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Functional calculus and multi-analytic models on regular Λ-polyballs ☆

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ABSTRACT

In a recent paper, we introduced the standard k-tuple $\mathbf{S} := (\mathbf{S}_1, \dots, \mathbf{S}_k)$ of pure row isometries $\mathbf{S}_i := [S_{i,1} \cdots S_{i,n_i}]$ acting on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$, where $\mathbb{F}_n^+$ is the unital free semigroup with n generators, and showed that $\mathbf{S}$ is the universal k-tuple of doubly Λ-commuting row isometries, i.e.

$$S_{i,s}^* S_{j,t} = \overline{\lambda_{ij}(s,t)} S_{j,t} S_{i,s}^*$$

for every $i, j \in \{1, \dots, k\}$ with $i \neq j$ and every $s \in \{1, \dots, n_i\}$, $t \in \{1, \dots, n_j\}$, where $\Lambda_{ij} := [\lambda_{i,j}(s,t)]$ is an $n_i \times n_j$-matrix with the entries in $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$ and $\Lambda_{j,i} = \Lambda_{i,j}^*$. It was also proved that the set of all k-tuples $T := (T_1, \dots, T_k)$ of row operators $T_i := [T_{i,1} \cdots T_{i,n_i}]$ acting on a Hilbert space $\mathcal{H}$ which admit $\mathbf{S}$ as universal model, i.e. there is a Hilbert space $\mathcal{D}$ such that $\mathcal{H}$ is jointly co-invariant for all operators $S_{i,s} \otimes I_{\mathcal{D}}$ and

$$T_{i,s}^* = (S_{i,s}^* \otimes I_{\mathcal{D}})|_{\mathcal{H}}, \quad i \in \{1, \dots, k\} \quad \text{and} \quad s \in \{1, \dots, n_i\},$$

consists of the pure elements of a set $\mathbf{B}_\Lambda(\mathcal{H})$ which was called the regular Λ-polyball. The goal of the present paper is to introduce and study noncommutative Hardy spaces associated with the regular Λ-polyball, to develop a functional calculus on noncommutative Hardy spaces for the completely non-coisometric (c.n.c.) k-tuples in $\mathbf{B}_\Lambda(\mathcal{H})$, and to study the characteristic functions and the associated multi-analytic models for the c.n.c. elements in the regular Λ-polyball. In addition, we show that the characteristic function is a complete unitary invariant for the class of c.n.c. k-tuples in $\mathbf{B}_\Lambda(\mathcal{H})$. These results extend the corresponding classical results of Sz.-Nagy–Foiaş for contractions and the noncommutative versions for row contractions. In the particular case when $n_1 = \cdots = n_k = 1$ and $\Lambda_{ij} = 1$, we obtain a functional calculus and operator model theory in terms of characteristic functions for k-tuples of contractions satisfying Brehmer condition.

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Contents

0. Introduction

In a recent paper [16], inspired by the work of De Jeu and Pinto [5], and J. Sarkar [18], we studied the structure of the k-tuples of doubly Λ-commuting row isometries and the C^*-algebras they generate from the point of view of noncommutative multivariable operator theory.

Given row isometries $V_i := [V_{i,1} \cdots V_{i,n_i}]$, $i \in \{1, \ldots, k\}$, i.e. $V_{i,s}^* V_{i,t} = \delta_{st} I$, we say that $V := (V_1, \ldots, V_k)$ is a k-tuple of *doubly Λ-commuting row isometries* if

$$V_{i,s}^* V_{j,t} = \overline{\lambda_{ij}(s,t)} V_{j,t} V_{i,s}^*$$

for every $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and every $s \in \{1, \ldots, n_i\}$, $t \in \{1, \ldots, n_j\}$, where $\Lambda_{ij} := [\lambda_{i,j}(s,t)]$ is an $n_i \times n_j$-matrix with the entries in the torus $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$ and $\Lambda_{j,i} = \Lambda_{i,j}^*$.

We obtained Wold decompositions and used them to classify the k-tuples of doubly Λ-commuting row isometries up to a unitary equivalence. We proved that there is a one-to-one correspondence between the unitary equivalence classes of k-tuples of doubly Λ-commuting row isometries and the enumerations of 2^k unitary equivalence classes of unital representations of the twisted Λ-tensor algebras $\otimes_{i \in A^c}^{\Lambda} \mathcal{O}_{n_i}$, as A is any subset of $\{1, \ldots, k\}$, where $\mathcal{O}_{n_i}$ is the Cuntz algebra with n_i generators (see [4]). The algebra $\otimes_{i \in A^c}^{\Lambda} \mathcal{O}_{n_i}$ can be seen as a twisted tensor product of Cuntz algebras. We remark that, when $n_1 = \cdots = n_k = 1$, the corresponding algebras are higher-dimensional noncommutative tori which are studied in noncommutative differential geometry (see [20], [3], [6], and the appropriate references there in). We should mention that C^*-algebras generated by isometries with twisted commutation relations have been studied in the literature in various particular cases (see [7], [17], [8], and [22]).

We introduced in [16] the standard k-tuple $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ of doubly Λ-commuting pure row isometries $\mathbf{S}_i := [S_{i,1} \cdots S_{i,n_i}]$ acting on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$, where $\mathbb{F}_n^+$ is the unital free semigroup with n generators, and proved that the universal C^*-algebra generated by a k-tuple of doubly Λ-commuting row isometries is $*$-isomorphic to the C^*-algebra $C^*(\{S_{i,s}\})$. The regular Λ-polyball $\mathbf{B}_\Lambda(\mathcal{H})$ was introduced as the set of all k-tuples of row contractions $T_i = [T_{i,1} \ldots T_{i,n_i}]$, i.e. $T_{i,1}T_{i,1}^* + \cdots + T_{i,n_i}T_{i,n_i}^* \leq I$, such that

$$T_{i,s} T_{j,t} = \lambda_{ij}(s,t) T_{j,t} T_{i,s}$$

for every $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and every $s \in \{1, \ldots, n_i\}$, $t \in \{1, \ldots, n_j\}$, and such that

$$\Delta_{rT}(I) := (id - \Phi_{rT_k}) \circ \cdots \circ (id - \Phi_{rT_1})(I) \geq 0, \qquad r \in [0, 1),$$

where $\Phi_{rT_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by $\Phi_{rT_i}(X) := \sum_{s=1}^{n_i} r^2 T_{i,s} X T_{i,s}^*$. We proved that a k-tuple $T := (T_1, \ldots, T_k)$ of row operators $T_i := [T_{i,1} \ldots T_{i,n_i}]$, acting on a Hilbert space $\mathcal{H}$, admits $\mathbf{S}$ as universal model, i.e. there is a Hilbert space $\mathcal{D}$ such that $\mathcal{H}$ is jointly co-invariant for $S_{i,s} \otimes I_{\mathcal{D}}$ and

$$T_{i,s}^* = (S_{i,s}^* \otimes I_{\mathcal{D}})|_{\mathcal{H}}, \quad i \in \{1, \ldots, k\} \ \text{ and } \ s \in \{1, \ldots, n_i\},$$

if and only if T is a pure element of $\mathbf{B}_\Lambda(\mathcal{H})$.

The goal of the present paper is to continue the work in [16] and develop a multivariable functional calculus for k-tuples of Λ-commuting row contractions on noncommutative Hardy spaces associated with regular Λ-polyballs. We also study the characteristic functions and the associated multi-analytic models for the elements of $\mathbf{B}_\Lambda(\mathcal{H})$. Many of the techniques developed in [16] and [15] are refined and used in the present paper.

In Section 1, we present some preliminaries on noncommutative Berezin transforms associated with Λ-polyballs which are very useful in the next sections. In Section 2, we introduce the noncommutative Hardy algebra $F^\infty(\mathbf{B}_\Lambda)$ which can be seen as a noncommutative multivariable version of the Hardy algebra $H^\infty(\mathbb{D})$. We prove that $F^\infty(\mathbf{B}_\Lambda)$ is WOT- (resp. SOT-, w*-) closed and

$$F^\infty(\mathbf{B}_\Lambda) = \overline{\mathcal{P}(\{S_{i,s}\})}^{\mathrm{SOT}} = \overline{\mathcal{P}(\{S_{i,s}\})}^{\mathrm{WOT}} = \overline{\mathcal{P}(\{S_{i,s}\})}^{\mathrm{w}^*},$$

where $\mathcal{P}(\{S_{i,s}\})$ is the algebra of all polynomials in $S_{i,s}$ and the identity. Moreover, we show that $F^\infty(\mathbf{B}_\Lambda)$ is the sequential SOT-(resp. WOT-, w*-) closure of $\mathcal{P}(\{S_{i,s}\})$. Using noncommutative Berezin transforms associated with Λ-polyballs, we prove that each element $A \in F^\infty(\mathbf{B}_\Lambda)$ has a unique formal Fourier representation

$$\varphi(\{S_{i,s}\}) = \sum_{(\beta_1,\ldots,\beta_k)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1}\ldots S_{k,\beta_k}$$

such that, for all $r \in [0,1)$, $\varphi(\{rS_{i,s}\})$ is in the Λ-polyball algebra $\mathcal{A}(\mathbf{B}_\Lambda)$, the normed closed non-self-adjoint algebra generated by the isometries $S_{i,s}$ and the identity. Moreover, we prove that

$$A = \text{SOT-}\lim_{r\to 1} \varphi(\{rS_{i,s}\})$$

and

$$\|A\| = \sup_{0\le r<1} \|\varphi(\{rS_{i,s}\})\| = \lim_{r\to 1} \|\varphi(\{rS_{i,s}\})\|.$$

In Section 3, we prove the existence of an $F^\infty(\mathbf{B}_\Lambda)$-functional calculus for the completely non-coisometric (c.n.c.) elements T in the Λ-polyball $\mathbf{B}_\Lambda$ which extends the Sz.-Nagy–Foias functional calculus for c.n.c. contractions [19] and the functional calculus for c.n.c. row contractions [14]. In this case, we prove that if $\varphi(\{S_{i,s}\})$ is the Fourier representation of $A \in F^\infty(\mathbf{B}_\Lambda)$, then

$$\Psi_T(A) := \text{SOT-}\lim_{r\to 1} \varphi(\{rT_{i,s}\})$$

exists and defines a unital completely contractive homomorphism $\Psi_T : F^\infty(\mathbf{B}_\Lambda) \to B(\mathcal{H})$ which is WOT-(resp. SOT-, w*-) continuous on bounded sets.

Section 4 is dedicated to the set $Hol(\mathbf{B}_\Lambda^\circ)$ of free holomorphic functions on the open Λ-polyball $\mathbf{B}_\Lambda^\circ(\mathcal{H})$, which is the interior of $\mathbf{B}_\Lambda(\mathcal{H})$. We introduce the algebra $H^\infty(\mathbf{B}_\Lambda^\circ)$ of all $\varphi \in Hol(\mathbf{B}_\Lambda^\circ)$ such that

$$\|\varphi\|_\infty := \sup \|\varphi(\{X_{i,s}\})\| < \infty,$$

where the supremum is taken over all $\{X_{i,s}\} \in \mathbf{B}_\Lambda^\circ(\mathcal{H})$ and any Hilbert space. $H^\infty(\mathbf{B}_\Lambda^\circ)$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$ and has an operator space structure in the sense of Ruan (see [10], p. 181). Using noncommutative Berezin transforms, we show that the algebra of bounded free holomorphic functions $H^\infty(\mathbf{B}_\Lambda^\circ)$ is completely isometric isomorphic to the noncommutative Hardy algebra

$F^\infty(\mathbf{B}_\Lambda)$ introduced in Section 2. We also introduce the algebra $A(\mathbf{B}_\Lambda^\circ)$ of all functions $f \in Hol(\mathbf{B}_\Lambda^\circ)$ such that the map $\mathbf{B}_\Lambda^\circ(\mathcal{H}) \ni X \mapsto f(X) \in B(\mathcal{H})$ has a continuous extension to $\mathbf{B}_\Lambda(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. It turns out that $A(\mathbf{B}_\Lambda^\circ)$ is a Banach algebra with pointwise multiplication and the norm $\|\cdot\|_\infty$ and has an operator space structure. We conclude this section by showing that $A(\mathbf{B}_\Lambda^\circ)$ is completely isometric isomorphic to the noncommutative Λ-polyball algebra $\mathcal{A}(\mathbf{B}_\Lambda)$.

In Section 5, we show that a k-tuple $T = (T_1, \ldots, T_k)$ in the noncommutative Λ-polyball $\mathbf{B}_\Lambda(\mathcal{H})$ admits a characteristic function if and only if

$$\Delta_{\mathbf{S}\otimes I}(I - K_T K_T^*) \geq 0,$$

where K_T is the noncommutative Berezin kernel associated with T and

$$\Delta_{\mathbf{S}\otimes I} := (id - \Phi_{\mathbf{S}_1\otimes I}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k\otimes I}).$$

We provide a model theorem for the class of completely non-coisometric k-tuple of operators in $\mathbf{B}_\Lambda(\mathcal{H})$ which admit characteristic functions, and show that the characteristic function is a complete unitary invariant for this class of k-tuples. These are generalizations of the corresponding classical results [19] and of the noncommutative versions obtained in [11].

We remark that in the particular case when $n_1 = \cdots = n_k = 1$ and $\Lambda_{ij} = 1$, we obtain a functional calculus and operator model theory for k-tuples of contractions satisfying Brehmer condition [2] (see also [19]).

1. Preliminaries on regular Λ-polyballs and noncommutative Berezin transforms

In this section, we introduce the standard k-tuple $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ of doubly Λ-commuting pure row isometries $\mathbf{S}_i := [S_{i,1} \cdots S_{i,n_i}]$ and present some preliminaries results on noncommutative Berezin transforms associated with Λ-polyballs.

For each $i, j \in \{1, \ldots, k\}$ with $i \neq j$, let $\Lambda_{ij} := [\lambda_{i,j}(s,t)]$, where $s \in \{1, \ldots, n_i\}$ and $t \in \{1, \ldots, n_j\}$ be an $n_i \times n_j$-matrix with the entries in the torus $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$, and assume that $\Lambda_{j,i} = \Lambda_{i,j}^*$. Given row isometries $V_i := [V_{i,1} \cdots V_{i,n_i}]$, $i \in \{1, \ldots, k\}$, we say that $V = (V_1, \ldots, V_k)$ is a k-tuple of *doubly Λ-commuting row isometries* if

$$V_{i,s}^* V_{j,t} = \overline{\lambda_{ij}(s,t)} V_{j,t} V_{i,s}^*$$

for every $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and every $s \in \{1, \ldots, n_i\}$, $t \in \{1, \ldots, n_j\}$. We remark that the relation above implies that

$$V_{i,s} V_{j,t} = \lambda_{ij}(s,t) V_{j,t} V_{i,s}.$$

For each $i \in \{1, \ldots, k\}$, let $\mathbb{F}_{n_i}^+$ be the unital free semigroup with generators $g_1^i, \ldots, g_{n_i}^i$ and neutral element g_0^i. The length of $\alpha \in \mathbb{F}_{n_i}^+$ is defined by $|\alpha| = 0$ if $\alpha = g_0^i$ and $|\alpha| = m$ if $\alpha = g_{p_1}^i \cdots g_{p_m}^i \in \mathbb{F}_{n_i}^+$, where $p_1, \ldots, p_m \in \{1, \ldots, n_i\}$. If $T_i := [T_{i,1} \cdots T_{i,n_i}]$, we use the notation $T_{i,\alpha} := T_{i,p_1} \cdots T_{i,p_m}$ and $T_{i,g_0^i} := I$.

