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Accumulation Point and C. T. Yang's Theorem in L-fuzzy Topological Spaces*

Mao-kang Luo, Ji-hua Liang, and Ying-ming Liu

Institute of Mathematics, Sichuan University, Chengdu 610064, China

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The investigations on accumulation point in fuzzy topological spaces (L-fuzzy topological spaces) have lasted for more than 16 years (e.g., see [9], [11], [6], [15], [17], [20], [13], [4], [1], [18], [16], [5]). Amongst these researches, the notion introduced in [9] even yielded the fuzzy form of the famous C. T. Yang's Theorem. But, however, the main basic problem, making the derivation operator derived from it preserve finite joins, is still unsolved; even only inclusion order of subsets cannot be preserved. Compared with the fundamental position of this concept standing at in topology, this shortage is obvious. With a new operation possessing many quite nice properties, the so-called "quasi-difference" between two L-fuzzy subsets, a reasonable definition is introduced for this basic notion in this paper which possesses the wanted properties and, applying its nice property, the L-fuzzy form of the C. T. Yang's Theorem is proved.

Key Words: accumulation point; quasi-difference; adherent point; L-fuzzy topological space

1. PRELIMINARIES

For convenience, we introduce some frequently used notions and results in L-fuzzy topological spaces.

In the sequel, X always stands for a nonempty ordinary set, L always means a completely distributive lattice with an order-reversing involution $' : L \to L$, and call every lattice of this kind an *F-lattice*. The smallest element and the largest elements of L are denoted by 0_L and 1_L, or 0 and 1 for short, respectively. For every $A \in L^X$, denote $\text{supp}(A) = \{x \in X :$

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$A(x) > 0\}$; call it the *support* of A. Also define $A' \in L^X$ by $A'(x) = A(x)'$ for every $x \in X$. For every $a \in L$, let $\underline{a}$ denote the constant mapping from X to L with value a.

Note that the partial order $\leq$ on L naturally induces a pointwise order $\leq$ on L^X as follows: For every pair $U, V \in L^X$,

$$U \leq V \quad \Leftrightarrow \quad \forall x \in X, U(x) \leq V(x).$$

This order is also a partial order, and it makes L^X to be an F-lattice. So for convenience, we always consider that every subfamily $\mathscr{A} \subset L^X$ has been equipped with the relative order in L^X.

DEFINITION 1.1. A relation $\leq$ on a set D is called *directed*, if for every finite $D_0 \subset D$, there exists $d_0 \in D$ such that $d \leq d_0$ for every $d \in D_0$. A set D equipped with a directed preorder $\leq$ is called a *directed set* or *up-directed set*. The dual notion *down-directed set* is dually defined.

DEFINITION 1.2. $\forall a \in L$, denote

$$\uparrow a = \{b \in L \colon b \geq a\}, \qquad \downarrow a = \{b \in L \colon L\, b \leq a\}.$$

$\alpha \in L$ is called *join-irreducible*, if for every two $a, b \in L$,

$$\alpha \leq a \vee b \quad \Rightarrow \quad \alpha \leq a \text{ or } \alpha \leq b;$$

every nonzero join-irreducible element is called a *molecule*. For every $A \subset L$, denote the set of all the molecules contained in A by $M(A)$.

Note that for a nonempty ordinary set X, an F-lattice L and an L-fuzzy subset $A \in L^X$, according to the symbol defined above, $\downarrow A$ is the set of all the elements in L^X smaller than A. So the set of all the molecules of L^X smaller than A is just

$$M(\downarrow A) = \{x_\lambda : x \in X, \lambda \in M(L), \lambda \leq A(x)\}.$$

DEFINITION 1.3. $\forall x \in X$, $a \in L$, denote the L-fuzzy subset taking value a at x and value 0 at other points of X by x_a; call it an *L-fuzzy point* on X. For every $\mathscr{A} \subset L^X$, denote the set of all the L-fuzzy points on X contained in $\mathscr{A}$ by $\mathrm{pt}(\mathscr{A})$.

$\delta \subset L^X$ is called an *L-fuzzy topology* on X, if δ is closed under arbitrary joins and finite meets, especially, $\underline{0}, \underline{1} \in \delta$. Call (L^X, δ) an *L-fuzzy topological space*, or call it an L-fts for short. Every $U \in \delta$ is called an *open subset* in (L^X, δ), and every $P \in L^X$ such that $P' \in \delta$ is called a *closed subset* in (L^X, δ). Denote the family of all the closed subsets in (L^X, δ) by δ'.

