological complexity. In order to study the spreading and vanishing phenomenon, many mathematical models have been established. The logistic equation, competition and prey-predator models with local diffusions and

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free boundaries have been studied widely by many authors, please refer to, for example, [6] for the logistic equation, [10,24], [7,11,22,23,29] for the competition models, [18,21,26,27] for the prey-predator models, and the references therein.

It is well known that random dispersal or local diffusion describes the movements of organisms between adjacent spatial locations. It has been increasingly recognized that the movements and interactions of some organisms can occur between non-adjacent spatial locations. The evolution of nonlocal diffusion has attracted a lot of attentions for both theoretically and empirically; see [1,2,15] and references therein. An extensively used nonlocal diffusion operator to replace the local diffusion term $d\Delta u$ (the Laplacian operator in $\mathbb{R}^N$) is given by

$$d(J * u - u)(t,x) := d\left(\int_{\mathbb{R}^N} J(x-y)u(t,y)dy - u(t,x)\right).$$

To describe the spatial spreading of species in the nonlocal diffusion processes, recently, the authors of [4] studied the following free boundary problem of Fisher-KPP nonlocal diffusion model:

$$\begin{cases} u_t = d\displaystyle\int_{g(t)}^{h(t)} J(x-y)u(t,y)\mathrm{d}y - du(t,x) + f(t,x,u), & t>0,\ g(t)<x<h(t),\\ u(t,g(t)) = u(t,h(t)) = 0, & t>0,\\ h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ u(0,x) = u_0(x),\ h(0) = -g(0) = h_0, & |x|\le h_0, \end{cases} \tag{1.1}$$

where $x = g(t)$ and $x = h(t)$ are free boundaries to be determined together with $u(t,x)$, which is always assumed to be identically 0 for $x \in \mathbb{R}\setminus[g(t),h(t)]$; d, μ and h_0 are positive constants. The kernel function $J : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

(J) $J(0) > 0,\ J(x) \ge 0,\ \displaystyle\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$, J is symmetric, and $\sup_{\mathbb{R}} J < \infty$.

The reaction function $f(t,x,u)$ has logistic structure. It was shown in [4] that the problem (1.1) has a unique global solution. Furthermore, the spreading-vanishing dichotomy about free boundary problems of local diffusive logistic equation ([6]) holds true for the nonlocal diffusive problem (1.1) when $f(t,x,u) = f(u)$. However, from [4, Remark 1.4] we know that when $d \le f'(0)$, spreading happens no matter how small h_0, μ and u_0 are. This is very different from the spreading-vanishing criteria for the local diffusion models.

Recently, Du et al. [8] studied the semi-wave and spreading speed of the problem (1.1). They found a threshold condition on the kernel function J such that spreading grows linearly in time exactly when this condition holds, which is achieved by completely solving the associated semi-wave problem that determines this linear speed; when the kernel function violates this condition, they showed that accelerating spreading happens.

Motivated by the papers [4] and [10,16,22,24,29] (two species local diffusion systems with common free boundary), the authors of [9] studied the following free boundary problem of nonlocal diffusive system

$$\begin{cases} u_{it} = d_i \int\limits_{g(t)}^{h(t)} J_i(x-y)u_i(t,y)\mathrm{d}y - d_i u_i + f_i(t,x,u_1,u_2), & t>0,\ g(t)<x<h(t), \\ u_i(t,g(t)) = u_i(t,h(t)) = 0, & t \ge 0, \\ h'(t) = \sum_{i=1}^{2} \mu_i \int\limits_{g(t)}^{h(t)} \int\limits_{h(t)}^{\infty} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t \ge 0, \\ g'(t) = -\sum_{i=1}^{2} \mu_i \int\limits_{g(t)}^{h(t)} \int\limits_{-\infty}^{g(t)} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t \ge 0, \\ u_i(0,x) = u_{i0}(x), \quad h(0) = -g(0) = h_0, & |x| \le h_0, \\ i = 1,\ 2. \end{cases} \tag{1.2}$$

They proved the existence and uniqueness of global solution, a spreading-vanishing dichotomy and obtained the criteria for spreading and vanishing.

Kao et al. [12] studied the competition model in which one diffusion is local and the other one is non-local:

$$\begin{cases} u_t = d_1 \Delta u + u(a-u-v), & t>0,\ x \in \Omega, \\ v_t = d_2 \int\limits_{\Omega} J(x-y)v(t,y)\mathrm{d}y - d_2 v + v(a-u-v), & t>0,\ x\in\Omega. \end{cases}$$

Recently, Li et al. [14] investigated the following free boundary problem

$$\begin{cases} u_t = d_1 \int\limits_{-\infty}^{\infty} J(x-y)u(t,y)\mathrm{d}y - d_1 u + f_1(t,x,u,v), & t>0,\ -\infty<x<\infty, \\ v_t = d_2 v_{xx} + f_2(t,x,u,v), & t>0,\ g(t)<x<h(t), \\ v=0, \quad g'(t) = -\mu v_x, & t\ge 0,\ x=g(t), \\ v=0, \quad h'(t) = -\mu v_x, & t\ge 0,\ x = h(t), \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \\ v(0,x) = v_0(x), & x\in[-h_0,h_0], \\ -g(0) = h(0) = h_0 > 0. \end{cases}$$

Motivated by the above mentioned works, it will be interesting to study the free boundary problems with nonlocal and local diffusions. Based on the deductions of free boundary conditions in [3] and [4], it is reasonable to study the following free boundary problems:

$$\begin{cases} u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J(x,y)u(t,y)\mathrm{d}y - d_1 u + f_1(t,x,u,v), & t>0,\ g(t)<x<h(t), \\ v_t = d_2 v_{xx} + f_2(t,x,u,v), & t>0,\ g(t)<x<h(t), \\ u(t,g(t)) = u(t,h(t)) = v(t,g(t)) = v(t,h(t)) = 0, & t \ge 0, \\ h'(t) = -\mu v_x(t,h(t)) + \rho \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x, & t \ge 0, \\ g'(t) = -\mu v_x(t,g(t)) - \rho \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x, & t \ge 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ h(0) = -g(0) = h_0 > 0, & |x| \le h_0, \end{cases} \tag{1.3}$$

where $J(x,y) = J(x-y)$; $[-h_0, h_0]$ represents the initial population range of the species u and v; $x = g(t)$ and $x = h(t)$ are the free boundaries to be determined together with $u(t,x)$ and $v(t,x)$, which are always assumed to be identically 0 for $x \in \mathbb{R} \setminus [g(t), h(t)]$; d_i and μ, ρ are positive constants.

Denote by $C^{1-}(\Omega)$ the Lipschitz continuous function space in Ω. We assume that the initial functions u_0, v_0 satisfy

$$(u_0, v_0) \in C^{1-}([-h_0, h_0]) \times W_p^2(-h_0, h_0),\ u_0(\pm h_0) = v_0(\pm h_0) = 0,\ u_0,\ v_0 > 0 \ \text{ in } \ (-h_0, h_0) \tag{1.4}$$

with $p > 3$. The kernel function J is supposed to satisfy

(J1) The condition **(J)** holds, and $J \in C^{1-}(\mathbb{R})$.

It follows from **(J)** that there exist constants $\bar{\varepsilon} \in (0, h_0/4)$ and $\delta_0 > 0$ such that

$$J(x,y) > \delta_0 \quad \text{if} \quad |x-y| < \bar{\varepsilon}. \tag{1.5}$$

The growth terms $f_i : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are assumed to be continuous and satisfy

(f) $f_1(t,x,0,v) = f_2(t,x,u,0) = 0$, $f_i(t,x,u,v)$ is differentiable with respect to $u, v \in \mathbb{R}^+$, and for any $c_1, c_2 > 0$, there exists a constant $L(c_1, c_2) > 0$ such that

$$|f_i(t,x,u_1,v_1) - f_i(t,x,u_2,v_2)| \le L(c_1,c_2)(|u_1-u_2| + |v_1-v_2|),\ i = 1,2$$

for all $u_1, u_2 \in [0, c_1]$, $v_1, v_2 \in [0, c_2]$ and all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$;

(f1) There exist $k_0 > 0$ and $r > 0$ such that for all $v \ge 0$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, there hold: $f_1(t,x,u,v) < 0$ when $u > k_0$, $f_1(t,x,u,v) \le ru$ when $0 < u \le k_0$;

(f2) For the given $k > 0$, there exists $\Theta(k) > 0$ such that $f_2(t,x,u,v) < 0$ for $0 \le u \le k$, $v \ge \Theta(k)$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$;

(f3) $f_{ix}(t,x,u,v)$ is continuous and for any $c_1, c_2 > 0$, there exists a constant $L^*(c_1,c_2) > 0$ such that

$$|f_i(t,x,u,v) - f_i(t,y,u,v)| \le L^*(c_1,c_2)|x-y|, \quad i = 1,\ 2$$

for all $u \in [0, c_1]$, $v \in [0, c_2]$ and all $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$.

The condition **(f)** implies

$$|f_1(t,x,u,v)| \le L(c_1,c_2)u, \quad |f_2(t,x,u,v)| \le L(c_1,c_2)v$$

for all $u \in [0,c_1]$, $v \in [0,c_2]$ and all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

Except where otherwise stated, we always assume that **(f)-(f3)** hold, the kernel function J satisfies **(J1)** and u_0, v_0 satisfy the condition (1.4) throughout this paper. We write $\|\phi,\varphi\| \le M$ means that $\|\phi\| \le M$, $\|\varphi\| \le M$.

Since this paper is very long, and the techniques used in the first part are rather different from those in the second part, it is divided into two separate parts. Part I here is mainly concerned with the existence, uniqueness, regularity and estimates of global solution. Part II ([17]) focuses on the spreading-vanishing dichotomy, criteria of spreading and vanishing, and long time behavior of solution when spreading happens.

Before ending this section we should mention that when this article is finished, we find that Cao et al. ([5]) studied a nonlocal diffusion Lotka-Volterra type competition model that consisting of a native species and an invasive species in a one-dimensional habitat with free boundaries, Zhao et al. ([28]) investigated an epidemic model with nonlocal diffusion and free boundaries.