Consider the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ with the standard basis $\{\chi_{(\alpha_1, \ldots, \alpha_k)}\}$, where $\alpha \in \mathbb{F}_{n_1}^+, \ldots, \alpha_k \in \mathbb{F}_{n_k}^+$. For each $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$, we define the row operator $\mathbf{S}_i := [S_{i,1} \cdots S_{i,n_i}]$, where $S_{i,s}$ is defined on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ by setting

$$\begin{aligned} S_{i,s}&\left(\chi_{(\alpha_1,\ldots,\alpha_k)}\right) \\ &:= \begin{cases} \chi_{(g_s^i\alpha_1,\alpha_2,\ldots,\alpha_k)}, & \text{if } i=1 \\ \boldsymbol{\lambda}_{i,1}(s,\alpha_1)\cdots\boldsymbol{\lambda}_{i,i-1}(s,\alpha_{i-1})\chi_{(\alpha_1,\ldots,\alpha_{i-1},g_s^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)}, & \text{if } i\in\{2,\ldots,k\} \end{cases} \end{aligned} \tag{1.1}$$

for all $\alpha_1 \in \mathbb{F}_{n_1}^+, \ldots, \alpha_k \in \mathbb{F}_{n_k}^+$, where, for each $j \in \{1, \ldots, k\}$,

$$\boldsymbol{\lambda}_{i,j}(s, \beta) := \begin{cases} \prod_{b=1}^{q} \lambda_{i,j}(s, j_b), & \text{if } \beta = g_{j_1}^j \cdots g_{j_q}^j \in \mathbb{F}_{n_j}^+ \\ 1, & \text{if } \beta = g_0^j. \end{cases}$$

Let $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$ and note that relation (1.1) implies

$$\begin{aligned} &S_{i,s}^* \left(\chi_{(\alpha_1, \ldots, \alpha_k)}\right) \\ &\quad = \begin{cases} \overline{\boldsymbol{\lambda}_{i,1}(s, \alpha_1)} \cdots \overline{\boldsymbol{\lambda}_{i,i-1}(s, \alpha_{i-1})} \chi_{(\alpha_1, \ldots, \alpha_{i-1}, \beta_i, \alpha_{i+1}, \ldots, \alpha_k)}, & \text{if } \alpha_i = g_s^i \beta_i \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{1.2}$$

for any $\alpha_1 \in \mathbb{F}_{n_1}^+, \ldots, \alpha_k \in \mathbb{F}_{n_k}^+$. Hence, we deduce that

$$\begin{aligned} &\sum_{s=1}^{n_i} S_{i,s} S_{i,s}^* \left(\chi_{(\alpha_1, \ldots, \alpha_k)}\right) \\ &\quad = \begin{cases} |\boldsymbol{\lambda}_{i,1}(s, \alpha_1)|^2 \cdots |\boldsymbol{\lambda}_{i,i-1}(s, \alpha_{i-1})|^2 \chi_{(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k)}, & \text{if } |\alpha_i| \geq 1 \\ 0, & \text{otherwise} \end{cases} \\ &\quad = \begin{cases} \chi_{(\alpha_1, \ldots, \alpha_k)}, & \text{if } |\alpha_i| \geq 1 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

which shows that $[S_{i,1} \cdots S_{i,n_i}]$ is a row isometry for every $i \in \{1, \ldots, k\}$. In [16], we showed that, if $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and any $s \in \{1, \ldots, n_i\}$, $t \in \{1, \ldots, n_j\}$, then

$$S_{i,s}^* S_{j,t} = \overline{\lambda_{i,j}(s,t)} S_{j,t} S_{i,s}^*. \tag{1.3}$$

Consequently, $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ is a k-tuple of doubly Λ-commuting row isometries.

Given row contractions $T_i := [T_{i,1} \cdots T_{i,n_i}]$, $i \in \{1, \ldots, k\}$, acting on a Hilbert space $\mathcal{H}$, we say that $T = (T_1, \ldots, T_k)$ is a k-tuple of Λ-*commuting row contractions* if

$$T_{i,s} T_{j,t} = \lambda_{ij}(s,t) T_{j,t} T_{i,s} \tag{1.4}$$

for every $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and every $s \in \{1, \ldots, n_i\}$, $t \in \{1, \ldots, n_j\}$. We say that T is in the *regular* Λ-*polyball*, which we denote by $\mathbf{B}_\Lambda(\mathcal{H})$, if T is a Λ-commuting tuple and

$$\Delta_{rT}(I) := (id - \Phi_{rT_k}) \circ \cdots \circ (id - \Phi_{rT_1})(I) \geq 0, \qquad r \in [0, 1),$$

where $\Phi_{rT_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by $\Phi_{rT_i}(X) := \sum_{s=1}^{n_i} r^2 T_{i,s} X T_{i,s}^*$. We remark that, due to the Λ-commutation relation (1.4), we have $\Phi_{T_i} \circ \Phi_{T_j}(X) = \Phi_{T_j} \circ \Phi_{T_i}(X)$ for any $i, j \in \{1, \ldots, k\}$ and $X \in B(\mathcal{H})$.

Let $T = (T_1, \ldots, T_k)$ be a k-tuple in the *regular* Λ-*polyball* $\mathbf{B}_\Lambda(\mathcal{H})$. We define the noncommutative Berezin kernel

$$K_T : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}(T),$$

by setting

$$K_T h := \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} \chi_{(\beta_1,\dots,\beta_k)} \otimes \Delta_T(I)^{1/2} T_{k,\beta_k}^* \cdots T_{1,\beta_1}^* h, \qquad h \in \mathcal{D}(T),$$

where $\Delta_T(I) := (id - \Phi_{T_k}) \circ \cdots \circ (id - \Phi_{T_1})(I)$ and $\mathcal{D}(T) := \overline{\Delta_T(I)\mathcal{H}}$.

The first theorem is an extension of the corresponding result from [16] for pure k-tuples in $\mathbf{B}_\Lambda(\mathcal{H})$.

Theorem 1.1. *Let $T = (T_1, \dots, T_k)$ be a k-tuple in the regular Λ-polyball $\mathbf{B}_\Lambda(\mathcal{H})$. Then the following statements hold.*

(i) *The noncommutative Berezin kernel K_T is a contraction and*

$$K_T^* K_T = \lim_{p_k\to\infty} \dots \lim_{p_1\to\infty} (id - \Phi_{T_k}^{p_k}) \circ \cdots \circ (id - \Phi_{T_1}^{p_1})(I),$$

where the limits are in the weak operator theory.

(ii) *For every $i \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_i\}$,*

$$K_T T_{i,s}^* = \left(S_{i,s}^* \otimes I_{\mathcal{D}(T)}\right) K_T.$$

Proof. For each $i \in \{1, \dots, k\}$, we set

$$\Delta_{(T_i,T_{i-1},\dots,T_1)}(I) := (id - \Phi_{T_i}) \circ \cdots \circ (id - \Phi_{T_1})(I)$$

and remark that, due to the fact that T_i is a row contraction, $A_i := \lim_{q_i\to\infty} \Phi_{T_i}^{q_i+1}(I)$ exists in the weak operator theory. Using the fact that $\Phi_{T_i} \circ \Phi_{T_j}(X) = \Phi_{T_j} \circ \Phi_{T_i}(X)$ for all $i, j \in \{1, \dots, k\}$ and $X \in B(\mathcal{H})$, we deduce that

$$\begin{aligned}
\sum_{q_k=0}^{\infty} \Phi_{T_k}^{q_k}[\Delta_{(T_k,\dots,T_1)}(I)] &= \lim_{p_k\to\infty} \sum_{q_k=0}^{p_k} \left\{ \Phi_{T_k}^{q_k}[\Delta_{(T_{k-1},\dots,T_1)}(I)] - \Phi_{T_k}^{q_k+1}[\Delta_{(T_{k-1},\dots,T_1)}(I)] \right\} \\
&= \Delta_{(T_{k-1},\dots,T_1)}(I) - \lim_{p_k\to\infty} \Phi_{T_k}^{p_k+1}[\Delta_{(T_{k-1},\dots,T_1)}(I))] \\
&= \Delta_{(T_{k-1},\dots,T_1)}(I) - \Delta_{(T_{k-1},\dots,T_1)}\left(\lim_{p_k\to\infty} \Phi_{T_k}^{p_k+1}(I) \right) \\
&= \Delta_{(T_{k-1},\dots,T_1)}(I - A_k).
\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}
&\sum_{q_{k-1}=0}^{\infty} \Phi_{T_{k-1}}^{q_{k-1}} \left(\sum_{q_k=0}^{\infty} \Phi_{T_k}^{q_k}[\Delta_{(T_k,\dots,T_1)}(I)] \right) \\
&\quad= \sum_{q_{k-1}=0}^{\infty} \Phi_{T_{k-1}}^{q_{k-1}} \left(\Delta_{(T_{k-1},\dots,T_1)}(I - A_k) \right) \\
&\quad= \lim_{p_{k-1}\to\infty} \sum_{q_{k-1}=0}^{p_{k-1}} \left\{ \Phi_{T_{k-1}}^{q_{k-1}}[\Delta_{(T_{k-2},\dots,T_1)}(I - A_k)] - \Phi_{T_{k-1}}^{q_{k-1}+1}[\Delta_{(T_{k-2},\dots,T_1)}(I - A_k)] \right\} \\
&\quad= \Delta_{(T_{k-2},\dots,T_1)}(I - A_k) - \lim_{p_{k-1}\to\infty} \Phi_{T_{k-1}}^{p_{k-1}+1}[\Delta_{(T_{k-2},\dots,T_1)}(I - A_k)] \\
&\quad= \Delta_{(T_{k-2},\dots,T_1)}(I - A_k) - \Delta_{(T_{k-2},\dots,T_1)}\left((I - A_k) \lim_{p_{k-1}\to\infty} \Phi_{T_{k-1}}^{p_{k-1}+1}(I) \right) \\
&\quad= \Delta_{(T_{k-2},\dots,T_1)}[(I - A_k)(I - A_{k-1})].
\end{aligned}$$

Continuing this process, we obtain

$$\sum_{q_1=0}^{\infty} \Phi_{T_1}^{q_1}\left(\sum_{q_2=0}^{\infty} \Phi_{T_2}^{q_2}\left(\cdots \sum_{q_k=0}^{\infty} \Phi_{T_k}^{q_k}[\Delta_{(T_k,\ldots,T_1)}(I)]\cdots\right)\right) = (I-A_k)\cdots(I-A_1),$$

where the convergence of the series is in the weak operator topology. Since we can rearrange the series of positive terms, we obtain

$$\sum_{q_1,\ldots,q_k=0}^{\infty} \Phi_{T_1}^{q_1}\circ\cdots\circ\Phi_{T_k}^{q_k}[\Delta_{(T_k,\ldots,T_1)}(I)] = (I-A_k)\cdots(I-A_1).$$

Using this relation, one can see that

$$\begin{aligned}\langle K_T^*K_Th,h\rangle &= \left\langle \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} T_{1,\beta_1}\cdots T_{k,\beta_k}\Delta_T(I)T_{k,\beta_k}^*\cdots T_{1,\beta_1}^*h,h\right\rangle \\ &= \langle (I-A_k)\cdots(I-A_1)h,h\rangle\end{aligned}$$

for any $h\in\mathcal{H}$, which proves item (i).

Now, we prove item (ii). Note that, for every $h,h'\in\mathcal{H}$,

$$\begin{aligned}\left\langle K_TT_{i,s}^*h,\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h'\right\rangle &= \left\langle \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} \chi_{(\beta_1,\ldots,\beta_k)}\otimes\Delta_T(I)^{1/2}T_{k,\beta_k}^*\cdots T_{1,\beta_1}^*T_{i,s}^*h,\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h'\right\rangle \\ &= \left\langle \Delta_T(I)^{1/2}T_{k,\alpha_k}^*\cdots T_{1,\alpha_1}^*T_{i,s}^*h,h'\right\rangle \\ &= \left\langle h,T_{i,s}T_{1,\alpha_1}\cdots T_{i-1,\alpha_{i-1}}T_{i,\alpha_i}\cdots T_{k,\alpha_k}\Delta_T(I)^{1/2}h'\right\rangle \\ &= \overline{\boldsymbol{\lambda}_{i,1}(s,\alpha_1)}\cdots\overline{\boldsymbol{\lambda}_{i,i-1}(s,\alpha_{i-1})}\left\langle h,T_{1,\alpha_1}\cdots T_{i-1,\alpha_{i-1}}T_{i,g_s^i\alpha_i}\cdots T_{k,\alpha_k}\Delta_T(I)^{1/2}h'\right\rangle\end{aligned}$$

for all $\alpha_1\in\mathbb{F}_{n_1}^+,\ldots,\alpha_k\in\mathbb{F}_{n_k}^+$ where, for all $j\in\{1,\ldots,k\}$,

$$\boldsymbol{\lambda}_{i,j}(s,\beta) := \begin{cases}\prod_{b=1}^q \lambda_{i,j}(s,j_b) & \text{if } \beta=g_{j_1}^j\cdots g_{j_q}^j\in\mathbb{F}_{n_j}^+\\ 1 & \text{if } \beta=g_0^j.\end{cases} \tag{1.5}$$

Due to the definition of the noncommutative Berezin kernel K_T and using relation (1.2), we obtain

$$\begin{aligned}&\left\langle \left(S_{i,s}^*\otimes I\right)K_Th,\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h'\right\rangle \\ &= \left\langle S_{i,s}^*(\chi_{(\alpha_1,\ldots,\alpha_{i-1},g_s^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)})\otimes\Delta_T(I)^{1/2}T_{k,\alpha_k}^*\cdots T_{i+1,\alpha_{i+1}}^*T_{i,g_s^i\alpha_i}^*T_{i-1,\alpha_{i-1}}^*\cdots T_{1,\alpha_1}^*h,\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h'\right\rangle \\ &= \overline{\boldsymbol{\lambda}_{i,1}(s,\alpha_1)}\cdots\overline{\boldsymbol{\lambda}_{i,i-1}(s,\alpha_{i-1})}\left\langle h,T_{1,\alpha_1}\cdots T_{i-1,\alpha_{i-1}}T_{i,g_s^i\alpha_i}\cdots T_{k,\alpha_k}\Delta_T(I)^{1/2}h'\right\rangle.\end{aligned}$$

Consequently, we obtain

$$\left\langle \left(S_{i,s}^*\otimes I\right)K_Th,\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h'\right\rangle = \left\langle h,T_{1,\alpha_1}\cdots T_{i-1,\alpha_{i-1}}T_{i,g_s^i\alpha_i}\cdots T_{k,\alpha_k}\Delta_T(I)^{1/2}h'\right\rangle$$

and conclude that item (ii) holds. The proof is complete. □

Note that due to the doubly Λ-commutativity relations (1.3) satisfied by the standard shift $\mathbf{S} = (\mathbf{S}_1, \ldots, \mathbf{S}_n)$ and the fact that $S_{i,s}^* S_{i,t} = \delta_{st} I$ for every $i \in \{1, \ldots, k\}$ and $s, t \in \{1, \ldots, n_i\}$, and every polynomial in $\{S_{i,s}\}$ and $\{S_{i,s}^*\}$ is a finite sum the form

$$p(\{S_{i,s}\}, \{S_{i,s}^*\}) = \sum a_{(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_m)} S_{i_1,\alpha_1} \cdots S_{i_p,\alpha_p} S_{j_1,\beta_1}^* \cdots S_{j_m,\beta_m}^*,$$

where $\alpha_1 \in \mathbb{F}_{n_{i_1}}^+, \ldots, \alpha_p \in \mathbb{F}_{n_{i_p}}^+$ and $\beta_1 \in \mathbb{F}_{n_{j_1}}^+, \ldots, \beta_m \in \mathbb{F}_{n_{j_m}}^+$. We define

$$p(\{T_{i,s}\}, \{T_{i,s}^*\}) := \sum a_{(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_m)} T_{i_1,\alpha_1} \cdots T_{i_p,\alpha_p} T_{j_1,\beta_1}^* \cdots T_{j_m,\beta_m}^*$$

and note that the definition is correct due to the following von Neumann inequality obtained in [16], i.e.

$$\|p(\{T_{i,s}\}, \{T_{i,s}^*\})\| \leq \|p(\{S_{i,s}\}, \{S_{i,s}^*\})\|$$

for every k-tuple $T = (T_1, \ldots, T_k)$ in the *regular* Λ-*polyball*, which extends the classical result [21] and the noncommutative version for row contractions [13].

The Λ-polyball algebra $\mathcal{A}(\mathbf{B}_\Lambda)$ is the normed closed non-self-adjoint algebra generated by the isometries $S_{i,s}$, where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, and the identity. We denote by $C^*(\{S_{i,s}\})$ the C^*-algebra generated by the isometries $S_{i,s}$ We prove in [16] that if $T \in \mathbf{B}_\Lambda(\mathcal{H})$, then the map

$$\Psi_T(f) := \lim_{r\to 1} K_{rT}^*[f \otimes I]K_{rT}, \qquad f \in C^*(\{S_{i,s}\}),$$

where the limit is in the operator norm topology, is a is completely contractive linear map. Moreover, its restriction to the Λ-polyball algebra $\mathcal{A}(\mathbf{B}_\Lambda)$ is a completely contractive homomorphism. If, in addition, T is a pure k-tuple, i.e., for each $i \in \{1, \ldots, k\}$, $\Phi_{T_i}^p(I) \to 0$, as $p \to \infty$, then $\Psi_T(f) = K_T^*[f \otimes I]K_T$. We call the map Ψ_T the *noncommutative Berezin transform* at T associated with the Λ-polyball.

2. Noncommutative Hardy spaces associated with regular Λ-polyballs

In this section, we introduce the noncommutative Hardy algebra $F^\infty(\mathbf{B}_\Lambda)$, which can be seen as a noncommutative multivariable version of the Hardy algebra $H^\infty(\mathbb{D})$, and prove some basic properties.