DEFINITION 1.4. Let (L^X, δ) be an L-fts, $A \in L^X$. Define, respectively, the *interior* A° and the *closure* A^- of A as

$$A^\circ = \bigvee \{U \in \delta : U \leq A\}, \quad A^- = \bigwedge \{P \in \delta' : P \geq A\}.$$

For every two $A, B \in L^X$, say, A *quasi-coincides with* B, denoted by $A\hat{q}B$, if there exists $x \in X$ such that $A(x) \nleq B'(x)$; otherwise, denote it by $A \neg \hat{q}B$.

Denote the set of all the points in X, at which $A\hat{q}B$, by $A \mathbin{/\!\!\!\wedge\!\!\backslash} B$, i.e.,

$$A \mathbin{/\!\!\!\wedge\!\!\backslash} B = \left\{x \in X\colon x_{A(x)}\hat{q}B\right\}.$$

Let $x_a \in \mathrm{pt}(L^X)$, $U \in \delta$, $P \in \delta'$. U is called a *quasi-coincident neighborhood* of x_a, if $x_a\hat{q}U$; denote the family of all the quasi-coincident neighborhoods of x_a by $\mathscr{Q}(x_a)$, called the *quasi-coincident neighborhood system* of x_a in (L^X, δ).

According to the previous stipulation, every quasi-coincident neighborhood system $\mathscr{Q}(x_a)$ in (L^X, δ) is equipped with the relative order in L^X.

DEFINITION 1.5. Define a relation $\preccurlyeq$ on L as follows: For every two $a, b \in L$, $a \preccurlyeq b$ if and only if for every $C \subset L$ such that $\bigvee C \geq b$, there exists $c \in C$ such that $a \leq c$. Denote $\beta(a) = \{b \in L\colon b \preccurlyeq a\}$. Every subset $D \subset \beta(a)$ satisfying $\bigvee D = a$ is called a *minimal set* of a in L.

THEOREM 1.6. [7, 12, 13, 19]. *Let L be a complete lattice. Then the following conditions are equivalent*:

(i) *L is completely distributive.*

(ii) *Every element of L has a minimal set.*

(iii) *Every element of L has a minimal set consisting of molecules in L.* ∎

COROLLARY 1.7. *Every element in a completely distributive lattice can be represented as a join of molecules.*

THEOREM 1.8. *Let (L^X, δ) be an L-fts. Then for every $x_\lambda \in M(L^X)$, $\mathscr{Q}(x_\lambda)$ is a down-directed set in L^X and $\underline{0} \notin \mathscr{Q}(x_a)$.*

PROPOSITION 1.9. *Let (L^X, δ) be an L-fts, $A, B, C \in L^X, \{A_t : t \in T\} \subset L^X$, $x \in X$, $a \in L \setminus \{0\}$. Then*

(i) $\{x \in X : A(x) \nleq B(x)'\}$.

(ii) $A\hat{q}B$ *at* $x \Leftrightarrow x \in A \mathbin{/\!\!\!\wedge\!\!\backslash} B$.

(iii) $A\hat{q}B \Leftrightarrow B \nleq A'$.

(iv) $A \leq B \Rightarrow A \mathbin{/\!\!\!\wedge\!\!\backslash} C \subset B \mathbin{/\!\!\!\wedge\!\!\backslash} C$.

(v) $A \mathbin{/\!\!\!\wedge\!\!\backslash} \bigvee_{t \in T} A_t = \bigcup_{t \in T} A \mathbin{/\!\!\!\wedge\!\!\backslash} A_t$.

(vi) $A\hat{q} \bigvee_{t \in T} A_t \Leftrightarrow \exists t \in T,\ A\hat{q}A_t$.

(vii) $A \leq B,\ C\hat{q}A \Rightarrow C\hat{q}B$. ∎

2. QUASIDIFFERENCE

DEFINITION 2.1. Let L^X be an L-fuzzy space, $A, B \in L^X$. Define the *quasi-difference* of A and B, denoted by $A \setminus\!\setminus B$, as

$$A \setminus\!\setminus B = \bigvee \{x_\lambda \in M(\downarrow A) \colon B(x) = 0 \text{ or } \lambda \not\geq B(x) > 0\};$$

particularly, $\forall x_a \in \mathrm{pt}(L^X)$,

$$A \setminus\!\setminus x_a = \bigvee \{y_\lambda \in M(L^X) : y_\lambda \leq A,\ y \neq x\} \vee \bigvee \{x_\lambda \in M(L^X) : a \not\leq \lambda \leq A(x)\}.$$

Remark 2.2. Obviously, $A \setminus\!\setminus B = A \setminus B$ if $L = \{0,1\}$. For $L = [0,1]$ (or a chain) and $B \in L^X$ such that $\mathrm{supp}(B) = X$, $A \setminus\!\setminus B = A \wedge B$; in particular, $A \setminus\!\setminus X = A$.

