

Twin Solutions to Singular Dirichlet Problems

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The existence of two nonnegative solutions to Dirichlet second order boundary value problems is established in this paper. Our nonlinearity may be singular at $y = 0$, $t = 0$, and/or $t = 1$. © 1999 Academic Press

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1. INTRODUCTION

In this paper we establish the existence of two nonnegative solutions to the singular second order Dirichlet problem

$$\begin{aligned} y''(t) + \phi(t)[g(y(t)) + h(y(t))] &= 0, & 0 < t < 1, \\ y(0) = y(1) &= 0; \end{aligned} \tag{1.1}$$

here our nonlinear term $g + h$ may be singular at $y = 0$. All the papers in the literature, except one [2] to our knowledge, discuss the existence of one solution to (1.1) in the case when $g + h$ is singular at $y = 0$. In [2] we used Krasnoselski's fixed point theorem in a cone together with a Leray–Schauder alternative to establish the existence of two solutions to both the singular (n, p) and $(p, n - p)$ focal boundary value problems. It is possible using quite a different cone (i.e., the cone in this paper) to extend the



ideas in [2] (using the ideas also in this paper) to guarantee the existence of two (or more) solutions to (1.1). However, some strong integrability conditions (similar to those in [2]) have to be assumed on ϕ and $g + h$. In this paper by using a more general fixed point theorem (which was established in [3] using degree theory and in [7] using the essential map approach) than that of Krasnoselski, we are able to establish the existence of two solutions to (1.1) under very general assumptions (in particular no integrability assumptions need to be assumed on $g + h$). In [1] for example we established the existence of one solution to (1.1) using a Leray–Schauder alternative and in this paper we note that by adding one extra assumption we are able to guarantee the existence of two solutions. Also we would like to remark that the theory presented in this paper for the Dirichlet problem could be extended (in an obvious way) so that the results in [2] could be improved for (1, 1) focal boundary value problems.

For the remainder of this section we present some results from the literature which will be needed in Section 2. First we state the fixed point result we will use in Section 2 (see [3, 7]).

THEOREM 1.1. *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E . Also, r, R are constants with $0 < r < R$. Suppose $A: \overline{\Omega_R} \cap K \rightarrow K$ (here $\Omega_R = \{x \in E : \|x\| < R\}$) is a continuous, compact map and assume the conditions*

$$x \neq \lambda A(x) \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial_E \Omega_r \cap K \quad (1.2)$$

and

$$\begin{aligned} &\text{there exists a } v \in K \setminus \{0\} \text{ with } x \neq A(x) + \delta v \\ &\text{for any } \delta > 0 \text{ and } x \in \partial_E \Omega_R \cap K. \end{aligned} \quad (1.3)$$

hold. Then A has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

Remark 1.1. In Theorem 1.1 if (1.2) and (1.3) are replaced by

$$x \neq \lambda A(x) \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial_E \Omega_R \cap K \quad (1.2)^*$$

and

$$\begin{aligned} &\text{there exists a } v \in K \setminus \{0\} \text{ with } x \neq A(x) + \delta v \\ &\text{for any } \delta > 0 \text{ and } x \in \partial_E \Omega_r \cap K \end{aligned} \quad (1.3)^*$$

then A also has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

THEOREM 1.2. *Let $E = (E, \|\cdot\|)$ be a Banach space, let $K \subset E$ be a cone, and let $\|\cdot\|$ be increasing (strictly) with respect to K . Also, r, R are constants with $0 < r < R$. Suppose $A: \overline{\Omega_R} \cap K \rightarrow K$ (here $\Omega_R = \{x \in E : \|x\| < R\}$) is*

a continuous, compact map and assume the conditions

$$x \neq \lambda A(x) \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial_E \Omega_r \cap K \quad (1.4)$$

and

$$\|Ax\| \geq \|x\| \quad \text{for } x \in \partial_R \Omega_R \cap K. \quad (1.5)$$

hold. Then A has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

Proof. Notice (1.5) guarantees that (1.3) is true. This is a standard argument and for completeness we supply it here. Suppose there exists $v \in K \setminus \{0\}$ with $x = A(x) + \delta v$ for some $\delta > 0$ and $x \in \partial_E \Omega_R \cap K$. Then since $\|\cdot\|$ is increasing with respect to K we have, since $\delta v \in K$,