According to relations (1.1) and (1.5), for each $i \in \{1, \ldots, k\}$ and $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we have

$$S_{i,g_s^i}(\chi_{\boldsymbol{\alpha}}) = \boldsymbol{\mu}_i(g_s^i, \boldsymbol{\alpha}) \chi_{(\alpha_1,\ldots,\alpha_{i-1},g_s^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)},$$

where

$$\boldsymbol{\mu}_i(g_s^i, \boldsymbol{\alpha}) := \boldsymbol{\lambda}_{i,1}(s, \alpha_1) \cdots \boldsymbol{\lambda}_{i,i-1}(s, \alpha_{i-1}).$$

Consequently, if $\gamma_i := g_{i_1}^i \cdots g_{i_p}^i \in \mathbb{F}_{n_i}^+$, then

$$S_{i,\gamma^i}(\chi_{\boldsymbol{\alpha}}) = \boldsymbol{\mu}_i(\gamma_i, \boldsymbol{\alpha}) \chi_{(\alpha_1,\ldots,\alpha_{i-1},\gamma_i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)},$$

where

$$\boldsymbol{\mu}_i(\gamma_i, \boldsymbol{\alpha}) := \boldsymbol{\mu}_i(g_{i_1}^i, \boldsymbol{\alpha}) \cdots \boldsymbol{\mu}_i(g_{i_p}^i, \boldsymbol{\alpha}).$$

Given $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we deduce that

$$S_{1,\gamma_1}\cdots S_{k,\gamma_k}(\chi_{\boldsymbol{\alpha}})=\boldsymbol{\mu}(\boldsymbol{\gamma},\boldsymbol{\alpha})\chi_{(\gamma_1\alpha_1,\ldots,\gamma_k\alpha_k)}$$

where

$$\boldsymbol{\mu}(\boldsymbol{\gamma},\boldsymbol{\alpha}):=\boldsymbol{\mu}_1(\gamma_1,\boldsymbol{\alpha})\cdots\boldsymbol{\mu}_k(\gamma_k,\boldsymbol{\alpha}).$$

Let $\{c_{(\beta_1,\ldots,\beta_k)}\}_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}$ be a sequence of complex numbers such $\sum|c_{(\beta_1,\ldots,\beta_k)}|^2<\infty$ and consider the formal series

$$\varphi(\{S_{i,s}\}):=\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}c_{(\beta_1,\ldots,\beta_k)}S_{1,\beta_1}\ldots S_{1,\beta_k}.$$

Set $\mathbf{g}_0:=(g_0^1,\ldots,g_0^k)$ and note that $\boldsymbol{\mu}(\boldsymbol{\beta},\mathbf{g_0})\in\mathbb{T}$ and

$$\begin{aligned}\varphi(\{S_{i,s}\})(\chi_{\mathbf{g}_0}):=&\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}c_{(\beta_1,\ldots,\beta_k)}S_{1,\beta_1}\ldots S_{1,\beta_k}(\chi_{\mathbf{g}_0})\\=&\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}c_{(\beta_1,\ldots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\mathbf{g_0})\chi_{(\boldsymbol{\beta_1},\ldots,\boldsymbol{\beta_k})}\end{aligned}$$

is an element in $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$. Similarly, for each $\boldsymbol{\gamma}:=(\gamma_1,\ldots,\gamma_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, we have $\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma})\in\mathbb{T}$ and

$$\varphi(\{S_{i,s}\})(\chi_{\boldsymbol{\gamma}})=\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}c_{(\beta_1,\ldots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma})\chi_{(\beta_1\gamma_1,\ldots,\beta_k\gamma_k)}$$

is an element in $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$. Now, let $\mathcal{P}$ be the linear span of the vectors $\{\chi_{\boldsymbol{\gamma}}\}_{\boldsymbol{\gamma}}$, assume that

$$\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(\{S_{i,s}\})p\|<\infty.$$

In this case, there is a unique operator $A\in B(\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+))$ such that $Ap=\varphi(\{S_{i,s}\})p$ for any $p\in\mathcal{P}$. We say that $\varphi(\{S_{i,s}\})$ is the formal Fourier series associated A. We denote by $F^\infty(\mathbf{B}_\Lambda)$ the set of all operators A obtained in this manner.

Theorem 2.1. *Let $\mathcal{P}(\{S_{i,s}\})$ be the algebra of all polynomials in $S_{i,s}$ and the identity, where $i\in\{1,\ldots,k\}$, and $s\in\{1,\ldots,n_i\}$. Then the noncommutative Hardy space $F^\infty(\mathbf{B}_\Lambda)$ is WOT- (resp. SOT-, w*-) closed and*

$$F^\infty(\mathbf{B}_\Lambda)=\overline{\mathcal{P}(\{S_{i,s}\})}^{\text{SOT}}=\overline{\mathcal{P}(\{S_{i,s}\})}^{\text{WOT}}=\overline{\mathcal{P}(\{S_{i,s}\})}^{\text{w}^*}.$$

Moreover, $F^\infty(\mathbf{B}_\Lambda)$ is the sequential SOT-(resp. WOT-, w-) closure of $\mathcal{P}(\{S_{i,s}\})$.*

Proof. First, we prove that the noncommutative Hardy space $F^\infty(\mathbf{B}_\Lambda)$ is WOT- (resp. SOT-) closed. Let $\{A_\iota\}_\iota$ be a net in $F^\infty(\mathbf{B}_\Lambda)$ and assume that WOT-$\lim_\iota A_\iota=A\in B(\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+))$ If $\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}c^\iota_{(\beta_1,\ldots,\beta_k)}S_{1,\beta_1}\ldots S_{1,\beta_k}$ is the formal Fourier series of A_ι, then

$$\left\langle A\chi_{\mathbf{g}_0},\chi_{(\beta_1,\ldots,\beta_k)}\right\rangle=\lim_\iota\left\langle A_\iota\chi_{\mathbf{g}_0},\chi_{(\beta_1,\ldots,\beta_k)}\right\rangle=\lim_\iota c^\iota_{(\beta_1,\ldots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\mathbf{g}_0).$$

Define $c_{(\beta_1,\ldots,\beta_k)}:=\frac{1}{\boldsymbol{\mu}(\boldsymbol{\beta},\mathbf{g}_0)}\left\langle A\chi_{\mathbf{g}_0},\chi_{(\beta_1,\ldots,\beta_k)}\right\rangle$ and note that $\lim_\iota c^\iota_{(\beta_1,\ldots,\beta_k)}=c_{(\beta_1,\ldots,\beta_k)}$. On the other hand, we have

$$\begin{aligned}\left\langle A\chi_{(\gamma_1,\dots,\gamma_k)},\chi_{(\beta_1\gamma_1,\dots,\beta_k\gamma_k)}\right\rangle &= \lim_\iota \left\langle A_\iota\chi_{(\gamma_1,\dots,\gamma_k)},\chi_{(\beta_1\gamma_1,\dots,\beta_k\gamma_k)}\right\rangle\\ &= \lim_\iota c^\iota_{(\beta_1,\dots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma})\\ &= c_{(\beta_1,\dots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma}).\end{aligned}$$

Note that

$$\sum_{(\beta_1,\dots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} |c_{(\beta_1,\dots,\beta_k)}|^2 = \sum_{(\beta_1,\dots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} |\left\langle A\chi_{\mathbf{g}_0},\chi_{(\beta_1,\dots,\beta_k)}\right\rangle|^2 = \|A\chi_{\mathbf{g}_0}\|^2<\infty$$

and consider the formal series

$$\varphi(\{S_{i,s}\}) := \sum_{(\beta_1,\dots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} c_{(\beta_1,\dots,\beta_k)}S_{1,\beta_1}\dots S_{1,\beta_k}.$$

Using the results above, one can see that

$$\begin{aligned}\left\langle A\chi_{(\gamma_1,\dots,\gamma_k)},\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle &= \lim_\iota \left\langle A_\iota\chi_{(\gamma_1,\dots,\gamma_k)},\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle\\ &= \lim_\iota\left\langle \sum_{(\beta_1,\dots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} c^\iota_{(\beta_1,\dots,\beta_k)}S_{1,\beta_1}\dots S_{1,\beta_k}\chi_{(\gamma_1,\dots,\gamma_k)},\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle\\ &= \lim_\iota\left\langle \sum_{(\beta_1,\dots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} c^\iota_{(\beta_1,\dots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma})\chi_{(\beta_1\gamma_1,\dots,\beta_k\gamma_k)},\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle\\ &= \begin{cases}\lim_\iota c^\iota_{(\beta_1,\dots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma}), & \text{if } (\alpha_1,\dots,\alpha_k)=(\beta_1\gamma_1,\dots,\beta_k\gamma_k)\\ 0, & \text{otherwise}\end{cases}\\ &= \begin{cases}c_{(\beta_1,\dots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma}), & \text{if } (\alpha_1,\dots,\alpha_k)=(\beta_1\gamma_1,\dots,\beta_k\gamma_k)\\ 0, & \text{otherwise}\end{cases}\\ &= \left\langle \varphi(\{S_{i,s}\})\chi_{(\gamma_1,\dots,\gamma_k)},\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle\end{aligned}$$

for all $(\gamma_1,\dots,\gamma_k),(\alpha_1,\dots,\alpha_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}$. Consequently, we have

$$\left\langle Ap,\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle = \left\langle \varphi(\{S_{i,s}\})p,\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle$$

for all $p\in\mathcal{P}$. Hence, we deduce that

$$\|Ap\|^2 = \sum_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} |\left\langle Ap,\chi_{(\alpha_1,\dots,\alpha_k)}\right\rangle|^2 = \|\varphi(\{S_{i,s}\})p\|^2$$

which implies $\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(\{S_{i,s}\})p\| = \|A\|$. This shows that $A\in F^\infty(\mathbf{B}_\Lambda)$ and $\varphi(\{S_{i,s}\})$ is its formal Fourier representation.

Now, we prove that any operator in $F^\infty(\mathbf{B}_\Lambda)$ is the SOT-limit of a sequence of polynomials in $S_{i,s}$ and the identity. For each $m\in\mathbb{Z}$, define the completely contractive linear map $Q_m : B(\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}))\to B(\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}))$ by setting

$$Q_m(T) := \sum_{n\ge\max\{0,-m\}} P_nTP_{n+m},$$

where P_n, $n \geq 0$, is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ onto the span of all vectors $\chi_{(\beta_1,\ldots,\beta_k)}$ such that $|\beta_1| + \cdots + |\beta_k| = n$, where $\beta_i \in \mathbb{F}_{n_i}^+$. Consider the Cesaro operators on $B(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+))$ defined by

$$C_n(T) := \sum_{|m|<n} \left(1 - \frac{|m|}{n}\right) Q_m(T), \qquad n \geq 1.$$

One can easily see that these operators are completely contractive and SOT-$\lim_{n\to\infty} C_n(T) = T$. Now, let $T \in F^\infty(\mathbf{B}_\Lambda)$ have the formal Fourier representation

$$\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1} \ldots S_{1,\beta_k}.$$

Using the definition of the isometries $S_{i,s}$ we deduce that

$$P_{n+m} T P_m = \left(\sum_{\substack{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=n}} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1} \ldots S_{1,\beta_k} \right) P_m$$

for all $n, m \geq 0$. On the other hand, we have $P_m T P_{n+m} = 0$ if $n \geq 1$ and $m \geq 0$. Consequently, we have

$$C_n(T) = \sum_{0\leq p\leq n-1} \left(1 - \frac{p}{n}\right) \left(\sum_{\substack{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=p}} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1} \ldots S_{1,\beta_k} \right)$$

and SOT-$\lim_{n\to\infty} C_n(T) = T$. This shows that T is the SOT-limit of a sequence of polynomials in $S_{i,s}$ and the identity. Consequently, T is also the WOT-(resp. w*-) limit of a sequence of polynomials in $S_{i,s}$ and the identity. Denoting by $\mathcal{P}(\{S_{i,s}\})$ the algebra of all polynomials in $S_{i,s}$ and the identity, we deduce that

$$F^\infty(\mathbf{B}_\Lambda) \subset \overline{\mathcal{P}(\{S_{i,s}\})}^{\text{SOT}} \subset \overline{\mathcal{P}(\{S_{i,s}\})}^{\text{WOT}}.$$

Since $\mathcal{P}(\{S_{i,s}\}) \subset F^\infty(\mathbf{B}_\Lambda)$ and $F^\infty(\mathbf{B}_\Lambda)$ is WOT-closed, we have $\overline{\mathcal{P}(\{S_{i,s}\})}^{\text{WOT}} \subset F^\infty(\mathbf{B}_\Lambda)$. Therefore,

$$F^\infty(\mathbf{B}_\Lambda) = \overline{\mathcal{P}(\{S_{i,s}\})}^{\text{SOT}} = \overline{\mathcal{P}(\{S_{i,s}\})}^{\text{WOT}}.$$

Due to the results above, we also have $F^\infty(\mathbf{B}_\Lambda) \subset \overline{\mathcal{P}(\{S_{i,s}\})}^{\text{w}^*}$. Moreover, since $F^\infty(\mathbf{B}_\Lambda)$ is a convex subset of $B(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+))$, we know that $F^\infty(\mathbf{B}_\Lambda)$ is w*-closed if and only if it is WOT sequential closed. Due to the results above, we conclude that $F^\infty(\mathbf{B}_\Lambda)$ is w*-closed. Since $\mathcal{P}(\{S_{i,s}\}) \subset F^\infty(\mathbf{B}_\Lambda)$, we have $\overline{\mathcal{P}(\{S_{i,s}\})}^{\text{w}^*} \subset F^\infty(\mathbf{B}_\Lambda)$ and conclude that $F^\infty(\mathbf{B}_\Lambda) = \overline{\mathcal{P}(\{S_{i,s}\})}^{\text{w}^*}$. The proof is complete. □

Corollary 2.2. *The noncommutative Hardy algebra $F^\infty(\mathbf{B}_\Lambda)$ is a Banach algebra.*

Theorem 2.3. *Let $A \in F^\infty(\mathbf{B}_\Lambda)$ have a formal Fourier representation*

$$\varphi(\{S_{i,s}\}) = \sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1} \ldots S_{k,\beta_k}.$$

Then $\varphi(\{rS_{i,s}\}) \in \mathcal{A}(\mathbf{B}_\Lambda)$, *for all* $r \in [0,1)$,

$$A = \text{SOT-}\lim_{r\to 1} \varphi(\{rS_{i,s}\})$$

and

$$\|A\| = \sup_{0\le r<1} \|\varphi(\{rS_{i,s}\})\| = \lim_{r\to 1} \|\varphi(\{rS_{i,s}\})\|.$$

Proof. Since Φ_{S_i} is a completely positive linear map with $\|\Phi_{S_i}(I)\| \le 1$, we have

$$\Phi_{S_1}^{p_1} \circ \cdots \circ \Phi_{S_k}^{p_k}(I) \le \|\Phi_{S_k}^{p_k}(I)\| \cdots \|\Phi_{S_1}^{p_1}(I)\| I \le \|\Phi_{S_k}(I)\|^{p_k} \cdots \|\Phi_{S_1}(I)\|^{p_1} I \le I$$

for all $p_1, \dots p_k \in \mathbb{N}$. Consequently, for every $r \in [0,1)$, we have

$$\begin{aligned}
&\sum_{p=0} r^p \left\| \sum_{\substack{p_1,\dots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\dots,|\beta_k|=p_k}} c_{(\beta_1,\dots,\beta_k)} S_{1,\beta_1}\dots S_{k,\beta_k} \right\| \\
&\quad \le \sum_{p=0} r^p \sum_{\substack{p_1,\dots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \left(\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\dots,|\beta_k|=p_k}} |c_{(\beta_1,\dots,\beta_k)}|^2 \right)^{1/2} \|\Phi_{S_1}^{p_1} \circ \cdots \circ \Phi_{S_k}^{p_k}(I)\|^{1/2}\| \\
&\quad \le \sum_{p=0} r^p \sum_{\substack{p_1,\dots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \left(\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\dots,|\beta_k|=p_k}} |c_{(\beta_1,\dots,\beta_k)}|^2 \right)^{1/2} \\
&\quad = \left(\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots\beta_k\in\mathbb{F}_{n_k}^+} |c_{(\beta_1,\dots,\beta_k)}|^2 \right)^{1/2} \left(\sum_{p=0} r^p \sum_{\substack{p_1,\dots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} 1 \right) \\
&\quad = \left(\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots\beta_k\in\mathbb{F}_{n_k}^+} |c_{(\beta_1,\dots,\beta_k)}|^2 \right)^{1/2} \sum_{p=0}^\infty r^p \binom{p+k-1}{k-1} < \infty.
\end{aligned}$$

This shows that

$$\varphi(\{rS_{i,s}\}) := \sum_{p=0}^\infty \sum_{\substack{(\beta_1,\dots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} r^{|\beta_1|+\cdots+|\beta_k|} c_{(\beta_1,\dots,\beta_k)} S_{1,\beta_1}\dots S_{k,\beta_k}$$

converges in the operator norm topology and, consequently, $\varphi(\{rS_{i,s}\}) \in \mathcal{A}(\mathbf{B}_\Lambda)$.