2. Existence, uniqueness, regularity and estimates of global solution of (1.3)

For convenience, we first introduce some notations. Let $L(u_0)$ and $L(J)$ be the Lipschitz constants of u_0 and J, respectively. Let $k_0, \Theta(\cdot)$ be given in **(f1)**, **(f2)**. Denote

$$k_1 = \max\left\{\|u_0\|_\infty,\, k_0\right\}, \quad k_2 = \max\left\{\|v_0\|_\infty,\, \Theta(k_1)\right\}, \quad L = L(k_1,k_2),$$

$$L^* = L^*(k_1,k_2), \quad k_3 = \max\left\{\frac{1}{h_0},\ \sqrt{\frac{L}{2d_2}},\ \frac{\|v_0'\|_{C([-h_0,h_0])}}{k_2}\right\},$$

$$x(t,y) = \frac{(h(t)-g(t))y + h(t) + g(t)}{2}, \quad y(t,x) = \frac{2x - g(t) - h(t)}{h(t)-g(t)},$$

$$\xi(t) = \frac{4}{(h(t)-g(t))^2}, \quad \zeta(t,y) = \frac{h'(t)+g'(t)}{h(t)-g(t)} + \frac{(h'(t)-g'(t))y}{h(t)-g(t)},$$

$$\Sigma = [-1,1], \quad \Pi_s = [0,s] \times \Sigma, \quad R(t) = \mu k_3 + 2(h_0\rho k_1 + \mu k_3)e^{\rho k_1 t}.$$

For the given $T > 0$, define

$$\mathbb{H}^T = \left\{h \in C^1([0,T]) \ : \ h(0) = h_0,\ 0 < h'(t) \le R(t)\right\},$$

$$\mathbb{G}^T = \left\{g \in C^1([0,T]) \ : -g \in \mathbb{H}^T\right\}.$$

And for $g \in \mathbb{G}^T$, $h \in \mathbb{H}^T$, define

$$D^T_{g,h} = \left\{(t,x) \in \mathbb{R}^2 : \ 0 < t \le T,\ g(t) < x < h(t)\right\},$$

$$\mathbb{X}^T_1 = \mathbb{X}^T_{u_0,g,h} = \left\{\varphi \in C(\overline{D}^T_{g,h}) : \ 0 \le \varphi \le k_1,\ \varphi\big|_{t=0} = u_0,\ \varphi\big|_{x=g(t),h(t)} = 0\right\},$$

$$\mathbb{X}^T_2 = \mathbb{X}^T_{v_0,g,h} = \left\{\varphi \in C(\overline{D}^T_{g,h}) : \ 0 \le \varphi \le k_2,\ \varphi\big|_{t=0} = v_0,\ \varphi\big|_{x=g(t),h(t)} = 0\right\},$$

as well as

$$\mathbb{X}^T_{g,h} := \mathbb{X}^T_1 \times \mathbb{X}^T_2.$$

The following theorem is our main result in this part.

Theorem 2.1. *The problem (1.3) has a unique local solution (u, v, g, h) defined on $[0, T]$ for some $0 < T < \infty$. Moreover, $(g, h) \in \mathbb{G}^T \times \mathbb{H}^T$, $(u, v) \in \mathbb{X}^T_{g,h}$ and*

$$u \in C^{1,1-}(\overline{D}^T_{g,h}), \qquad v \in W^{1,2}_p(D^T_{g,h}), \tag{2.1}$$

$$0 < u \le k_1, \quad 0 < v \le k_2 \quad \text{in} \quad D^T_{g,h}, \tag{2.2}$$

$$0 < -v_x(t, h(t)), \ v_x(t, g(t)) \le k_3, \quad 0 < t \le T, \tag{2.3}$$

where $u \in C^{1,1-}(\overline{D}^T_{g,h})$ means that u is differentiable continuously in $t \in [0, T]$ and is Lipschitz continuous in $x \in [g(t), h(t)]$.

If we further assume that

(f4) *For any given τ, l, c_1, $c_2 > 0$, there exists a constant $\bar{L}(\tau, l, c_1, c_2)$ such that*

$$\|f_2(\cdot, x, u, v)\|_{C^{\frac{\alpha}{2}}([0,\tau])} \le \bar{L}(\tau, l, c_1, c_2) \tag{2.4}$$

for all $x \in [-l, l]$, $u \in [0, c_1]$, $v \in [0, c_2]$.

Then the solution (u, v, g, h) exists globally. Moreover, for any given $\tau > 0$, (2.2) and (2.3) hold with T replaced by τ, and

$$g, \, h \in C^{1+\alpha/2}([0, \tau]), \quad u \in C^{1,1-}(\overline{D}^\tau_{g,h}), \quad v \in C^{1+\alpha/2,\, 2+\alpha}((0, \tau] \times [g(t), h(t)]). \tag{2.5}$$

For the classical competition and prey-predator models

$$\textit{Competition Model}: \quad f_1 = u(a - u - bv), \quad f_2 = v(1 - v - cu), \tag{2.6}$$

$$\textit{Prey-predator Model}: \quad f_1 = u(a - u - bv), \quad f_2 = v(1 - v + cu), \tag{2.7}$$

the conditions **(f)**–**(f4)** hold, where a, b, c are positive constants.

Due to the presence of the nonlocal diffusion and local diffusion, the methods that solve the local diffusion models are not applicable any more and the arguments for the nonlocal system developed in [4,9] are far from sufficient for the present stage, the proofs of Theorem 2.1 are highly non trivial. Our approach to prove Theorem 2.1 is based on the fixed point theorem. Some new ideas and delicate calculations are given in the proof of Theorem 2.1.

The proof of Theorem 2.1 will be divided into several lemmas because it is too long. Throughout this paper we use C, C', C_i and C'_i to represent general constants, which may not be the same in different places.

We first state the following *Maximum Principle* which will be used frequently in our analysis.

Lemma 2.2 *(Maximum Principle [4, Lemma 2.2]). Assume that J satisfies* **(J)** *and d is a positive constant, and $(r, \eta) \in \mathbb{G}^T \times \mathbb{H}^T$. Suppose that $\psi, \psi_t \in C(\overline{D}^T_{\eta,r})$ and fulfill, for some $\varrho \in L^\infty(D^T_{\eta,r})$,*

$$\begin{cases} \psi_t \ge d \displaystyle\int_{\eta(t)}^{r(t)} J(x, y)\psi(t, y)\mathrm{d}y - d\psi + \varrho\psi, & (t, x) \in D^T_{\eta,r}, \\ \psi(t, \eta(t)) \ge 0, \ \psi(t, r(t)) \ge 0, & 0 \le t \le T, \\ \psi(0, x) \ge 0, & |x| \le h_0. \end{cases}$$

Then $\psi \geq 0$ *on* $\overline{D}^T_{\eta,r}$. *Moreover, if* $\psi(0,x) \not\equiv 0$ *in* $[-h_0,h_0]$, *then* $\psi > 0$ *in* $D^T_{\eta,r}$.

Lemma 2.3. *For any* $T > 0$ *and* $(g,h) \in \mathbb{G}^T \times \mathbb{H}^T$, *the problem*

$$\begin{cases} u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J(x,y)u(t,y)\mathrm{d}y - d_1 u + f_1(t,x,u,v), & (t,x) \in D^T_{g,h}, \\ v_t = d_2 v_{xx} + f_2(t,x,u,v), & (t,x) \in D^T_{g,h}, \\ u(t,g(t)) = u(t,h(t)) = v(t,g(t)) = v(t,h(t)) = 0, & 0 \leq t \leq T, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & |x| \leq h_0 \end{cases} \tag{2.8}$$

admits a unique solution $(u_{g,h}, v_{g,h}) \in \mathbb{X}^T_{g,h}$, *and* $(u_{g,h}, v_{g,h})$ *satisfies* (2.2) *and* (2.3). *Moreover,* $v_{g,h} \in W^{1,2}_p(D^T_{g,h})$.

Proof. *Step 1*: For $\tilde{u} \in \mathbb{X}^s_1$ with $0 < s \leq T$, consider the following initial-boundary value problem

$$\begin{cases} v_t = d_2 v_{xx} + f_2(t,x,\tilde{u},v), & (t,x) \in D^s_{g,h}, \\ v(t,g(t)) = v(t,h(t)) = 0, & 0 \leq t \leq s, \\ v(0,x) = v_0(x), & |x| \leq h_0. \end{cases} \tag{2.9}$$

Let $z(t,y) = v(t,x(t,y))$, $\tilde{w}(t,y) = \tilde{u}(t,x(t,y))$. It follows from (2.9) that

$$\begin{cases} z_t = d_2\xi(t)z_{yy} + \zeta(t,y)z_y + f^*_2(t,y,\tilde{w},z), & 0 < t \leq s, \ |y| < 1, \\ z(t,\pm 1) = 0, & 0 \leq t \leq s, \\ z(0,y) = v_0(h_0 y) =: z_0(y), & |y| \leq 1, \end{cases} \tag{2.10}$$

where $f^*_2(t,y,\tilde{w},z) = f_2(t,x(t,y),\tilde{w},z)$. Note that $(g,h) \in \mathbb{G}_{h_0,s} \times \mathbb{H}_{h_0,s}$, we have $\xi \in C([0,s])$, $\zeta \in C(\Pi_s)$ and

$$\|\xi\|_{L^\infty((0,s))} \leq 1/h_0^2, \quad \|\zeta\|_{L^\infty(\Pi_s)} \leq 2R(s)/h_0 \leq 2R(T)/h_0.$$

It is easy to see that $\tilde{w} \in C(\Pi_s)$ and $0 \leq \tilde{w} \leq k_1$. Notice that $z_0(y) \in \mathring{W}^1_2(\Sigma)$. By the upper and lower solutions method and L^2 theory ([13, Ch. III, Theorem 6.1]) we can show that the problem (2.10) has a unique solution $z \in W^{1,2}_2(\Pi_s)$, and $z \in C^{\alpha/2,\alpha}(\Pi_s)$ by the embedding theorem. Moreover, $0 \leq z \leq k_2$ in Π_s by the weak maximum principle. Hence, the problem (2.9) admits a unique solution $v \in \mathbb{X}^s_2$.

Step 2: For $0 < s \leq T$, let v be the unique solution of (2.9) and consider

$$\begin{cases} u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J(x,y)u(t,y)\mathrm{d}y - d_1 u + f_1(t,x,u,v), & (t,x) \in D^s_{g,h}, \\ u(t,g(t)) = u(t,h(t)) = 0, & 0 \leq t \leq s, \\ u(0,x) = u_0(x), & |x| \leq h_0. \end{cases} \tag{2.11}$$

Thanks to [4, Lemma 2.3], this problem admits a unique solution u which satisfies $0 < u \leq k_1$ for $(t,x) \in [0,s] \times (g(t),h(t))$. It is easily seen that $u \in \mathbb{X}^s_1$. Define a mapping $\mathcal{F}_s : \mathbb{X}^s_1 \to \mathbb{X}^s_1$ by

$$\mathcal{F}_s \tilde{u} = u.$$

If $\mathcal{F}_s \tilde{u} = \tilde{u}$, then $(\tilde{u}, v)$ solves (2.8) in $D^s_{g,h}$.