PROPOSITION 2.3. *Let L^X be an L-fuzzy space, $A, B, C \in L^X$, $\{A_t\colon t \in T\} \subset L^X$, $x_a \in \mathrm{pt}(L^X)$. Then the following conclusions hold*:

(i) $A \setminus\!\setminus B \leq A$.

(ii) $A \setminus\!\setminus \underline{0} = A$.

(iii) $1_L \notin M(L) \Rightarrow A \setminus\!\setminus \underline{1} = A$.

(iv) $A \leq B \Rightarrow A \setminus\!\setminus C \leq B \setminus\!\setminus C$.

(v) $(\bigvee_{t \in T} A_t) \setminus\!\setminus C = \bigvee_{t \in T} (A_t \setminus\!\setminus C)$.

(vi) $\forall x \in \mathrm{supp}(B)$, $B(x) \not\leq A(x) \Rightarrow A \setminus\!\setminus B = A$.

(vii) $A \wedge B = \underline{0} \Rightarrow A \setminus\!\setminus B = A$.

(viii) $x_a \not\leq A \Rightarrow A \setminus\!\setminus x_a = A$.

(ix) $\mathrm{supp}(B) = \mathrm{supp}(C)$, $B \leq C \Rightarrow A \setminus\!\setminus B \leq A \setminus\!\setminus C$.

Proof. (i) By the definition.

(ii) By $\bigvee M(\downarrow A) = A$.

(iii) Since $1_L \notin M(L)$, $\forall x_\lambda \in A$, $\lambda \not\geq 1_L = \underline{1}(x) > 0_L$ always hold, so $A \leq A \setminus\!\setminus \underline{1}$. By (i), $A \setminus\!\setminus \underline{1} = A$.

(iv) By the definition of quasi-difference.

(v) By (iv), $\bigvee_{t \in T} (A_t \setminus\!\setminus C) \leq (\bigvee_{t \in T} A_t) \setminus\!\setminus C$.

Simply denote the condition "$C(x) = 0$ or $\lambda \not\geq C(x) > 0$" by "$P(x_\lambda) > 0$." Suppose $x_\lambda \in M(\downarrow \bigvee_{t \in T} A_t)$ and $P(x_\lambda) > 0$.

If $C(x) = 0$,

$$\begin{aligned}\Big(\bigvee_{t\in T}(A_t \mathbin{\backslash\backslash} C)\Big)(x) &= \bigvee_{t\in T}\bigvee\left\{\mu \in M(\downarrow A_t(x)) : P(x_\mu) > 0\right\}\\ &= \bigvee_{t\in T}\bigvee M(\downarrow A_t(x))\\ &= \bigvee_{t\in T} A_t(x) \geq \lambda.\end{aligned}$$

If $\lambda \not\geq C(x) > 0$, $\forall t \in T$, denote $a_t = \lambda \wedge A_t(x)$. Since $\lambda \not\geq C(x)$, $\forall t \in T$, $\mu \in M(\downarrow a_t) \Rightarrow \mu \not\geq C(x)$. So we have

$$\begin{aligned}\Big(\bigvee_{t\in T}(A_t \mathbin{\backslash\backslash} C)\Big)(x) &= \bigvee_{t\in T}\bigvee\left\{\mu \in M(\downarrow A_t(x)) : P(x_\mu) > 0\right\}\\ &= \bigvee_{t\in T}\bigvee\left\{\mu \in M(\downarrow A_t(x)) : \mu \not\geq C(x) > 0\right\}\\ &\geq \bigvee_{t\in T}\bigvee\left\{\mu \in M(\downarrow a_t) : \mu \not\geq C(x) > 0\right\}\\ &= \bigvee_{t\in T}\bigvee M(\downarrow a_t)\\ &= \bigvee_{t\in T} a_t\\ &= \bigvee_{t\in T}(\lambda \wedge A_t(x))\\ &= \lambda \wedge \bigvee_{t\in T} A_t(x)\\ &= \lambda.\end{aligned}$$

So $(\bigvee_{t\in T}(A_t \mathbin{\backslash\backslash} C))(x) \geq \lambda$ is always true. Since the join of all these x_λ's is just equal to $(\bigvee_{t\in T} A_t) \mathbin{\backslash\backslash} C$, we get $\bigvee_{t\in T}(A_t \mathbin{\backslash\backslash} C) \geq (\bigvee_{t\in T} A_t) \mathbin{\backslash\backslash} C$.