The next step is to show that

$$\|\varphi(\{rS_{i,s}\})\| \le \|A\|, \qquad r \in [0,1). \tag{2.1}$$

For each $n \in \mathbb{N}$, set

$$q_n(\{S_{i,s}\}) := \sum_{p=0}^{n} \sum_{\substack{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1}\ldots S_{k,\beta_k}$$

and note that

$$\varphi(\{rS_{i,s}\})^*\chi_{(\alpha_1,\ldots\alpha_k)} = q_n(\{rS_{i,s}\})^*\chi_{(\alpha_1,\ldots\alpha_k)}, \qquad r\in[0,1),$$

and

$$A^*\chi_{(\alpha_1,\ldots\alpha_k)} = q_n(\{S_{i,s}\})^*\chi_{(\alpha_1,\ldots\alpha_k)}$$

for all $(\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$ with $|\alpha_1|+\cdots+|\alpha_k|\leq n$. According to Theorem 1.1, the noncommutative Berezin transform $K_{rS}:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$ satisfies the relation $K_{rS}(rS_{i,s}^*) = (S_{i,s}^*\otimes I)K_{rS}$ for every $i\in\{1,\ldots,k\}$ and $s\in\{1,\ldots,n_i\}$. Let $\boldsymbol{\gamma}:=(\gamma_1,\ldots,\gamma_k)$, $\boldsymbol{\sigma}:=(\sigma_1,\ldots,\sigma_k)$, and $\boldsymbol{\omega}:=(\omega_1,\ldots,\omega_k)$ be in $\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. Due to the definition of $S_{i,s}$ we have $S_{k,\beta_k}^*\cdots S_{1,\beta_1}^*\chi_{\boldsymbol{\gamma}}=0$ if $|\beta_1|+\cdots+|\beta_k|>|\gamma_1|+\cdots+|\gamma_k|$. Using the relations above and the definition of K_{rS} and taking $n\geq|\gamma_1|+\cdots+|\gamma_k|$, we obtain

$$\begin{aligned}
&\langle K_{rS}\varphi(\{rS_{i,s}\})^*\chi_{\boldsymbol{\gamma}},\chi_{\boldsymbol{\sigma}}\otimes\chi_{\boldsymbol{\omega}}\rangle\\
&=\langle K_{rS}q_n(\{rS_{i,s}\})^*\chi_{\boldsymbol{\gamma}},\chi_{\boldsymbol{\sigma}}\otimes\chi_{\boldsymbol{\omega}}\rangle\\
&=\langle (q_n(\{S_{i,s}\})^*\otimes I)K_{rS}\chi_{\boldsymbol{\gamma}},\chi_{\boldsymbol{\sigma}}\otimes\chi_{\boldsymbol{\omega}}\rangle\\
&=\left\langle (q_n(\{S_{i,s}\})^*\otimes I)\left(\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}\chi_{(\beta_1,\ldots,\beta_k)}\otimes r^{|\beta_1+\cdots|\beta_k|}\Delta_{rS}(I)^{1/2}S_{k,\beta_k}^*\cdots S_{1,\beta_1}^*\chi_{\boldsymbol{\gamma}}\right),\chi_{\boldsymbol{\sigma}}\otimes\chi_{\boldsymbol{\omega}}\right\rangle\\
&=\sum_{\boldsymbol{\beta}:=(\beta_1,\ldots,\beta_k)\in\ \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} r^{|\beta_1+\cdots+|\beta_k|}\langle q_n(\{S_{i,s}\})^*\chi_{\boldsymbol{\beta}},\chi_{\boldsymbol{\sigma}}\rangle\left\langle S_{k,\beta_k}^*\cdots S_{1,\beta_1}^*\chi_{\boldsymbol{\gamma}},\Delta_{rS}(I)^{1/2}\chi_{\boldsymbol{\omega}}\right\rangle\\
&=\sum_{\boldsymbol{\beta}:=(\beta_1,\ldots,\beta_k)\in\ \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} r^{|\beta_1+\cdots+|\beta_k|}\langle A^*\chi_{\boldsymbol{\beta}},\chi_{\boldsymbol{\sigma}}\rangle\left\langle S_{k,\beta_k}^*\cdots S_{1,\beta_1}^*\chi_{\boldsymbol{\gamma}},\Delta_{rS}(I)^{1/2}\chi_{\boldsymbol{\omega}}\right\rangle\\
&=\langle (A^*\otimes I)K_{rS}\chi_{\boldsymbol{\gamma}},\chi_{\boldsymbol{\sigma}}\otimes\chi_{\boldsymbol{\omega}}\rangle
\end{aligned}$$

for all $r\in[0,1)$. Since A and $\varphi(\{rS_{i,s}\})$ are bounded operators, we deduce that

$$K_{rS}\varphi(\{rS_{i,s}\})^* = (A^*\otimes I)K_{rS}, \qquad r\in[0,1).$$

Since K_{rS} is an isometry, we have $\varphi(\{rS_{i,s}\}) = K_{rS}^*(A\otimes I)K_{rS}$ and

$$\|\varphi(\{rS_{i,s}\})\|\leq\|A\|, \qquad r\in[0,1), \tag{2.2}$$

which proves relation (2.1). Consequently, taking into account that

$$A\chi_{\boldsymbol{\alpha}} = \lim_{r\to1}\varphi(\{rS_{i,s}\}\chi_{\boldsymbol{\alpha}}, \qquad (\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+,$$

we conclude that $A=\text{SOT-}\lim_{r\to1}\varphi(\{rS_{i,s}\})$.

To prove the last part of the theorem, let $0<r_1<r_2<1$. Since $\varphi(\{r_2S_{i,s}\})\in\mathcal{A}(\mathbf{B}_\Lambda)$, inequality (2.2) applied to $A=\varphi(\{r_2S_{i,s}\})$ implies $\|\varphi(\{rr_2S_{i,s}\})\|\leq\|\varphi(\{r_2S_{i,s}\})|$ for any $r\in[0,1)$. Taking $r=\frac{r_1}{r_2}$, we deduce that $\|\varphi(\{r_1S_{i,s}\})\|\leq\|\varphi(\{r_2S_{i,s}\})|$. The rest of the proof is straightforward. $\square$

3. Functional calculus

In this section, we prove the existence of an $F^\infty(\mathbf{B}_\Lambda)$-functional calculus for the completely non-coisometric (c.n.c.) elements in the Λ-polyball. This extends the Sz.-Nagy–Foiaş functional calculus for c.n.c. contractions and the functional calculus for c.n.c. row contractions.

First, we consider the case of pure k-tuples in the regular Λ-polyball.

Theorem 3.1. *Let $T = (T_1, \ldots, T_k)$ be a pure k-tuple in the regular Λ-polyball $\mathbf{B}_\Lambda(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, and let $\Psi_T : F^\infty(\mathbf{B}_\Lambda) \to B(\mathcal{H})$ be defined by*

$$\Psi_T(A) := K_T^*(A \otimes I)K_T, \qquad A \in F^\infty(\mathbf{B}_\Lambda),$$

where K_T is the noncommutative Berezin kernel associated with T. Then the following statements hold.

(i) *Ψ_T is WOT-(resp. SOT-) continuous on bounded sets.*
(ii) *Ψ_T is a unital completely contractive homomorphism which is w*-continuous.*
(iii) *If*

$$\varphi(\{S_{i,s}\}) = \sum_{(\beta_1,\ldots,\beta_k)\in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1} \ldots S_{1,\beta_k}$$

is the formal Fourier representation of $A \in F^\infty(\mathbf{B}_\Lambda)$, then

$$\Psi_T(A) = \text{SOT-}\lim_{r\to 1} \varphi(\{rT_{i,s}\})$$

and $\Psi_T(p(\{S_{i,s}\})) = p(\{T_{i,s}\})$ for any polynomial $p(\{S_{i,s}\}) \in \mathcal{P}(\{S_{i,s}\})$.

Proof. Let $\{A_\iota\}_\iota$ be a bounded net in $F^\infty(\mathbf{B}_\Lambda)$. Then WOT-$\lim_\iota A_\iota = 0$ if and only if w*-$\lim_\iota A_\iota = 0$. The latter relation implies WOT-$\lim_\iota A_\iota \otimes I_\mathcal{H} = 0$ and w*-$\lim_\iota A_\iota \otimes I_\mathcal{H} = 0$. Now, it is clear that WOT-$\lim_\iota K_T^*(A_\iota \otimes I_\mathcal{H})K_T = 0$, thus Ψ_T is WOT-continuous. Since the map $A \mapsto A \otimes I_\mathcal{H}$ is SOT-continous on bounded sets, so is Ψ_T.

To prove (ii), note first that a net $\{A_\iota\}_\iota$ in $F^\infty(\mathbf{B}_\Lambda)$ converges to 0 in the w*-topology if and only if $A_\iota \otimes I_\mathcal{H} \to 0$ in the w*-topology. This implies that Ψ_T is continuous in the w*-topology.

On the other hand, since T is a pure k-tuple, the noncommutative Berezin kernel K_T is an isometry. Due to Theorem 1.1, we have

$$[\Psi_T(A_{ij})]_{m\times m} = \operatorname{diag}_m(K_T^*)\,[A_{ij} \otimes I]_{m\times m} \operatorname{diag}_m(K_T)$$

which implies

$$\left\|[\Psi_T(A_{ij})]_{m\times m}\right\| \le \left\|[A_{ij}]_{m\times m}\right\|$$

for every matrix $[A_{ij} \otimes I]_{m\times m}$ with entries in $F^\infty(\mathbf{B}_\Lambda)$. This proves that Ψ_T is a unital completely contractive linear map.

Due to Theorem 1.1, Ψ_T is a homomorphism on the algebra of polynomial $\mathcal{P}(\{S_{i,s}\})$ which, due to Theorem 2.1, is sequentially WOT-dense in $F^\infty(\mathbf{B}_\Lambda)$. Since Ψ_T is WOT- continuous on bounded sets and using the principle of uniform boundedness, one can easily see that Ψ_T is a homomorphism on $F^\infty(\mathbf{B}_\Lambda)$. This completes the proof of item (ii).

Now, we prove part (iii) of the theorem. According to Theorem 2.3, we have

$$A = \text{SOT-}\lim_{r\to 1} \varphi(\{rS_{i,s}\}) \quad \text{and} \quad \|A\| = \sup_{0\le r<1} \|\varphi(\{rS_{i,s}\})\|.$$

Since the map $X \mapsto X \otimes I_{\mathcal{H}}$ is SOT-continuous on bounded sets, we have

$$K_T^*(A \otimes I_{\mathcal{H}})K_T = \text{SOT-}\lim_{r\to 1} K_T^*(\varphi(\{rS_{i,s}\}) \otimes I_{\mathcal{H}})K_T. \tag{3.1}$$

On the other hand,

$$\varphi(\{rS_{i,s}\}) = \sum_{(\beta_1,\ldots,\beta_k)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} c_{(\beta_1,\ldots,\beta_k)} r^{|\beta_1|+\cdots+|\beta_k|} S_{1,\beta_1}\ldots S_{1,\beta_k}$$

is in $\mathcal{A}(\mathbf{B}_\Lambda)$ and the convergence is in the operator norm. Setting

$$q_n(\{rS_{i,s}\}) := \sum_{\substack{(\beta_1,\ldots,\beta_k)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|\le n}} c_{(\beta_1,\ldots,\beta_k)} r^{|\beta_1|+\cdots+|\beta_k|} S_{1,\beta_1}\ldots S_{1,\beta_k},$$

we have $\varphi(\{rS_{i,s}\}) = \lim_{n\to\infty} q_n(\{rS_{i,s}\})$. Using the von Neumann type inequality

$$\|q_n(\{rT_{i,s}\}) - q_m(\{rT_{i,s}\})\| \le \|q_n(\{rS_{i,s}\}) - q_m(\{rS_{i,s})\}\|, \tag{3.2}$$

we also deduce that $\varphi(\{rT_{i,s}\}) = \lim_{n\to\infty} q_n(\{rT_{i,s}\})$ in the norm topology. Consequently,

$$K_T^*(\varphi(\{rS_{i,s}\}) \otimes I_{\mathcal{H}})K_T = \lim_{n\to\infty} K_T^*(q_n(\{rS_{i,s}\}) \otimes I_{\mathcal{H}})K_T = \lim_{n\to\infty} q_n(\{rT_{i,s}\}) = \varphi(\{rT_{i,s}\}).$$

Hence, and using relation (3.1), we obtain

$$\Psi_T(A) = K_T^*(A \otimes I_{\mathcal{H}})K_T = \text{SOT-}\lim_{r\to 1} \varphi(\{rT_{i,s}\}).$$

The fact that $\Psi_T(p(\{S_{i,s}\})) = p(\{T_{i,s}\})$ for any polynomial $p(\{S_{i,s}\}) \in \mathcal{P}(\{S_{i,s}\})$ is due to Theorem 1.1. The proof is complete. $\square$

Lemma 3.2. *Let $T := (T_1,\ldots,T_k) \in \mathbf{B}_\Lambda(\mathcal{H})$ and let $A \in F^\infty(\mathbf{B}_\Lambda)$ have the Fourier representation*

$$\varphi(\{S_{i,s}\}) := \sum_{p=0}^{\infty} \sum_{\substack{(\beta_1,\ldots,\beta_k)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} c_{(\beta_1,\ldots,\beta_k)} S_{1,\beta_1}\ldots S_{k,\beta_k}.$$

Then the series defining $\varphi(\{rT_{i,s}\}$, $r\in[0,1)$, is convergent in the operator norm topology and

$$\varphi(\{rT_{i,s})\} = K_{rT}^*(A \otimes I_{\mathcal{H}})K_{rT}, \qquad r\in[0,1),$$

where K_T is the noncommutative Berezin kernel of T.

Proof. The fact that the series defining $\varphi(\{rT_{i,s})\}$, $r\in[0,1)$, is convergent in the operator norm topology follows from the proof of Theorem 3.1, where we showed that $\varphi(\{rT_{i,s}\}) = \lim_{n\to\infty} q_n(\{rT_{i,s}\})$. Moreover, if $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|q_n(\{rtS_{i,s}\}) - \varphi(\{rtT_{i,s}\})\| \leq \|q_n(\{rS_{i,s}\}) - \varphi(\{rT_{i,s}\})\| < \frac{\epsilon}{3}$$

for every $t \in [0,1]$ and $n \geq N$. Let $\delta \in (0,1)$ be such that

$$\|q_N(\{rtS_{i,s}\}) - \|q_N(\{rS_{i,s}\}))\| < \frac{\epsilon}{3}, \qquad t \in [\delta, 1).$$

Now, we can see that

$$\begin{aligned}\|\varphi(\{rS_{i,s}\}) - \varphi(\{rtS_{i,s}\})\| &\leq \|\varphi(\{rS_{i,s}\}) - q_N(\{rS_{i,s}\})\| + \|q_N(\{rS_{i,s}\}) - q_N(\{rtS_{i,s}\})\| \\ &= \|q_N(\{rtS_{i,s}\}) - \varphi(\{rtS_{i,s}\})\| < \epsilon\end{aligned}$$

for every $t \in [\delta, 1)$. This shows that $\varphi(\{rS_{i,s}\}) = \lim_{t\to 1} \varphi(\{rtS_{i,s}\})$ in the operator norm. On the other hand, as we saw in the proof of Theorem 3.1,

$$\varphi(\{rtS_{i,s}\}) = K_{rT}^*(\varphi(\{tS_{i,s}\}) \otimes I_{\mathcal{H}})K_{rT}, \qquad r, t \in [0,1).$$

Using the fact that $X \mapsto X \otimes I_{\mathcal{H}}$ is SOT-continuous on bounded sets and, due to Theorem 2.3, $A = \text{SOT-}\lim_{t\to 1} \varphi(\{tS_{i,s}\})$, we pass to the limit in the relation above as $t \to 1$ and obtain

$$\varphi(\{rT_{i,s})\} = K_{rT}^*(A \otimes I_{\mathcal{H}})K_{rT}, \qquad r \in [0,1).$$

The proof is complete. $\square$

We say that $T := (T_1, \ldots, T_k) \in \mathbf{B}_\Lambda(\mathcal{H})$ is a completely non-coisometric k-tuple if there is no $h \in \mathcal{H}$, $h \neq 0$, such that

$$\left\langle (id - \Phi_{T_k}^{p_k}) \circ \cdots \circ (id - \Phi_{T_1}^{p_1})(I)h, h \right\rangle = 0$$

for all $(p_1, \ldots, p_k) \in \mathbb{N}^k$. We saw in the proof of Theorem 1.1 that

$$(id - \Phi_{T_k}^{p_k}) \circ \cdots \circ (id - \Phi_{T_1}^{p_1})(I) = \sum_{s_1}^{p_1 - 1} \Phi_{T_1} \circ \cdots \circ \left(\sum_{s_k = 1}^{p_k - 1} \Phi_{T_k} \circ (\Delta_T(I)) \right).$$

This shows that the sequence $\left\{(id - \Phi_{T_k}^{p_k}) \circ \cdots \circ (id - \Phi_{T_1}^{p_1})(I)\right\}_{(p_1,\ldots,p_k)\in\mathbb{N}^k}$ is increasing and, consequently, T is completely non-coisometric if and only if there is no $h \in \mathcal{H}$, $h \neq 0$, such that

$$\lim_{p_k\to\infty} \cdots \left\langle \lim_{p_1\to\infty} (id - \Phi_{T_k}^{p_k}) \circ \cdots \circ (id - \Phi_{T_1}^{p_1})(I)h, h \right\rangle = 0.$$

Note that each pure k-tuple is completely non-coisometric.