Step 3: We shall prove that $\mathcal{F}_s$ has a fixed point in $\mathbb{X}_1^s$ provided s small enough. Evidently, $\mathbb{X}_1^s$ is a closed bounded subset of $C(\overline{D}_{g,h}^s)$. Let $\tilde{u}_1, \tilde{u}_2 \in \mathbb{X}_1^s$ and $u_i = \mathcal{F}_s \tilde{u}_i$ with $i = 1, 2$. Let v_i be the unique solution of (2.9) with $\tilde{u}_i$. Then $(u_i, v_i) \in \mathbb{X}^s$. Notice that u_i satisfies

$$\begin{cases} u_{i,t} = d_1 \displaystyle\int_{g(t)}^{h(t)} J(x,y) u_i(t,y) \mathrm{d}y - d_1 u_i + f_1(t,x,u_i,v_i), & t_x < t \le s, \ g(t) < x < h(t), \\ u_i(t_x, x) = \tilde{u}_0(x), & g(s) < x < h(s), \end{cases}$$

where

$$\tilde{u}_0(x) = \begin{cases} 0, & |x| > h_0, \\ u_0(x), & |x| \le h_0, \end{cases} \qquad t_x = \begin{cases} t_{x,g} & \text{if } \ x \in [g(s), -h_0), \ x = g(t_{x,g}), \\ 0 & \text{if } \ |x| \le h_0, \\ t_{x,h} & \text{if } \ x \in (h_0, h(s)], \ x = h(t_{x,h}). \end{cases}$$

Let $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, $u = u_1 - u_2$ and $v = v_1 - v_2$, we have

$$\begin{cases} u_t + a(t,x) u = d_1 \displaystyle\int_{g(t)}^{h(t)} J(x,y) u(t,y) \mathrm{d}y + b(t,x) v, & t_x < t \le s, \ g(t) < x < h(t), \\ u(t_x, x) = 0, & g(s) < |x| < h(s), \end{cases} \tag{2.12}$$

where

$$a(t,x) = d_1 - \int_0^1 f_{1,u}(t,x,u_2 + (u_1 - u_2)\tau, v_2) \mathrm{d}\tau,$$

$$b(t,x) = \int_0^1 f_{1,v}(t,x,u_1, v_2 + (v_1 - v_2)\tau) \mathrm{d}\tau.$$

Recall **(f)**, there holds that $\|a, b\|_\infty \le d_1 + L =: L_1$. It follows from (2.12) that, for $x \in (g(t), h(t))$ and $t_x < t \le s$,

$$u(t,x) = e^{-\int_{t_x}^t a(\tau,x)\mathrm{d}\tau} \int_{t_x}^t e^{\int_{t_x}^l a(\tau,x)\mathrm{d}\tau} \left(d_1 \int_{g(l)}^{h(l)} J(x,y) u(l,y) \mathrm{d}y + b(l,x) v(l,x) \right) \mathrm{d}l.$$

Due to $(g(t), h(t)) \subset (g(s), h(s))$ for $t_x < t \le s$, this implies that

$$|u(t,x)| \le e^{2L_1 s} \left(d_1 \|u\|_{C(\overline{D}_{g,h}^s)} s + L_1 \int_{t_x}^t |v(l,x)| \mathrm{d}l \right). \tag{2.13}$$

Note that v satisfies

$$\begin{cases} v_t = d_2 v_{xx} + a_0(t,x) v + b_0(t,x) \tilde{u}, & (t,x) \in D_{g,h}^s, \\ v(t, g(t)) = v(t, h(t)) = 0, & 0 \le t \le s, \\ v(0,x) = 0, & |x| \le h_0, \end{cases}$$

where

$$a_0(t,x)=\int_0^1 f_{2,v}(t,x,\tilde u_1, v_2+(v_1-v_2)\tau)\mathrm{d}\tau,$$

$$b_0(t,x)=\int_0^1 f_{2,u}(t,x,\tilde u_2+(\tilde u_1-\tilde u_2)\tau, v_2)\mathrm{d}\tau.$$

Clearly, $\|a_0,b_0\|_\infty\le L$. Let

$$\tilde w(t,y)=\tilde u(t,x(t,y)),\quad \tilde z(t,y)=v(t,x(t,y)),\quad \tilde a_0(t,y)=a_0(t,x(t,y)),\quad \tilde b_0(t,y)=b_0(t,x(t,y)).$$

It is easy to see that $\tilde z$ satisfies

$$\begin{cases}\tilde z_t=d_2\xi(t)\tilde z_{yy}+\zeta(t,y)\tilde z_y+\tilde a_0\tilde z+\tilde b_0\tilde w, & 0<t\le s,\ |y|<1,\\ \tilde z(t,\pm1)=0, & 0\le t\le s,\\ \tilde z(0,y)=0, & |y|\le 1.\end{cases}$$

Thanks to the parabolic L^p theory, one can obtain that, with $p>3$ and $\alpha=1-3/p$,

$$\|\tilde z\|_{W^{1,2}_p(\Pi_s)}\le C\|\tilde w\|_{C(\Pi_s)}=C\|\tilde u\|_{C(\overline{D}^s_{g,h})}.$$

Using the arguments in the proof of [19, Theorem 1.1] we have

$$[\tilde z,\ \tilde z_y]_{C^{\alpha/2,\alpha}(\Pi_s)}\le C'\|\tilde z\|_{W^{1,2}_p(\Pi_s)}\le C'C\|\tilde u\|_{C(\overline{D}^s_{g,h})},\tag{2.14}$$

where C' is independent of s^{-1}, and $[\,\cdot\,]_{C^{\frac{\alpha}{2},\alpha}(\Pi_s)}$ is the Hölder semi-norm. It follows from $\tilde z(0,y)=0$ that $\|\tilde z\|_{L^\infty(\Pi_s)}\le[\tilde z]_{C^{\alpha/2,\alpha}(\Pi_s)}s^{\alpha/2}$. Thus we have, for $t_x\le t\le s\le 1$,

$$\int_{t_x}^{t}|v(l,x)|\mathrm{d}l\le\int_0^s\|\tilde z\|_{L^\infty(\Pi_s)}\mathrm{d}l\le s[\tilde z]_{C^{\alpha/2,\alpha}(\Pi_s)}\le sC'C\|\tilde u\|_{C(\overline{D}^s_{g,h})}.$$

Inserting this into (2.13) gives

$$|u(t,x)|\le e^{2L_1s}\big(d_1s\|u\|_{C(\overline{D}^s_{g,h})}+L_1C'Cs\|\tilde u\|_{C(\overline{D}^s_{g,h})}\big).$$

Taking s small enough such that

$$d_1se^{2L_1s}\le 1/2,\quad L_1C'Cse^{2L_1s}\le 1/4.$$

Then $\|u\|_{C(\overline{D}^s_{g,h})}\le\frac12\|\tilde u\|_{C(\overline{D}^s_{g,h})}$. The contraction mapping theorem shows that $\mathcal{F}_s$ has a unique fixed point u in $\mathbb{X}^s_1$. Let z be the unique solution of (2.10) with $\tilde w(t,y)$ replaced by $w(t,y)=u(t,x(t,y))$.

Step 4: The local existence and uniqueness of solution (u,v) of (2.8). From the above analysis, the function $v(t,x)=z(t,y(t,x))$ solves (2.9) with $\tilde u$ replaced by u and $v\in\mathbb{X}^s_2$. Hence, $(u,v)\in\mathbb{X}^s_{g,h}$ solves (2.8) with T replaced by s. Moreover, from the above arguments we know that any solution (U,V) of (2.8) in $(0,s]$ satisfies $(U,V)\in\mathbb{X}^s_{g,h}$. Hence, (u,v) is the unique solution of (2.8) in $(0,s]$.

Step 5: We finally show that the unique solution (u, v) of (2.8) can be extended to $D^T_{g,h}$. It is clear that $u(s, x) \in C([g(s), h(s)])$, $0 \le u(s, x) \le k_1$, $0 \le v(s, x) \le k_2$ and

$$u(s, g(s)) = u(s, h(s)) = v(s, g(s)) = v(s, g(s)) = 0.$$

Same as the above, let $z(t, y) = v(t, x(t, y))$, $w(t, y) = u(t, x(t, y))$. Since $z_0(y) = v_0(h_0 y) \in W^2_p(\Sigma)$ and $p > 3$, where $\Sigma = [-1, 1]$, applying the L^p theory to (2.10) and the uniqueness of weak solution, we have $z \in W^{1,2}_p(\Pi_s) \hookrightarrow C^{(1+\alpha)/2,1+\alpha}(\Pi_s)$. And so $z(s, \cdot) \in \overset{\circ}{W}{}^1_2(\Sigma)$. Note that in the above Steps 1, 2, 3 we only used $u_0 \in C([-h_0, h_0])$, $z_0(y) \in \overset{\circ}{W}{}^1_2(\Sigma)$ without using $u_0 \in C^{1-}([-h_0, h_0])$ and $z_0 \in W^2_p(\Sigma)$. We can apply the above Steps 1, 2, 3 to (2.8) but with initial time $t = 0$ replaced by $t = s$ to get an $\bar{s} > s$ and a unique $(\hat{u}, \hat{z})$ which satisfies

$$\begin{cases} \hat{u}_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J(x, y)\hat{u}(t, y)\mathrm{d}y - d_1\hat{u} + f_1(t, x, \hat{u}, \hat{v}), & s < t \le \bar{s},\ g(t) < x < h(t), \\ \hat{u}(t, g(t)) = \hat{u}(t, h(t)) = 0, & s \le t \le \bar{s}, \\ \hat{u}(s, x) = u(s, x), & g(s) \le x \le h(s), \end{cases}$$

and

$$\begin{cases} \hat{z}_t = d_2\xi(t)\hat{z}_{yy} + \zeta(t, y)\hat{z}_y + f^*_2(t, y, \hat{w}, \hat{z}), & s < t \le \bar{s},\ |y| < 1, \\ \hat{z}(t, \pm 1) = 0, & s \le t \le \bar{s}, \\ \hat{z}(s, y) = v(s, x(s, y)), & |y| \le 1 \end{cases}$$