(vi) Suppose $B(x) \not\leq A(x)$ for every $x \in \mathrm{supp}(B)$. $\forall x_\lambda \in A$. If $B(x) = 0$, we have $x_\lambda \in A \mathbin{\backslash\backslash} B$. If $B(x) > 0$, then we cannot have $\lambda \geq B(x)$; otherwise we should have $A(x) \geq \lambda \geq B(x) > 0$, i.e., $x \in \mathrm{supp}(B)$ and $B(x) \leq A(x)$. This contradicts with the supposition. So $\lambda \not\geq B(x) > 0$, $x_\lambda \in A \mathbin{\backslash\backslash} B$. By (i), $A \mathbin{\backslash\backslash} B = A$.

(vii), (viii) Both are straightforwardly deduced from (vi).

(ix) Prove straight. ∎

3. ACCUMULATION POINT

DEFINITION 3.1. Let (L^X, δ) be an L-fts, $A \in L^X$, $x_\lambda \in M(L^X)$.

x_λ is called an *adherent point* of A, if for every $U \in \mathscr{Q}(x, \lambda)$, U quasi-coincides with A, i.e., $U\hat{q}A$.

x_λ is called an *accumulation point* of A, if x_λ is an adherent point of $A \setminus\setminus x_\lambda$; that is to say, for every $U \in \mathscr{Q}(x, \lambda)$, U quasi-coincides with the quasi-difference of A and x_λ, i.e., $U\hat{q}(A \setminus\setminus x_\lambda)$.

Denote the set of all accumulation points of A in (L^X, δ) by $\mathrm{Acu}(A)$ or $\mathrm{Acu}_\delta(A)$ to emphasize the concerning topology δ.

Define the *derived set* of A as $\bigvee \mathrm{Acu}(A)$, denoted by A^d.

Remark 3.2. (1) Note that by the virtue of Proposition 2.3(i) and Proposition 1.9(iv), every accumulation point x_λ of A is an adherent point of A.

(2) Since every molecule in an L-fts is certainly an adherent point of every molecule which has the same supporting point and a greater height, therefore, naturally, when the notion of accumulation point is defined, the higher molecules with same support point should be removed. But whether or not the lower molecules at the same support point "approximating" the considered molecule reflects the structure of the value domain, so they have enough reason to be preserved. This is just the motivation of defining quasi-difference and then defining accumulation point. Certainly, under this definition, a molecule can be an accumulation point of itself if it is the join of the lower molecules, for example, if $L = [0, 1]$. Since $\{0, 1\}$ has no this property, the preceding definition returns the ordinary correspondent one when $L = \{0, 1\}$.

THEOREM 3.3. *Let (L^X, δ) be an L-fts, $A, B \in L^X$. Then*

(i) *$M(\downarrow A^-)$ is just the set of all the adherent points of A.*

(ii) *$A^- = \bigvee\{x_\lambda \in M(L^X) : x_\lambda$ is an adherent point of $A\}$.*

(iii) *$A \le B \Rightarrow$ every adherent point of A is an adherent point of B.*

Proof. (i) Suppose $x_\lambda \in M(\downarrow A^-)$. $\forall U \in \mathscr{Q}(x_\lambda)$. If $A \le U'$, then $x_\lambda \le A^- \le U'^- = U'$, $x_\lambda \le U'$, contradicts with $U \in \mathscr{Q}(x_\lambda)$. So $A \not\le U'$, x_λ is an adherent point of A.

Suppose x_λ is an adherent point of A. If $x_\lambda \not\le A^-$, then $A^{-\prime} \in \mathscr{Q}(x_\lambda)$. Since x_λ is an adherent point of A, $A \not\le A^-$. But this is a contradiction. So $x_\lambda \le A^-$.