The main result of this section is the following

Theorem 3.3. *Let $T := (T_1, \ldots, T_k) \in \mathbf{B}_\Lambda(\mathcal{H})$ be a completely non-coisometric tuple. Then*

$$\Psi_T(A) := \text{SOT-}\lim_{r\to 1} K_{rT}^*(A \otimes I_{\mathcal{H}})K_{rT}, \qquad A \in F^\infty(\mathbf{B}_\Lambda),$$

exists and defines a linear map $\Psi_T : F^\infty(\mathbf{B}_\Lambda) \to B(\mathcal{H})$ with the following properties.

(i) *If $\varphi(\{S_{i,s}\})$ is the Fourier representation of $A \in F^\infty(\mathbf{B}_\Lambda)$, then*

$$\Psi_T(A) = \text{SOT-}\lim_{r\to 1} \varphi(\{rT_{i,s}\}).$$

(ii) *Ψ_T is WOT-(resp. SOT-, w*-) continuous on bounded sets.*

(iii) *Ψ_T is a unital completely contractive homomorphism.*

Proof. Let $A \in F^\infty(\mathbf{B}_\Lambda)$ have the Fourier representation

$$\varphi(\{S_{i,s}\}) := \sum_{p=0}^{\infty} \sum_{\substack{(\beta_1,\dots,\beta_k)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=p}} c_{(\beta_1,\dots,\beta_k)} S_{1,\beta_1}\dots S_{k,\beta_k}.$$

According to Theorem 1.1, we have

$$T_{i,s}K_T^* = K_T^*(S_{i,s}\otimes I_{\mathcal{H}})$$

for all $i \in \{1,\dots,k\}$ and $s \in \{1,\dots,n_i\}$, where K_T is the noncommutative Berezin kernel of T. Since the series $\varphi(\{rS_{i,s}\})$, $r \in [0,1)$, is convergent in the operator norm, so is $\varphi(\{rT_{i,s}\})$. To see this, it is enough to use relation (3.2), where

$$q_n(\{rS_{i,s}\}) := \sum_{\substack{(\beta_1,\dots,\beta_k)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|\le n}} c_{(\beta_1,\dots,\beta_k)} r^{|\beta_1|+\cdots+|\beta_k|} S_{1,\beta_1}\dots S_{1,\beta_k}.$$

Now, note that

$$q_n(\{rT_{i,s}\})K_T^* = K_T^*(q_n(\{rS_{i,s}\})\otimes I_{\mathcal{H}}).$$

Taking $n\to\infty$, we deduce that

$$\varphi(\{rT_{i,s}\})K_T^* = K_T^*(\varphi(\{rS_{i,s}\})\otimes I_{\mathcal{H}}). \tag{3.3}$$

On the other hand, due to Theorem 2.3, we have

$$A\otimes I_{\mathcal{H}} = \text{SOT-}\lim_{r\to 1}\varphi(\{rS_{i,s}\})\otimes I_{\mathcal{H}}.$$

Using the later relation in (3.3), we deduce that the map $\Omega : \operatorname{range} K_T^* \to \mathcal{H}$ defined by

$$\Omega(K_T^* f) := \lim_{r\to 1}\varphi(\{rT_{i,s}\})K_T^* f, \qquad f \in \ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}_T,$$

is well-defined, linear, and

$$\begin{aligned}\|\Omega K_T^* f\| &\le \limsup_{r\to 1}\|\varphi(\{rT_{i,s}\})\|\|K_T^* f\| \\ &\le \limsup_{r\to 1}\|\varphi(\{rS_{i,s}\})\|\|K_T^* f\| \\ &\le \|A\|\|K_T^* f\|.\end{aligned}$$

Since T is a completely non-coisometric tuple, Theorem 1.1 shows that $K_T^* K_T$ is a one-to-one operator, which implies $\overline{\text{range}\, K_T^*} = \mathcal{H}$. Due to inequalities above, Ω has a unique extension $\widetilde{\Omega}$ to a bounded operator on $\mathcal{H}$ with $\|\widetilde{\Omega}\| \leq \|A\|$.

In what follows, we show that

$$\widetilde{\Omega}h = \lim_{r\to 1} \varphi(\{rT_{i,s}\})h, \qquad h \in \mathcal{H}. \tag{3.4}$$

Fix $h \in \mathcal{H}$ and let $\{h_k\}_{k=1}^\infty \subset \text{range}\, K_T^*$ such that $h_k \to h$ as $k \to \infty$. Since $\|\varphi(\{rT_{i,s}\})\| \leq \|\varphi(\{rS_{i,s}\})\| \leq \|A\|$ for every $r \in [0, 1)$, we deduce that

$$\begin{aligned}
\|\widetilde{\Omega}h - \varphi(\{rS_{i,s}\})h\| &\leq \|\widetilde{\Omega}h - \widetilde{\Omega}h_k\| + \|\widetilde{\Omega}h_k - \varphi(\{rT_{i,s}\})h_k\| + \|\varphi(\{rT_{i,s}\})h_k - \varphi(\{rT_{i,s}\})h\| \\
&\leq \|\widetilde{\Omega}\|\|h - h_k\| + \|\widetilde{\Omega}h_k - \varphi(\{rT_{i,s}\})h_k\| + \|\varphi(\{rT_{i,s}\})\|\|h_k - h\| \\
&\leq 2\|A\|\|h - h_k\| + \|\widetilde{\Omega}h_k - \varphi(\{rT_{i,s}\})h_k\|.
\end{aligned}$$

Using the fact that $\widetilde{\Omega}h_k - \lim_{r\to 1} \varphi(\{rT_{i,s}\})h_k$, we deduce relation (3.4). According to Lemma 3.2, we have

$$\varphi(\{rT_{i,s})\} = K_{rT}^*(A \otimes I_{\mathcal{H}})K_{rT}, \qquad r \in [0, 1).$$

Consequently, taking $r \to 1$ and using relation (3.4), we obtain

$$\widetilde{\Omega} = \text{SOT-}\lim_{r\to 1} K_{rT}^*(A \otimes I_{\mathcal{H}})K_{rT},$$

which shows that $\Psi_T(A) = \widetilde{\Omega}$. Therefore, item (i) holds. To prove part (ii), let $[A_{pq}]_{m\times m}$ be a matrix with entries in $F^\infty(\mathbf{B}_\Lambda)$ and let $\varphi_{pq}(\{S_{i,s}\}$ be the Fourier representation of A_{pq}. Lemma 3.2 shows that

$$[\varphi_{pq}(\{rT_{i,s})\}]_{m\times m} = \text{diag}_m(K_{rT}^*)[A_{pq} \otimes I_{\mathcal{H}}]_{m\times m}\text{diag}_m(K_{rT}), \qquad r \in [0, 1).$$

On the other hand, since K_{rT} is an isometry, we deduce that

$$\|[\varphi_{pq}(\{rT_{i,s})\}]_{m\times m}\| \leq \|[A_{pq} \otimes I_{\mathcal{H}}]_{m\times m}\|, \qquad r \in [0, 1), m \in \mathbb{N}.$$

Since $\Psi_T(A_{pq}) = \text{SOT-}\lim_{r\to 1} \varphi_{pq}(\{rT_{i,s}\})$, we deduce that Ψ_T is a completely contractive linear map. Now, using that fact that Ψ_T is a homomorphism on the algebra of polynomials $\mathcal{P}(\{S_{i,s}\})$ and that $F^\infty(\mathbf{B}_\Lambda)$ is the sequential WOT-closure of $\mathcal{P}(\{S_{i,s}\})$ (see Theorem 2.1), one can use the WOT-continuity of Ψ_T on bounded sets to deduce that Ψ_T is a homomorphism on $F^\infty(\mathbf{B}_\Lambda)$.

Now, we prove part (iii). Due to the proof of part (i), we have $\|\Psi_T(A)\| \leq \|A\|$ for all $A \in F^\infty(\mathbf{B}_\Lambda)$. On the other hand, taking $r \to 1$ in relation (3.3) we obtain

$$\Psi_T(A)K_T^* = K_T^*(A \otimes I_{\mathcal{H}}), \qquad A \in F^\infty(\mathbf{B}_\Lambda). \tag{3.5}$$

Let $\{A_\iota\}$ be a bounded net in $F^\infty(\mathbf{B}_\Lambda)$ such that $A_\iota \to A \in F^\infty(\mathbf{B}_\Lambda)$ in the WOT (resp. SOT). Then $A_\iota \otimes I_{\mathcal{H}} \to A \otimes I_{\mathcal{H}}$ in the WOT (resp. SOT). Due to relation (3.5), we have $\Psi_T(A_\iota)K_T^* = K_T^*(A_\iota \otimes I_{\mathcal{H}})$. Since $\overline{\text{range}\, K_T^*} = \mathcal{H}$ and $\{\Psi_T(A_\iota)\}_\iota$ is a bounded net, we can easily see that $\Psi_T(A_\iota) \to \Psi_T(A)$ in the WOT (resp. SOT). The proof is complete. $\square$

4. Free holomorphic functions on regular Λ-polyballs

In this section, we introduce the algebra $H^\infty(\mathbf{B}_\Lambda^\circ)$ of bounded free holomorphic functions on the interior of $\mathbf{B}_\Lambda(\mathcal{H})$, for any Hilbert space $\mathcal{H}$, and prove that it is completely isometric isomorphic to the noncommutative Hardy algebra $F^\infty(\mathbf{B}_\Lambda)$ introduced in Section 2. We also introduce the algebra $A(\mathbf{B}_\Lambda^\circ)$ and show that it is completely isometric isomorphic to the noncommutative Λ-polyball algebra $\mathcal{A}(\mathbf{B}_\Lambda)$.

If $A \in B(\mathcal{H})$ is an invertible positive operator, we write $A > 0$. Recall that if $X \in \mathbf{B}_\Lambda(\mathcal{H})$, then

$$\Delta_X(I) := (id - \Phi_{X_k}) \circ \cdots \circ (id - \Phi_{X_1})(I).$$

Proposition 4.1. *The set*

$$\mathbf{B}_\Lambda^\circ(\mathcal{H}) := \{X \in \mathbf{B}_\Lambda(\mathcal{H}) : \ \Delta_X(I) > 0\}$$

is relatively open in $\mathbf{B}_\Lambda(\mathcal{H})$ and

$$\overline{\mathbf{B}_\Lambda^\circ(\mathcal{H})} = \mathbf{B}_\Lambda(\mathcal{H}).$$

Moreover, the interior of $\mathbf{B}_\Lambda(\mathcal{H})$ coincides with $\mathbf{B}_\Lambda^\circ(\mathcal{H})$.

Proof. Let $X = (X_1, \ldots, X_k) \in \mathbf{B}_\Lambda^\circ(\mathcal{H})$ and assume that $\Delta_X(I) > cI$ for some $c > 0$. If $d \in (0, c)$, then there exists $\epsilon > 0$ such that for all $Y = (Y_1, \ldots Y_k) \in \mathbf{B}_\Lambda(\mathcal{H})$ with $\|X_i - Y_i\| < \epsilon$ for $i \in \{1, \ldots, k\}$, we have

$$-dI \le \Delta_Y(I) - \Delta_X(I) \le dI.$$

Hence,

$$\Delta_Y(I) = (\Delta_Y(I) - \Delta_X(I)) + \Delta_X(I) \ge (c - d)I > 0$$

and, consequently, $Y \in \mathbf{B}_\Lambda^\circ(\mathcal{H})$. Therefore, $\mathbf{B}_\Lambda^\circ(\mathcal{H})$ is a relatively open set in $\mathbf{B}_\Lambda(\mathcal{H})$.

Now, we prove that $\overline{\mathbf{B}_\Lambda^\circ(\mathcal{H})} = \mathbf{B}_\Lambda(\mathcal{H})$. To prove the inclusion $\overline{\mathbf{B}_\Lambda^\circ(\mathcal{H})} \subset \mathbf{B}_\Lambda(\mathcal{H})$, let $Y = (Y_1, \ldots Y_k) \in \overline{\mathbf{B}_\Lambda(\mathcal{H})}$, and let $Y^{(n)} = (Y_1^{(n)}, \ldots Y_k^{(n)}) \in \mathbf{B}_\Lambda(\mathcal{H})$ be a sequence such that $Y^{(n)} \to Y$, as $n \to \infty$, in the norm topology of $B(\mathcal{H})^{n_1 + \cdots + n_k}$. Since, for every $i, j \in \{1, \ldots, k\}$ with $i \ne j$ and every $s \in \{1, \ldots, n_i\}$, $t \in \{1, \ldots, n_j\}$,

$$Y_{i,s}^{(n)} Y_{j,t}^{(n)} = \lambda_{ij}(s,t) Y_{j,t}^{(n)} Y_{i,s}^{(n)},$$

taking $n \to \infty$, we obtain $Y_{i,s} Y_{j,t} = \lambda_{ij}(s,t) Y_{j,t} Y_{i,s}$. On the other hand, we have

$$(id - \Phi_{rY_k^{(n)}}) \circ \cdots \circ (id - \Phi_{rY_1^{(n)}})(I) \ge 0, \qquad r \in [0,1), n \in \mathbb{N},$$

which implies

$$(id - \Phi_{rY_k}) \circ \cdots \circ (id - \Phi_{rY_1})(I) \ge 0, \qquad r \in [0,1).$$

Consequently, $Y \in \mathbf{B}_\Lambda(\mathcal{H})$.

Now, we prove the inclusion $\mathbf{B}_\Lambda(\mathcal{H}) \subset \overline{\mathbf{B}_\Lambda^\circ(\mathcal{H})}$. Let $Y \in \mathbf{B}_\Lambda(\mathcal{H})$ and $r \in [0,1)$. According to Lemma 4.3 from [16], if $i, j \in \{1, \ldots, k\}$ with $i \ne j$, then $S_{i,s}$ commutes with $S_{j,\alpha} S_{j,\alpha}^*$ for any $s \in \{1, \ldots, n_i\}$ and $\alpha \in \mathbb{F}_{n_j}^+$. Moreover, for any $i_1, \ldots, i_p$ distinct elements in $\{1, \ldots, k\}$ and $\alpha_1 \in \mathbb{F}_{n_{i_1}}^+, \ldots, \alpha_p \in \mathbb{F}_{n_{i_p}}^+$,

$$\left(S_{i_1,\alpha_1}S^*_{i_1,\alpha_1}\right)\cdots\left(S_{i_p,\alpha_p}S^*_{i_p,\alpha_p}\right)=S_{i_1,\alpha_1}\cdots S_{i_p,\alpha_p}S^*_{i_p,\alpha_p}\cdots S^*_{i_1,\alpha_1}.$$

Consequently, we deduce that

$$(id-\Phi_{rS_k})\circ\cdots\circ(id-\Phi_{rS_1})(I)=\prod_{i=1}^k(I-\Phi_{rS_i}(I))\geq\prod_{i=1}^k(1-r^2)I.$$

Applying Theorem 1.1 when $X=tY$, $t\in[0,1)$, we obtain

$$\begin{aligned}(id-\Phi_{rtY_k})\circ\cdots\circ(id-\Phi_{rtY_1})(I)&=K^*_{tY}\left[(id-\Phi_{rS_k})\circ\cdots\circ(id-\Phi_{rS_1})(I)\right]K_{tY}\\&\geq\prod_{i=1}^k(1-r^2)I.\end{aligned}$$

Here, we use the fact that tY is a pure k–tuple and K_{tY} is an isometry. Taking $t\to 1$, we get

$$(id-\Phi_{rY_k})\circ\cdots\circ(id-\Phi_{rY_1})(I)\geq\prod_{i=1}^k(1-r^2)I$$

which shows that $rY\in\mathbf{B}^\circ_\Lambda(\mathcal{H})$ for all $r\in[0,1)$. Hence, it is clear that $Y\in\overline{\mathbf{B}^\circ_\Lambda(\mathcal{H})}$.