as well as $\hat{u}, \hat{v} \in C([s, \bar{s}] \times [g(t), h(t)])$, where $\hat{v}(t, x(t, y)) = \hat{z}(t, y)$, $\hat{w}(t, y) = \hat{u}(t, x(t, y))$. Set $u(t, x) = \hat{u}(t, x), z(t, y) = \hat{z}(t, y)$ for $t \in [s, \bar{s}], g(t) \le x \le h(t)$, $|y| \le 1$. Clearly, $u \in C(\overline{D}{}^{\bar{s}}_{g,h})$ solves (2.11) with (s, v) replaced by $(\bar{s}, v)$, where $v(t, x) = z(t, y(t, x))$; z is a weak solution of (2.10) with $(s, \tilde{w})$ replaced by $(\bar{s}, w)$, where $w(t, y) = u(t, x(t, y))$. Therefore $(u, v) \in \mathbb{X}^{\bar{s}}_{g,h}$ and solves (2.8) in $(0, \bar{s}]$. Applying the L^p theory to (2.10) with $(s, \tilde{w})$ replaced by $(\bar{s}, w)$ and the uniqueness of weak solution, we have $z \in W^{1,2}_p(\Pi_{\bar{s}}) \hookrightarrow C^{(1+\alpha)/2,1+\alpha}(\Pi_{\bar{s}})$. Hence, $z(\bar{s}, \cdot) \in \overset{\circ}{W}{}^1_2(\Sigma)$. From the arguments in the above Steps 1, 2, 3 we see that $\bar{s}$ depends only on d_i, k_i, h_0, $i = 1, 2$. By repeating this process finitely many times, the solution (u, v) will be uniquely extended to $D^T_{g,h}$ and $(u, v) \in \mathbb{X}^T_{g,h}$.

Thanks to Lemma 2.2, we have $u > 0$ in $D^T_{g,h}$. And, it follows from the parabolic maximum principle for the strong solution that $v > 0$ in $D^T_{g,h}$. Hence, we get (2.2). Since $v > 0$ in $D^T_{g,h}$ and $v(t, h(t)) = v(t, g(t)) = 0$, we have $v_x(t, h(t)) < 0$ and $v_x(t, g(t)) > 0$ (see the proof of [20, Theorem 1.1, pp. 2597]). Recall $0 \le v \le k_2$ and $f_2(t, x, u, v) \le Lv$. By using the similar arguments in the proof of [25, Lemma 2.1] (cf. [19, Lemma 2.1]), one can easily show that

$$0 < -v_x(t, h(t)), v_x(t, g(t)) \le \max\left\{\frac{1}{h_0},\ \sqrt{\frac{L}{2d_2}},\ \frac{\|v_0'\|_{C([-h_0,h_0])}}{k_2}\right\} = k_3.$$

This implies (2.3). In view of (1.4) and the parabolic L^p theory we have $v \in W^{1,2}_p(D^T_{g,h})$ for all $p > 1$. The proof is complete. □

According to Lemma 2.3, for any $T > 0$ and $(g, h) \in \mathbb{G}^T \times \mathbb{H}^T$, there exists a unique $(u, v) = (u_{g,h}, v_{g,h}) \in \mathbb{X}^T_{g,h}$ that solves (2.8), and (2.2) holds. For $0 < t \le T$, define the mapping

$$\mathcal{G}(g, h) = (\tilde{g}, \tilde{h})$$

by

$$\tilde{h}(t) = h_0 - \mu \int_0^t v_x(\tau, h(\tau))\mathrm{d}\tau + \rho \int_0^t \int_{g(\tau)}^{h(\tau)} \int_{h(\tau)}^{\infty} J(x,y)u(\tau,x)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau,$$

$$\tilde{g}(t) = -h_0 - \mu \int_0^t v_x(\tau, g(\tau))\mathrm{d}\tau - \rho \int_0^t \int_{g(\tau)}^{h(\tau)} \int_{-\infty}^{g(\tau)} J(x,y)u(\tau,x)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau.$$

We shall show that $\mathcal{G}$ maps a suitable closed subset Γ_T of $\mathbb{G}^T \times \mathbb{H}^T$ into itself and is a contraction mapping provided T sufficiently small.

Lemma 2.4. *There exists a closed subset $\Gamma_T \subset \mathbb{G}^T \times \mathbb{H}^T$ such that $\mathcal{G}(\Gamma_T) \subset \Gamma_T$.*

Proof. Let $(g,h) \in \mathbb{G}^T \times \mathbb{H}^T$. Then $\tilde{g}, \tilde{h} \in C^1([0,T])$ and for $0 < t \le T$,

$$\tilde{h}'(t) = -\mu v_x(t, h(t)) + \rho \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x,$$

$$\tilde{g}'(t) = -\mu v_x(t, g(t)) - \rho \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x.$$

It follows that

$$[\tilde{h}(t) - \tilde{g}(t)]' = -\mu\big[v_x(t,h(t)) - v_x(t,g(t))\big] + \rho \int_{g(t)}^{h(t)} \left[\int_{h(t)}^{\infty} + \int_{-\infty}^{g(t)}\right] J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x. \tag{2.15}$$

Taking

$$0 < \varepsilon_0 < \min\left\{\bar{\varepsilon},\ \frac{8\mu k_3}{\rho k_1}\right\}, \quad M = 2h_0 + \frac{\varepsilon_0}{4}, \quad 0 < T_0 \le \frac{\varepsilon_0}{4\big(2\mu k_3 + \rho k_1 M\big)}$$

such that $h(T_0) - g(T_0) \le M$. Let $\bar{R} = \mu k_3 + \rho k_1 M$. Then, due to (2.2), (2.3) and (2.15), we have

$$[\tilde{h}(t) - \tilde{g}(t)]' \le 2\mu k_3 + \rho k_1[h(T_0) - g(T_0)] \le 2\mu k_3 + \rho k_1 M.$$

This implies

$$\tilde{h}(t) - \tilde{g}(t) \le 2h_0 + t\big(2\mu k_3 + \rho k_1 M\big) \le M, \quad t \in [0, T_0]. \tag{2.16}$$

Similarly, we can show that

$$\tilde{h}'(t) \le \bar{R}, \quad -\tilde{g}'(t) \le \bar{R}, \quad t \in [0, T_0]. \tag{2.17}$$

It is easily verified that

$$h(t) \in [h_0, h_0 + \varepsilon_0/4], \quad g(t) \in [-h_0 - \varepsilon_0/4, -h_0], \quad t \in [0, T_0]. \tag{2.18}$$

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Since (u,v) solves (2.8), due to **(f)-(f2)** and (2.2) we have

$$\begin{cases} u_t \geq -d_1 u - Lu, & (t,x) \in D_{g,h}^{T_0}, \\ u(t,g(t)) = u(t,h(t)) = 0, & 0 \leq t \leq T_0, \\ u(0,x) = u_0(x), & |x| \leq h_0, \end{cases}$$

which implies that

$$u(t,x) \geq e^{-(d_1+L)t} u_0(x) \geq e^{-(d_1+L)T_0} u_0(x), \quad t \in (0,T_0], \ |x| \leq h_0.$$

This combined with (1.5) and (2.18) allows us to derive

$$\begin{aligned} \rho \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x,y) u(t,x) \mathrm{d}y \mathrm{d}x &\geq \rho \int_{h(t)-\frac{\varepsilon_0}{2}}^{h(t)} \int_{h(t)}^{h(t)+\frac{\varepsilon_0}{2}} J(x,y) u(t,x) \mathrm{d}y \mathrm{d}x \\ &\geq \rho e^{-(d_1+L)T_0} \int_{h_0-\frac{\varepsilon_0}{4}}^{h_0} \int_{h_0+\frac{\varepsilon_0}{4}}^{h_0+\frac{\varepsilon_0}{2}} J(x,y) u_0(x) \mathrm{d}y \mathrm{d}x \\ &\geq \frac{\varepsilon_0}{4} \delta_0 \rho e^{-(d_1+L)T_0} \int_{h_0-\frac{\varepsilon_0}{4}}^{h_0} u_0(x) \mathrm{d}x \\ &=: \rho c_0, \quad t \in (0,T_0]. \end{aligned}$$

Similarly,

$$-\rho \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x,y) u(t,x) \mathrm{d}y \mathrm{d}x \leq -\frac{\varepsilon_0}{4} \delta_0 \rho e^{-(d_1+L)T_0} \int_{-h_0}^{-h_0+\frac{\varepsilon_0}{4}} u_0(x) \mathrm{d}x =: -\rho c_0^*.$$

Thus, by (2.3),

$$\tilde{h}'(t) \geq \rho c_0, \quad \tilde{g}'(t) \leq -\rho c_0^*, \quad t \in [0,T_0]. \tag{2.19}$$

Moreover, by the definitions of $\bar{R}, R(t)$ and the choice of ε_0, we know that

$$\bar{R} \leq \mu k_3 + 2(h_0 \rho k_1 + \mu k_3) \leq \mu k_3 + 2(h_0 \rho k_1 + \mu k_3) e^{\rho k_1 t} = R(t)$$

for all $t \in [0,T_0]$. Noticing that

$$\rho c_0 \leq \rho e^{-(d_1+L)T_0} \int_{h_0-\frac{\varepsilon_0}{4}}^{h_0} \int_{h_0+\frac{\varepsilon_0}{4}}^{h_0+\frac{\varepsilon_0}{2}} J(x,y) u_0(x) \mathrm{d}y \mathrm{d}x \leq \rho h_0 k_1,$$

one has

$$\rho c_0, \ \rho c_0^* \leq \rho h_0 k_1 < \bar{R} \leq R(t), \quad t \in [0,T_0].$$

For $0 < T \leq T_0$, we define

$$\Gamma_T = \{(g,h) \in \mathbb{G}^T \times \mathbb{H}^T : \rho c_0 \le h'(t) \le \bar{R},\ -\bar{R} \le g'(t) \le -\rho c_0^*,\ h(T) - g(T) \le M\}.$$

It follows from the above analysis that $\mathcal{G}(\Gamma_T) \subset \Gamma_T$. □

In the following we show that $\mathcal{G}$ is a contraction mapping on Γ_T when T is small.