(ii) By (i),

$$A^- = \bigvee M(\downarrow A^-)$$
$$= \bigvee\left\{x_\lambda \in M(L^X) : x_\lambda \in \mathrm{Acu}(A)\right\}.$$

(iii) By (i). ∎

THEOREM 3.4. *Let (L^X, δ) be an L-fts, $A, B \in L^X$, $\{A_t\colon t \in T\} \subset L^X$. Then*

(i) $A^- = A \vee A^d$.

(ii) $A \leq B \Rightarrow A^d \leq B^d$.

(iii) $(A \vee B)^d = A^d \vee B^d$.

(iv) $\bigvee_{t \in T} A_t^d \leq (\bigvee_{t \in T} A_t)^d$.

Proof. (i) By Remark 3.2 and Theorem 3.3(ii), $A \vee A^d \leq A^-$. $\forall x_\lambda \in M(\downarrow A^-)$.

If $x_\lambda \in A$, then we have had $x_\lambda \leq A \vee A^d$ already. If $x_\lambda \notin A$, by Theorem 3.3(i), x_λ is an adherent point of A, by Proposition 2.3(viii), $\forall U \in \mathscr{Q}(x_\lambda)$, $U \mathrel{\hat{\wedge}} (A \setminus\setminus x_\lambda) = U \mathrel{\hat{\wedge}} A \neq \emptyset$, x_λ is an accumulation point of A, we still have $x_\lambda \leq A^d \leq A \vee A^d$. So $A^- = A \vee A^d$.

(ii) Suppose x_λ is an accumulation point of A, then $\forall U \in \mathscr{Q}(x_\lambda)$, by Proposition 2.3(iv) and Proposition 1.9(iv),

$$U \mathrel{\hat{\wedge}} (B \setminus\setminus x_\lambda) \supset U \mathrel{\hat{\wedge}} (A \setminus\setminus x_\lambda) \neq \emptyset,$$

x_λ is also an accumulation point of B, $x_\lambda \in B^d$. So $A^d \leq B^d$.

(iii) By (ii), $A^d \vee B^d \leq (A \vee B)^d$. Suppose x_λ is an accumualtion point of $A \vee B$. If x_λ is not an accumulation point of either A or B, then $\exists U_0, U_1 \in \mathscr{Q}(x_\lambda)$ such that $U_0 \mathrel{\hat{\wedge}} (A \setminus\setminus x_\lambda) = U_1 \mathrel{\hat{\wedge}} (B \setminus\setminus x_\lambda) = \emptyset$. Take $U = U_0 \wedge U_1 \in \mathscr{Q}(x_\lambda)$, since x_λ is an accumulation point of $A \vee B$,

$$\begin{aligned}
&\big(U_0 \mathrel{\hat{\wedge}} (A \setminus\setminus x_\lambda)\big) \cup \big(U_1 \mathrel{\hat{\wedge}} (B \setminus\setminus x_\lambda)\big) \\
&\qquad \supset \big(U \mathrel{\hat{\wedge}} (A \setminus\setminus x_\lambda)\big) \cup \big(U \mathrel{\hat{\wedge}} (B \setminus\setminus x_\lambda)\big) && \text{(by Proposition 1.9(iv))} \\
&\quad = U \mathrel{\hat{\wedge}} \big((A \setminus\setminus x_\lambda) \vee (B \setminus\setminus x_\lambda)\big) && \text{(by Proposition 1.9(v))} \\
&\quad = U \mathrel{\hat{\wedge}} \big((A \vee B) \setminus\setminus x_\lambda\big) && \text{(by Proposition 2.3(v))} \\
&\quad \neq \emptyset.
\end{aligned}$$

This is a contradiction.

(iv) By (ii). ∎

Remark 3.5. Obviously, the concept of accumulation point and the relative derivation operation are fundamental in topology. In our paper [9], we considered their fuzzy form in I^X $(I = [0,1])$ as follows: For every $A \in I^X$ and every fuzzy point $e = x_\lambda$ $(x \in X,\ \lambda \in (0,1])$, if for every quasi-coincident neighborhood G of e, there exists a point $y \in X$, $y \neq x$ such that $A(y) + G(y) > 1$, then e is called an accumulation point of A. In other words, G and $A \setminus \{x\}$ are quasi-coincident. According to this definition we have given a theory of accumulation point in fuzzy topology. In particular, we define the derived set

$$A^d = \bigvee \{x_\lambda : x_\lambda \text{ is an accumulation point of } A\}$$

and yielded the fuzzy form of the famous C. T. Yang's Theorem about A^d.