Now, we prove the last part of the proposition. If $X\in Int(\mathbf{B}_\Lambda(\mathcal{H}))$, the interior of $\mathbf{B}_\Lambda(\mathcal{H})$, then there exists $r_0\in(0,1)$ such that $\frac{1}{r_0}X\in\mathbf{B}_\Lambda(\mathcal{H})$. Hence, $X\in r_0\mathbf{B}_\Lambda(\mathcal{H})$. Thus $X=r_0Y$ for some $Y\in\mathbf{B}_\Lambda(\mathcal{H})$. We proved above that $r_0Y\in\mathbf{B}^\circ_\Lambda(\mathcal{H})$. Consequently, $Int(\mathbf{B}_\Lambda(\mathcal{H}))\subset\mathbf{B}^\circ_\Lambda(\mathcal{H})$. Since $\mathbf{B}^\circ_\Lambda(\mathcal{H})$ is relatively open in $\mathbf{B}_\Lambda(\mathcal{H})$, we conclude that $Int(\mathbf{B}_\Lambda(\mathcal{H}))=\mathbf{B}^\circ_\Lambda(\mathcal{H})$. The proof is complete. □

Corollary 4.2. $\mathbf{B}^\circ_\Lambda(\mathcal{H})=\bigcup_{0\leq r<1}r\mathbf{B}_\Lambda(\mathcal{H})$.

For each $i\in\{1,\ldots,k\}$, let $Z_i=(Z_{i,1},\ldots,Z_{i,n_i})$ be an n_i-tuple of noncommutative indeterminates subject to the relations

$$Z_{i,s}Z_{j,t}=\lambda_{ij}(s,t)Z_{j,t}Z_{i,s}$$

for every $i,j\in\{1,\ldots,k\}$ with $i\neq j$ and every $s\in\{1,\ldots,n_i\}$, $t\in\{1,\ldots,n_j\}$. We set $Z_{i,\alpha}:=Z_{i,p_1}\cdots Z_{i,p_m}$ if $\alpha=g^i_{p_1}\cdots g^i_{p_m}\in\mathbb{F}^+_{n_i}$, where $p_1,\ldots,p_m\in\{1,\ldots,n_i\}$ and $Z_{i,g^i_0}:=1$. If $\boldsymbol{\beta}:=(\beta_1,\ldots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}$, we denote $Z_{\boldsymbol{\beta}}:=Z_{1,\beta_1}\cdots Z_{k,\beta_k}$ and $a_{\boldsymbol{\beta}}:=a_{(\beta_1\ldots\beta_k)}\in\mathbb{C}$. A formal power series

$$\varphi:=\sum_{\boldsymbol{\beta}\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}}a_{\boldsymbol{\beta}}Z_{\boldsymbol{\beta}},\qquad a_{\boldsymbol{\beta}}\in\mathbb{C},$$

in indeterminates $Z_{i,s}$, where $i\in\{1,\ldots,k\}$ and $s\in\{1,\ldots,n_i\}$, is called free holomorphic function on $\mathbf{B}^\circ_\Lambda$ if the series

$$\varphi(\{X_{i,s}\}):=\sum_{p=0}^\infty\sum_{\substack{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_k)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}\\|\beta_1|+\cdots+|\beta_k|=p}}a_{\boldsymbol{\beta}}X_{\boldsymbol{\beta}}$$

is convergent in the operator norm topology for any $X\in\mathbf{B}^\circ_\Lambda(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. We remark that the coefficients of a free holomorphic functions on $\mathbf{B}^\circ_\Lambda$ are uniquely determined by its representation on an infinite dimensional separable Hilbert space. Indeed, assume that $\varphi(\{rS_{i,s}\})=0$ for any $r\in[0,1)$, where $\mathbf{S}=(\mathbf{S}_1,\ldots,\mathbf{S}_k)$ is the universal model associated with the Λ-polyball $\mathbf{B}_\Lambda$. Using relation (1.1), we obtain

$$\begin{aligned}0 &= \langle \varphi(\{rS_{i,s}\})\chi_{\mathbf{g}_0}, S_{1,\alpha_1}\cdots S_{k,\alpha_k}\chi_{\mathbf{g}_0}\rangle \\ &= r^{|\alpha_1|+\cdots+|\alpha_k|}\left\langle \sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\beta_1,\ldots,\beta_k)}\boldsymbol{\mu}(\boldsymbol{\beta},\mathbf{g_0})\chi_{(\beta_\mathbf{1},\ldots,\beta_\mathbf{k})}, \boldsymbol{\mu}(\boldsymbol{\alpha},\mathbf{g_0})\chi_{(\alpha_\mathbf{1},\ldots,\alpha_\mathbf{k})}\right\rangle \\ &= r^{|\alpha_1|+\cdots+|\alpha_k|}a_{(\alpha_1,\ldots,\alpha_k)}|\boldsymbol{\mu}(\boldsymbol{\alpha},\mathbf{g}_0)|^2 = r^{|\alpha_1|+\cdots+|\alpha_k|}a_{(\alpha_1,\ldots,\alpha_k)}.\end{aligned}$$

Hence, $a_{(\alpha_1,\ldots,\alpha_k)} = 0$, which proves our assertion. We denote by $Hol(\mathbf{B}_\Lambda^\circ)$ the set of all free holomorphic functions on $\mathbf{B}_\Lambda^\circ$.

Proposition 4.3. *Let $\mathbf{S} = (\mathbf{S}_1,\ldots,\mathbf{S}_k)$ be the universal model associated with the Λ-polyball $\mathbf{B}_\Lambda$. Then $\varphi := \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}}Z_{\boldsymbol{\beta}}$ is in $Hol(\mathbf{B}_\Lambda^\circ)$ if and only if the series*

$$\varphi(\{rS_{i,s}\}) := \sum_{p=0}^\infty \sum_{\substack{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} r^{|\beta_1|+\cdots+|\beta_k|}a_{\boldsymbol{\beta}}\mathbf{S}_{\boldsymbol{\beta}}$$

is convergent in the operator norm topology for all $r\in[0,1)$.

Proof. The direct implication is obvious. Note that the converse of the proposition is due to Theorem 3.1. □

We remark that $Hol(\mathbf{B}_\Lambda^\circ)$ is an algebra. Let $H^\infty(\mathbf{B}_\Lambda^\circ)$ be the set of all $\varphi\in Hol(\mathbf{B}_\Lambda^\circ)$ such that

$$\|\varphi\|_\infty := \sup\|\varphi(\{X_{i,s}\})\| < \infty,$$

where the supremum is taken over all $\{X_{i,s}\}\in\mathbf{B}_\Lambda^\circ(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that $H^\infty(\mathbf{B}_\Lambda^\circ)$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$. There is an operator space structure on $H^\infty(\mathbf{B}_\Lambda^\circ)$, in the sense of Ruan (see [10], p. 181), if we define the norms $\|\cdot\|_m$ on $M_{m\times m}(H^\infty(\mathbf{B}_\Lambda^\circ))$ by setting

$$\|[\varphi_{uv}]_{m\times m}\|_m := \sup\|[\varphi_{uv}(\{X_{i,s}\})]_{m\times m}\|,$$

where the supremum is taken over all $\{X_{i,s}\}\in\mathbf{B}_\Lambda^\circ(\mathcal{H})$ and any Hilbert space. We remark that if $\varphi\in Hol(\mathbf{B}_\Lambda^\circ)$ and $r\in[0,1)$, then φ is continuous on $r\mathbf{B}_\Lambda(\mathcal{H})$ and

$$\|\varphi(\{X_{i,s}\})\| \le \|\varphi(\{rS_{i,s}\})\|$$

for every $\{X_{i,s}\}\in r\mathbf{B}_\Lambda(\mathcal{H})$. Moreover, the series defining $\varphi(\{X_{i,s}\})$ converges uniformly on $r\mathbf{B}_\Lambda(\mathcal{H})$ in the operator norm topology.

Given $A\in F^\infty(\mathbf{B}_\Lambda)$ and a Hilbert space $\mathcal{H}$, we define the *noncommutative Berezin transform* associated with the regular Λ-polyball $\mathbf{B}_\Lambda^\circ(\mathcal{H})$ to be the map $\mathbf{B}[A]:\mathbf{B}_\Lambda^\circ(\mathcal{H})\to B(\mathcal{H})$ defined by

$$\mathbf{B}[A](X) := K_X^*[A\otimes I_\mathcal{H}]K_X, \qquad X\in\mathbf{B}_\Lambda^\circ(\mathcal{H}).$$

Theorem 4.4. *The map $\Gamma: H^\infty(\mathbf{B}_\Lambda^\circ)\to F^\infty(\mathbf{B}_\Lambda)$ defined by*

$$\Gamma\left(\sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}}Z_{\boldsymbol{\beta}}\right) := \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}}\mathbf{S}_{\boldsymbol{\beta}}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $f\in Hol(\mathbf{B}_\Lambda^\circ)$, then the following statements are equivalent.

(i) $f \in H^\infty(\mathbf{B}_\Lambda^\circ)$;
(ii) $\sup_{1\leq r<1} \|f(\{rS_{i,s}\})\| < \infty$;
(iii) *there exists* $A \in F^\infty(\mathbf{B}_\Lambda)$ *with* $f = \mathbf{B}[A]$, *where* $\mathbf{B}$ *is the noncommutative Berezin transform associated with the* Λ*-polyball* $\mathbf{B}_\Lambda^\circ$.

In this case, we have

$$\Gamma(f) = \text{SOT-}\lim_{r\to 1} f(\{rS_{i,s}\} \quad \textit{and} \quad \Gamma^{-1}(f) = \mathbf{B}[A].$$

Moreover, $\|\Gamma(f)\| = \sup_{1\leq r<1} \|f(\{rS_{i,s}\})\|$.

Proof. Let $f = \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}} Z_{\boldsymbol{\beta}}$ be in $H^\infty(\mathbf{B}_\Lambda^\circ)$. Since $r\mathbf{S} \in \mathbf{B}_\Lambda^\circ(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+))$ for all $r \in [0,1)$, the series

$$f(\{rS_{i,s}\}) := \sum_{p=0}^\infty \sum_{\substack{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} r^{|\beta_1|+\cdots+|\beta_k|} a_{\boldsymbol{\beta}} \mathbf{S}_{\boldsymbol{\beta}}$$

is convergent in the operator norm topology for all $r \in [0,1)$ and $M := \sup_{1\leq r<1} \|f(\{rS_{i,s}\})\| < \infty$. Consequently, for every $r \in [0,1)$ and $\gamma \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we have

$$f(\{rS_{i,s}\})(\chi_{\boldsymbol{\gamma}}) = \sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\beta_1,\ldots,\beta_k)} r^{|\beta_1|+\cdots+|\beta_k|} \boldsymbol{\mu}(\boldsymbol{\beta},\boldsymbol{\gamma}) \chi_{(\beta_1\gamma_1,\ldots,\beta_k\gamma_k)}$$

and

$$\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} |a_{(\beta_1,\ldots,\beta_k)}|^2 r^{2(|\beta_1|+\cdots+|\beta_k|)} = \|f(\{rS_{i,s}\})(\chi_{g_0})\|^2 < M^2.$$

Hence, $\sum_{(\beta_1,\ldots,\beta_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} |a_{(\beta_1,\ldots,\beta_k)}|^2 < M^2$ and, for every noncommutative polynomial $p \in \mathcal{P}$ in $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$, we have $f(\{rS_{i,s}\})p \to f(\{S_{i,s}\})p$ as $r \to 1$. Since $\sup_{1\leq r<1} \|f(\{rS_{i,s}\})\| < \infty$, we deduce that $\sup_{p\in\mathcal{P},\|p\|\leq 1} \|f(\{S_{i,s}\})p\| < \infty$. Consequently, $\sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}} S_{\boldsymbol{\beta}}$ is the Fourier series of an element $A \in F^\infty(\mathbf{B}_\Lambda)$ which, according to Theorem 2.3, satisfies the relation $A = \text{SOT-}\lim_{r\to 1} f(\{rS_{i,s}\}$ and $\|A\| = \sup_{1\leq r<1} \|f(\{rS_{i,s}\})\|$. This proves that Γ is a well-defined isometric linear map. The fact that Γ is surjective is due to Theorem 2.3 and the fact that $\|\varphi(\{X_{i,s}\})\| \leq \|\varphi(\{rS_{i,s}\})\|$ for any $\{X_{i,s}\} \in r\mathbf{B}_\Lambda(\mathcal{H})$. Passing to matrices, we can use similar techniques to show that Γ is a completely isometric isomorphism. The rest of the proof follows from Theorem 2.3 and Theorem 3.1. The proof is complete. $\square$

Denote by $A(\mathbf{B}_\Lambda^\circ)$ the set of all functions $f \in Hol(\mathbf{B}_\Lambda^\circ)$ such that the map $\mathbf{B}_\Lambda^\circ(\mathcal{H}) \ni X \mapsto f(X) \in B(\mathcal{H})$ has a continuous extension to $\mathbf{B}_\Lambda(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. Using standards arguments, we can show that $A(\mathbf{B}_\Lambda^\circ)$ is a Banach algebra with pointwise multiplication and the norm $\|\cdot\|_\infty$. It also has an operator space structure with respect to the norms $\|\cdot\|_m$, $m \in \mathbb{N}$, defined after Proposition 4.3. One can prove the following result.

Theorem 4.5. *The map* $\Gamma : A(\mathbf{B}_\Lambda^\circ) \to \mathcal{A}(\mathbf{B}_\Lambda)$ *defined by*

$$\Gamma\left(\sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}} Z_{\boldsymbol{\beta}}\right) := \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{\boldsymbol{\beta}} S_{\boldsymbol{\beta}}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $f \in Hol(\mathbf{B}_\Lambda^\circ)$, then the following statements are equivalent.

(i) $f \in A(\mathbf{B}_\Lambda^\circ)$;
(ii) $\lim_{r\to 1} f(\{rS_{i,s}\})$ *exists in the operator norm topology;*
(iii) *there exists* $A \in \mathcal{A}(\mathbf{B}_\Lambda)$ *with* $f = \mathbf{B}[A]$, *where* $\mathbf{B}$ *is the noncommutative Berezin transform.*

In this case, we have

$$\Gamma(f) = \text{SOT-}\lim_{r\to 1} f(\{rS_{i,s}\} \quad \textit{and} \quad \Gamma^{-1}(f) = \mathbf{B}[A].$$

Proof. Using Theorem 4.4, Theorem 4.9 from [16], and an approximation argument, one can complete the proof. □

5. Characteristic functions and multi-analytic models

In this section, we characterize the elements in the noncommutative Λ-polyball which admit a characteristic functions. We provide a model theorem for the class of completely non-coisometric k-tuple of operators in $\mathbf{B}_\Lambda(\mathcal{H})$ which admit characteristic functions, and show that the characteristic function is a complete unitary invariant for this class of k-tuples.

An operator $A : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ is called *multi-analytic* with respect to the universal model $\mathbf{S} = (\mathbf{S}_1, \ldots, \mathbf{S}_k)$, $\mathbf{S}_i = (S_{i,1}, \ldots, S_{i,n_i})$, associated with the Λ-polyball $\mathbf{B}_\Lambda$ if

$$A(S_{i,s} \otimes I_\mathcal{H}) = (S_{i,s} \otimes I_\mathcal{K})A$$

for every $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. If, in addition, A is a partial isometry, we call it an *inner multi-analytic* operator. The support of A is the smallest reducing subspace $\operatorname{supp}(A) \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$ under all the operators $S_{i,s}$, containing the co-invariant subspace $\overline{A^*(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K})}$. According to Theorem 5.1 from [16], we have

$$\operatorname{supp}(A) = \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L},$$

where $\mathcal{L} := (\mathbf{P}_\mathbb{C} \otimes I_\mathcal{H})\overline{A^*(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K})}$ and $\mathbf{P}_\mathbb{C}$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ onto $\mathbb{C}$ which is identified to the subspace $\mathbb{C}\chi_{(g_1^0,\ldots g_k^0)}$ of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$.

In [16], we proved the following Beurling type factorization result which extends the corresponding result, when $k = 1$, from [12].