$$\|x\| = \|Ax + \delta v\| > \|Ax\| \geq \|x\|,$$

a contradiction. The result now follows from Theorem 1.1. ■

Remark 1.2. In Theorem 1.2, if (1.4) and (1.5) are replaced by

$$x \neq \lambda A(x) \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial_R \Omega_R \cap K \quad (1.4)^*$$

and

$$\|Ax\| \geq \|x\| \quad \text{for } x \in \partial_E \Omega_r \cap K, \quad (1.5)^*$$

then A has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

In this paper $E = (C[0, 1], |\cdot|_0)$ (here $|u|_0 = \sup_{t \in [0, 1]} |u(t)|$, $u \in C[0, 1]$) will be our Banach space and

$$K = \{y \in C[0, 1] : y(t) \geq 0 \text{ for } t \in [0, 1] \text{ and } y(t) \text{ is concave on } [0, 1]\}. \quad (1.6)$$

Let $\theta: [0, 1] \times [0, 1] \rightarrow [0, \infty)$ be defined by

$$\theta(t, s) = \begin{cases} \frac{t}{s} & \text{if } 0 \leq t \leq s, \\ \frac{1-t}{1-s} & \text{if } s \leq t \leq 1. \end{cases}$$

The following result is easy to prove and is well known.

THEOREM 1.3. *Let $y \in K$ (as in (1.6)). Then there exists $t_0 \in [0, 1]$ with $y(t_0) = |y|_0$ and*

$$y(t) \geq \theta(t, t_0)|y|_0 \geq t(1-t)|y|_0 \quad \text{for } t \in [0, 1].$$

Proof. The existence of t_0 is immediate. Now if $0 \leq t \leq t_0$ then since $y(t)$ is concave on $[0, 1]$ we have

$$y(t) = y\left(\left(1 - \frac{t}{t_0}\right)0 + \frac{t}{t_0}t_0\right) \geq \left(1 - \frac{t}{t_0}\right)y(0) + \frac{t}{t_0}y(t_0);$$

that is,

$$y(t) \geq \frac{t}{t_0}y(t_0) = \theta(t, t_0)|y|_0 \geq t(1 - t)|y|_0.$$

A similar argument establishes the result if $t_0 \leq t \leq 1$. ■

Finally in this section we state the existence result established in [1] for the problem

$$\begin{aligned} y''(t) + \phi(t)f(t, y(t)) &= 0, & 0 < t < 1, \\ y(0) = y(1) &= 0. \end{aligned} \quad (1.7)$$

THEOREM 1.4. *Suppose the conditions*

$$\phi \in C(0, 1) \text{ with } \phi > 0 \text{ on } (0, 1) \quad \text{and} \quad \int_0^1 t(1 - t)\phi(t) dt < \infty, \quad (1.8)$$

$$\lim_{t \rightarrow 0^+} t^2(1 - t)\phi(t) = 0 \quad \text{if } \int_0^1 (1 - t)\phi(t) dt = \infty, \quad (1.9)$$

$$\lim_{t \rightarrow 1^-} t(1 - t)^2\phi(t) = 0 \quad \text{if } \int_0^1 t\phi(t) dt = \infty,$$

$$f: [0, 1] \times (0, \infty) \rightarrow (0, \infty) \text{ is continuous}, \quad (1.10)$$

$$\left\{ \begin{array}{l} f(t, y) \leq g(y) + h(y) \text{ on } [0, 1] \times (0, \infty) \text{ with } g > 0 \\ \text{continuous and nonincreasing on } (0, \infty), h \geq 0 \\ \text{continuous on } [0, \infty) \text{ and } h/g \text{ nondecreasing on } (0, \infty) \end{array} \right\}, \quad (1.11)$$

$$\left\{ \begin{array}{l} \text{for each constant } H > 0 \text{ there exists } \psi_H \text{ continuous on } [0, 1] \\ \text{and positive on } (0, 1) \text{ such that } f(t, y) \geq \psi_H(t) \\ \text{on } [0, 1] \times (0, H) \end{array} \right\}, \quad (1.12)$$

and

$$\text{there exists a constant } r > 0 \text{ with } \frac{1}{\{1 + h(r)/g(r)\}} \int_0^r \frac{du}{g(y)} > b_0 \quad (1.13)$$

are satisfied; here

$$b_0 = \max \left\{ 2 \int_0^{1/2} t(1-t)\phi(t) dt, 2 \int_{1/2}^1 t(1-t)\phi(t) dt \right\}. \quad (1.14)$$

Then (1.7) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $|y|_0 < r$.