But in the above definition of accumulation point x_λ, we did not take account of lower points x_μ ($\mu < \lambda$) which can also "approximate" x_λ. Thus, e.g., take A as a fuzzy point y_ρ, then any y_μ is not an accumulation point of A. Hence the derivation operation $(\ )^d$ does not preserve finite join, even the inclusion order of subsets.

Remark 3.6. In ordinary set theory, the point is the smallest element of a set. Contrasting with this situation, in fuzzy set theory, a fuzzy point may contain other smaller fuzzy points. Because of this difference, a join of some fuzzy points may contain some other fuzzy points different from those original ones. Therefore, it is natural that a molecule—an L-fuzzy point x_λ contained in the derived set A^d of an L-fuzzy set A—perhaps is not an accumulation point of A. For example, take X as a singleton $\{x\}$, $L = \{0, a', a, 1\}$ to be a chain, $0 < a' < a < 1$, and let $\underline{0}$ be mapped to $\underline{1}$ and a to a'; we obtain an order-reversing involution $'$ and then an F-lattice L. Define $\delta = \{\underline{0}, x_a, \underline{1}\}$, $A = x_a$, then $x_1 \in \mathrm{Acu}(A)$, and hence $A^d = x_1$. But $x_a \in A^d$ is not an accumulation point of A.

4. C. T. YANG'S THEOREM

In general topology, there is a theorem as follows: In a topological space, if the derived set of each singleton is closed, so then is the derived set of each subset. This theorem is called C. T. Yang's Theorem, not difficult to be proved in general topology. Some authors have tried to prove it in L-fuzzy topology (e.g., relative results in [13]), but since the derivation operations used were not very reasonable, only some partial results were obtained. Now we shall prove that the L-fuzzy form of C. T. Yang's Theorem still holds for L-fuzzy points in L-fuzzy topology:

THEOREM 4.1 (C. T. Yang's Theorem). *Let (L^X, δ) be an L-fts. If the derived set of each L-fuzzy point in (L^X, δ) is a closed subset, then the derived set of each L-fuzzy subset of (L^X, δ) is a closed subset.*

Proof. Suppose the derived set of each L-fuzzy point is closed. $A \in L^X$. To prove A^d is closed, we need only prove $A^{dd} \le A^d$, because in this case by Theorem 3.4(i) we shall have $(A^d)^- = A^d \vee A^{dd} = A^d$. So let $x_\lambda \in \mathrm{Acu}(A^d)$, we need only prove $x_\lambda \le A^d$.

Suppose it is not true, i.e., $x_\lambda \nleq A^d$. Denote $A(x) = a$, by Theorem 3.4(ii), $x_a^d \leq A^d$. By $x_\lambda \nleq A^d$, $x_\lambda \nleq x_a^d$. $\forall U \in \mathscr{Q}(x_\lambda)$, since x_a^d is a closed subset, by Theorem 1.8(ii) and $x_\lambda \nleq x_a^d$, $V = (U' \vee x_a^d)' = U \wedge (x_a^d)' \in \mathscr{Q}(x_\lambda)$. So we have

$$\begin{aligned}(A \setminus\setminus x_\lambda) \vee x_a = &\bigvee \{y_\mu \in M(\downarrow A) : y \neq x\} \\ &\vee \bigvee \{x_\mu : \mu \in M(\downarrow A(x)) \setminus \uparrow\lambda\} \vee x_{A(x)} \\ = &A. \end{aligned} \tag{4.1}$$

By $x_\lambda \nleq A^d$ and Prop. 2.3(viii), $A^d \setminus\setminus x_\lambda = A^d$, by $V \in \mathscr{Q}(x_\lambda)$ and $x_\lambda \in \mathrm{Acu}(A^d)$, $A^d \setminus\setminus x_\lambda \nleq V'$. So by Theorem 3.4(iii) and equality (4.1),

$$(A \setminus\setminus x_\lambda)^d \vee x_a^d = ((A \setminus\setminus x_\lambda) \vee x_a)^d = A^d = A^d \setminus\setminus x_\lambda \nleq V' = U' \vee x_a^d,$$
$$(A \setminus\setminus x_\lambda)^d \nleq U',$$

there exists $y_\mu \in \mathrm{Acu}(A \setminus\setminus x_\lambda)$ such that $y_\mu \nleq U'$, $U \in \mathscr{Q}(y_\mu)$. So by $y_\mu \in \mathrm{Acu}(A \setminus\setminus x_\lambda)$, $(A \setminus\setminus x_\lambda) \setminus\setminus y_\mu \nleq U'$, by Proposition 2.3(i), $A \setminus\setminus x_\lambda \nleq U'$. Since U was arbitrarily taken out from $\mathscr{Q}(x_\lambda)$, it means $x_\lambda \in \mathrm{Acu}(A)$, $x_\lambda \leq A^d$; this is a contradiction. ∎