Theorem 5.1. *Let* $\mathbf{S} = (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ *be the universal model associated with the Λ-polyball and let Y be a selfadjoint operator on the Hilbert space* $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$. *Then the following statements are equivalent.*

(i) *There is a multi-analytic operator* $A : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ *such that*

$$Y = AA^*.$$

(ii) $(id - \Phi_{\mathbf{S}_1 \otimes I_\mathcal{K}}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I_\mathcal{K}})(Y) \geq 0$, *where the completely positive maps* $\Phi_{\mathbf{S}_i \otimes I_\mathcal{K}}$ *are defined in Section 1.*

We recall [16] the construction of the operator A in part (i) of Theorem 5.1. Consider the subspace $\mathcal{G} := \overline{Y^{1/2}\left(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}\right)}$ and set

$$C_{i,s}(Y^{1/2}g) := Y^{1/2}(S_{i,s}^* \otimes I_\mathcal{K})g$$

for every $g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$, $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. The operator $C_{i,s}$ is well-defined on the range of $Y^{1/2}$ and can be extended by continuity to the space $\mathcal{G}$. Setting $M_{i,s} := C_{i,s}^*$, we note that $M = (M_1, \ldots, M_k)$, where $M_i = (M_{i,1}, \ldots, M_{i,n_i})$, is a pure element in the regular Λ-polyball $\mathbf{B}_\Lambda(\mathcal{G})$. Consequently, the associated noncommutative Berezin kernel $K_M : \mathcal{G} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_M(I)\mathcal{G}}$ is an isometry and

$$K_M M_{i,s}^* = \left(S_{i,s}^* \otimes I_\mathcal{G}\right) K_M$$

for every $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. One can see that the map

$$A := Y^{1/2} K_M^* : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_M(I)\mathcal{G}} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$$

is a multi-analytic operator and $Y = AA^*$.

Following the classical result of Beurling [1], we say that $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ is a *Beurling type jointly invariant subspace* under the operators $S_{i,s} \otimes I_\mathcal{K}$, where $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$, if there is an inner multi-analytic operator $\Psi : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ such that

$$\mathcal{M} = \Psi\left(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}\right).$$

In what follows, we use the notation $((\mathbf{S}_1 \otimes I_\mathcal{K})|_\mathcal{M}, \ldots, (\mathbf{S}_k \otimes I_\mathcal{K})|_\mathcal{M})$, where

$$(\mathbf{S}_i \otimes I_\mathcal{K})|_\mathcal{M} := ((S_{i,1} \otimes I_\mathcal{K})|_\mathcal{M}, \ldots, (S_{i,n_i} \otimes I_\mathcal{K})|_\mathcal{M}), \qquad i \in \{1, \ldots, k\}.$$

We proved in [16] the following characterization of the Beurling type jointly invariant subspaces under the universal model of the regular Λ-polyball. For a related result, in the commutative case, we refer the reader to [9].

Theorem 5.2. *Let $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ be a jointly invariant subspace under $S_{i,s} \otimes I_\mathcal{K}$, where $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. Then the following statements are equivalent.*

(i) *$\mathcal{M}$ is a Beurling type jointly invariant subspace.*
(ii) *$(id - \Phi_{\mathbf{S}_1 \otimes I_\mathcal{K}}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I_\mathcal{K}})(\mathbf{P}_\mathcal{M}) \geq 0$, where $\mathbf{P}_\mathcal{M}$ is the orthogonal projection onto $\mathcal{M}$.*
(iii) *The k-tuple $((\mathbf{S}_1 \otimes I_\mathcal{K})|_\mathcal{M}, \ldots, (\mathbf{S}_k \otimes I_\mathcal{K})|_\mathcal{M})$ is doubly Λ-commuting.*
(iv) *There is an isometric multi-analytic operator $\Psi : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ such that*

$$\mathcal{M} = \Psi\left(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}\right).$$

We say that two multi-analytic operators $A : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}_1 \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}_2$ and $A' : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}_1' \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}_2'$ coincide if there are two unitary operators $u_j \in B(\mathcal{K}_j, \mathcal{K}_j')$, $j = 1, 2$, such that

$$A'(I_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)} \otimes u_1) = (I_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)} \otimes u_2)A.$$

Lemma 5.3. *Let $A_s : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}_s \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$, $s = 1, 2$, be multi-analytic operators with respect to $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ such that $A_1 A_1^* = A_2 A_2^*$. Then there is a unique partial isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$A_1 = A_2(I_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)} \otimes V),$$

where $I_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)} \otimes V$ *is an inner multi-analytic operator with initial space* $\operatorname{supp}(A_1)$ *and final space* $\operatorname{supp}(A_2)$. *In particular, the multi-analytic operators* $A_1|_{\operatorname{supp}(A_1)}$ *and* $A_2|_{\operatorname{supp}(A_2)}$ *coincide.*

Proof. Using the definition of the universal model $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$, one can easily prove that $(id - \Phi_{\mathbf{S}_1}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k})(I) = \mathbf{P}_{\mathbb{C}}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ onto $\mathbb{C}1 \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$. Since A_1, A_2 are multi-analytic operators with respect to $\mathbf{S}$ and $A_1A_1^* = A_2A_2^*$, we deduce that

$$\begin{aligned}
\|(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_1})A_1^* f\|^2 &= \langle A_1(id - \Phi_{\mathbf{S}_1 \otimes I}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I})(I)A_1^* f, f\rangle \\
&= \langle (id - \Phi_{\mathbf{S}_1 \otimes I}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I})(A_1A_1^*)f, f\rangle \\
&= \langle (id - \Phi_{\mathbf{S}_1 \otimes I}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I})(A_2A_2^*)f, f\rangle \\
&= \langle A_2(id - \Phi_{\mathbf{S}_1 \otimes I}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I})(I)A_2^* f, f\rangle \\
&= \|(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_2})A_2^* f\|^2
\end{aligned}$$

for all $f \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$. Define $\mathcal{L}_s := \overline{(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_s})A_s^*(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K})}$, $s = 1, 2$, and consider the unitary operator $U : \mathcal{L}_1 \to \mathcal{L}_2$ defined by

$$U(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_1})A_1^* f := (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_2})A_2^* f, \qquad f \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}.$$

Now, we can extend U to a partial isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ with initial space $\mathcal{L}_1 = \operatorname{supp}(A_1)$ and final space $\mathcal{L}_2 = \operatorname{supp}(A_2)$. Moreover, we have $A_1V^* = A_2|_{\mathbb{C} \otimes \mathcal{H}_2}$. Since A_1, A_2 are multi-analytic operators with respect to $\mathbf{S}$, we deduce that $A_1(I_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)} \otimes V^*) = A_2$. The last part of the lemma is obvious. □

We say that $T = (T_1, \ldots, T_k) \in \mathbf{B}_\Lambda(\mathcal{H})$ has characteristic function if there is a Hilbert space $\mathcal{E}$ and a multi-analytic operator $\Psi : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{E} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}$ with respect to $S_{i,j}$, $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, such that

$$K_T K_T^* + \Psi\Psi^* = I,$$

where $K_T : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}(T)$ is the noncommutative Berezin kernel associated with T. According to Lemma 5.3, if there is a characteristic function for $T \in \mathbf{B}_\Lambda(\mathcal{H})$, then it is essentially unique.

Theorem 5.4. *A k-tuple $T = (T_1, \ldots, T_k)$ in the noncommutative Λ-polyball $\mathbf{B}_\Lambda(\mathcal{H})$ admits a characteristic function if and only if*

$$\Delta_{\mathbf{S} \otimes I}(I - K_T K_T^*) \geq 0,$$

where K_T is the noncommutative Berezin kernel associated with T and

$$\Delta_{\mathbf{S} \otimes I} := (id - \Phi_{\mathbf{S}_1 \otimes I}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k \otimes I}).$$

If, in addition, T is a pure k-tuple in $\mathbf{B}_\Lambda(\mathcal{H})$, then the following statements are equivalent.

(i) *T admits a characteristic function.*
(ii) *$(K_T\mathcal{H})^\perp$ is a Beurling type invariant subspace under all the operators $S_{i,s} \otimes I$.*
(iii) *The k-tuple $(\mathbf{S} \otimes I)|_{(K_T\mathcal{H})^\perp}$ is doubly Λ-commuting.*

(iv) *There is a Beurling type invariant subspace $\mathcal{M}$ under $S_{i,s}\otimes I_{\mathcal{D}}$ for some Hilbert space $\mathcal{D}$ such that $T_{i,s}^*=(S_{i,s}^*\otimes I_{\mathcal{D}})|_{\mathcal{M}^\perp}$ for any $i\in\{1,\dots,k\}, s\in\{1,\dots,n_i\}$ and*

$$\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}=\bigvee_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}(\mathbf{S}_{\boldsymbol{\alpha}}\otimes I_{\mathcal{D}})\mathcal{M}^\perp.$$

Proof. Assume that T has characteristic function. Then there is a multi-analytic operator Ψ such that $K_TK_T^*+\Psi\Psi^*=I$. Since Ψ is a multi-analytic operator and, $\Delta_{\mathbf{S}\otimes I}(I)=\mathbf{P}_{\mathbb{C}}\otimes I$, we have

$$\Delta_{\mathbf{S}\otimes I}(I-K_TK_T^*)=\Delta_{\mathbf{S}\otimes I}(\Psi\Psi^*)=\Psi\Delta_{\mathbf{S}\otimes I}(I)\Psi^*=\Psi(\mathbf{P}_{\mathbb{C}}\otimes I)\Psi^*\geq 0.$$

In order to prove the converse, we apply Theorem 5.1 to the operator $Y=I-K_TK_T^*$.

To prove the second part of the theorem, note that if T is a pure k-tuple in $\mathbf{B}_\Lambda$, then the Berezin kernel K_T is an isometry and $I-K_TK_T^*=\mathbf{P}_{\mathcal{M}}$, where $\mathbf{P}_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}:=(K_T\mathcal{H})^\perp$. Using the first part of the theorem and applying Theorem 5.2, one obtains the equivalences of the items (i), (ii), and (iii).

Due to Theorem 5.6 from [16], if $T=(T_1,\dots,T_k)$ is a pure k-tuple in the regular Λ-polyball and

$$K_T:\mathcal{H}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\overline{\Delta_T(I)(\mathcal{H})},$$

is the corresponding noncommutative Berezin kernel, then the dilation provided by Theorem 1.1 is minimal, i.e.

$$\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\overline{\Delta_T(I)(\mathcal{H})}=\bigvee_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}(\mathbf{S}_{\boldsymbol{\alpha}}\otimes I_{\mathcal{D}(T)})K_T\mathcal{H}.$$

Moreover, this dilation is unique up to an isomorphism. Setting $\mathcal{M}:=(K_T\mathcal{H})^\perp$, $\mathcal{D}:=\mathcal{D}(T):=\overline{\Delta_T(I)(\mathcal{H})}$, and identifying $\mathcal{H}$ with $K_T\mathcal{H}$, we conclude that (ii) $\implies$ (iv). Now, we prove the implication (iv) $\implies$ (ii).

Assume that $T\in\mathbf{B}_\Lambda(\mathcal{H})$ is a pure element and that there is a Beurling type invariant subspace $\mathcal{M}$ under $S_{i,s}\otimes I_{\mathcal{D}}$ such that $T_{i,s}^*=(S_{i,s}^*\otimes I_{\mathcal{D}})|_{\mathcal{M}^\perp}$ and $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}=\bigvee_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}(\mathbf{S}_{\boldsymbol{\alpha}}\otimes I_{\mathcal{D}})\mathcal{M}^\perp$. Using the uniqueness of the dilation provided by the noncommutative Berezin kernel associate with T, we deduce that there is a unitary operator $\Omega:\mathcal{D}(T)\to\mathcal{D}$ such that $(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)}\otimes\Omega)K_T\mathcal{H}=\mathcal{M}^\perp$. Hence, $(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)}\otimes\Omega)K_TK_T^*(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)}\otimes\Omega^*)=\mathbf{P}_{\mathcal{M}^\perp}$. Since $\mathcal{M}$ is a Beuling type invariant subspace, there is an inner multi-analytic operator $\Psi:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}$ such that

$$\mathbf{P}_{\mathcal{M}}=\Psi\Psi^*.$$

Now, one can easily see that

$$I-K_TK_T^*=\Phi\Phi^*,$$

where $\Phi:=(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)}\otimes\Omega^*)\Psi$ is an inner multi-analytic operator. The proof is complete. □

If T has characteristic function, the multi-analytic operator A provided by Theorem 5.1 when $Y=I-K_TK_T^*$, which we denote by Θ_T, is called the *characteristic function* of T. More precisely, due to the remarks following Theorem 5.1, one can see that Θ_T is the multi-analytic operator

$$\Theta_T:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\overline{\Delta_{M_T}(I)(\mathcal{M}_T)}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\overline{\Delta_T(I)(\mathcal{H})}$$

defined by $\Theta_T := (I - K_T K_T^*)^{1/2} K_{M_T}^*$, where

$$K_T : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}$$

is the noncommutative Berezin kernel associated with $T \in \mathbf{B}_\Lambda(\mathcal{H})$ and

$$K_{M_T} : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_{M_T}(I)(\mathcal{M}_T)}$$

is the noncommutative Berezin kernel associated with $M_T \in \mathbf{B}_\Lambda(\mathcal{M}_T)$. Here, we have

$$\mathcal{M}_T := \overline{\text{range}\,(I - K_T K_T^*)}$$

and $M_T := (M_1, \ldots, M_k)$ is the k-tuple with $M_i := (M_{i,1}, \ldots, M_{i,n_i})$ and $M_{i,s} \in B(\mathcal{M}_T)$ given by $M_{i,s} := A_{i,s}^*$, where $A_{i,s} \in B(\mathcal{M}_T)$ is uniquely defined by

$$A_{i,s}\left[(I - K_T K_T^*)^{1/2} f\right] := (I - K_T K_T^*)^{1/2}(S_{i,s} \otimes I)f$$

for all $f \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}$. According to Theorem 5.1, we have $K_T K_T^* + \Theta_T \Theta_T^* = I$.

Theorem 5.5. *Let $T = (T_1, \ldots, T_k)$ be a k-tuple in $\mathbf{B}_\Lambda(\mathcal{H})$ which admits characteristic function. Then T is pure if and only if the characteristic function Θ_T is an inner multi-analytic operator. Moreover, in this case $T = (T_1, \ldots, T_k)$ is unitarily equivalent to $G = (G_1, \ldots, G_k)$, where $G_i := (G_{i,1}, \ldots, G_{i,n_i})$ is defined by*

$$G_{i,s} := \mathbf{P}_{H_T}\,(S_{i,s} \otimes I)\,|_{H_T}, \quad i \in \{1, \ldots, k\}, s \in \{1, \ldots, n_i\},$$

and $\mathbf{P}_{H_T}$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}$ onto

$$H_T := \left\{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}\right\} \ominus \text{range}\,\Theta_T.$$

Proof. Assume that T is a pure k-tuple in $\mathbf{B}_\Lambda(\mathcal{H})$ which admits characteristic function. Theorem 1.1 shows that

$$K_T^* K_T = \lim_{p_k \to \infty} \ldots \lim_{p_1 \to \infty} (id - \Phi_{T_k}^{p_k}) \circ \cdots \circ (id - \Phi_{T_1}^{p_1})(I), \tag{5.1}$$

where the limits are in the weak operator theory. Consequently, since T is a pure k-tuple, the noncommutative Berezin kernel associated with T, i.e.,

$$K_T : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}$$

is an isometry. Moreover, the subspace $K_T\mathcal{H}$ is coinvariant under the operators $S_{i,s} \otimes I_{\overline{\Delta_T(I)(\mathcal{H})}}$, $i \in \{1, \ldots, k\}$, $s \in \{1, \ldots, n_i\}$, and $T_{i,s} = K_T^*(S_{i,s} \otimes I_{\overline{\Delta_T(I)(\mathcal{H})}})K_T$. Since $K_T K_T^*$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \overline{\Delta_T(I)(\mathcal{H})}$ onto $K_T\mathcal{H}$ and $K_T K_T^* + \Theta_T \Theta_T^* = I$, we deduce that Θ_T is a partial isometry and $K_T\mathcal{H} = H_T$. Taking into account that K_T is an isometry, we can identify $\mathcal{H}$ with $K_T\mathcal{H}$. Therefore, $T = (T_1, \ldots, T_k)$ is unitarily equivalent to $G = (G_1, \ldots, G_k)$.