Lemma 2.5. *The mapping $\mathcal{G}$ is contraction on Γ_T when T is small.*

Proof. For $(g_i, h_i) \in \Gamma_T$ with $0 < T \le \min\{T_0, 1\}$, let

$$\Omega_T = D^T_{g_1,h_1} \cup D^T_{g_2,h_2},\quad u_i = u_{g_i,h_i},\quad v_i = v_{g_i,h_i},\quad \mathcal{G}(g_i,h_i) = (\tilde{g}_i, \tilde{h}_i),\quad i = 1,2,$$
$$u = u_1 - u_2,\quad v = v_1 - v_2,\ g = g_1 - g_2,\quad h = h_1 - h_2,\quad \tilde{g} = \tilde{g}_1 - \tilde{g}_2,\quad \tilde{h} = \tilde{h}_1 - \tilde{h}_2.$$

Note that $(u_i, v_i) \in \mathbb{X}^T_{g_i,h_i}$. By Lemma 2.3, $v_i \in W^{1,2}_p(D^T_{g_i,h_i})$ with $p > 3$. Make the zero extension of u_i, v_i in $([0,T] \times \mathbb{R}) \setminus D^T_{g_i,h_i}$ for $i = 1,2$. It is easy to see that

$$\begin{aligned} |\tilde{h}'(t)| \le\ & \mu|v_{1,x}(t,h_1(t)) - v_{2,x}(t,h_2(t))| \\ & + \rho\left|\int_{g_1(t)}^{h_1(t)}\int_{h_1(t)}^{\infty} J(x,y)u_1(t,x)\mathrm{d}y\mathrm{d}x - \int_{g_2(t)}^{h_2(t)}\int_{h_2(t)}^{\infty} J(x,y)u_2(t,x)\mathrm{d}y\mathrm{d}x\right| \\ =:\ & \mu\phi_1(t) + \rho\phi_2(t). \end{aligned} \tag{2.20}$$

Step 1: The estimation of $\phi_1(t)$. It follows from (2.8) that, for $i = 1,2$,

$$\begin{cases} v_{i,t} = d_2 v_{i,xx} + f_2(t,x,u_i,v_i), & (t,x) \in D^T_{g_i,h_i}, \\ v_i(t,g_i(t)) = v_i(t,h_i(t)) = 0, & 0 \le t \le T, \\ v_i(0,x) = 0, & |x| \le h_0. \end{cases} \tag{2.21}$$

For $i = 1,2$, let

$$x_i(t,y) = \frac{1}{2}[(h_i(t) - g_i(t))y + h_i(t) + g_i(t)],$$

and define

$$w_i(t,y) = u_i(t,x_i(t,y)),\quad z_i(t,y) = v_i(t,x_i(t,y)),\quad f_2^i(t,y,u,v) = f_2(t,x_i(t,y),u,v)$$

for $t \in [0,T]$, $y \in \Sigma$ and $u, v \in \mathbb{R}^+$. Then (2.21) turns into

$$\begin{cases} z_{i,t} = d_2\xi_i(t)z_{i,yy} + \zeta_i(t,y)z_{i,y} + f_2^i(t,y,w_i,z_i), & 0 < t \le T,\ |y| < 1, \\ z_i(t,-1) = z_i(t,1) = 0, & 0 \le t \le T, \\ z_i(0,y) = v_0(h_0 y) =: z_0(y), & |y| \le 1, \end{cases} \tag{2.22}$$

where $\xi_i(t)$ and $\zeta_i(t,y)$ are the same as $\xi(t)$ and $\zeta(t,y)$ in there g,h are replaced by g_i,h_i. Making use of $(g_i,h_i) \in \Gamma_T$ and (2.2), we have

$$\|\xi_i\|_{L^\infty((0,T))} \le 1/h_0^2,\quad \|\zeta_i\|_{L^\infty(\Pi_T)} \le 2\bar{R}/h_0,\quad \|f_2^i\|_{L^\infty(\Pi_T)} \le C_0 \tag{2.23}$$

for $i=1,2$, where C_0 depends only on k_1,k_2. By the parabolic L^p theory, $z_i\in W^{1,2}_p(\Pi_T)$ and

$$\|z_i\|_{W^{1,2}_p(\Pi_T)}\le C. \tag{2.24}$$

Same as (2.14) we have $[z_i,\, z_{i,y}]_{C^{\alpha/2,\alpha}(\Pi_T)}\le C_1$, where C_1 is independent of T^{-1}. This implies

$$\|z_{i,y}\|_{C(\Pi_T)}\le\|z_0'(y)\|_{C(\Sigma)}+C_1T^{\alpha/2}\le\|z_0'(y)\|_{C(\Sigma)}+C_1.$$

Extend $z_i(t,y)=0$ for $|y|\ge1$. Then $z_{i,y}\in L^\infty([0,T]\times\mathbb{R})$ and

$$\|z_{i,y}\|_{L^\infty([0,T]\times\mathbb{R})}\le\|z_0'(y)\|_{C(\Sigma)}+C_1:=C_2. \tag{2.25}$$

Let $z=z_1-z_2$, $w=w_1-w_2$, $\xi=\xi_1-\xi_2$ and $\zeta=\zeta_1-\zeta_2$. It follows from (2.22) that

$$\begin{cases} z_t-d_2\xi_1(t)z_{yy}-\zeta_1(t,y)z_y-a(t,y)z & \\ \qquad =d_2\xi(t)z_{2,yy}+\zeta(t,y)z_{2,y}+b(t,y)+c(t,y)w, & 0<t\le T,\ |y|<1,\\ z(t,\pm1)=0, & 0\le t\le T,\\ z(0,y)=0, & |y|\le1,\end{cases} \tag{2.26}$$

where

$$a(t,y)=\int_0^1 f^1_{2,v}(t,y,w_1,z_2+(z_1-z_2)\tau)\mathrm{d}\tau,$$

$$b(t,y)=f^1_2(t,y,w_1,z_2)-f^2_2(t,y,w_1,z_2),$$

$$c(t,y)=\int_0^1 f^2_{2,u}(t,y,w_2+(w_1-w_2)\tau,z_2)\mathrm{d}\tau.$$

Note that $(g_i,h_i)\in\Gamma_T$. It follows that

$$\|\xi\|_{L^\infty((0,T))}\le\frac{A}{h_0^4}\|g,\,h\|_{C([0,T])},\quad \|\zeta\|_{L^\infty(\Pi_T)}\le\frac{\bar R+A}{h_0^2}\|g,\,h\|_{C^1([0,T])}$$

with $A=h_0+\varepsilon_0/4$, and

$$\|a,\,c\|_{L^\infty(\Pi_T)}\le L,\quad \|b\|_{L^\infty(\Pi_T)}\le L^*\|g,\,h\|_{C([0,T])}.$$

Recall (2.23), (2.24), applying the parabolic L^p theory to (2.26), one can obtain

$$\|z\|_{W^{1,2}_p(\Pi_T)}\le C_3\big(\|g,\,h\|_{C^1([0,T])}+\|w\|_{C(\Pi_T)}\big),$$

where C_3 depends on $h_0,\bar R,k_1,k_2,k_3,\varepsilon_0$. Same as (2.14), one has

$$[z]_{C^{\alpha/2,\alpha}(\Pi_T)}+[z_y]_{C^{\alpha/2,\alpha}(\Pi_T)}\le C_4\big(\|g,\,h\|_{C^1([0,T])}+\|w\|_{C(\Pi_T)}\big), \tag{2.27}$$

where $C_4>0$ is independent of T^{-1}. We claim that, for T small enough,

$$\|w\|_{C(\Pi_T)}\le C\big(\|u\|_{C(\overline{\Omega}_T)}+\|g,h\|_{C([0,T])}\big). \tag{2.28}$$

Because the proof of (2.28) is very long, it will be treated as a separate lemma (Lemma 2.6). It follows from (2.27) and (2.28) that

$$[z]_{C^{\alpha/2,\alpha}(\Pi_T)} + [z_y]_{C^{\alpha/2,\alpha}(\Pi_T)} \le C_5\big(\|g,\, h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big), \tag{2.29}$$

noticing $z_y(0,1) = 0$. One has, by (2.29),

$$|z_y(t,1)|_{C([0,T])} \le C_5 T^{\alpha/2}\big(\|g,\, h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big). \tag{2.30}$$

As $h(0) = g(0) = 0$, it is easy to see that

$$|h(t)| \le t\|h'\|_{C([0,T])} \le t\|h\|_{C^1([0,T])}, \quad |g(t)| \le t\|g\|_{C^1([0,T])}. \tag{2.31}$$

As $v_{i,x}(t,h_i(t)) = \frac{2z_{i,y}(t,1)}{h_i(t)-g_i(t)}$, $i = 1,2$, it follows from (2.3) that $|z_{2,y}(t,1)| \le k_3M/2 := B$. Making use of (2.30) and (2.31) we have

$$\begin{aligned}
\phi_1(t) &= |v_{1,x}(t,h_1(t)) - v_{2,x}(t,h_2(t))| \\
&= \left|\frac{2[z_{1,y}(t,1) - z_{2,y}(t,1)]}{h_1(t)-g_1(t)} + 2z_{2,y}(t,1)\frac{g(t)-h(t)}{[h_1(t)-g_1(t)][h_2(t)-g_2(t)]}\right| \\
&\le \frac{1}{h_0}|z_y(t,1)| + 2|z_{2,y}(t,1)|\frac{|h(t)|+|g(t)|}{4h_0^2} \\
&\le \frac{1}{h_0}|z_y(t,1)| + 2|z_{2,y}(t,1)|\frac{t\|h\|_{C^1([0,T])} + t\|g\|_{C^1([0,T])}}{4h_0^2} \\
&\le \frac{1}{h_0}C_5T^{\alpha/2}\big(\|g,\, h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big) + \frac{B}{2h_0^2}T\|g,\, h\|_{C^1([0,T])} \\
&\le C_6T^{\alpha/2}\big(\|g,\, h\|_{C^1([0,T])} + \|u\|_{C(\bar{\Omega}_T)}\big).
\end{aligned} \tag{2.32}$$

Step 2: The estimation of $\phi_2(t)$. Inspired by the arguments in [4,9], using (2.31) we have

$$\begin{aligned}
\phi_2(t) &= \left|\int_{g_1(t)}^{h_1(t)}\int_{h_1(t)}^{\infty} J(x,y)u_1(t,x)\mathrm{d}y\mathrm{d}x - \int_{g_2(t)}^{h_2(t)}\int_{h_2(t)}^{\infty} J(x,y)u_2(t,x)\mathrm{d}y\mathrm{d}x\right| \\
&\le \int_{g_1(t)}^{h_1(t)}\int_{h_1(t)}^{\infty} J(x,y)|u(t,x)|\mathrm{d}y\mathrm{d}x \\
&\quad + \left|\left(\int_{g_1(t)}^{g_2(t)}\int_{h_1(t)}^{\infty} + \int_{h_2(t)}^{h_1(t)}\int_{h_1(t)}^{\infty} + \int_{g_2(t)}^{h_2(t)}\int_{h_1(t)}^{h_2(t)}\right) J(x,y)u_2(t,x)\mathrm{d}y\mathrm{d}x\right| \\
&\le 3h_0\|u\|_{C(\bar{\Omega}_T)} + k_1\|g\|_{C([0,T])} + 2k_1\|h\|_{C([0,T])} \\
&\le C_7\big(\|u\|_{C(\bar{\Omega}_T)} + T\|g,\, h\|_{C^1([0,T])}\big).
\end{aligned} \tag{2.33}$$

Step 3: The estimation of $\|u\|_{C(\bar{\Omega}_T)}$. Fixed $(s,x) \in \Omega_T$.