Furthermore, since points in L-fuzzy topological spaces can be represented as joins of molecules, so the following question is interesting:

OPEN QUESTION 4.1. Is C. T. Yang's Theorem still true if the word "point" in its condition "the derived set of each L-fuzzy point is closed" is replaced by "molecule?"

Although we have not had a perfect answer to the preceding question yet, but we can prove that it is true for some kind of L's, this is just the Theorem 4.4 in the sequel.

For convenience, if the word "point" in C. T. Yang's Theorem is replaced by "molecule" but the theorem is still true, the C. T. Yang's Theorem holds for molecules."

LEMMA 4.2. *Let L be a complete lattice; then the join of every directed set of molecules in L is still a molecule.*

Proof. Suppose $D \subset M(L)$ is a directed set in L; denote $\zeta = \bigvee D$. If $\zeta \notin M(L)$, then $\exists a, b < \zeta$ such that $a \vee b = \zeta$. Since $\zeta = \bigvee D$, $\exists \alpha, \beta \in D$ such that $\alpha \nleq a$, $\beta \nleq b$. Since D is directed, $\exists \gamma \in D$ such that $\gamma \geq \alpha, \beta$. So $\gamma \nleq a, b$. But $\gamma \in M(L)$, so $\gamma \nleq a \vee b = \zeta = \bigvee D$, this is a contradiction. ∎

Since a chain in a complete lattice is naturally a directed set, so by Zorn's Lemma, the following corollary holds:

COROLLARY 4.3. *Let L be a complete lattice, $a \in L$, $\lambda \in M(L)$, $\lambda \leq a$. Then $M(\uparrow\lambda \cap \downarrow a)$ contains at least one maximal element.* ∎

THEOREM 4.4. *Let L be an F-lattice such that for every $a \in L$ and every molecule $\lambda \leq a$, $M(\uparrow\lambda \cap \downarrow a)$ contains at most finite number of maximal elements, (L^X, δ) be an L-fts. Then C. T. Yang's Theorem holds for molecules.*

Proof. By Theorem 4.1, we need only prove that under the given condition, the derived set of each L-fuzzy point is closed.

$\forall x_a \in \mathrm{pt}(L^X)$: For every $\lambda \in M(\downarrow a)$, denote $K_\lambda = M(\uparrow\lambda \cap \downarrow a)$, and suppose the set of all the maximal elements of K_λ is F_λ. First of all, we shall prove the following implication:

$$\forall \alpha \in K_\lambda \Rightarrow \exists \gamma_\alpha \in F_\lambda,\ \gamma_\alpha \geq \alpha. \tag{4.2}$$

By the virtue of Kuratowski's Lemma, for every $\alpha \in K_\lambda$, there exists a maximal chain C_α in K_λ such that $\alpha \in C_\alpha$. Let $\gamma_\alpha = \bigvee C_\alpha$; then by the virtue of Lemma 4.2, γ_α is still a molecule. So $\gamma_\alpha \in K_\alpha$. γ_α is clearly a maximal element of K_λ, so $\gamma_\alpha \in F_\lambda$ and $\gamma_\alpha \geq \alpha$. (4.2) is true.

Now note that $F_\lambda^x = \{x_\mu\colon \mu \in F_\lambda\}$ is just the set of all the maximal elements of $K_\lambda^x = \{x_\mu\colon \mu \in K_\lambda\}$, by the virtue of implication (4.2), we have $\bigvee K_\lambda^x = \bigvee F_\lambda^x$, since F_λ is a finite set, by Theorem 3.4(iii),

$$\left(\bigvee K_\lambda^x\right)^d = \left(\bigvee F_\lambda^x\right)^d = \left(\bigvee\left\{x_\mu\colon \mu \in F_\lambda\right\}\right)^d = \bigvee\left\{x_\mu^d\colon \mu \in F_\lambda\right\}. \tag{4.3}$$