Remark 1.3. In [1] we showed $|y|_0 \leq r$. In fact $|y|_0 \neq r$ from the argument in Theorem 2.2.

2. SINGULAR PROBLEMS

In this section we examine the singular Dirichlet problem

$$\begin{aligned} y''(t) + \phi(t)[g(y(t)) + h(y(t))] &= 0, & 0 < t < 1, \\ y(0) = y(1) &= 0. \end{aligned} \quad (2.1)$$

THEOREM 2.1. *Suppose the conditions*

$$\phi \in C(0, 1) \text{ with } \phi > 0 \text{ on } (0, 1) \quad \text{and} \quad \int_0^1 t(1-t)\phi(t) dt < \infty \quad (2.2)$$

$$\lim_{t \rightarrow 0^+} t^2(1-t)\phi(t) = 0 \quad \text{if } \int_0^1 (1-t)\phi(t) dt = \infty, \quad (2.3)$$

$$\lim_{t \rightarrow 1^-} t(1-t)^2\phi(t) = 0 \quad \text{if } \int_0^1 t\phi(t) dt = \infty,$$

$$g > 0 \text{ is continuous and nonincreasing on } (0, \infty), \quad (2.4)$$

$$h \geq 0 \text{ is continuous on } [0, \infty) \text{ with } h/g \text{ nondecreasing on } (0, \infty), \quad (2.5)$$

and

$$\text{there exists a constant } r > 0 \text{ with } \frac{1}{\{1 + h(r)/g(r)\}} \int_0^r \frac{du}{g(u)} > b_0 \quad (2.6)$$

are satisfied; here

$$b_0 = \max \left\{ 2 \int_0^{1/2} t(1-t)\phi(t) dt, 2 \int_{1/2}^1 t(1-t)\phi(t) dt \right\}.$$

Then (2.1) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $|y|_0 < r$.

Proof. The result follows from Theorem 1.4 with $f(t, u) = g(u) + h(u)$. Notice (1.12) is clearly satisfied with $\psi_H(t) = g(H)$. ■

THEOREM 2.2. Assume (2.2)–(2.6) hold. Choose $a \in (0, 1/2)$ and fix it, and suppose there exists $R > r$ with

$$\frac{Rg(a(1-a)R)}{g(R)g(a(1-a)R) + g(R)h(a(1-a)R)} \leq \int_a^{1-a} G(\sigma, s)\phi(s) ds; \quad (2.7)$$

here $0 \leq \sigma \leq 1$ is such that

$$\int_a^{1-a} G(\sigma, s)\phi(s) ds = \sup_{t \in [0, 1]} \int_a^{1-a} G(t, s)\phi(s) ds \quad (2.8)$$

and

$$G(t, s) = \begin{cases} (1-t)s, & 0 \leq s \leq t, \\ (1-s)t, & t \leq s \leq 1. \end{cases}$$

Then (2.1) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $r < |y|_0 \leq R$.

Proof. To show the existence of the solution described in the statement of Theorem 2.2, we will apply Theorem 1.2. First however choose $\epsilon > 0$ and $\epsilon < r$ with

$$\frac{1}{\{1 + h(r)/g(r)\}} \int_{\epsilon}^r \frac{du}{g(u)} > b_0. \quad (2.9)$$

Let $m_0 \in \{1, 2, \dots\}$ be chosen so that $1/m_0 < \epsilon$ and $1/m_0 < a(1-a)R$ and let $N_0 = \{m_0, m_0 + 1, \dots\}$. We first show that

$$\begin{aligned} y''(t) + \phi(t)[g(y(t)) + h(y(t))] &= 0, & 0 < t < 1, \\ y(0) = y(1) &= 1/m \end{aligned} \quad (2.10)^m$$

has a solution y_m for each $m \in N_0$ with $y_m > 1/m$ on $(0, 1)$ and $r \leq |y_m|_0 \leq R$. To show (2.10)^m has such a solution for each $m \in N_0$, we will look at

$$\begin{aligned} y''(t) + \phi(t)[g^*(y(t)) + h(y(t))] &= 0, & 0 < t < 1, \\ y(0) = y(1) &= 1/m \end{aligned} \quad (2.11)^m$$

with

$$g^*(u) = \begin{cases} g(u), & u \geq 1/m, \\ g(1/m), & 0 \leq u \leq 1/m. \end{cases}$$

Remark 2.1. Notice $g^*(u) \leq g(u)$ for $u > 0$.