Case 1: $x \in (g_1(s), h_1(s)) \setminus (g_2(s), h_2(s))$. In this case, either $g_1(s) < x \le g_2(s)$ or $h_2(s) \le x < h_1(s)$ and $u_2(s,x) = v_2(s,x) = 0$. For $h_0 < h_2(s) \le x < h_1(s)$, there is $0 < s_1 < s$ such that $x = h_1(s_1)$. Clearly,

$h_2(t) \le h_2(s) \le x = h_1(s_1) < h_1(s)$ and $g_1(t) < h_1(s_1) = x \le h_1(t)$ for $t \in [s_1, s]$. Hence, $u_2(t,x) = 0$ for $t \in [s_1, s]$ and $u_1(s_1, x) = 0$. Integrating the equation of u_1 from s_1 to s gives

$$\begin{aligned}
|u(s,x)| = u_1(s,x) &= \int_{s_1}^{s} \left(d_1 \int_{g_1(t)}^{h_1(t)} J(x,y) u_1(t,y)\mathrm{d}y - d_1 u_1 + f_1(t,x,u_1,v_1) \right) \mathrm{d}t \\
&\le (s-s_1)(d_1+L)k_1 \\
&\le (\rho c_0)^{-1}[h_1(s) - h_1(s_1)](d_1+L)k_1 \\
&\le (\rho c_0)^{-1}(d_1+L)k_1[h_1(s) - h_2(s)] \\
&\le C_8 \|h_1 - h_2\|_{C([0,T])}.
\end{aligned}$$

When $g_1(s) < x \le g_2(s)$, by using the similar arguments, it is easy to derive that $|u(s,x)| = u_1(s,x) \le C_8' \|g\|_{C([0,s])}$. Therefore, $|u(s,x)| \le C_9 \|g,\, h\|_{C([0,s])}$ with $C_9 = \max\{C_8, C_8'\}$. This combined with (2.31) allows us to derive

$$|u(s,x)| \le C_9 T \|g,\, h\|_{C^1([0,s])}. \tag{2.34}$$

Case 2: $x \in (g_2(s), h_2(s)) \setminus (g_1(s), h_1(s))$. Parallel to the case 1 we have (2.34).

Case 3: $x \in (g_1(s), h_1(s)) \cap (g_2(s), h_2(s))$. If $x \in (g_1(t), h_1(t)) \cap (g_2(t), h_2(t))$ for all $0 < t < s$, then

$$\begin{aligned}
u_t(t,x) &= u_{1t}(t,x) - u_{2t}(t,x) \\
&= d_1 \int_{g_1(t)}^{h_1(t)} J(x,y) u(t,y)\mathrm{d}y + d_1 \left(\int_{g_1(t)}^{g_2(t)} + \int_{h_2(t)}^{h_1(t)} \right) J(x,y) u_2(t,y)\mathrm{d}y \\
&\quad -d_1 u(t,x) + f_1(t,x,u_1,v_1) - f_1(t,x,u_2,v_2).
\end{aligned} \tag{2.35}$$

Notice that

$$|f_1(t,x,u_1,v_1) - f_1(t,x,u_2,v_2)| \le L(|u| + |v|),$$

and $u(0,x) = u_1(0,x) - u_2(0,x) = 0$. Integrating (2.35) from 0 to s yields

$$\begin{aligned}
|u(s,x)| &\le T\big((2d_1+L)\|u\|_{C(\bar\Omega_T)} + d_1 k_1 \|J\|_\infty \|g,\, h\|_{C([0,s])}\big) + L \int_0^s |v(t,x)|\mathrm{d}t \\
&\le TC_{10}\big(\|u\|_{C(\bar\Omega_T)} + \|g,\, h\|_{C^1([0,T])}\big) + L \int_0^s |v(t,x)|\mathrm{d}t.
\end{aligned} \tag{2.36}$$

If there is $0 < t < s$ such that $x \notin (g_1(t), h_1(t)) \cap (g_2(t), h_2(t))$, then we can choose the largest $t_0 \in (0,t)$ such that

$$x \in (g_1(t), h_1(t)) \cap (g_2(t), h_2(t)), \quad \forall\, t_0 < t \le s, \tag{2.37}$$

and

$$x \in (g_1(t_0), h_1(t_0)) \setminus (g_2(t_0), h_2(t_0)), \quad \text{or} \quad x \in (g_2(t_0), h_2(t_0)) \setminus (g_1(t_0), h_1(t_0)).$$

It follows from the conclusions of Case 1 and Case 2 that $|u(t_0,x)| \le C_9\|g,\, h\|_{C([0,s])}$. Thus,

$$|u(t_0,x)| \le C_9 s\|g,\, h\|_{C^1([0,s])} \le C_9 T\|g,\, h\|_{C^1([0,T])}$$

by (2.31). Note that (2.35) holds for any $t_0 < t \le s$ due to (2.37). Integrating (2.35) from t_0 to s we have

$$\begin{aligned}|u(s,x)| &\le |u(t_0,x)| + T\big((2d_1+L)\|u\|_{C(\bar\Omega_T)} + d_1k_1\|J\|_\infty\|g,\, h\|_{C([0,s])}\big) + L\int_{t_0}^{s}|v(t,x)|\mathrm{d}t \\ &\le C_{11}T\big(\|u\|_{C(\bar\Omega_T)} + \|g,\, h\|_{C^1([0,T])}\big) + L\int_{t_0}^{s}|v(t,x)|\mathrm{d}t. \qquad (2.38)\end{aligned}$$

Now we estimate $\int_{t_0}^{s}|v(t,x)|\mathrm{d}t$ and $\int_{0}^{s}|v(t,x)|\mathrm{d}t$. Let

$$y_i = y_i(t,x) = \frac{2x - h_i(t) - g_i(t)}{h_i(t) - g_i(t)}, \quad i = 1, 2.$$

Then

$$x = \frac{(h_i(t) - g_i(t))y_i + h_i(t) + g_i(t)}{2},$$

and due to (2.37) we have $y_i(t,x) \in \Sigma$. Moreover,

$$\|y_1(\cdot,x) - y_2(\cdot,x)\|_{C([t_0,s])} \le \frac{h_0+\varepsilon_0/4}{h_0^2}\|g,\, h\|_{C([0,T])} = \frac{A}{h_0^2}\|g,\, h\|_{C([0,T])}. \qquad (2.39)$$

Clearly, $z_i(t,y_i) = v_i(t,x)$ for $t_0 < t \le s$. Note that $z(0,y) = z_1(0,y) - z_2(0,y) = 0$, we have that, for any $(t,y) \in \Pi_T$,

$$|z(t,y)| = |z(t,y) - z(0,y)| \le t^{\alpha/2}[z]_{C^{\alpha/2,\alpha}(\Pi_T)}.$$

And so $\|z\|_{C(\Pi_T)} \le T^{\alpha/2}[z]_{C^{\alpha/2,\alpha}(\Pi_T)}$. Thanks to (2.25), (2.29) and (2.39), it educes that

$$\begin{aligned}\int_{t_0}^{s}|v(t,x)|\mathrm{d}t &= \int_{t_0}^{s}|z_1(t,y_1) - z_2(t,y_2)|\mathrm{d}t \\ &\le \int_{t_0}^{s}|z_1(t,y_1) - z_2(t,y_1)|\mathrm{d}t + \int_{t_0}^{s}|z_2(t,y_1) - z_2(t,y_2)|\mathrm{d}t \\ &\le T\|z\|_{C(\Pi_T)} + \int_{t_0}^{s}|y_1-y_2|\|z_{2,y}\|_{L^\infty([0,T]\times\mathbb{R})}\mathrm{d}t \\ &\le T\|z\|_{C(\Pi_T)} + T\|y_1-y_2\|_{C([t_0,s])}\|z_{2,y}\|_{L^\infty([0,T]\times\mathbb{R})} \\ &\le C_5T^{1+\alpha/2}\big(\|g,\, h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big) + \frac{AC_2}{h_0^2}T\|g,\, h\|_{C([0,T])} \\ &\le C_{12}T\big(\|g,\, h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big). \qquad (2.40)\end{aligned}$$

Similarly, one can find $C_{13} > 0$ such that

$$\int_0^s |v(t,x)|\mathrm{d}t \le C_{13}T\big(\|g,\,h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big). \tag{2.41}$$

Substituting the estimations (2.40) and (2.41) into (2.38) and (2.36), respectively, we have

$$|u(s,x)| \le C_{14}T\big(\|g,\,h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big). \tag{2.42}$$

The estimates (2.34) and (2.42) show that, for any case, the following holds:

$$|u(s,x)| \le C'_{14}T\big(\|g,\,h\|_{C^1([0,T])} + \|u\|_{C(\overline{\Omega}_T)}\big).$$

The arbitrariness of $(s,t) \in \Omega_T$ implies

$$\|u\|_{C(\overline{\Omega}_T)} \le 2C'_{14}T\|g,\,h\|_{C^1([0,T])} \tag{2.43}$$

provided $C'_{14}T \le 1/2$.