Conversely, assume that Θ_T is an inner multi-analytic operator. Since $K_T K_T^* + \Theta_T \Theta_T^* = I$, and Θ_T is a partial isometry, the noncommutative Berezin kernel K_T is a partial isometry. On the other hand, since T is completely non-coisometric, K_T is a one-to-one partial isometry and, consequently, an isometry. Due to Theorem 1.1, relation (5.1) holds. Hence, we deduce that T is a pure k-tuple in $\mathbf{B}_\Lambda(\mathcal{H})$. The proof is complete. $\square$

Now, we are able to provide a model theorem for the class of completely non-coisometric k-tuple of operators in $\mathbf{B}_\Lambda(\mathcal{H})$ which admit characteristic functions.

Theorem 5.6. *Let $T = (T_1, \dots, T_k)$ be a completely non-coisometric k-tuple in the Λ-polyball $\mathbf{B}_\Lambda(\mathcal{H})$ which admits characteristic function, and let $\mathbf{S} := (\mathbf{S}_1, \dots, \mathbf{S}_k)$ be the universal model associated to $\mathbf{B}_\Lambda(\mathcal{H})$. Set*

$$\mathcal{D} := \overline{\Delta_T(I)(\mathcal{H})}, \qquad \mathcal{D}_* := \overline{\Delta_{M_T}(I)(\mathcal{M}_T)},$$

and $D_{\Theta_T} := (I - \Theta_T^*\Theta_T)^{1/2}$, where Θ_T is the characteristic function of T. Then T is unitarily equivalent to $\mathbb{G} := (\mathbb{G}_1, \dots, \mathbb{G}_k) \in \mathbf{B}_\Lambda(\mathbb{H}_T)$, where $\mathbb{G}_i := (\mathbb{G}_{i,1}, \dots, \mathbb{G}_{i,n_i})$ and, for each $i \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_i\}$, $\mathbb{G}_{i,s}$ is a bounded operator acting on the Hilbert space

$$\begin{aligned}\mathbb{H}_T := &\left[\left(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}\right) \bigoplus \overline{D_{\Theta_T}(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_*)}\right] \\ &\ominus \left\{\Theta_T\varphi \oplus D_{\Theta_T}\varphi : \ \varphi \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_*\right\}\end{aligned}$$

and is uniquely defined by the relation

$$\left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) \mathbb{G}_{i,s}^* f = (S_{i,s}^* \otimes I_\mathcal{D}) \left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) f$$

for every $f \in \mathbb{H}_T$. Here, $\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}$ is the orthogonal projection of the Hilbert space

$$\mathbb{K}_T := \left(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}\right) \bigoplus \overline{D_{\Theta_T}(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_*)}$$

onto the subspace $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$ and $\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}$ is a one-to-one operator.

Proof. A straightforward computation reveals that the operator $\Psi : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_* \to \mathbb{K}_T$ defined by

$$\Psi\varphi := \Theta_T\varphi \oplus D_{\Theta_T}\varphi, \quad \varphi \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_*,$$

is an isometry and

$$\Psi^*(g \oplus 0) = \Theta_T^* g, \qquad g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}. \tag{5.2}$$

Consequently, we deduce that

$$\|g\|^2 = \|\mathbf{P}^{\mathbb{K}_T}_{\mathbb{H}_T}(g \oplus 0)\|^2 + \|\Psi\Psi^*(g \oplus 0)\|^2 = \|\mathbf{P}^{\mathbb{K}_T}_{\mathbb{H}_T}(g \oplus 0)\|^2 + \|\Theta_T^* g\|^2$$

for every $g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$, where $\mathbf{P}^{\mathbb{K}_T}_{\mathbb{H}_T}$ the orthogonal projection of $\mathbb{K}_T$ onto the subspace $\mathbb{H}_T$. Since

$$\|K_T^* g\|^2 + \|\Theta_T^* g\|^2 = \|g\|^2, \quad g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D},$$

we have

$$\|K_T^* g\| = \|\mathbf{P}^{\mathbb{K}_T}_{\mathbb{H}_T}(g \oplus 0)\|, \quad g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}. \tag{5.3}$$

Due to the fact that the k-tuple $T = (T_1, \dots, T_k)$ is completely non-coisometric in $\mathbf{B}_\Lambda(\mathcal{H})$, the noncommutative Berezin kernel K_T is a one-to-one operator. Thus $\overline{\text{range}\, K_T^*} = \mathcal{H}$. Let $f \in \mathbb{H}_T$ be with the property that $\left\langle f, \mathbf{P}_{\mathbb{H}_T}^{\mathbb{K}_T}(g \oplus 0)\right\rangle = 0$ for any $g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$. Due to the definition of $\mathbb{H}_T$ and the fact that $\mathbb{K}_T$ coincides with the closed span of all vectors $g \oplus 0$, for $g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$, and $\Theta_T \varphi \oplus D_{\Theta_T}\varphi$, for $\varphi \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$, we must have $f = 0$. Consequently,

$$\mathbb{H}_T = \left\{\mathbf{P}_{\mathbb{H}_T}^{\mathbb{K}_T}(g \oplus 0) : \ g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}\right\}.$$

Now, using relation (5.3), we deduce that there is a unitary operator $\Gamma : \mathcal{H} \to \mathbb{H}_T$ such that

$$\Gamma(K_T^* g) = \mathbf{P}_{\mathbb{H}_T}^{\mathbb{K}_T}(g \oplus 0), \qquad g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}. \tag{5.4}$$

For each $i \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_i\}$, we define the operator $\mathbb{G}_{i,s} : \mathbb{H}_T \to \mathbb{H}_T$ by relation

$$\mathbb{G}_{i,s} := \Gamma T_{i,s} \Gamma^*, \qquad i \in \{1, \dots, k\}, s \in \{1, \dots, n_i\}.$$

Since $T \in \mathbf{B}_\Lambda(\mathcal{H})$, we also have $\mathbb{G} \in \mathbf{B}_\Lambda(\mathcal{H})$. The next step is to show that

$$\left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) \mathbb{G}_{i,s}^* f = (S_{i,s}^* \otimes I_\mathcal{D}) \left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) f \tag{5.5}$$

for every $i \in \{1, \dots, k\}$, $s \in \{1, \dots, n_i\}$, and $f \in \mathbb{H}_T$. Taking into account relations (5.4) and (5.2), the fact that Ψ is an isometry and $\mathbf{P}_{\mathbb{H}_T}^{\mathbb{K}_T}(g \oplus 0) + \Psi\Psi^*(g \oplus 0) = g \oplus 0$, we obtain

$$\begin{aligned}
\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}} \Gamma K_T^* g &= \mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}} \mathbf{P}^{\mathbb{K}_T}_{\mathbb{H}_T}(g \oplus 0) \\
&= g - \mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}} \Psi\Psi^*(g \oplus 0) \\
&= g - \Theta_T \Theta_T^* g = K_T K_T^* g
\end{aligned}$$

for all $g \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$. Since $\overline{\text{range}\, K_T^*} = \mathcal{H}$, we obtain

$$\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}} \Gamma = K_T. \tag{5.6}$$

On the other hand, since T is a completely non-coisometric tuple, the noncommutative Berezin kernel K_T is one-to-one. Now, relation (5.6) implies

$$\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T} = K_T \Gamma^*$$

and shows that $\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}$ is a one-to-one operator acting from $\mathbb{H}_T$ to $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$. Hence, using relation (5.6) and Theorem 1.1, we deduce that

$$\begin{aligned}
\left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) \mathbb{G}_{i,s}^* \Gamma h &= \left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) \Gamma T_{i,s}^* h \\
&= K_T T_{i,s}^* h \\
&= \left(S_{i,s}^* \otimes I_\mathcal{D}\right) K_T h \\
&= \left(S_{i,s}^* \otimes I_\mathcal{D}\right) \left(\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}}|_{\mathbb{H}_T}\right) \Gamma h
\end{aligned}$$

for every $i \in \{1, \dots, k\}$, $s \in \{1, \dots, n_i\}$, and $h \in \mathcal{H}$. Therefore, relation (5.5) holds. We remark that, since the operator $\mathbf{P}^{\mathbb{K}_T}_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \mathcal{D}}|_{\mathbb{H}_T}$ is one-to-one, the relation (5.5) uniquely determines each operator $\mathbb{G}^*_{i,s}$ for all $i \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_i\}$. The proof is complete. □

Now, we show that the characteristic function Θ_T is a complete unitary invariant for the completely non-coisometric k-tuples in $\mathbf{B}_\Lambda(\mathcal{H})$ which admit characteristic functions.

Theorem 5.7. *Let $T := (T_1, \dots, T_k) \in \mathbf{B}_\Lambda(\mathcal{H})$ and $T' := (T'_1, \dots, T'_k) \in \mathbf{B}_\Lambda(\mathcal{H}')$ be two completely non-coisometric k-tuples which admit characteristic functions. Then T and T' are unitarily equivalent if and only if their characteristic functions Θ_T and $\Theta_{T'}$ coincide.*

Proof. To prove the direct implication of the theorem, assume that the k-tuples T and T' are unitarily equivalent. Let $W : \mathcal{H} \to \mathcal{H}'$ be a unitary operator such that $T_{i,s} = W^* T'_{i,s} W$ for every $i \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_i\}$. Note that $W \Delta_T(I) = \Delta_{T'}(I) W$ and $W\mathcal{D} = \mathcal{D}'$, where the subspaces $\mathcal{D}$ and $\mathcal{D}'$ are given by

$$\mathcal{D} := \overline{\Delta_T(I)(\mathcal{H})}, \qquad \mathcal{D}' := \overline{\Delta_{T'}(I)(\mathcal{H}')}.$$

On the other hand, using the definition of the noncommutative Berezin kernel associated with Λ-polyballs, it is easy to see that $(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W) K_T = K_{T'} W$. Consequently,

$$(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W)(I - K_T K_T^*)(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W) = I - K_{T'} K_{T'}^*$$

and $(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W)\mathcal{M}_T = \mathcal{M}_{T'}$, where $\mathcal{M}_T := \overline{\operatorname{range}(I - K_T K_T^*)}$ and $\mathcal{M}_{T'} := \overline{\operatorname{range}(I - K_{T'} K_{T'}^*)}$. As mentioned in the remarks preceding Theorem 5.5, $M_T := (M_1, \dots, M_k) \in \mathbf{B}_\Lambda(\mathcal{M}_T)$ is the k-tuple with $M_i := (M_{i,1}, \dots, M_{i,n_i})$ and $M_{i,s} \in B(\mathcal{M}_T)$, where $M_{i,s} := A^*_{i,s}$ and $A_{i,s} \in B(\mathcal{M}_T)$ is uniquely defined by relation

$$A_{i,s}\left[(I - K_T K_T^*)^{1/2} x\right] := (I - K_T K_T^*)^{1/2} (S_{i,s} \otimes I) x$$

for all $x \in \ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \overline{\Delta_T(I)(\mathcal{H})}$. In a similar manner, we define the k-tuple $M_{T'} \in \mathbf{B}_\Lambda(\mathcal{M}_{T'})$ and the operators $A'_{i,s} \in B(\mathcal{M}_{T'})$. It is easy to see that

$$\begin{aligned} A_{i,s}(I - K_T K_T^*)^{1/2} f &= (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W^*) A'_{i,s} (I - K_{T'} K_{T'}^*)^{1/2} (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W^*) f \\ &= (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W^*) A'_{i,s} (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W)(I - K_T K_T^*)^{1/2} f \end{aligned}$$

for all $f \in \ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \overline{\Delta_T(I)(\mathcal{H})}$. This implies

$$A_{i,s} = (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W^*) A'_{i,s} (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W).$$

It is clear now that $(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W)\mathcal{D}_* = \mathcal{D}'_*$, where $\mathcal{D}_* := \overline{\Delta_{M_T}(I)(\mathcal{M}_T)}$ and $\mathcal{D}'_* := \overline{\Delta_{M_{T'}}(I)(\mathcal{M}_{T'})}$. Define the unitary operators u and u' by setting

$$u := W|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}' \quad \text{and} \quad u_* := (I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes W)|_{\mathcal{D}_*} : \mathcal{D}_* \to \mathcal{D}'_*.$$

Straightforward calculations reveal that

$$(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes u)\Theta_T = \Theta_{T'}(I_{\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k})} \otimes u_*),$$

which proves the direct implication of the theorem. Conversely, assume that the characteristic functions of T and T' coincide. In this case, there exist unitary operators $u : \mathcal{D} \to \mathcal{D}'$ and $u_* : \mathcal{D}_* \to \mathcal{D}'_*$ such that

$$(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u)\Theta_T = \Theta_{T'}(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*).$$

Hence, we deduce that

$$D_{\Theta_T} = \left(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*\right)^* D_{\Theta_{T'}} \left(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*\right)$$

and

$$\left(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*\right) \overline{D_{\Theta_T}(\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}_*)} = \overline{D_{\Theta_{T'}}(\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}'_*)},$$

where $D_{\Theta_T} := (I - \Theta_T^*\Theta_T)^{1/2}$. Define now the unitary operator $U : \mathcal{K}_T \to \mathcal{K}_{T'}$ by setting

$$U := (I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u) \oplus (I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*).$$

It is easy to see that the operator $\Psi : \ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}_* \to \mathbb{K}_T$, defined by

$$\Psi\varphi := \Theta_T\varphi \oplus D_{\Theta_T}\varphi, \quad \varphi \in \ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}_*,$$

and the corresponding operator Ψ' satisfy the relations

$$U\Psi\left(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*\right)^* = \Psi' \tag{5.7}$$

and

$$\left(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u\right) P^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}} U^* = P^{\mathbb{K}_{T'}}_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}'}, \tag{5.8}$$

where $P^{\mathbb{K}_T}_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}}$ is the orthogonal projection of $\mathbb{K}_T$ onto $\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}$. On the other hand relation (5.7) implies

$$\begin{aligned} U\mathbb{H}_T &= U\mathbb{K}_T \ominus U\Psi(\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}_*) \\ &= \mathbb{K}_{T'} \ominus \Psi'(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u_*)(\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}_*) \\ &= \mathbb{K}_{T'} \ominus \Psi'(\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{D}'_*). \end{aligned}$$

Consequently, $U|_{\mathbb{H}_T} : \mathbb{H}_T \to \mathbb{H}_{T'}$ is a unitary operator. We remark that

$$(S_{i,s}^* \otimes I_{\mathcal{D}'})(I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u) = (I_{\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)} \otimes u)(S_{i,s}^* \otimes I_{\mathcal{D}}) \tag{5.9}$$

for every $i \in \{1,\ldots,k\}$ and $s \in \{1,\ldots,n_i\}$. Let $\mathbb{G} := (\mathbb{G}_1,\ldots,\mathbb{G}_n)$ and $\mathbb{G}' := (\mathbb{G}'_1,\ldots,\mathbb{G}'_n)$ be the model operators provided by Theorem 5.6 for T and T', respectively. Taking into account relations (5.8), (5.9), and relation (5.5) for T' and T, respectively, we deduce that

$$
\begin{aligned}
P^{\mathbb{K}_{T'}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}'} {\mathbb{G}'_{i,s}}^* Uf &= (S^*_{i,s}\otimes I_{\mathcal{D}'}) P^{\mathbb{K}_{T}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}} Ux \\
&= (S^*_{i,s}\otimes I_{\mathcal{D}'})(I_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})}\otimes u) P^{\mathbb{K}_{T}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}} f \\
&= (I_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})}\otimes u)(S^*_{i,s}\otimes I_{\mathcal{D}}) P^{\mathbb{K}_{T}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}} f \\
&= (I_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})}\otimes u) P^{\mathbb{K}_{T}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}} \mathbb{G}^*_i f \\
&= P^{\mathbb{K}_{T'}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}'} U\mathbb{G}^*_{i,s} f
\end{aligned}
$$

for every $f\in\mathbb{H}_T$, $i=\{1,\dots,k\}$, and $s\in\{1,\dots,n_i\}$. According to Theorem 5.6, $P^{\mathbb{K}_{T'}}_{\ell^2(\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k})\otimes\mathcal{D}'}|_{\mathbb{H}_{T'}}$ is a one-to-one operator. Consequently, the relations above imply $(U|_{\mathbb{H}_T})\,\mathbb{G}^*_{i,s} = (\mathbb{G}'_{i,s})^*\,(U|_{\mathbb{H}_T})$ for every $i\in\{1,\dots,k\}$ and $s\in\{1,\dots,n_i\}$. Using Theorem 5.6, we conclude that the k-tuples T and T' are unitarily equivalent. This completes the proof. $\square$

Corollary 5.8. *If $T:=(T_1,\dots,T_k)\in\mathbf{B}_\Lambda(\mathcal{H})$ is completely non-coisometric and has characteristic function $\Theta_T=0$, then T is unitarily equivalent to $(\mathbf{S}_1\otimes I_{\mathcal{D}},\dots,\mathbf{S}_k\otimes I_{\mathcal{D}})$ for some Hilbert space $\mathcal{D}$.*

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