Fix $m \in N_0$. Let $E = (C[0, 1], |\cdot|_0)$ and

$$K = \{u \in C[0, 1] : u(t) \geq 0 \text{ for } t \in [0, 1] \text{ and } u(t) \text{ is concave on } [0, 1]\}. \quad (2.12)$$

Clearly K is a cone of E . Let $A: K \rightarrow C[0, 1]$ be defined by

$$Ay(t) = \frac{1}{m} + \int_0^1 G(t, s) \phi(s) [g^*(y(s)) + h(y(s))] ds. \quad (2.13)$$

A standard argument [8] implies $A: K \rightarrow C[0, 1]$ is continuous and completely continuous. Next we show $A: K \rightarrow K$. If $u \in K$ then clearly $Au(t) \geq 0$ for $t \in [0, 1]$. Also notice that

$$(Au)''(t) \leq 0 \quad \text{on } (0, 1),$$

$$Au(0) = Au(1) = 1/m$$

so $Au(t)$ is concave on $[0, 1]$. Consequently $Au \in K$ so $A: K \rightarrow K$. Let

$$\Omega_1 = \{u \in C[0, 1] : |u_0| < r\} \quad \text{and} \quad \Omega_2 = \{u \in C[0, 1] : |u|_0 < R\}.$$

We first show

$$y \neq \lambda Ay \quad \text{for } \lambda \in [0, 1] \text{ and } y \in K \cap \partial\Omega_1. \quad (2.14)$$

Suppose this is false, i.e., suppose there exist $y \in K \cap \partial\Omega_1$ and $\lambda \in [0, 1]$ with $y = \lambda Ay$. We can assume $\lambda \neq 0$. Now since $y = \lambda Ay$ we have

$$\begin{aligned} y''(t) + \lambda \phi(t) [g^*(y(t)) + h(y(t))] &= 0, \quad 0 < t < 1, \\ y(0) = y(1) &= 1/m. \end{aligned} \quad (2.15)$$

Since $y'' \leq 0$ on $(0, 1)$ and $y > 1/m$ on $[0, 1]$ there exists $t_0 \in (0, 1)$ with $y' \geq 0$ on $(0, t_0)$, $y' \leq 0$ on $(t_0, 1)$, and $y(t_0) = |y|_0 = r$ (note $y \in K \cap \partial\Omega_1$). Also notice

$$g^*(y(t)) + h(y(t)) \leq g(y(t)) + h(y(t)) \quad \text{for } t \in (0, 1)$$

since g is nonincreasing on $(0, \infty)$. For $x \in (0, 1)$ we have

$$-y''(x) \leq g(y(x)) \left\{ 1 + \frac{h(y(x))}{g(y(x))} \right\} \phi(x). \quad (2.16)$$

Integrate from t ($t \leq t_0$) to t_0 to obtain

$$y'(t) \leq g(y(t)) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_t^{t_0} \phi(x) dx$$

and then integrate from 0 to t_0 to obtain

$$\int_{1/m}^r \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^{t_0} x \phi(x) dx.$$

Consequently

$$\int_{\epsilon}^r \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^{t_0} x \phi(x) dx$$

and so

$$\int_{\epsilon}^r \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(r)}{g(r)} \right\} \frac{1}{1-t_0} \int_0^{t_0} x(1-x) \phi(x) dx. \quad (2.17)$$

Similarly, if we integrate (2.16) from t_0 to ($t \geq t_0$) and then from t_0 to 1 we obtain

$$\int_{\epsilon}^r \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(r)}{g(r)} \right\} \frac{1}{t_0} \int_{t_0}^1 x(1-x) \phi(x) dx. \quad (2.18)$$

Now (2.17) and (2.18) imply

$$\int_{\epsilon}^r \frac{du}{g(u)} \leq b_0 \left\{ 1 + \frac{h(r)}{g(r)} \right\}, \quad (2.19)$$

where b_0 is as defined in (1.14). This contradicts (2.10) and consequently (2.14) is true.