Step 4: Inserting (2.43) into (2.32), (2.33) we get

$$\mu\phi_1(t) + \rho\phi_2(t) \le C_{15}T^{\alpha/2}\|g,\,h\|_{C^1([0,T])}, \quad \forall\ 0 < t \le T.$$

This combined with (2.20) implies

$$|\tilde{h}'(t)| \le C_{15}T^{\alpha/2}\|g,\,h\|_{C^1([0,T])}, \quad \forall\ 0 < t \le T.$$

Similarly,

$$|\tilde{g}'(t)| \le C_{16}T^{\alpha/2}\|g,\,h\|_{C^1([0,T])}, \quad \forall\ 0 < t \le T.$$

Moreover, as $\tilde{g}(0) = \tilde{h}(0) = 0$, it is easy to deduce that

$$\|\tilde{g}(t),\,\tilde{h}(t)\|_{C^1([0,T])} \le 2(C_{15} + C_{16})T^{\alpha/2}\|g,\,h\|_{C^1([0,T])} \le \frac{1}{2}\|g,\,h\|_{C^1([0,T])}$$

when T is small. Hence, $\mathcal{G}$ is a contraction mapping on Γ_T when T is small. □

Lemma 2.6. *The estimate (2.28) holds.*

Proof. To save space, let's assume $d_1 = 1$ here. For the fixed $(\tau, y) \in \Pi_T$, we set

$$x_i = x_i(\tau, y) = \frac{1}{2}[(h_i(\tau) - g_i(\tau))y + g_i(\tau) + h_i(\tau)], \quad i = 1, 2.$$

Then, $w_i(\tau, y) = u_i(\tau, x_i)$, $x_i \in [g_i(\tau), h_i(\tau)]$. The direct calculation yields

$$x_1 - x_2 = \frac{(h_2(\tau) - x_2)(g_1(\tau) - g_2(\tau))}{h_2(\tau) - g_2(\tau)} + \frac{(x_2 - g_2(\tau))(h_1(\tau) - h_2(\tau))}{h_2(\tau) - g_2(\tau)}, \tag{2.44}$$

which combined with the definition of Γ_T and (2.18) implies

$$|x_1 - x_2| \le \frac{M\varepsilon_0}{4h_0} \le \frac{3\varepsilon_0}{4} < h_0.$$

Hence, one of the following four cases must happen:

$$x_1, x_2 \in [-h_0, h_1(\tau)]; \quad x_1, x_2 \in [-h_0, h_2(\tau)]; \quad x_1, x_2 \in [g_1(\tau), h_0]; \quad x_1, x_2 \in [g_2(\tau), h_0].$$

Without loss of generality we may suppose that $h_1(\tau) \ge h_2(\tau)$ and $x_1, x_2 \in [-h_0, h_1(\tau)]$. For other cases, one can handle by the same way. Similar to Step 3 in the proof of Lemma 2.3, for this fixed τ and any $x \in [g_1(\tau), h_1(\tau)]$, we define

$$\tau_x = \begin{cases} \tau_{x,g_1} & \text{if } \ x \in [g_1(\tau), -h_0), \ x = g_1(\tau_{x,g_1}), \\ 0 & \text{if } \ |x| \le h_0, \\ \tau_{x,h_1} & \text{if } \ x \in (h_0, h_1(\tau)], \ x = h_1(\tau_{x,h_1}). \end{cases}$$

As $x_i \in [-h_0, h_1(\tau)]$, we have $\tau_{x_i} = \tau_{x_i,h_1}$ or $\tau_{x_i} = 0$, and $0 \le \tau_{x_i} \le \tau$, $i = 1, 2$. It is easy to get

$$\begin{aligned} |w_1(\tau, y) - w_2(\tau, y)| &\le |u_1(\tau, x_1) - u_1(\tau, x_2)| + |u_1(\tau, x_2) - u_2(\tau, x_2)| \\ &\le |u_1(\tau, x_1) - u_1(\tau, x_2)| + \|u\|_{C(\overline{\Omega}_T)}. \end{aligned} \tag{2.45}$$

We estimate $|u_1(\tau, x_1) - u_1(\tau, x_2)|$. Integrating the differential equation of u_1 from τ_x to τ gives

$$u_1(\tau, x) = u_1(\tau_x, x) + \int_{\tau_x}^{\tau} \left(\int_{g_1(s)}^{h_1(s)} J(x, y) u_1(s, y) \mathrm{d}y - u_1(s, x) + f_1(s, x, u_1, v_1) \right) \mathrm{d}s.$$

Denote $\tau_i = \tau_{x_i}$, $i = 1, 2$. Then τ_i depends on x_i. Without loss of generality we assume $\tau_1 \ge \tau_2$. Thus, for $\tau_1 \le t \le \tau$,

$$\begin{aligned} |u_1(t, x_1) - u_1(t, x_2)| \le |u_1(\tau_1, x_1) - u_1(\tau_2, x_2)| &+ \int_{\tau_1}^{t} \int_{g_1(s)}^{h_1(s)} |J(x_1, y) - J(x_2, y)| u_1(s, y) \mathrm{d}y\mathrm{d}s \\ &+ \int_{\tau_2}^{\tau_1} \int_{g_1(s)}^{h_1(s)} J(x_2, y) u_1(s, y) \mathrm{d}y\mathrm{d}s + \int_{\tau_1}^{t} |u_1(s, x_1) - u_1(s, x_2)| \mathrm{d}s \\ &+ \int_{\tau_2}^{\tau_1} u_1(s, x_2) \mathrm{d}s + \int_{\tau_2}^{\tau_1} |f_1(s, x_2, u_1(s, x_2), v_1(s, x_2))| \mathrm{d}y\mathrm{d}s \\ &+ \int_{\tau_1}^{t} |f_1(s, x_1, u_1(s, x_1), v_1(s, x_1)) - f_1(s, x_2, u_1(s, x_2), v_1(s, x_2))| \mathrm{d}y\mathrm{d}s. \end{aligned}$$

It follows from the conditions **(f)** and **(f3)** that

$$|f_1(s, x_2, u_1(s, x_2), v_1(s, x_2))| \le L|u_1(s, x_2)| \le Lk_1,$$

$$|f_1(s, x_1, u_1(s, x_1), v_1(s, x_1)) - f_1(s, x_2, u_1(s, x_2), v_1(s, x_2))|$$

$$\le L^*|x_1 - x_2| + L\big(|u_1(s, x_1) - u_1(s, x_2)| + |v_1(s, x_1) - v_1(s, x_2)|\big).$$

As g_i, h_i satisfy (2.16), i.e., $h_i(\tau) - g_i(\tau) \le M$ in $[0, T]$, using the condition **(J1)** we have

$$\begin{aligned}
|u_1(t,x_1) - u_1(t,x_2)| \le\ & |u_1(\tau_1,x_1) - u_1(\tau_2,x_2)| + Tk_1 M L(J)|x_1 - x_2| + k_1|\tau_1 - \tau_2| \\
& + T\|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,t])} + k_1|\tau_1 - \tau_2| \\
& + Lk_1|\tau_1 - \tau_2| + TL^*|x_1 - x_2| + TL\|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,t])} \\
& + L\int_{\tau_1}^{t} |v_1(s,x_1) - v_1(s,x_2)|\mathrm{d}s \\
\le\ & |u_1(\tau_1,x_1) - u_1(\tau_2,x_2)| + L\int_{\tau_1}^{t} |v_1(s,x_1) - v_1(s,x_2)|\mathrm{d}s \\
& + C\big(T|x_1 - x_2| + |\tau_1 - \tau_2| + T\|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,t])}\big)
\end{aligned} \tag{2.46}$$

for all $\tau_1 \le t \le \tau$. From (2.44), one has

$$|x_1 - x_2| \le \frac{M}{2h_0}\|g,h\|_{C([0,T])}. \tag{2.47}$$

In the following we estimate $|\tau_1 - \tau_2|$ and $|u_1(\tau_1,x_1) - u_1(\tau_2,x_2)|$.

Case 1: $\tau_i > 0$ for $i = 1, 2$. In this case, it is clear that $u_1(\tau_1,x_1) = u_1(\tau_2,x_2) = 0$. On the other hand, since $(g_1,h_1) \in \Gamma_T$, we have $h_1' \ge \rho c_0$ in $[0,\tau]$, and so

$$|\tau_1 - \tau_2| \le (\rho c_0)^{-1}|h_1(\tau_1) - h_1(\tau_2)| = (\rho c_0)^{-1}|x_1 - x_2|.$$

Case 2: $\tau_1 > 0$ and $\tau_2 = 0$. Then $x_2 \in [-h_0, h_0]$, $x_1 > h_0$, $u_1(\tau_1,x_1) = 0$. Let $L(u_0)$ be the Lipschitz constant of u_0. It follows that

$$\begin{aligned}
|\tau_1 - \tau_2| &= |\tau_1 - 0| \le (\rho c_0)^{-1}|h_1(\tau_1) - h_1(0)| = (\rho c_0)^{-1}|x_1 - h_0| \le (\rho c_0)^{-1}|x_1 - x_2|, \\
|u_1(\tau_1,x_1) - u_1(\tau_2,x_2)| &= |0 - u_0(x_2)| = |u_0(h_0) - u_0(x_2)| \le L(u_0)|h_0 - x_2| \le L(u_0)|x_1 - x_2|.
\end{aligned}$$

Case 3: $\tau_1 = \tau_2 = 0$, i.e., $x_1, x_2 \in [-h_0, h_0]$. Then $|\tau_1 - \tau_2| = 0$, and

$$|u_1(\tau_1,x_1) - u_1(\tau_2,x_2)| = |u_0(x_1) - u_0(x_2)| \le L(u_0)|x_2 - x_1|.$$

In a word,

$$|\tau_1 - \tau_2| + |u_1(\tau_1,x_1) - u_1(\tau_2,x_2)| \le [(\rho c_0)^{-1} + L(u_0)]|x_1 - x_2|. \tag{2.48}$$

Now we estimate $\int_{\tau_1}^{t} |v_1(s,x_1) - v_1(s,x_2)|\mathrm{d}s$. Let $y_i = \dfrac{2x_i - g_1(\tau) - h_1(\tau)}{h_1(\tau) - g_1(\tau)}$. Then $z_1(\tau,y_i) = v_1(\tau,x_i)$. Similar to the derivation of (2.40) we have

$$\begin{aligned}
\int_{\tau_1}^{t} |v_1(s,x_1) - v_1(s,x_2)|\mathrm{d}s &= \int_{\tau_1}^{t} |z_1(s,y_1) - z_1(s,y_2)|\mathrm{d}s \\
&\le T|y_1 - y_2| \cdot \|z_{1,y}\|_{L^\infty([0,T]\times\mathbb{R})} \\
&\le TC_{17}|x_1 - x_2|.
\end{aligned}$$