Now $\forall y_\lambda \in M(L^X)$ such that $y_\lambda \not\leq x_a^d$, we shall prove $\exists V_\lambda \in \mathscr{Q}(y_\lambda)$, such that $V_\lambda' \geq x_a^d$. If $y_\lambda \not\leq x_a$, then take $V_\lambda = (x_a \vee x_a^d)' = x_a^{-\prime} \in \delta$, by $y_\lambda \not\leq x_a^d$ and $y_\lambda \in M(L^X)$, we have $y_\lambda \not\leq x_a \vee x_a^d = V_\lambda'$, $V_\lambda \in \mathscr{Q}(y_\lambda)$ and, certainly, $V_\lambda' \geq x_a^d$. If $y_\lambda \leq x_a$, then $y = x$, $y_\lambda = x_\lambda$. Since $x_\lambda = y_\lambda \not\leq x_a^d$, $\exists U_\lambda \in \mathscr{Q}(x_\lambda)$ such that $U_\lambda \not\!\bowtie (x_a \setminus\!\setminus x_\lambda) = \varnothing$, i.e., $x_\lambda \leq U_\lambda'$. Using the previous symbols, we have

$$\begin{aligned}
&(x_a \setminus\!\setminus x_\lambda) \vee \bigvee K_\lambda^x \\
&\quad = \bigvee\left\{x_\mu\colon \mu \in M(\downarrow a),\ \mu \not\geq \lambda\right\} \\
&\qquad \vee \bigvee\left\{x_\mu\colon \mu \in M(\downarrow a),\ \mu \geq \lambda\right\} \quad \text{(by Theorem 1.6(i))} \\
&\quad = x_a \\
x_a^d &= \left((x_a \setminus\!\setminus x_\lambda) \vee \bigvee K_\lambda^x\right)^d \\
&= (x_a \setminus\!\setminus x_\lambda)^d \vee \left(\bigvee K^x\right)^d \qquad \text{(by Theorem 3.4(iii))} \\
&\leq (U_\lambda')^d \vee \left(\bigvee K_\lambda^x\right)^d
\end{aligned}$$

$$\leq (U'_\lambda)^- \vee (\bigvee K_\lambda^x)^d \qquad \text{(by Theorem 3.4(i))}$$

$$= U'_\lambda \vee \bigvee \{x_\mu^d \colon \mu \in F_\lambda\} \qquad \text{(by equality (4.3))}.$$

Then take $V_\lambda = (U'_\lambda \vee \bigvee\{x_\mu^d : \mu \in F_\lambda\})'$, since F_λ is a finite set and the derived set of each molecule is closed, $V_\lambda \in \delta$. Since $x_\lambda \not\leq x_a^d$, by Theorem 3.4(ii), $\forall \mu \in F_\lambda$, $x_\lambda \not\leq x_\mu^d$, and hence $x_\lambda \not\leq \bigvee\{x_\mu^d : \mu \in F_\lambda\}$ by $x_\lambda \in M(L^X)$ and the finiteness of the set F_λ. Since $U_\lambda \in \mathscr{Q}(x_\lambda)$, we still obtain $V_\lambda \in \mathscr{Q}(x_\lambda) = \mathscr{Q}(y_\lambda)$ and $V'_\lambda \geq x_a^d$.

At last, $\forall y_\lambda \in M(L^X)$ such that $y_\lambda \not\leq x_a^d$, as proved above, we can take $V_\lambda^y \in \mathscr{Q}(y_\lambda)$ such that $(V_\lambda^y)' \geq x_a^d$. Denote $C = \bigwedge\{(V_\lambda^y)' \colon y_\lambda \in M(L^X), y_\lambda \not\leq x_a^d\}$, then C is a closed subset and clearly $C \geq x_a^d$. If $C \not\leq x_a^d$, then by $\bigvee M(\downarrow C) = C$, $\exists y_\lambda \in M(\downarrow C)$, $y_\lambda \not\leq x_a^d$. But $y_\lambda \not\leq (V_\lambda^y)'$, and hence $y_\lambda \not\leq C$, this is a contradiction. So $x_a^d = C$ is a closed subset.

COROLLARY 4.1. *If L is a product lattice of a set of chains, then C. T. Yang's Theorem holds for molecules.*

COROLLARY 4.2. *If $L = [0,1]$, then C. T. Yang's Theorem holds for molecules.*

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