Next we show

$$|Ay|_0 \geq |y|_0 \quad \text{for } y \in K \cap \partial\Omega_2. \quad (2.20)$$

To see this let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$. Also since $y(t)$ is concave on $[0, 1]$ (since $y \in K$) we have from Theorem 1.3 that $y(t) \geq t(1-t)|y|_0 \geq t(1-t)R$ for $t \in [0, 1]$. Also for $s \in [a, 1-a]$ we have

$$g^*(y(s)) + h(y(s)) = g(y(s)) + h(y(s))$$

since $y(s) \geq a(1-a)R > 1/m_0$ for $s \in [a, 1-a]$. Note in particular that

$$y(s) \in [a(1-a)R, R] \quad \text{for } s \in [a, 1-a]. \quad (2.21)$$

With σ as defined in (2.8) we have, using (2.21) and (2.7),

$$\begin{aligned} Ay(\sigma) &= \frac{1}{m} + \int_0^1 G(\sigma, s) \phi(s) [g^*(y(s)) + h(y(s))] ds \\ &\geq \int_a^{1-a} G(\sigma, s) \phi(s) [g^*(y(s)) + h(y(s))] ds \\ &= \int_a^{1-a} G(\sigma, s) \phi(s) g(y(s)) \left\{ 1 + \frac{h(y(s))}{g(y(s))} \right\} ds \\ &\geq g(R) \left\{ 1 + \frac{h(a(1-a)R)}{g(a(1-a)R)} \right\} \int_a^{1-a} G(\sigma, s) \phi(s) ds \\ &\geq R = |y|_0, \end{aligned}$$

and so $|Ay|_0 \geq |y|_0$. Hence (2.21) is true.

Now Theorem 1.2 implies \mathcal{A} has a fixed point $y_m \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, i.e., $r \leq |y_m|_0 \leq R$. In fact $|y_m|_0 > r$ (note if $|y_m|_0 = r$ then following essentially the same argument from (2.16)–(2.19) will yield a contradiction). Consequently (2.11)^m (and also (2.10)^m) has a solution $y_m \in C[0, 1] \cap C^2(0, 1)$, $y_m \in K$, with

$$\frac{1}{m} \leq y_m(t) \quad \text{for } t \in [0, 1], r < |y_m|_0 \leq R \quad (2.22)$$

and (from Theorem 1.3, note $y_m \in K$)

$$y_m(t) \geq t(1-t)r \quad \text{for } t \in [0, 1]. \quad (2.23)$$

Next we will show

$$\{y_m\}_{m \in N_0} \text{ is a bounded, equicontinuous family on } [0, 1]. \quad (2.24)$$

Returning to (2.16) (with y replaced by y_m) we have

$$-y_m''(x) \leq g(y_m(x)) \left\{ 1 + \frac{h(R)}{g(R)} \right\} \phi(x) \quad \text{for } x \in (0, 1). \quad (2.25)$$

Now since $y_m'' \leq 0$ on $(0, 1)$ and $y_m \geq 1/m$ on $[0, 1]$ there exists $t_m \in (0, 1)$ with $y_m' \geq 0$ on $(0, t_m)$ and $y_m' \leq 0$ on $(t_m, 1)$. Integrate (2.25) from t ($t < t_m$) to t_m to obtain

$$\frac{y_m'(t)}{g(y_m(t))} \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_t^{t_m} \phi(x) dx. \quad (2.26)$$

On the other hand, integrate (2.25) from t_m to t ($t > t_m$) to obtain

$$\frac{-y'_m(t)}{g(y_m(t))} \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_{t_m}^t \phi(x) dx. \quad (2.27)$$

We now claim that there exist a_0 and a_1 with $a_0 > 0$, $a_1 < 1$, $a_0 < a_1$ with

$$a_0 < \inf\{t_m : m \in N_0\} \leq \sup\{t_m : m \in N_0\} < a_1. \quad (2.28)$$

Remark 2.2. Here t_m (as before) is the unique point in $(0, 1)$ with $y'_m(t_m) = 0$.