Substituting this and (2.48) into (2.46) and using (2.47), it yields that, for $\tau_1 \le t \le \tau$,

$$|u_1(t,x_1) - u_1(t,x_2)| \le C_{18}\big(\|g,h\|_{C([0,T])} + T\|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,t])}\big).$$

Thus we have

$$\|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,\tau])} \le C_{18}\big(\|g,h\|_{C([0,T])} + T\|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,\tau])}\big)$$

Taking T small such that $C_{18}T < 1/2$, then

$$|u(\tau,x_1) - u(\tau,x_2)| \le \|u_1(\cdot,x_1) - u_1(\cdot,x_2)\|_{C([\tau_1,\tau])} \le 2C_{18}\|g,h\|_{C([0,T])}.$$

Substituting this into (2.45) and by the arbitrariness of $(\tau,y) \in \Pi_T$, we get (2.28) immediately. □

Proof of Theorem 2.1. *Step 1: Local existence and uniqueness.* By Lemma 2.4 and Lemma 2.5 we see that $\mathcal{G}(\Gamma_T) \subset \Gamma_T$ and $\mathcal{G}$ is a contraction mapping on Γ_T when T is small. The *Contraction Mapping Theorem* shows that problem (1.3) admits a unique solution $(\hat u, \hat v, \hat g, \hat h)$ with $(\hat g, \hat h) \in \Gamma_T$. This solution is the unique solution of (1.3) if we can prove that $(g,h) \in \Gamma_T$ holds for any solution (u,v,g,h) of (1.3) defined for $t \in (0,T]$. Moreover, from the above arguments we see that $(\hat u, \hat v, \hat g, \hat h)$ satisfies (2.2) and (2.3).

Let (u,v,g,h) be an arbitrary solution of (1.3) defined in $(0,T]$. It follows that

$$h'(t) = -\mu v_x(t,h(t)) + \rho \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x,$$

$$g'(t) = -\mu v_x(t,g(t)) - \rho \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x.$$

It is easy to see from the above discussions that (2.2) and (2.3) hold. And hence

$$[h(t) - g(t)]' \le 2\mu k_3 + \rho k_1(h(t) - g(t)); \quad 0 < -g'(t), h'(t) \le \mu k_3 + \rho k_1(h(t) - g(t)).$$

The first inequality in the above implies $h(t) - g(t) \le 2[h_0 + \mu k_3/(\rho k_1)]e^{\rho k_1 t}$. So we have

$$[h(t) - g(t)]' \le 2\mu k_3 + 2(\rho k_1 h_0 + \mu k_3)e^{\rho k_1 t},$$
$$0 < h'(t), -g'(t) \le \mu k_3 + 2(\rho k_1 h_0 + \mu k_3)e^{\rho k_1 t} = R(t).$$

Therefore,

$$h(t) - g(t) \le 2h_0 + t\left(2\mu k_3 + 2(\rho k_1 h_0 + \mu k_3)e^{\rho k_1 t}\right), \quad \forall\ 0 < t \le T.$$

Shrink T small enough such that $T\big[2\mu k_3 + 2(\rho k_1 h_0 + \mu k_3)e^{\rho k_1 T}\big] \le \varepsilon_0/4$. Then $h(t) - g(t) \le M$ for $t \in [0,T]$. Furthermore, by using the proofs of (2.17) and (2.19), one can show that $\rho c_0 \le h'(t) \le \bar R$ and $-\bar R \le g'(t) \le -\rho c_0^*$ in $(0,T]$. Thus $(g,h) \in \Gamma_T$.

Step 2: Global existence and uniqueness. Assume that (2.4) holds. From Step 1, we know that the system (1.3) admits a unique solution (u,v,g,h) in some interval $(0,T]$.

Let $z(t,y)=v(t,x(t,y))$ and consider the problem

$$\begin{cases} z_t=d_2\xi(t)z_{yy}+\zeta(t,y)z_y+f_2^*(t,y,w,z), & 0<t\le T,\ |y|<1,\\ z(t,\pm1)=0, & 0\le t\le T,\\ z(0,y)=v_0(h_0y)=:z_0(y), & |y|\le 1, \end{cases} \tag{2.49}$$

where $w(t,y)=u(t,x(t,y))$, $f_2^*(t,y,w,z)=f_2(t,x(t,y),w,z)$. As $z_0(y)\in W_p^2(\Sigma)$, same as the above, $z\in W_p^{1,2}(\Pi_T)\hookrightarrow C^{(1+\alpha)/2,1+\alpha}(\Pi_T)$. Then $v_x\in C^{\alpha/2,\alpha}(\overline{D}_{g,h}^T)$. This combined with the assumptions **(f)** and **(f3)** implies that the function $F_1(t,x,u)=f_1(t,x,u,v(t,x))$ is differentiable with respect to x. Note that u satisfies

$$\begin{cases} u_t=d_1\int_{g(t)}^{h(t)}J(x,y)u(t,y)\mathrm{d}y-d_1u+f_1(t,x,u,v(t,x)), & t_x<t\le T,\ g(t)<x<h(t),\\ u(t_x,x)=\tilde u_0(x), & g(T)<x<h(T), \end{cases}$$

where

$$\tilde u_0(x)=\begin{cases} 0, & |x|>h_0,\\ u_0(x), & |x|\le h_0, \end{cases}\qquad t_x=\begin{cases} t_{x,g} & \text{if } x\in[g(T),-h_0),\ x=g(t_{x,g}),\\ 0 & \text{if } |x|\le h_0,\\ t_{x,h} & \text{if } x\in(h_0,h(T)],\ x=h(t_{x,h}). \end{cases}$$

View $G(t,x)=\int_{g(t)}^{h(t)}J(x,y)u(t,y)\mathrm{d}y$ as a known function. Then for $t\in[0,T]$, $t_x, u_0(x)$ and $G(t,x)$ are Lipschitz continuous in $x\in[g(t),h(t)]$. Using the continuous dependence of the solution with respect to the parameters we can show that for $t\in[0,T]$, $u(t,x)$ is Lipschitz continuous in $x\in[g(t),h(t)]$. Clearly, $u_t\in C(\overline{D}_{g,h}^T)$. This implies $u\in C^{1,1-}(\overline{D}_{g,h}^T)$ and hence $w\in C^{1,1-}(\overline{\Pi}_T)$.

It is easy to see that the function

$$\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty}J(x,y)u(t,x)\mathrm{d}y\mathrm{d}x$$

of t is differentiable. So $h'(t)\in C^{\alpha/2}([0,T])$ as $v_x(t,h(t))\in C^{\alpha/2}([0,T])$. Similarly, $g'(t)\in C^{\alpha/2}([0,T])$. Set $F_2(t,y,z)=f_2^*(t,y,w(t,y),z)$. Then, by using **(f4)** (or (2.4)), there hold

$$\xi\in C^{\alpha/2}([0,T]),\quad \zeta(\cdot,\cdot),\ F_2(\cdot,\cdot,z)\in C^{\alpha/2,\alpha}(\Pi_T).$$

By the interior Schauder theory we have $z\in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\Sigma)$ with $0<\varepsilon<T$, which implies $v(T,x)\in C^2([g(T),h(T)])$.

Recall that $u(T,x)$ is Lipschitz continuous in $x\in[g(T),h(T)]$. We can take $(u(T,x),v(T,x))$ as an initial function and $[g(T),h(T)]$ as the initial habitat and then use Step 1 to extend the solution from $t=T$ to some $T'>T$. Assume that $(0,T_0)$ is the maximal existence interval of (u,v,g,h) obtained by such extension process. We shall prove that $T_0=\infty$. Assume on the contrary that $T_0<\infty$.

Since $h',-g'>0$ in $(0,T_0)$, we can define $h(T_0)=\lim_{t\to T_0}h(t)$ and $g(T_0)=\lim_{t\to T_0}g(t)$. By the above arguments,

$$h(T_0)-g(T_0)\le 2h_0+T_0\big(2\mu k_3+2(\rho k_1h_0+\mu k_3)e^{\rho k_1T_0}\big).$$

In view of $0 < -v_x(t,h(t)), v_x(t,g(t)) \le k_3, 0 < u \le k_1, 0 < v \le k_2$ for $t \in (0,T_0)$, $h', g' \in L^\infty((0,T_0))$. Making use of Sobolev embedding theorem: $W^1_\infty((0,T_0)) \hookrightarrow C([0,T_0])$, we have $g,h \in C([0,T_0])$ with $g(T_0), h(T_0)$ defined as above. It follows from the parabolic L^p theory and Sobolev embedding theorem that $v \in C^{(1+\alpha)/2,1+\alpha}(\overline{D}^{T_0}_{g,h})$. These facts show that the first differential equation holds for $0 \le t \le T_0$. Similar to the above, $u \in C^{1,1-}(\overline{D}^{T_0}_{g,h})$, g', $h' \in C^{\alpha/2}([0,T_0])$. Consider the problem (2.49) with T replaced by T_0. Same as above, we can show that (2.49) has a unique solution $z \in W^{1,2}_p(\Pi_{T_0}) \cap C^{1+\alpha/2,2+\alpha}([\varepsilon,T_0]\times\Sigma)$. Consequently, $v(T_0,x) \in C^2([g(T_0),h(T_0)])$.

Due to $u(t,h(t)) = v(t,h(t)) = 0$ in $[0,T_0)$, it is easy to see that $u(T_0,h(T_0)) = v(T_0,h(T_0)) = 0$. Moreover, by the parabolic maximum principle and Lemma 2.2 we have $u(T_0,x) > 0, v(T_0,x) > 0$ for $x \in (g(T_0),h(T_0))$.

Therefore, we may treat $(u(T_0,x), v(T_0,x))$ as an initial function and $[g(T_0),h(T_0)]$ as the initial habitat and apply Step 1 to show that the solution of (1.3) can be extended to some $(0,\hat{T})$ with $\hat{T} > T_0$. This contradicts the definition of T_0. Hence, $T_0 = \infty$.

It follows from the above arguments that $(g,h) \in \mathbb{G}^T \times \mathbb{H}^T$, $(u,v) \in \mathbb{X}^T_{g,h}$, and (u,v,g,h) satisfies (2.2), (2.3) and (2.5). The proof is end. $\square$

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