We now show $\inf\{t_m : m \in N_0\} > 0$. If this is not true then there is a subsequence S of N_0 with $t_m \rightarrow 0$ as $m \rightarrow \infty$ in S . Now integrate (2.26) from 0 to t_m to obtain

$$\int_0^{y_m(t_m)} \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_0^{t_m} x \phi(x) dx + \int_0^{1/m} \frac{du}{g(u)} \quad \text{for } m \in S. \quad (2.29)$$

Since $t_m \rightarrow 0$ as $m \rightarrow \infty$ in S , we have from (2.29), that $y_m(t_m) \rightarrow 0$ as $m \rightarrow \infty$ in S . However, since the maximum of y_m on $[0, 1]$ occurs at t_m we have $y_m \rightarrow 0$ in $C[0, 1]$ as $m \rightarrow \infty$ in S . This contradicts (2.23). Consequently $\inf\{t_m : m \in N_0\} > 0$. A similar argument shows $\sup\{t_m : m \in N_0\} < 1$. Let a_0 and a_1 be chosen as in (2.28). Now (2.26), (2.27), and (2.28) imply

$$\frac{|y'_m(t)|}{g(y_m(t))} \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} v(t) \quad \text{for } t \in (0, 1), \quad (2.30)$$

where

$$v(t) = \int_{\min\{t, a_0\}}^{\max\{t, a_1\}} \phi(x) dx.$$

It is easy to see [7, p. 300] that $v \in L^1[0, 1]$. Let $I: [0, \infty) \rightarrow [0, \infty)$ be defined by

$$I(z) = \int_0^z \frac{du}{g(u)}.$$

Note I is an increasing map from $[0, \infty)$ onto $[0, \infty)$ (notice $I(\infty) = \infty$ since $g > 0$ is nonincreasing on $(0, \infty)$) with I continuous on $[0, A]$ for any $A > 0$. Notice

$$\{I(y_m)\}_{m \in N_0} \text{ is a bounded, equicontinuous family on } [0, 1]. \quad (2.31)$$

The equicontinuity follows from (here $t, s \in [0, 1]$)

$$|I(y_m(t)) - I(y_m(s))| = \left| \int_s^t \frac{y'_m(x)}{g(y_m(x))} dx \right| \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} \left| \int_s^t v(x) dx \right|.$$

This inequality, the uniform continuity of I^{-1} on $[0, I(R)]$, and

$$|y_m(t) - y_m(s)| = |I^{-1}(I(y_m(t))) - I^{-1}(I(y_m(s))))|$$

now establish (2.24).

The Arzela–Ascoli theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C[0, 1]$ with y_m converging uniformly on $[0, 1]$ to y as $m \rightarrow \infty$ through N . Also $y(0) = y(1) = 0$, $r \leq |y|_0 \leq R$, and $y(t) \geq t(1-t)r$ for $t \in [0, 1]$. In particular $y > 0$ on $(0, 1)$. Fix $t \in (0, 1)$ (without loss of generality assume $t \neq 1/2$). Now y_m , $m \in N$, satisfies the integral equation

$$\begin{aligned} y_m(x) &= y_m\left(\frac{1}{2}\right) + y'_m\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) \\ &\quad + \int_{1/2}^x (s-x)\phi(s)[g(y_m(s)) + h(y_m(s))] ds \end{aligned}$$

for $x \in (0, 1)$. Notice (take $x = 2/3$) that $\{y'_m(1/2)\}$, $m \in N$, is a bounded sequence since $rs(1-s) \leq y_m(s) \leq R$ for $s \in [0, 1]$. Thus $\{y'_m(1/2)\}_{m \in N}$ has a convergent subsequence; for convenience let $\{y'_m(1/2)\}_{m \in N}$ denote this subsequence also, and let $r_0 \in \mathbb{R}$ be its limit. Now for the above fixed t ,

$$\begin{aligned} y_m(t) &= y_m\left(\frac{1}{2}\right) + y'_m\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) \\ &\quad + \int_{1/2}^t (s-t)\phi(s)[g(y_m(s)) + h(y_m(s))] ds, \end{aligned}$$

and let $m \rightarrow \infty$ through N (we note here that $g + h$ is uniformly continuous on compact subsets of $[\min(1/2, t), \max(1/2, t)] \times (0, R]$) to obtain

$$y(t) = y\left(\frac{1}{2}\right) + r_0\left(t - \frac{1}{2}\right) + \int_{1/2}^t (s-t)\phi(s)[g(y(s)) + h(y(s))] ds.$$

We can do this argument for each $t \in (0, 1)$ and so $y''(t) + \phi(t)[g(y(t)) + h(y(t))] = 0$ for $0 < t < 1$. Finally it is easy to see that $|y|_0 > r$ (note if $|y|_0 = r$ then following essentially the argument from (2.16)–(2.19) will yield a contradiction). ■

Remark 2.3. If in (2.7) we have $R < r$ then (2.1) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $R \leq |y|_0 < r$. The argument is similar to that in Theorem 2.2 except here we use Remark 1.2.

Remark 2.4. It is also possible to use the ideas in Theorem 2.2 to discuss other boundary conditions, for example, $y'(0) = y(1) = 0$.

Remark 2.5. If we use Krasnoselski's fixed point theorem in a cone we need more than (2.2)–(2.6) and (2.7) to establish the existence of a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $r < |y|_0 \leq R$. This is because (2.14) is less restrictive than $|Ay|_0 \leq |y|_0$ for $y \in K \cap \partial\Omega_1$ (see also [2]).

THEOREM 2.3. Assume (2.2)–(2.6) and (2.7) hold. Then (2.1) has two solutions $y_1, y_2 \in C[0, 1] \cap C^2(0, 1)$ with $y_1 > 0$, $y_2 > 0$ on $(0, 1)$, and $|y_1|_0 < r < |y_2|_0 \leq R$.

Proof. The existence of y_1 follows from Theorem 2.1 and the existence of y_2 follows from Theorem 2.2. ■

EXAMPLE 2.1. The singular boundary value problem

$$\begin{aligned} y'' + \frac{1}{\alpha+1}(y^{-\alpha} + y^\beta + 1) &= 0 \quad \text{on } (0, 1), \\ y(0) = y(1) &= 0, \quad \alpha > 0, \beta > 1 \end{aligned} \quad (2.32)$$

has two solutions $y_1, y_2 \in C[0, 1] \cap C^2(0, 1)$ with $y_1 > 0$, $y_2 > 0$ on $(0, 1)$, and $|y_1|_0 < 1 < |y_2|_0$.

To see this we will apply Theorem 2.3 with $\phi = \frac{1}{\alpha+1}$, $g(u) = u^{-\alpha}$, and $h(u) = u^\beta + 1$. Clearly (2.2)–(2.5) hold. Also note

$$b_0 = \max \left\{ \frac{2}{\alpha+1} \int_0^{1/2} t(1-t) dt, \frac{2}{\alpha+1} \int_{1/2}^1 t(1-t) dt \right\} = \frac{1}{6(\alpha+1)}.$$

Consequently (2.6) holds (with $r = 1$) since

$$\begin{aligned} \frac{1}{\{1 + h(r)/g(r)\}} \int_0^r \frac{du}{g(u)} &= \frac{1}{(1 + r^{\alpha+\beta} + r^\alpha)} \left(\frac{r^{\alpha+1}}{\alpha+1} \right) \\ &= \frac{1}{3(\alpha+1)} > b_0 = \frac{1}{6(\alpha+1)}. \end{aligned}$$

Finally note (since $\beta > 1$, take $a = 1/4$) that

$$\begin{aligned} &\lim_{R \rightarrow \infty} \frac{Rg\left(\frac{3R}{16}\right)}{g(R)g\left(\frac{3R}{16}\right) + g(R)h\left(\frac{3R}{16}\right)} \\ &= \lim_{R \rightarrow \infty} \left(\frac{R^{\alpha+1} \left(\frac{3}{16}\right)^{-\alpha}}{\left(\frac{3}{16}\right)^{-\alpha} + \left(\frac{3}{16}\right)^\beta R^{\alpha+\beta} + R^\alpha} \right) = 0 \end{aligned}$$

so there exists $R > 1$ with (2.7) holding. The result now follows from Theorem 2.3.

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